ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
*Classe di Scienze*

O. A. OLEINIK
G. A. YOSIFIAN

**Boundary value problems for second order elliptic equations in unbounded domains and Saint-Venant's principle**

*Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4e série*, tome 4, nº 2 (1977), p. 269-290

<http://www.numdam.org/item?id=ASNSP_1977_4_4_2_269_0>

# Boundary Value Problems for Second Order Elliptic Equations in Unbounded Domains and Saint-Venant's Principle.

O. A. OLEINIK - G. A. YOSIFIAN (*)

*dedicated to Jean Leray*

A priori estimates similar to those known in the theory of elasticity as Saint-Venant's principle ([1], [2]) are obtained in this paper for solutions of linear second order elliptic equations. Immediately from these estimates follow uniqueness theorems for solutions of boundary value problems for second order elliptic equations in unbounded domains in classes of growing functions. By use of the estimates obtained here existence theorems for some boundary value problems in unbounded domains are also proved in this paper. A priori estimates of this kind, together with existence and uniqueness theorems, are also valid for some classes of second order equations with a non-negative characteristic form. The uniqueness theorems proved here may be considered as a generalisation of the well - known Phragmen-Lindelöf theorem for harmonic functions.

In a domain $\Omega \subset \boldsymbol{R}_x^n = (x_1, \ldots, x_n)$ consider an equation of the form

$$(1) \qquad L(u) \equiv \sum_{k,j=1}^{n} \left(a^{kj}(x)\, u_{x_k}\right)_{x_j} + \sum_{k=1}^{n} b^k(x)\, u_{x_k} + c(x)\, u = F(x)\,,$$

where $a^{kj} \equiv a^{jk}$, $\sum_{k,j=1}^{n} a^{kj}\xi_k\xi_j \geqslant 0$ for every $\xi \in \boldsymbol{R}_\xi^n$.

(*) Moscow University.
Pervenuto alla Redazione il 5 Luglio 1976.

We assume for simplicity that the coefficients $a^{kj}$, $b^j$, $c$ and also the functions $a^{kj}_{x_j}$, $\sum_{k=1}^{n} b^k_{x_k}$ are continuous in $\bar{\Omega}$. Let $\partial\Omega$ be the boundary of $\Omega$ and $\partial\Omega = \bar{\gamma}_1 \cup \bar{\gamma}_2 \cup \bar{\gamma}_3$ where $\gamma_1$, $\gamma_2$, $\gamma_3$ are mutually disjoint open subsets of $\partial\Omega$.

By $C^k(\mathfrak{D})$ is denoted the space of functions with continuous derivatives at interior points of $\mathfrak{D}$ up to the order $k$ which can be continuously extended to $\mathfrak{D}$.

On $\partial\Omega$ we shall consider the following boundary conditions

$$u\Big|_{\gamma_1} = \Psi_1\,, \qquad \frac{\partial u}{\partial \beta}\Big|_{\gamma_2} = \Psi_2\,, \qquad \left(\frac{\partial u}{\partial \beta} + au\right)\Big|_{\gamma_3} = \Psi_3\,, \tag{2}$$

where

$$\frac{\partial}{\partial \beta} = \sum_{k,j=1}^{n} a^{kj}\nu_k \frac{\partial}{\partial x_j}\,, \quad \nu = (\nu_1, \ldots, \nu_n)$$

is the unit outward normal to $\partial\Omega$, the function $a(x)$ is continuous on $\gamma_3$, $F \in L_2^{\text{loc}}(\Omega)$, $\Psi_j \in L_2^{\text{loc}}(\gamma_j)$, $j = 1, 2, 3$.

Consider a family of bounded subdomains $\Omega_\tau$ of $\Omega$ depending on the parameter $\tau = (\tau_1, \ldots, \tau_N)$ which ranges over the parallelepiped

$$G = \{\tau\colon 0 \leqslant \tau_l \leqslant \tau_l^0,\ l = 1, \ldots, N\}\,, \qquad \tau^0 = (\tau_1^0, \ldots, \tau_N^0)\,.$$

Suppose that $\Omega_\tau \subset \Omega_{\tau'}$, if $\tau_j \leqslant \tau_j'$ for $j = 1, \ldots, N$. The boundaries of the domains $\Omega_\tau$ and $\Omega$ are assumed piecewise smooth. Set $S_\tau = \partial\Omega_\tau \setminus \partial\Omega$.

