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Universal Flows in the Simplest Theories of Fluids.

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dedicated to Jean Leray

The constitutive relation of a class of fluids involves certain disposable or « empirical » constants or functions. A velocity field that satisfies the equations of motion for all fluids of the class, regardless of the values assigned to the disposable constants or functions which distinguish one such fluid from another, is said to be a *universal flow* for that class.

Universal flows are centrally important because they suggest experiments in which the velocity field is known, at least approximately, from the outset. For these flows the analysis of experimental data is not complicated by the need to determine at the same time an unknown velocity field. Many of the examples adduced in textbooks of fluid dynamics are universal flows. Usually the equations of motion are recast with greater or lesser labor in such a way as to show that they may be integrated for the pressure. As we shall see, *it is a trivial matter to effect this integration once and for all*, provided the flow be universal. It is equally trivial to determine whether or not a given vector field be a universal flow.

In this paper we shall characterize the universal flows for the three simplest kinds of *homogeneous incompressible fluids*. In addition, we develop a fundamental identity which is particularly useful in the characterization of universal flows for fluids of higher grade, and we give an application at grade 3. Namely, we use it to determine explicitly the complete set of universal shearings for a fluid of that grade.

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A flow which is universal for a certain kind of fluid must be universal also for every subclass of that kind. For example, a universal flow of viscous fluids is necessarily a universal flow of ideal fluids, though of course most universal flows of ideal fluids will not be flows of any viscous fluid subject to lamellar body force, let alone universal flows.

We apply this simple observation also. Namely, the shearings we prove to exhaust the class of universal shearings for fluids of grade 3 turn out to be universal flows for all simple fluids. Thus our analysis delivers effortlessly *all universal shearings* for the entire class of homogeneous incompressible simple fluids.

Henceforth the density ϱ will be an assignable constant, and the velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$, $\boldsymbol{x}$ being a place and t a time, will always be assumed solenoidal, so as to render isochoric any motion we consider. The body force will be assumed lamellar; we shall denote its potential, which may be time-dependent, by B, and we shall assume that B is single-valued. As was observed in effect by Euler, we might as well set $B \equiv 0$ as far as any general theorem is concerned, but since it is no trouble to retain an arbitrary B in the calculations, we shall do so.

1. – Case I: The Euler theory.

The equation of motion is

$$\boldsymbol{a} = -\operatorname{grad}(p/\varrho + B)\,, \tag{1}$$

$\boldsymbol{a}$ being the acceleration. Since ϱ is the only disposable constant in this constitutive class, the universal flows are those which satisfy (1) for every value of ϱ. As was remarked in effect by Euler, $\boldsymbol{v}$ satisfies (1) if and only if $\boldsymbol{a}$ has a single-valued potential, say

$$\boldsymbol{a} = -\operatorname{grad} A\,. \tag{2}$$

Equivalently, in view of a theorem of Kelvin's, *an isochoric velocity field is a flow of a homogeneous incompressible Euler fluid if and only if it is circulation-preserving; every flow of such a fluid is a universal flow.*

Of course, the usual test for the validity of (2) is its local equivalent

$$\operatorname{curl} \boldsymbol{a} = \boldsymbol{0}\,. \tag{3}$$

For any given velocity field, it is trivial to find out whether (3) holds. If (3) does not hold, the velocity field in question is not a flow of a homogeneous incompressible Euler fluid unless, of course, a suitable field of body force which is not lamellar be brought to bear. In particular, that velocity field cannot be imparted to any Euler fluid by any system of normal pressure impulses on any bounding surface. If (3) does hold, it is an elementary matter to determine the acceleration-potential A. If A is single-valued, then comparison of (1) and (2) determines the pressure uniquely to within an arbitrary function h of time alone:

$$p/\varrho = A - B + h\,. \tag{4}$$

Pressures belonging to this family, and such pressures alone, suffice to effect the flow. Thus, when the body force is assigned, the pressure is determined to within an arbitrary function of time alone by the acceleration-potential.

Circulation-preserving flows have many special properties. In particular, the vorticity theorems of Helmholtz and Kelvin as well as numerous theorems of Bernoullian type hold for them (1). By way of example, we recall here one of the latter. In a general motion the vortex-lines and the stream-lines do not sweep out a family of surfaces. If they do so, those surfaces are called *Lamb surfaces*. Lamb's Bernoullian theorem asserts that *in any steady circulation-preserving motion, Lamb surfaces* $\mathcal{S}$ *exist everywhere, and*

$$A + \tfrac{1}{2}|\boldsymbol{v}|^2 = f(\mathcal{S})\,. \tag{5}$$

When we apply this result to homogeneous incompressible Euler fluids, we conclude from (4) that

$$p/\varrho + \tfrac{1}{2}|\boldsymbol{v}|^2 + B = f(\mathcal{S})\,; \tag{6}$$

that is, the classical « Bernoulli Theorem » holds on each Lamb surface. In circulation-preserving flows having unsteady vorticity fields, Lamb surfaces generally do not exist.

As we have said already in somewhat different words, *the universal flows of any class of simple fluids must be universal flows of Euler fluids.* Hence (2) is a *necessary condition* for universal flows: *all universal flows are*

(1) C. TRUESDELL, *The Kinematics of Vorticity*, Indiana University Science Series no. 19, Bloomington, Indiana, 1954; see Chapters VII-IX. A less complete but in some ways improved treatment of these matters is given by C. TRUESDELL and R. TOUPIN, *The Classical Field Theories*, in Flügge's *Handbuch der Physik*, Berlin etc., Springer, 1960; see §§ 105-138.

circulation-preserving (2). In particular, the Bernoullian theorem (5) will hold for any steady universal solution, but generally (6) will not, since it depends upon the particular determination (4) of the pressure, which is appropriate only to Euler fluids.

2. – Case II: The Navier-Stokes theory.