We assume that $S_\tau = \bigcup_{l=1}^{N} S_{\tau_l}$ where $S_{\tau_l}$ is a connected smooth hypersurface with the boundary $\bar{S}_{\tau_l} \setminus S_{\tau_l}$ on $\partial\Omega$, and also that there exist positive continuous functions $h_l(\tau_l, x)$, $l = 1, \ldots, N$, such that for any non-negative continuous function $v(x)$

$$\frac{\partial}{\partial \tau_l} \int_{\Omega_\tau \setminus \Omega_0} v\, dx = \int_{S_{\tau_l}} v(x) h_l(\tau_l, x)\, dS\,, \qquad l = 1, \ldots, N\,, \tag{3}$$

where $dS$ is the element of area on $S_{\tau_l}$. We use the notation

$$Q(u, u) \equiv \sum_{k,j=1}^{n} a^{kj} u_{x_k} u_{x_j} + \left(\tfrac{1}{2} \sum_{k=1}^{n} b^k_{x_k} - c\right) u^2\,.$$

From now on it will be assumed that $\frac{1}{2}\sum_{k=1}^{n} b^k_{x_k} - c \geqslant 0$ in $\Omega_{\tau_0}$ and that for some non-negative integers $q$, $p$ and any $\tau \in G$

$$\bar{S}_{\tau_l} \cap \gamma_1 \neq \emptyset, \qquad l = 1, \dots, q;$$

$$\bar{S}_{\tau_l} \cap \gamma_1 = \emptyset, \qquad c - \tfrac{1}{2}\sum_{k=1}^{n} b^k_{x_k} \neq 0 \quad \text{on } S_{\tau_l}, \qquad l = q+1, \dots, p;$$

$$\bar{S}_{\tau_l} \cap \partial\Omega \subset \gamma_2, \qquad l = p+1, \dots, N.$$

Let

$$\lambda_l(\tau_l) = \inf_{v \in \mathcal{M}_1} \left\{ \int_{S_{\tau_l}} Q(v, v) h_l \, dS \left[ \int_{S_{\tau_l}} \sum_{k,j=1}^{n} a^{kj} \nu_k \nu_j h_l^{-1} v^2 \, dS \right]^{-1} \right\}, \qquad l = 1, \dots, p, \tag{4}$$

where $\mathcal{M}_1$ is the set of functions $v(x)$ of class $C^1(\bar{\Omega}_{\tau_0})$ which vanish identically on $\bar{S}_{\tau_l} \cap \bar{\gamma}_1$ and satisfy the condition

$$\int_{S_{\tau_l}} \sum_{k,j=1}^{n} a^{kj} \nu_k \nu_j h_l^{-1} v^2 \, dS \neq 0.$$

Let

$$\lambda_l(\tau_l) = \inf_{v \in \mathcal{M}_2} \left\{ \int_{S_{\tau_l}} Q(w, w) h_l \, dS \left[ \int_{S_{\tau_l}} w^2 \, dS \right]^{-1} \right\}, \qquad l = p+1, \dots, N, \tag{5}$$

where $\mathcal{M}_2$ is the set of functions $w(x)$ such that $w(x) \in C^1(\bar{\Omega}_{\tau_0})$, $w \not\equiv 0$ on $S_{\tau_l}$ and $\int_{S_{\tau_l}} w \, dS = 0$.

We shall always suppose that $\lambda_l(\tau_l) \neq 0$ when $\tau \in G$, $l = 1, \dots, N$. For any fixed $l$ such that either $0 \leqslant l \leqslant q$ or $N \geqslant l \geqslant p+1$, it is easy to show that if the quadratic form $\sum_{k,j=1}^{n} a^{kj} \xi_k \xi_j$ is positive-definite for every $x \in \bar{S}_{\tau_l}$ then $\lambda_l(\tau_l)$ is not smaller than the first positive eigenvalue of a certain second order elliptic boundary value problem on $S_{\tau_l}$.

It is also assumed in what follows that $\frac{1}{2}\sum_{k=1}^{n} b^k \nu_k \leqslant 0$ on $S_{\tau_l}$ for $l = p+1, \dots, N$ where $(\nu_1, \dots, \nu_n)$ is the unit outward normal to $\partial\Omega_\tau$.