The equation of motion is

$$\boldsymbol{a} = -\operatorname{grad}(p/\varrho + B) + \nu \operatorname{div} \boldsymbol{A}_1 ; \tag{7}$$

$\boldsymbol{A}_1$ is the first Rivlin-Ericksen tensor (see Appendix), and ν, the kinematic viscosity, has been assumed constant. The universal flows of a Navier-Stokes fluid are the velocity fields for which a pressure field p can be found such as to satisfy (7) for all values of ϱ and ν. By setting $\nu = 0$ we recover (2), a necessary condition. Hence for universal solutions (7) becomes

$$-\operatorname{grad}(A - B - p/\varrho) = \nu \operatorname{div} \boldsymbol{A}_1 . \tag{8}$$

Thus follows the necessary condition

$$\operatorname{div} \boldsymbol{A}_1 = -\operatorname{grad} P_1 , \tag{9}$$

P_1 being a scalar field. If this condition as well as (2) is satisfied, so is (7), and then, provided P_1 be single-valued, the pressure required to effect the corresponding universal flow is given by

$$p/\varrho = A - B - \nu P_1 + h , \tag{10}$$

h being an arbitrary function of t only.

The result we have just derived is a classical one, due essentially to Craig (1880) (3); the treatments of fluid mechanics along traditional lines

(2) Equivalent to (3) is the d'Alembert-Euler condition

$$\partial_t \boldsymbol{w} + \operatorname{curl}(\boldsymbol{w} \times \boldsymbol{v}) = \boldsymbol{0} .$$

Much of the recent literature on non-linearly viscous fluids concerns « creeping flow », which is defined by the « approximation » $\operatorname{curl}(\boldsymbol{w} \times \boldsymbol{v}) \simeq \boldsymbol{0}$. In other words, as far as steady flow is concerned, the theory of universal « creeping flows » simply neglects the essential condition (3) altogether.

(3) Cf. § 49.2 of TRUESDELL's *Kinematics of Vorticity*, where references to the work of Craig and others on this subject are given.

usually phrase it in terms of the vorticity $\boldsymbol{w}$. Since $\boldsymbol{v}$ is solenoidal, $\operatorname{div} \boldsymbol{A}_1 = \Delta \boldsymbol{v} = -\operatorname{curl} \boldsymbol{w}$, so (9) is equivalent to

$$\operatorname{curl} \boldsymbol{w} = \operatorname{grad} P_1 , \tag{11}$$

and P_1 is called a *flexion-potential*. Thus *an isochoric flow of a viscous fluid of uniform viscosity and density is universal if and only if it is a circulation-preserving flow that has a flexion-potential.* Obviously the flexion-potential is a harmonic function. The pressure is determined in terms of it by (10): *To effect a universal flow of a Navier-Stokes fluid, it is necessary and sufficient to add a pressure field of amount* $-\nu\varrho P_1$ *to a pressure field that would effect the same flow in an Euler fluid.*

A class of isochoric flows which have a flexion-potential is furnished by the following formulae in rectangular Cartesian components:

$$v_x = f(y, z) , \quad v_y = 0 , \quad v_z = 0 , \tag{12}$$

provided

$$\Delta f = -C , \quad \text{a constant} . \tag{13}$$

Thus

$$P_1 = Cx . \tag{14}$$

Not only to these flows satisfy (9), but also they are accelerationless and hence circulation-preserving; thus they are universal solutions for the Navier-Stokes theory. They are called *shearings*, and they are frequently used in studies of viscometry. They belong to the kinematic class called *steady viscometric flows*, a class which has been characterized and has been studied with great attention (4).

In particular, they are applied to determine the flow in straight tubes. In this application f is rendered unique by the requirement that the velocity vector shall vanish on the curves which delimit the cross-sections of the tube. The velocity field (12) is complex-lamellar, and the vortex-lines in the planes $x =$ const. are the isovels in those planes. The Lamb surfaces are the right cylinders erected upon the isovels; they are the surfaces

(4) Cf. §§ 106 and 109 of TRUESDELL - NOLL'S, *The non-linear theories of mechanics*, Flügge's *Handbuch der Physik*, III/3, Berlin etc., Springer-Verlag, 1965, and W. L. YIN - A. C. PIPKIN, *Kinematics of viscometric flow*, Arch. Rational Mech. Anal., **37** (1970), pp. 111-135.

$|\boldsymbol{v}| = \text{const.}$, and they are material surfaces which move rigidly (5). Of course, $A = 0$, so (10) becomes

$$p/\varrho = -B - \nu Cx + h . \tag{15}$$

Little is known in general about the class of flows that have a flexion-potential. The problem of determining all universal Navier-Stokes flows is unsolved (6).

In the case of *plane* flows perpendicular to the z-axis of a rectangular Cartesian coordinate system the isochoric velocity field may be represented in terms of d'Alembert's stream function $\psi = \psi(x, y, t)$:

$$v_x = \frac{\partial \psi}{\partial y}, \quad v_y = -\frac{\partial \psi}{\partial x}, \quad v_z = 0 . \tag{16}$$

In order that the flow be universal, ψ must satisfy both (3) and (9), which imply, respectively, that

$$\frac{\partial \Delta \psi}{\partial t} + \boldsymbol{v} \cdot \operatorname{grad} \Delta \psi = 0 , \quad \Delta^2 \psi = 0 ; \tag{17}$$

the latter condition expresses the requirement, which follows from (9), that $\operatorname{curl} \operatorname{div} \boldsymbol{A}_1 = \boldsymbol{0}$. A more explicit characterization of the complete class of stream functions which satisfy (16) and (17) is not known.

(5) Much more generally, Yin - Pipkin in *op. cit.* have shown that any viscometric flow may be visualized as the sliding of a family of material surfaces which move isometrically.

(6) Marris at first used a more restricted definition of « universal solution », according to which not only the flow but also the pressure is required to be independent of ν. Thus he required that $\operatorname{div} \boldsymbol{A}_1 = \boldsymbol{0}$, equivalently, $\operatorname{curl} \boldsymbol{w} = \boldsymbol{0}$, instead of the weaker conditions (9) and (11). Even so he was able to characterize only certain special subclasses of such solutions. Cf. A. W. Marris, *Steady non-rectilinear complex-lamellar universal motions of a Navier-Stokes fluid*, Arch. Rational Mech. Anal., **41** (1971), pp. 354-362; *Steady rectilinear universal motions of a Navier-Stokes fluid*, ibid., **48** (1972), pp. 379-396; *Steady universal motions of a Navier-Stokes fluid: the case when the velocity magnitude is constant on a Lamb surface*, ibid., **50** (1973), pp. 99-117. In more recent work, *On the general impossibility of controllable axi-symmetric Navier-Stokes motions*, Arch. Rational Mech. Anal., in press, Marris and M. G. Aswani have shown that the axially symmetric Navier-Stokes flows universal in the sense we employ here are exhausted by two classes: those in which the ratio of vorticity to the distance from the axis is constant throughout the flow, and those whose stream-lines in the generating plane are parallel straight lines.

3. – Case III: The theory of the fluid of grade 2.

The equation of motion is

$$\boldsymbol{a} = -\operatorname{grad}(p/\varrho + B) + \nu \operatorname{div} \boldsymbol{A}_1 + \lambda_1 \operatorname{div} \boldsymbol{A}_2 + \lambda_2 \operatorname{div} \boldsymbol{A}_1^2, \tag{18}$$

in which $\boldsymbol{A}_2$ is the second Rivlin-Ericksen tensor (see Appendix), and λ_1 and λ_2, which have been assumed to be constants, are second-grade analogues of the kinematic viscosity: $\lambda_1 \equiv \alpha_1/\varrho$, $\lambda_2 \equiv \alpha_2/\varrho$, α_1 and α_2 being the second-grade constitutive coefficients of the fluid. The universal flows of a fluid of second grade are the velocity fields for which a pressure field can be found such as to satisfy (18) for all values of ϱ, ν, λ_1, and λ_2. By setting $\lambda_1 = \lambda_2 = 0$ we recover (2) and (9), so both these conditions are necessary. Therefore for a universal solution (18) becomes

$$-\operatorname{grad}(A - B - \nu P_1 - p/\varrho) = \lambda_1 \operatorname{div} \boldsymbol{A}_2 + \lambda_2 \operatorname{div} \boldsymbol{A}_1^2. \tag{19}$$

Giesekus (7) has discovered the following identity: If $\boldsymbol{v}$ is isochoric and if (9) holds, then

$$\operatorname{div}(\boldsymbol{A}_2 - \boldsymbol{A}_1^2) = -\operatorname{grad}(\dot{P}_1 - \tfrac{1}{4}\operatorname{tr} \boldsymbol{A}_1^2), \tag{20}$$

$\dot{P}_1$ being the material time derivative of P_1: Therefore (19) becomes

$$-\operatorname{grad}[A - B - \nu P_1 + \lambda_2(\dot{P}_1 - \tfrac{1}{4}\operatorname{tr} \boldsymbol{A}_1^2) - p/\varrho] = (\lambda_1 + \lambda_2) \operatorname{div} \boldsymbol{A}_2. \tag{21}$$

Hence follows the necessary condition

$$\operatorname{div} \boldsymbol{A}_2 = -\operatorname{grad} P_2, \tag{22}$$

in which P_2 is a scalar field. Conversely, if this condition as well as (2) and (9) is satisfied, and if P_1 and P_2 are single-valued, then (18) is satisfied, and the pressure required to effect the corresponding universal flow is given by

$$p/\varrho = A - B - \nu P_1 + \lambda_2(\dot{P}_1 - \tfrac{1}{4}\operatorname{tr} \boldsymbol{A}_1^2) - (\lambda_1 + \lambda_2) P_2 + h, \tag{23}$$

h being an arbitrary function of t only.

(7) H. Giesekus, *Die simultane Translations- und Rotationsbewegung einer Kugel in einer elastoviskosen Flüssigkeit*, Rheologica Acta, **3** (1963), pp. 59-71; see § 2.4 and Anhang II.

We may express the result in another way: *The universal flows of fluids of second grade are those universal Navier-Stokes flows that satisfy* (22) *for some single-valued* P_2, *and those alone.*

For an example of universal flows of fluids of the second grade, we easily show that the flows given by (12) satisfy (22) if they satisfy (13), and that

$$P_2 = 2Cf - |\operatorname{grad} f|^2 . \tag{24}$$

We note that $|f|$ is the speed and $|\operatorname{grad} f| = |\boldsymbol{w}|$, the magnitude of the vorticity. Now by (14) and (12), $\dot{P}_1 = Cf$; also $\operatorname{tr} \boldsymbol{A}_1^2 = 2|\operatorname{grad} f|^2$; and so (23) yields

$$p/\varrho = -B - \nu Cx - (2\lambda_1 + \lambda_2)(Cf - \tfrac{1}{2}|\operatorname{grad} f|^2) + h . \tag{25}$$

For the fluid of second grade, then, all the shearings (12) are universal, but a secondary pressure field proportional to $C|\boldsymbol{v}| - \frac{1}{2}|\boldsymbol{w}|^2$ is required to produce them.