Set $B_l(x, \tau_l) = \max\left\{0, \sum_{k=1}^{n} b^k(x) \nu_k\right\}$ for $x \in S_{\tau_l}$, $l = 1, \dots, p$. Suppose that

$$\int_{S_{\tau_l}} B_l(x, \tau_l) v^2 \, dS \leqslant \mu_l(\tau_l) \int_{S_{\tau_l}} Q(v, v) h_l \, dS, \qquad l = 1, \dots, p, \tag{6}$$

for any $v$ of class $C^1(\bar{\Omega}_{\tau_0})$ vanishing identically on $\bar{S}_{\tau_l} \cap \bar{\gamma}_1$. Let

$$\eta_l(\tau_l) = \sup_{x \in S_{\tau_l}} \sum_{k,j=1}^{n} a^{kj} \nu_k \nu_j h_l^{-1}.$$

Denote by $A_l(\tau_l)$, $l=1,\dots,N$, the functions satisfying the following conditions

$$
(7)\qquad \begin{cases} A_l(\tau_l)\geqslant(\lambda_l(\tau_l))^{-\frac{1}{2}}+\mu_l(\tau_l) & \text{for } l=1,\dots,p\,;\\ A_l(\tau_l)\geqslant[\lambda_l(\tau_l)]^{-\frac{1}{2}}(\eta_l(\tau_l))^{\frac{1}{2}} & \text{for } l=p+1,\dots,N\,;\\ [A_l(\tau_l)]^{-1}\in L_1[0,\tau_l^0]\,, & \text{for } l=1,\dots,N\,. \end{cases}
$$

Let $B$ be a bounded subdomain of $\Omega$. The boundary $\partial B$ of $B$ is assumed piecewise smooth. For a set $\Gamma$ which belongs to $\partial B$ denote by $V(\Gamma, B)$ the functional completion of the space, formed by all functions $v(x)$ such that $v(x)\in C^1(\bar{B})$ and $v\equiv 0$ on $\Gamma$, with respect to the norm

$$
\|v\|^2_{H^1(B)}=\int_B\Big(v^2+\sum_{i=1}^n|v_{x_i}|^2\Big)\,dx\,.
$$

Let $\Gamma_1$, $\Gamma_2$, $\Gamma_3$ be mutually disjoint subsets of $\partial B$. (It is possible that $\bar{\Gamma}_1\cup\bar{\Gamma}_2\cup\bar{\Gamma}_3\neq\partial B$). Consider the boundary conditions

$$
(8)\qquad u\Big|_{\Gamma_1}=0\,,\qquad \frac{\partial u}{\partial\beta}\Big|_{\Gamma_2}=\Psi_2\,,\qquad \Big(\frac{\partial u}{\partial\beta}+au\Big)\Big|_{\Gamma_3}=0\,,
$$

where

$$
\frac{\partial}{\partial\beta}=\sum_{k,j=1}^n a^{kj}\nu_k\frac{\partial}{\partial x_k}\,,\quad \nu=(\nu_1,\dots,\nu_n)
$$

is the unit outward normal to $\partial B$, $a(x)$ is a continuous function on $\Gamma_3$, and $\Psi_2$ is a continuous function on $\Gamma_2$.

We say that $u(x)$ is a weak solution of equation (1) satisfying boundary conditions (8), if $u\in V(\Gamma_1, B)$ and for any $v\in V\big(\partial B\setminus(\Gamma_2\cup\Gamma_3), B\big)$ the integral identity holds

$$
(9)\quad \int_B\Big(-\sum_{k,j=1}^n a^{kj}u_{x_k}v_{x_j}+\sum_{k=1}^n b^k u_{x_k}v+cuv\Big)\,dx+\int_{\Gamma_2}\Psi_2 v\,dS-\int_{\Gamma_3}auv\,dS=\int_B Fv\,dx.
$$

THEOREM 1. *Let $u(x)$ be a weak solution of equation* (1) *in $\Omega_{\tau^0}$ satisfying boundary conditions* (8) *on $\Gamma_j=\partial\Omega_{\tau^0}\cap\gamma_j$, $j=1,2,3$. Suppose that $\Psi_2\equiv 0$ on $\Gamma_2$, $F\equiv 0$ in $\Omega_{\tau^0}$, $u\in C^1(\bar{\Omega}_{\tau^0}\setminus\Omega_0)$,*

$$
(10)\qquad \begin{cases} \displaystyle\sum_{k=1}^n b^k\nu_k\leqslant 0 \quad \text{on}\quad \bar{\gamma}_2\cap\bar{\Omega}_{\tau^0}\,; \qquad \sum_{k=1}^n \tfrac{1}{2}b^k\nu_k-a\leqslant 0 \quad\text{on}\quad \bar{\gamma}_3\cap\bar{\Omega}_{\tau^0}\,.\\ \displaystyle\sum_{k=1}^n b^k\nu_k\leqslant 0 \quad\text{on}\quad S_{\tau_l} \qquad \text{for } l=p+1,\dots,N\,; \end{cases}
$$

$$
(11)\qquad \int_{S_{\tau_l}}\sum_{k,j=1}^n a^{kj}u_{x_k}\nu_j\,dS=0 \qquad \text{for }\ \tau\in G,\ l=p+1,\dots,N\,.
$$

*Then the following estimate holds*

$$\int_{\Omega_0} Q(u, u)\, dx \leqslant \exp[-R] \int_{\Omega_{\varphi(R)}} Q(u, u)\, dx , \tag{12}$$

*where* $\varphi(R) = \big(\varphi_1(R), \ldots, \varphi_N(R)\big)$, $\varphi_l(R)$ *is the inverse function to*

$$R = \phi_l(\tau_l) \equiv \int_0^{\tau_l} [A_l(\eta)]^{-1} d\eta , \qquad l = 1, \ldots, N .$$

PROOF. Fix an arbitrary $\tau \in G$ such that $0 < \tau_j < \tau_j^0$, $j = 1, \ldots, N$, and take a function $\psi_\varrho(x)$ which depends on a small parameter $\varrho$, $0 < \varrho \leqslant 1$, and possesses the following properties:

$$\psi_\varrho(x) \in C^\infty(\bar{\Omega}_{\tau^0}) ,$$
$$\psi_\varrho(x) = 1 \quad \text{for } x \in \bar{\Omega}_\tau ,$$

$\psi_\varrho(x) = 0$ if the distance from $x$ to $\Omega_\tau$ is greater than $\varrho/2$, $0 \leqslant \psi \leqslant 1$ in $\bar{\Omega}_{\tau^0}$, $\psi_\varrho = 0$ on $S_{\tau_l}$, $|\partial\psi/\partial x_k| \leqslant M\varrho^{-1}$ in $\bar{\Omega}_{\tau^0}$ (the constant $M$ is independent of $\varrho$).

First we shall prove that for any function $g(x)$, continuous in $\bar{\Omega}_{\tau^0} \setminus \Omega_0$,

$$\lim_{\varrho \to 0} \int_{\Omega_{\tau}^0} g \psi_{\varrho x_j}\, dx = -\int_{S_\tau} g \nu_j\, dS , \qquad j = 1, \ldots, n , \tag{13}$$

where $\nu = (\nu_1, \ldots, \nu_n)$ is the unit outward normal to $\partial\Omega_\tau$.

Let $\{g_m(x)\}$ be a sequence of functions of class $C^1(\bar{\Omega}_{\tau^0} \setminus \Omega_0)$ such that $\max\limits_{\bar{\Omega}_\tau^0 \setminus \Omega_\tau} |g - g_m| \to 0$ as $m \to \infty$. Applying the integration by parts we get

$$\int_{\Omega_\tau^0} g \psi_{\varrho x_j}\, dx \equiv \int_{\Omega_\tau^0 \setminus \Omega_\tau} g_m \psi_{\varrho x_j}\, dx + \int_{\Omega_\tau^0 \setminus \Omega_\tau} (g - g_m) \psi_{\varrho x_j}\, dx =$$
$$= -\int_{S_\tau} g_m \nu_j\, dS + \int_{\sigma_\varrho} g_m \nu_j \psi_\varrho\, dS - \int_{\Omega_\tau^0 \setminus \Omega_\tau} g_{m x_j} \psi_\varrho\, dx + \int_{\Omega_\tau^0 \setminus \Omega_\tau} (g - g_m) \psi_{\varrho x_j}\, dx$$

where $\sigma_\varrho = \partial\Omega \cap \operatorname{supp} \psi_{\varrho x_j}$. Taking into account the properties of $\psi_\varrho(x)$, we can deduce that