As a second example we consider the plane flows (16) and show that *the set of universal flows of this class for fluids of grade* 2 *is the same as that for Navier-Stokes fluids* (8). Toward this end we first note that the following identities are readily constructed from (16):

$$\begin{cases} \operatorname{div} \boldsymbol{A}_1^2 = \operatorname{grad}\,[2|\operatorname{grad}\operatorname{grad}\psi|^2 - (\Delta\psi)^2] , \\ \operatorname{tr} \boldsymbol{A}_1^2 = 2[2|\operatorname{grad}\operatorname{grad}\psi|^2 - (\Delta\psi)^2] . \end{cases} \tag{26}$$

Thus, it follows from (20) that if (9), or equivalently $(17)_2$, holds, then $\operatorname{div} \boldsymbol{A}_2$ has a potential representation of the form (22), and P_2 is given by

$$P_2 = \dot{P}_1 - \tfrac{3}{2}[2|\operatorname{grad}\operatorname{grad}\psi|^2 - (\Delta\psi)^2] . \tag{27}$$

Therefore, the universal plane flows of fluids of grade 2 are subject to no further restrictions beyond those already given in (17), so our assertion is established. From (23) we see that the pressure of a plane universal flow

(8) This result generalizes a theorem of R. TANNER, *Plane creeping flows of incompressible second-order fluids*, Physics of Fluids, **9** (1966), pp. 1246-1247, which proves that steady plane « creeping flows » of Navier-Stokes fluids are also such flows for fluids of second grade. For steady plane creeping flows Tanner drops the acceleration in the equation of motion, which in terms of the present work implies that he does not consider the restriction $(17)_1$ on the stream function.

of this class is given by

$$(28)\quad p/\varrho = A - B - \nu P_1 - \lambda_1 \dot{P}_1 + \tfrac{1}{2}(3\lambda_1 + 2\lambda_2)[2|\operatorname{grad}\operatorname{grad}\psi|^2 - (\Delta\psi)^2] + h\,.$$

Looking back at (21) we may note two obvious results of a more special kind. First, *for the particular fluids of second grade such that* $\lambda_1 + \lambda_2 = 0$, *the class of universal flows is exactly the same as that for Navier-Stokes fluids* (9). Second, putting $\lambda_1 = 0$ shows that *the universal flows of a Reiner-Rivlin fluid with constant response functions are universal flows for fluids of the second grade.* There are many papers (10) on Reiner-Rivlin fluids of this special kind, and the theorem just stated shows that in so far as those papers treat universal flows, their results may be generalized effortlessly to fluids of the second grade. One has only to cast into the form (23) with $\lambda_1 = 0$ the pressure function obtained for the particular Reiner-Rivlin flow, then replace λ_2 by $\lambda_1 + \lambda_2$ as the coefficient of P_2.

4. – Case IV: Partial results for the theory of the fluid of grade 3.

The equation of motion is

$$(29)\quad \begin{aligned} \boldsymbol{a} = &- \operatorname{grad}(p/\varrho + B) + \nu \operatorname{div} \boldsymbol{A}_1 + \lambda_1 \operatorname{div} \boldsymbol{A}_2 + \lambda_2 \operatorname{div} \boldsymbol{A}_1^2 \\ &+ \eta_1 \operatorname{div} \boldsymbol{A}_3 + \eta_2 \operatorname{div}(\boldsymbol{A}_1\boldsymbol{A}_2 + \boldsymbol{A}_2\boldsymbol{A}_1) + \eta_3 \operatorname{div}[(\operatorname{tr}\boldsymbol{A}_2)\boldsymbol{A}_1]\,. \end{aligned}$$

Here $\boldsymbol{A}_3$ is the third Rivlin-Ericksen tensor, and η_1, η_2, and η_3 are related as follows to the third-grade constitutive coefficients of the fluid, denoted by Truesdell and Noll as β_1, β_2, and β_3: $\eta_k = \beta_k/\varrho$, $k = 1, 2, 3$. For universal flows we still have the necessary conditions (2), (9), and (22), and after use of the Giesekus identity (20) we obtain the following reduced form of (29):

$$(30)\quad \begin{aligned} &- \operatorname{grad}[A - B - \nu P_1 + \lambda_2(\dot{P}_1 - \tfrac{1}{4}\operatorname{tr}\boldsymbol{A}_1^2) - (\lambda_1 + \lambda_2)P_2 - p/\varrho] \\ &\qquad = \eta_1 \operatorname{div} \boldsymbol{A}_3 + \eta_2 \operatorname{div}(\boldsymbol{A}_1\boldsymbol{A}_2 + \boldsymbol{A}_2\boldsymbol{A}_1) + \eta_3 \operatorname{div}[(\operatorname{tr}\boldsymbol{A}_2)\boldsymbol{A}_1]\,. \end{aligned}$$

(9) A thermodynamic basis for the condition $\lambda_1 + \lambda_2 = 0$ has recently been given by J. E. DUNN - R. L. FOSDICK, *Thermodynamics, stability and boundedness of fluids of complexity 2 and fluids of second grade*, Arch. Rational Mech. Anal., **56** (1974), pp. 191-252.

(10) This literature is cited in footnote (2), page 479 of the article by C. TRUESDELL - W. NOLL, *The non-linear field theories of mechanics*, Flügge's *Handbuch der Physik*, III/3, Berlin etc., Springer, 1965.

Presently we shall prove a fundamental identity which when appropriately specialized yields not only Giesekus' identity (20) but also the result that *if* $\boldsymbol{v}$ *is isochoric and if both* (9) *and* (22) *hold, then*

$$(31) \qquad \begin{aligned} &\operatorname{div}[(\operatorname{tr} \boldsymbol{A}_2)\boldsymbol{A}_1 - (\boldsymbol{A}_1\boldsymbol{A}_2 + \boldsymbol{A}_2\boldsymbol{A}_1)] \\ &\quad = -\operatorname{grad}\,(\ddot{P}_1 - \dot{P}_2 - \tfrac{1}{4}\,\overline{\operatorname{tr}\boldsymbol{A}^2}^{\,\boldsymbol{\cdot}} + \tfrac{5}{6}\operatorname{tr}\boldsymbol{A}_1^3)\,. \end{aligned}$$