$$\Big| \int_{\Omega_\tau^0} g \psi_{\varrho x_j}\, dx + \int_{S_\tau} g_m \nu_j\, dS \Big| \leqslant M_1(\varrho) \max |g_m| + M_2(\varrho) \max |g_{m x_j}| + K_1 \max |g - g_m| ,$$

where $M_j(\varrho) \to 0$, $j = 1, 2$ as $\varrho \to 0$, $K_1 = \text{const}$, all maximums being taken over the set $\operatorname{supp} \psi_{\varrho x_j}$. Hence, we obtain relation (13). Integral identity (9) for $B = \Omega_{\tau^0}$, $v = u\psi_\varrho$ may be written in the form

$$(14)\qquad \int\limits_{\Omega_{\tau^0}}\Big[-\sum_{k,j=1}^{n}(a^{kj}u_{x_k}u_{x_j}\psi_\varrho + a^{kj}u_{x_k}u\psi_{\varrho x_j}) + \tfrac{1}{2}\sum_{k=1}^{n}b^k(u^2)_{x_k}\psi_\varrho + cu^2\psi_\varrho\Big]dx - \\ -\int\limits_{\partial\Omega_{\tau^0}\cap\gamma_3} au^2\psi_\varrho\,dS = 0\,.$$

Integrating by parts the terms $\frac{1}{2}\int\limits_{\Omega_{\tau^0}}\sum_{k=1}^{n}b^k(u^2)_{x_k}\psi_\varrho\,dx$ and letting $\varrho$ tend to zero, one obtains from (14), according to (13), the following:

$$\int\limits_{\Omega_\tau} Q(u,u)\,dx = \int\limits_{S_\tau}\Big(\sum_{k,j=1}^{n}a^{kj}u_{x_k}u\nu_j + \tfrac{1}{2}\sum_{k=1}^{n}b^k\nu_k u^2\Big)dS + \\ + \tfrac{1}{2}\int\limits_{\partial\Omega_\tau\cap\gamma_2}\sum_{k=1}^{n}b^k\nu_k u^2\,dS + \int\limits_{\partial\Omega_\tau\cap\gamma_3}\Big(-a + \tfrac{1}{2}\sum_{k=1}^{n}b^k\nu_k\Big)u^2\,dS\,.$$

Taking into account conditions (10), (11) of theorem 1 we get

$$\int\limits_{\Omega_\tau} Q(u,u)\,dx \leqslant \sum_{l=1}^{p}\int\limits_{S_{\tau_l}}\sum_{k,j=1}^{n}a^{kj}\nu_j u_{x_k}u\,dS + \\ + \sum_{l=p+1}^{N}\int\limits_{S_{\tau_l}}\sum_{k,j=1}^{n}a^{kj}\nu_j u_{x_k}(u+c_l)\,dS + \sum_{l=1}^{p}\int\limits_{S_{\tau_l}}\sum_{k=1}^{n}\tfrac{1}{2}b^k\nu_k u^2\,dS \leqslant \\ \sum_{l=1}^{p}\Big(\int\limits_{S_{\tau_l}}\sum_{k,j=1}^{n}a^{kj}u_{x_k}u_{x_j}h_l\,dS\Big)^{\frac{1}{2}}\Big(\int\limits_{S_{\tau_l}}\sum_{k,j=1}^{n}a^{kj}\nu_j\nu_k h_l^{-1}u^2\,dS\Big)^{\frac{1}{2}} + \\ \sum_{l=p+1}^{N}\Big(\int\limits_{S_{\tau_l}}\sum_{k,j=1}^{n}a^{kj}u_{x_k}u_{x_j}h_l\,dS\Big)^{\frac{1}{2}}\Big(\int\limits_{S_{\tau_l}}\sum_{k,j=1}^{n}a^{kj}\nu_k\nu_j(u+c_l)^2h_l^{-1}\,dS\Big)^{\frac{1}{2}} + \sum_{l=1}^{p}\int\limits_{S_{\tau_l}}B_l(\tau_l,x)u^2\,dS\,,$$

where $c_l$ are constants which we choose to satisfy the conditions:

$$\int\limits_{S_{\tau_l}}(u+c_l)\,dS = 0\,,\qquad l = p+1,\ldots,N\,.$$

These inequalities together with (4), (5), (6), (7) yield

$$(15)\qquad \int\limits_{\Omega_\tau} Q(u,u)\,dx \leqslant \sum_{l=1}^{p}[\lambda_l(\tau_l)]^{-\frac{1}{2}}\int\limits_{S_{\tau_l}}Q(u,u)h_l\,dS + \\ + \sum_{l=1}^{p}\mu_l(\tau_l)\int\limits_{S_{\tau_l}}Q(u,u)h_l\,dS + \sum_{l=p+1}^{N}[\lambda_l^{-1}(\tau_l)\eta_l(\tau_l)]^{\frac{1}{2}}\int\limits_{S_{\tau_l}}Q(u,u)h_l\,dS \leqslant \\ \leqslant \sum_{l=1}^{N}A_l(\tau_l)\int\limits_{S_{\tau_l}}Q(u,u)h_l\,dS\,.$$

Set $E(\tau) \equiv \int\limits_{\Omega_\tau}Q(u,u)\,dx.$

It follows from relations (3) and (15) that

$$E(\tau) \leqslant \sum_{l=1}^{N} A_l(\tau_l) \frac{\partial E}{\partial \tau_l}. \tag{16}$$

From (16) and the definition of the function $\varphi(R) = (\varphi_1(R), \ldots, \varphi_N(R))$ we have $E(\varphi(\sigma)) \leqslant (d/d\sigma) E(\varphi(\sigma))$ and, therefore, $(d/d\sigma)\ln E(\varphi(\sigma)) \geqslant 1$.

Integration of this inequality over the segment $(0, R)$ yields $E(0) \leqslant \exp[-R] E(\varphi(R))$. Thus theorem 1 is proved.

From theorem 1 immediately follows theorem 2, which is similar to Saint-Venant's principle, well-known in the theory of elasticity (see [1], [2]).

THEOREM 2. (Saint-Venant's principle). *Suppose that the domain $\Omega$ is bounded and that for any $l$ ($l = 1, \ldots, N$) the hypersurfaces $S_{\tau_l^0}$ and $\partial\Omega$ confine a domain $G_l$ which does not intersect with $\Omega_{\tau^0}$. Let $c \equiv 0$, $b^k \equiv 0$, $k = 1, \ldots, n$, in $\Omega$, $F \equiv 0$ in $\Omega_{\tau}$, and let $\gamma_1$, $\gamma_3$ be empty sets. If $\Psi_2 \equiv 0$ on $\gamma_2 \cap \partial\Omega_{\tau^0}$,*

$$\int_{G_l} F\,dx - \int_{\partial\Omega \cap \partial G_l} \Psi_2\,dS = 0, \qquad l = 1, \ldots, N, \tag{17}$$

*and if $u(x)$ is a weak solution of equation (1) in $\Omega$ satisfying boundary conditions (2) on $\partial\Omega$ and belonging to the class $C^1(\bar{\Omega}\setminus\Omega_0) \cap C^2(\Omega\setminus\bar{\Omega}_0)$, then the following estimate holds*

$$\int_{\Omega_0} \sum_{k,j=1}^{n} a^{kj} u_{x_k} u_{x_j}\,dx \leqslant \exp[-R] \int_{\Omega_{\varphi(R)}} \sum_{k,j=1}^{n} a^{kj} u_{x_k} u_{x_j}\,dx, \tag{18}$$

*where $\varphi(R) = (\varphi_1(R), \ldots, \varphi_N(R))$, $\varphi_l(R)$ is the inverse function to*

$$R = \phi_l(\tau_l) \equiv \int_0^{\tau_l} [A_l(\eta)]^{-1}\,d\eta, \qquad l = 1, \ldots, N,$$

*and $A_l(\tau_l)$, $l = 1, \ldots, N$, satisfy conditions (7) with $p = 0$.*

PROOF. Relation (17) implies that assumption (11) of theorem 1 is valid. We have, in fact,

$$\int_{G_l} F\,dx = \int_{G_l} L(u)\,dx = \int_{\partial G_l} \sum_{k,j=1}^{n} a^{kj} u_{x_k} \nu_j\,dS = \int_{\partial\Omega \cap \partial G_l} \Psi_2\,dS + \int_{S_{\tau_l^0}} \sum_{k,j=1}^{n} a^{kj} u_{x_k} \nu_j\,dS,$$

and, therefore, due to (17)

$$\int\limits_{S_{\tau_l^0}} \sum_{k,j=1}^{n} a^{kj} u_{x_k} \nu_j \, dS = 0 , \qquad l = 1, \dots, N .$$

Let $\tau' = (\tau_1^0, \dots, \tau_{l-1}^0, \tau_l, \tau_{l+1}^0, \dots, \tau_N^0)$ where $0 \leqslant \tau_l < \tau_l^0$.
Then

$$0 = \int\limits_{\Omega_{\tau^0} \setminus \Omega_{\tau'}} L(u)\, dx = \int\limits_{S_{\tau_l^0}} \sum_{k,j=1}^{n} a^{kj} u_{x_k} \nu_j \, dS - \int\limits_{S_{\tau_l}} \sum_{k,j=1}^{n} a^{kj} u_{x_k} \nu_j \, dS$$

and consequently (11) holds. It is easy to see that the other assumptions of theorem 1 follow from those of theorem 2. Therefore (12) is true and so is (18). Theorem 2 is proved.

Note that if the coefficients of (1), the boundary $\partial\Omega$ and the functions $F, \Psi_2$ are sufficiently smooth, then according to the well-known results ([3], [4]) the weak solution $u(x)$ of problem (1), (2) belongs to the class $C^1(\bar{\Omega} \setminus \Omega_0) \cap \cap C^2(\Omega \setminus \bar{\Omega}_0)$, provided that the conditions of theorem 2 are satisfied and equation (1) is elliptic in $\Omega$.

We shall now consider some particular cases of theorem 2.

1) Let $\Omega = \omega \times \{x_n \colon 0 < x_n < T\}$, where the domain $\omega \subset \boldsymbol{R}_{x'}^{n-1} = (x_1, \dots, x_{n-1})$. Let $\sigma = \text{const} > 0$, $\sigma + \tau_1^0 < T$,

$$\Omega_{\tau_1} = \omega \times \{x_n \colon 0 < x_n < \sigma + \tau_1\} .$$

Suppose that

$$a_0 |\xi|^2 \leqslant \sum_{k,j=1}^{n} a^{kj} \xi_k \xi_j \leqslant a_1 |\xi|^2 , \qquad a_0, a_1 = \text{const} > 0 . \tag{19}$$

In that case inequality (18) becomes

$$\int\limits_{\Omega_0} \sum_{k,j=1}^{n} a^{kj} u_{x_k} u_{x_j} \, dx \leqslant \exp[-\tau_1 \Lambda] \int\limits_{\Omega_{\tau_1}} \sum_{k,j=1}^{n} a^{kj} u_{x_k} u_{x_j} \, dx , \tag{20}$$

where $\Lambda = [\lambda a_0 / a_1]^{\frac{1}{2}}$, $\lambda$ is equal to the smallest positive eigenvalue of the following Neuman problem

$$\Delta_{x'} v + \lambda v = 0 \quad \text{in } \omega , \qquad \left.\frac{\partial v}{\partial \nu}\right|_{\partial\omega} = 0 .$$

It is evident that $G_1 = \omega \times \{x_n \colon \sigma + \tau_1^0 < x_n < T\}$. If $F \equiv 0$ in $G_1$, $\partial u / \partial \beta \equiv 0$ on $\partial\Omega$ for $x_n < T$, and $\partial u / \partial \beta = \Psi_2$ for $x_n = T$, then relations (17), which provide that (18) holds, take the form

$$\int\limits_{\partial\Omega\cap\{x:x_n=T\}} \Psi_2 dS = 0 . \tag{21}$$

The estimate (20) and the condition (21) correspond to Saint-Venant's principle in its simplest form (see [1], [2]).

2) Suppose that $\Omega$ belongs to the half-space $\{x: x_n > 0\}$ and that the intersection of $\Omega$ with the plane $\{x: x_n = \tau_1 + \sigma\}$, $\sigma = \text{const} > 0$, is a domain $S_{\tau_1}$ such that the first positive eigenvalue for the problem

$$\Delta_{x'} u + \lambda u = 0 \quad \text{on } S_{\tau_1}, \qquad \left.\frac{\partial u}{\partial \nu}\right|_{\partial S_{\tau_1}} = 0 \tag{22}$$

equals $\lambda(\tau_1)$. We assume that $a_0(\tau_1)|\xi|^2 \leqslant \sum\limits_{k,j=1}^{n} a^{kj}\xi_k\xi_j \leqslant a_1(\tau_1)|\xi|^2$ on $S_{\tau_1}$, where $a_0(\tau_1) > 0$, $a_1(\tau_1) > 0$. Let $\Omega_{\tau_1} = \Omega \cap \{x: 0 < x_n < \sigma + \tau_1\}$.