As an immediate consequence of this we see that (30) reduces to

$$(32) \qquad \begin{aligned} &-\operatorname{grad}\,[A - B - \nu P_1 + \lambda_2(\dot{P}_1 - \tfrac{1}{4}\operatorname{tr}\boldsymbol{A}_1^2) - (\lambda_1 + \lambda_2)P_2 \\ &\qquad + \eta_2(\ddot{P}_1 - \dot{P}_2 - \tfrac{1}{4}\,\overline{\operatorname{tr}\boldsymbol{A}_1^2}^{\,\boldsymbol{\cdot}} + \tfrac{5}{6}\operatorname{tr}\boldsymbol{A}_1^3) - p/\varrho] \\ &\qquad = \eta_1 \operatorname{div}\boldsymbol{A}_3 + (\eta_2 + \eta_3)\operatorname{div}[(\operatorname{tr}\boldsymbol{A}_2)\boldsymbol{A}_1]\,. \end{aligned}$$

Hence the universal flows of fluids of third grade must satisfy not only (2), (9) and (22) but also

$$(33) \qquad \begin{cases} \operatorname{curl}\operatorname{div}\boldsymbol{A}_3 = \boldsymbol{0}\,, \\ \operatorname{curl}\operatorname{div}[(\operatorname{tr}\boldsymbol{A}_2)\boldsymbol{A}_1] = \boldsymbol{0}\,. \end{cases}$$

These conditions are, in general, independent as we shall see later. First, however, we establish the following

FUNDAMENTAL IDENTITY. *Let* $\boldsymbol{A}$ *be any symmetric tensor field such that*

$$(34) \qquad \operatorname{div}\boldsymbol{A} = -\operatorname{grad}\Phi\,,$$

and define $\boldsymbol{B}$ *as follows:*

$$(35) \qquad \boldsymbol{B} \equiv \dot{\boldsymbol{A}} + \boldsymbol{A}\operatorname{grad}\boldsymbol{v} + (\operatorname{grad}\boldsymbol{v})^T\boldsymbol{A}\,.$$

Then, if $\boldsymbol{v}$ *is isochoric,*

$$(36) \qquad \operatorname{grad}\dot{\Phi} = -\operatorname{div}\boldsymbol{B} + \tfrac{1}{2}\boldsymbol{A}\cdot(\operatorname{grad}\boldsymbol{A}_1) + \operatorname{div}(\boldsymbol{A}\boldsymbol{A}_1)\,.$$

To see this we introduce a potential Φ and a symmetric $\boldsymbol{A}$ such as to satisfy (34) and then observe from the definition of material time differentiation that

$$\operatorname{grad}\dot{\Phi} = -\operatorname{div}\dot{\boldsymbol{A}} + \operatorname{div}[(\operatorname{grad}\boldsymbol{A})\boldsymbol{v}] - \operatorname{grad}[(\operatorname{div}\boldsymbol{A})\cdot\boldsymbol{v}]\,.$$

By inserting (35) into this last we find that

$$\operatorname{grad} \dot{\Phi} = -\operatorname{div} \boldsymbol{B} + \operatorname{div} [\boldsymbol{A} \operatorname{grad} \boldsymbol{v} + (\operatorname{grad} \boldsymbol{v})^T \boldsymbol{A} + (\operatorname{grad} \boldsymbol{A}) \boldsymbol{v}] - \operatorname{grad} [(\operatorname{div} \boldsymbol{A}) \cdot \boldsymbol{v}],$$

and with the aid of the readily established identities

$$\begin{aligned}
\operatorname{grad} [(\operatorname{div} \boldsymbol{A}) \cdot \boldsymbol{v}] &= (\operatorname{grad} \operatorname{div} \boldsymbol{A})^T \boldsymbol{v} + (\operatorname{grad} \boldsymbol{v})^T \operatorname{div} \boldsymbol{A}, \\
\operatorname{div} [(\operatorname{grad} \boldsymbol{v})^T \boldsymbol{A}] &= (\operatorname{grad} \boldsymbol{v})^T \operatorname{div} \boldsymbol{A} + \boldsymbol{A} \cdot [\operatorname{grad} (\operatorname{grad} \boldsymbol{v})^T], \\
\operatorname{div} [(\operatorname{grad} \boldsymbol{A}) \boldsymbol{v}] &= (\operatorname{grad} \operatorname{div} \boldsymbol{A}) \boldsymbol{v} + \operatorname{div} [\boldsymbol{A} (\operatorname{grad} \boldsymbol{v})^T],
\end{aligned}$$

together with the hypothesis that $\boldsymbol{A}$ is symmetric, we complete the proof of (36).

Clearly, if we set $\boldsymbol{A} = \boldsymbol{A}_1$ in (36) and observe that $\boldsymbol{A}_1 \cdot \operatorname{grad} \boldsymbol{A}_1 = \frac{1}{2} \operatorname{grad} \operatorname{tr} \boldsymbol{A}_1^2$, then (35) shows that $\boldsymbol{B} = \boldsymbol{A}_2$, and we obtain the Giesekus identity (20) with Φ to be identified with P_1 since (9) and (34) are equivalent statements.

In order to establish (31) we take $\boldsymbol{A} \equiv \boldsymbol{A}_1^2$ in the fundamental identity. It is clear that (34) is satisfied since by (20) and (22) we have

$$\operatorname{div} \boldsymbol{A}_1^2 = -\operatorname{grad} (P_2 - \dot{P}_1 + \tfrac{1}{4} \operatorname{tr} \boldsymbol{A}_1^2).$$

Thus, we take $\Phi = P_2 - \dot{P}_1 + \frac{1}{4} \operatorname{tr} \boldsymbol{A}_1^2$ and rewrite (36) as

$$\operatorname{grad} (\dot{P}_2 - \ddot{P}_1 + \tfrac{1}{4} \dot{\overline{\operatorname{tr} \boldsymbol{A}_1^2}}) = -\operatorname{div} (\boldsymbol{B} - \boldsymbol{A}_1^3) + \tfrac{1}{6} \operatorname{grad} (\operatorname{tr} \boldsymbol{A}_1^3). \tag{37}$$

Here $\boldsymbol{B}$ is given by (35) as

$$\boldsymbol{B} = \dot{\overline{\boldsymbol{A}_1^2}} + \boldsymbol{A}_1^2 \operatorname{grad} \boldsymbol{v} + (\operatorname{grad} \boldsymbol{v})^T \boldsymbol{A}_1^2,$$

or, equivalently,

$$\boldsymbol{B} = \dot{\overline{\boldsymbol{A}_1^2}} + \boldsymbol{A}_1^3 + \boldsymbol{A}_1^2 \boldsymbol{W} - \boldsymbol{W} \boldsymbol{A}_1^2,$$

$\boldsymbol{W}$ being the spin:

$$\boldsymbol{W} \equiv \tfrac{1}{2} [\operatorname{grad} \boldsymbol{v} - (\operatorname{grad} \boldsymbol{v})^T]. \tag{38}$$

The second Rivlin-Ericksen tensor may also be written in the form

$$\boldsymbol{A}_2 = \dot{\boldsymbol{A}}_1 + \boldsymbol{A}_1^2 + \boldsymbol{A}_1 \boldsymbol{W} - \boldsymbol{W} \boldsymbol{A}_1, \tag{39}$$

and we readily obtain the identity

$$B - A_1^3 = A_1A_2 + A_2A_1 - 2A_1^3 . \tag{40}$$

Now, since $\operatorname{tr} A_1 = 0$, the Hamilton-Cayley theorem yields

$$A_1^3 = \tfrac{1}{2}(\operatorname{tr} A_1^2)A_1 + \tfrac{1}{3}(\operatorname{tr} A_1^3)\mathbf{1} ,$$

while from (39) we see that

$$\operatorname{tr} A_2 = \operatorname{tr} A_1^2 .$$

Hence, using (40), we obtain

$$B - A_1^3 = A_1A_2 + A_2A_1 - (\operatorname{tr} A_2)A_1 - \tfrac{2}{3}(\operatorname{tr} A_1^3)\mathbf{1} ,$$

and by inserting this result into (37) we obtain (31), which we have shown to lead to the necessary conditions (33) for universal flows.

We remark that the fundamental identity can be used to generate special identities which may be useful in analysis of universal flows of fluids of higher grade. For example, from the above expression of the Hamilton-Cayley theorem for A_1 and the equation following it we see that if $(33)_2$ holds, then $\operatorname{div} A_1^3$ is the gradient of a potential, and we may take $A = A_1^3$ in the fundamental identity. We shall not carry the matter further here, but this approach leads to a useful relation in the theory of fluids of grade 4.

We are presently unable to exhibit all the solutions of (2), (9), (22), and (33) and so determine all universal flows of the homogeneous incompressible fluid of grade 3. We record here a set of equivalent local conditions:

$$\begin{aligned}
&\operatorname{curl} \boldsymbol{a} = \mathbf{0} ,\\
&\operatorname{curl} \operatorname{div} A_1 = \mathbf{0} ,\\
&\operatorname{curl} \operatorname{div} A_2 = \mathbf{0} ,\\
&\operatorname{curl} \operatorname{div} [(\operatorname{tr} A_2)A_1] = \mathbf{0} ,\\
&\operatorname{curl} \operatorname{div} A_3 = \mathbf{0} .
\end{aligned}$$

5. – Remarks on universal solutions for fluids of higher grade.

The more general is the constitutive class considered, the fewer universal solutions there are. To show that a certain flow is not universal for all simple fluids, we need only show that it is not universal for all fluids in a certain subclass. At present the determination of all universal flows for homogeneous incompressible simple fluids is an open problem. We could attempt to solve it by determining the entire class of universal flows for fluids of some low grade, such as 3 or 4. If we could exhibit these flows, it would then be a simple matter to see if they were indeed universal for all simple fluids. For discovery, this approach is more systematic then the method of trial and error, which has been used to discover such universal solutions as are now known, and it is mathematically easier than would be a frontal attack upon the equations of motion of a simple fluid in general. We expect that at grade 3 or grade 4 all universal flows would be found.

We shall now illustrate our approach by carrying it through for the special class of shearings, defined by (12). It is known that a shearing is universal for homogeneous incompressible simple fluids if and only if [11] f reduces to an affine function of y and z or of Arctan (y/z). We shall show now that these shearings constitute also the complete set of universal shearings for the fluid of grade 3. That is, as far as shearings are concerned, in order to exhibit all universal flows it suffices to find the universal flows for a fluid of grade 3.

6. – Universal shearings for fluids of grade 3.

The shearings of (12) are viscometric and, of course, for them $\boldsymbol{A}_3 \equiv \mathbf{0}$. Thus $(33)_1$ is trivially satisfied, and $(33)_2$ reduces to the requirement that

$$D\cdot(|Df|^2 Df) = 2K \tag{41}$$

for some K, the operator D being the gradient in the $y - z$ plane and $D\cdot$ being the corresponding divergence. We see from this that in general the two conditions (33) are independent, as we remarked earlier.

The universal shearings of fluids of third grade are characterized by the common solutions of (13) and (41), which we now determine. We proceed in the following two steps: First we show that if $C = 0$ in (13) then $K = 0$ in (41), and in this case the only common solutions of (13) and (41) are the simple shearings and the shearings of fanned planes. Then we show that $C = 0$ always.

[11] Sufficiency of the former alternative has been known for a long time. Sufficiency for the latter alternative was proved by YIN - PIPKIN, *op. cit.* Their term for « universal » is « completely controllable ». They show that the only other universal viscometric flows for simple fluids are Couette flows with uniform shear rates.

It is advantageous for the remaining analysis to introduce the complex variable $\zeta \equiv y + iz$ with conjugate $\bar{\zeta} = y - iz$ so that $f = f(\zeta, \bar{\zeta})$, and to transform (13) and (41) accordingly. In so doing we replace (13) by its general solution in terms of an analytic function F:

$$f = -\tfrac{1}{4} C\zeta\bar{\zeta} + F(\zeta) + \overline{F(\zeta)}\,, \tag{42}$$

In addition, (41) becomes

$$f_{\zeta\zeta}(f_{\bar{\zeta}})^2 + f_{\bar{\zeta}\bar{\zeta}}(f_{\zeta})^2 - Cf_{\zeta} f_{\bar{\zeta}} = K/4, \tag{43}$$

where subscripts denote partial differentiation. Thus our analysis from this point onward concerns whether f as given by (42) can also satisfy (43).

We first suppose that $C = 0$. In this case (42) and (43) yield

$$F''(\bar{F}')^2 + \bar{F}''(F')^2 = K/4\,, \tag{44}$$

where primes denote ordinary differentiation, and by differentiating this result with respect to ζ we obtain

$$F'''(\bar{F}')^2 + \bar{F}''((F')^2)' = 0\,. \tag{44A}$$

Now, if either $F' = 0$ or $F'' = 0$, then $K = 0$, and using (42) we find either $f =$ const. or $f = ay + bz$, respectively, a and b being real constants. If, on the contrary, neither F' nor F'' vanish, we may rewrite (44A) as

$$\frac{F'''}{((F')^2)'} + \frac{\bar{F}''}{(\bar{F}')^2} = 0\,,$$

and conclude that

$$\frac{F'''}{((F')^2)'} = a_1\,, \qquad \frac{F''}{(F')^2} = -\bar{a}_1\,, \tag{45}$$

a_1 being a complex constant. By integrating the first of these and then making use of the second we find that

$$(F')^2(a_1 + \bar{a}_1) = a_2\,,$$

a_2 being a second complex constant. Thus, since we have assumed that $F' \neq$ const. it follows that $a_1 + \bar{a}_1 = a_2 = 0$. Consequently, a_1 is purely imaginary, i.e., $a_1 = ia$, where a is real, and by integrating $(45)_2$ we reach

$$F' = \frac{1}{a_3 - ia\zeta},$$

in which a_3 is a complex constant. Since $F' \neq$ constant, then $a \neq 0$ and a second integration gives

$$F = -\frac{1}{ia}\log(ia\zeta - a_3),$$

which, with (42) and the hypothesis that $C = 0$, yields

$$f = -\frac{2}{a}\theta$$

where, for convenience, we have introduced the polar form $a_3 - ia\zeta = re^{i\theta}$. By appeal to (44) it is easy to show for this F the constant K must vanish. Thus, when $C = 0$ we have shown that $K = 0$ and that there are only two possible cases when (42) can satisfy (43); these possibilities correspond to simple shearing and to shearing of fanned planes, the latter being relative to an appropriately rotated and translated coordinate system.

We now show that C *must vanish*. As a first step toward this end we insert (42) into (43) and so obtain

$$(46)\qquad F''\left(\frac{C^2}{16}\zeta^2 - \frac{C}{2}\zeta\bar{F}' + (\bar{F}')^2\right) + \bar{F}''\left(\frac{C^2}{16}\bar{\zeta}^2 - \frac{C}{2}\bar{\zeta}F' + (F')^2\right)$$
$$- C\left(\frac{C^2}{16}\zeta\bar{\zeta} - \frac{C}{4}\bar{\zeta}\bar{F}' - \frac{C}{4}\zeta F' + \bar{F}'F'\right) = K/4\,.$$

By differentiation of (46) with respect to ζ and $\bar{\zeta}$ we obtain

$$(47)\qquad F'''\left(-\frac{C}{2}\zeta\bar{F}'' + 2\bar{F}'\bar{F}''\right) + \bar{F}'''\left(-\frac{C}{2}\bar{\zeta}F'' + 2F'F''\right)$$
$$-2C\bar{F}''F'' - C^3/16 = 0\,;$$

differentiation of (47) with respect to ζ and $\bar{\zeta}$ produces

$$(48)\qquad \frac{F^{\text{iv}}}{F'''}\left(-\frac{C}{2}\zeta + 2\frac{(\bar{F}'\bar{F}'')'}{\bar{F}'''}\right) + \frac{\bar{F}^{\text{iv}}}{\bar{F}'''}\left(-\frac{C}{2}\bar{\zeta} + 2\frac{(F'F'')'}{F'''}\right) - 3C = 0\,,$$

provided $F'''' \neq 0$. If, on the other hand, $F''''=0$, then (47) clearly shows that

$$C(2\bar{F}''F''+C^2/16)=0\,,$$

which holds if and only if $C=0$. Thus if we are to find a case in which $C\neq 0$ we must assume that $F''''\neq 0$. We then differentiate (48) repeatedly with respect to ζ and $\bar{\zeta}$ to obtain

$$\left(\frac{F^{\text{iv}}}{F''''}\right)'\left(\frac{(\bar{F}'\bar{F}'')'}{\bar{F}''''}\right)'+\left(\frac{\bar{F}^{\text{iv}}}{\bar{F}''''}\right)'\left(\frac{(F'F'')'}{F''''}\right)'=0\,. \tag{49}$$

Suppose first that

$$\left(\frac{(F'F'')'}{F''''}\right)'=0\,;$$

then (49) yields

$$\frac{(F'F'')'}{F''''}=a_1\,, \tag{50}$$

where a_1 is a complex constant, and by integration we obtain

$$F'F''=a_1F''+a_2\,, \tag{51}$$

where a_2 is a complex constant, and by subsequent use of (47) it follows that

$$F''''\left(\left(-\frac{C}{2}\zeta+2\bar{a}_1\right)\bar{F}''+2\bar{a}_2\right)+\bar{F}''''\left(\left(-\frac{C}{2}\bar{\zeta}+2a_1\right)F''+2a_2\right)-2C\bar{F}''F''-C^3/16=0\,. \tag{52}$$

If we now differentiate (52) with respect to $\bar{\zeta}$, divide the result by $F''\bar{F}''''$, which does not vanish under the present assumptions, and then differentiate the remaining equation first with respect to $\bar{\zeta}$ and then with respect to ζ, we find that

$$2a_2\left(\frac{\bar{F}^{\text{iv}}}{\bar{F}''''}\right)'\left(\frac{1}{F''}\right)'=0\,. \tag{53}$$

Thus, since $F''''\neq 0$, one or both of the following alternatives must hold: $a_2=0$, $(\bar{F}^{\text{iv}}/\bar{F}'''')'=0$. However, from (51) the former alternative is incompatible with the hypothesis that $F''''\neq 0$. Thus $a_2\neq 0$ and the latter alternative implies that

$$\frac{\bar{F}^{\text{iv}}}{\bar{F}''''}=a_3\,, \tag{54}$$

where a_3 is a complex constant. This, together with (48) and (50) leads to

$$a_3 C = 0\,,$$

which shows that either $a_3 = 0$ or $C = 0$. If $a_3 = 0$, we see from (48) and (54) that $C = 0$ as well.

The only case that remains to be disposed of is that in which

$$\left(\frac{(F'F'')'}{F'''}\right)' \neq 0\,;$$

then (49) yields

$$\left(\frac{F^{\text{iv}}}{F'''}\right)' \Big/ \left(\frac{(F'F'')'}{F'''}\right)' = ia\,. \tag{55}$$

Integrating this equation twice yields

$$F''' = iaF'F'' + b_1 F'' + b_2\,, \tag{56}$$

where b_1 and b_2 are complex constants. By substituting this result into (47) we find an equation of the same form as (52), in which, if we assume for the moment that $a \neq 0$, a_1 and a_2 are replaced by $-2b_1/ia$ and $-2b_2/ia$, respectively. Our analysis of (52) applies again here and we find that *either* $b_2 = 0$ *or* $(\bar{F}^{\text{iv}}/\bar{F}''')' = 0$. The latter alternative is not possible since by (49) it is incompatible with the assumption that $a \neq 0$. Thus $b_2 = 0$, and in this case (56) and (47) imply that

$$(iaF' + b_1)\left(-\frac{C}{2}\zeta + \frac{2\bar{b}_1}{ia}\right)F''\bar{F}'' + (-ia\bar{F}' + \bar{b}_1)\left(-\frac{C}{2}\bar{\zeta} - \frac{2b_1}{ia}\right)F''\bar{F}'' - 2CF''\bar{F}'' - C^3/16 = 0\,.$$

After dividing through this equation by $F''\bar{F}''$ and differentiating the result first with respect to ζ and then with respect to $\bar{\zeta}$ we readily obtain

$$C^3\left(\frac{1}{F''}\right)'\left(\frac{1}{\bar{F}''}\right)' = 0\,,$$

which shows that $C = 0$ since, by hypothesis, $F''' \neq 0$.

Our analysis of (56) has presumed that $a \neq 0$. To dispose of the only remaining possibility we suppose that $a = 0$ in (55), so that

$$\frac{F^{\text{iv}}}{F'''} = c_1\,, \tag{57}$$

where c_1 is a complex constant. Then we differentiate (47) with respect to $\bar{\zeta}$ and divide the result by $F''\bar{F}'''$ so as to obtain

$$\frac{F'''}{F''}\left(-\frac{C}{2}\zeta + 2\frac{(\bar{F}'\bar{F}'')'}{\bar{F}'''}\right) + \frac{\bar{F}^{\text{iv}}}{\bar{F}'''}\left(-\frac{C}{2}\bar{\zeta} + 2F'\right) - \frac{5}{2}C = 0\,; \tag{58}$$

then using (57) and differentiating first with respect to $\bar{\zeta}$ and then with respect to ζ, we find that

$$\left(\frac{F'''}{F''}\right)'\left(\frac{(\bar{F}'\bar{F}'')'}{\bar{F}'''}\right)' = 0\,. \tag{59}$$

Since the denominator in (55) is non-zero by hypothesis, we conclude from (59) that $(F'''/F'')' = 0$ and so with the aid of (57) obtain

$$\frac{F'''}{F''} = c_1\,. \tag{60}$$

However, upon placing this result and (57) into (58) and then differentiating twice with respect to ζ we finally reach

$$\bar{c}_1 F''' = 0\,,$$

which, by (60), is possible only if $F''' = 0$, contrary to hypothesis.

We have proved that $C = 0$ and so completed the determination of all universal shearings of the homogeneous incompressible fluid of grade 3.

Appendix: The Rivlin-Ericksen tensors.

The Rivlin-Ericksen tensors are often used so as to formulate a properly invariant constitutive description of materials of the differential type. If $\boldsymbol{v}$ denotes the velocity field and a superimposed dot denotes material time differentiation, then the Rivlin-Ericksen tensors may be defined according to the recursive scheme

$$\boldsymbol{A}_{n+1} \equiv \dot{\boldsymbol{A}}_n + \boldsymbol{A}_n \operatorname{grad}\boldsymbol{v} + (\operatorname{grad}\boldsymbol{v})^T\boldsymbol{A}_n\,, \qquad \boldsymbol{A}_0 \equiv 1\,. \qquad (n = 0, 1, 2 \ldots)\,.$$

Clearly $\boldsymbol{A}_1 = \operatorname{grad}\boldsymbol{v} + (\operatorname{grad}\boldsymbol{v})^T = 2\boldsymbol{D}$, where $\boldsymbol{D}$ is the stretching tensor often encountered in studies of classical viscous fluids.

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