From theorem 2 follows the estimate

$$\int\limits_{\Omega_0} \sum_{k,j=1}^{n} a^{kj} u_{x_k} u_{x_j} dx \leqslant \exp\left\{-\int\limits_0^{\tau_1} \left(\frac{a_0(\tau)\lambda(\tau)}{a_1(\tau)}\right)^{\frac12} d\tau\right\} \int\limits_{\Omega_{\tau_1}} \sum_{k,j=1}^{n} a^{kj} u_{x_k} u_{x_j} dx . \tag{23}$$

If $S_{\tau_1}$ is a ball of radius $f(\tau_1)$, then $\lambda(\tau_1) = K_1[f(\tau_1)]^{-2}$, where $K_1$ is the first positive eigenvalue of problem (22) when $S_{\tau_1}$ is a unit ball. According to formula (23) we have in that case

$$\int\limits_{\Omega_0} \sum_{k,j=1}^{n} a^{kj} u_{x_k} u_{x_j} dx \leqslant \exp\left\{-\int\limits_0^{\tau_1} \left(\frac{a_0(\tau)}{a_1(\tau)} K_1\right)^{\frac12} \frac{1}{f(\tau)} d\tau\right\} \int\limits_{\Omega_{\tau_1}} \sum_{k,j=1}^{n} a^{kj} u_{x_k} u_{x_j} dx . \tag{24}$$

It is of interest to note that if $\Omega$ is a cone, i.e. $f(\tau_1) = M_1(\tau_1 + \sigma)$, $M_1$, $\sigma = \text{const} > 0$, and if $a_0(\tau_1) = \text{const}$, $a_1(\tau_1) = \text{const}$, then

$$\exp\left\{-\int\limits_0^{\tau_1} \left(K_1 \frac{a_0(\tau)}{a_1(\tau)}\right)^{\frac12} \frac{1}{f(\tau)} d\tau\right\} = \left(\frac{\tau_1}{\sigma} + 1\right)^{-\theta}$$

where $\theta = (K_1 a_0 / a_1 M_1^2)^{\frac12}$.

This shows that the decay of the factor in (23) is not necessarily exponential, as $\tau_1$ tends to infinity.

The formula (24) is also valid when for any $\tau_1$ the set $S_{\tau_1}$ is a domain of $\boldsymbol{R}^{n-1}_{x'}$ such that the transformation of the coordinates $x'_j = x_j(f(\tau))^{-1}$, $j = 1, \dots, n-1$, maps $S_{\tau_1}$ onto a domain $\tilde{S}$ independent of $\tau_1$. The constant $K_1$ in that case is equal to the first positive eigenvalue of problem (22) in $\tilde{S}$.

As another corollary of theorem 1 we obtain

THEOREM 3. *Let $u(x)$ be a weak solution of equation* (I) *in $\Omega$ satisfying boundary conditions* (2) *on $\partial\Omega$. Suppose that*

$$\frac{1}{2}\sum_{k=1}^{n} b^k_{x_k} - c \geqslant c_l(\tau_l) > 0 ,$$

$$\sum_{k,j=1}^{n} a^{kj}(x)\xi_k\xi_j \leqslant a_l(\tau_l)|\xi|^2 \quad \text{for } x \in S_{\tau_l},\ \xi \in \boldsymbol{R}^n_\xi,\ l = 1, \dots, N; \quad p = N,$$

$$\sum_{k=1}^{n} b^k\nu_k \leqslant 0 \quad \text{on } S_\tau; \qquad \sum_{k=1}^{n} b^k\nu_k \leqslant 0 \quad \text{on } \gamma_2 \cap \bar{\Omega}_{\tau^0};$$

$$\sum_{k=1}^{n} \tfrac{1}{2} b^k\nu_k - a \leqslant 0 \quad \text{on } \gamma_3 \cap \bar{\Omega}_{\tau^0}; \qquad F \equiv 0 \quad \text{in } \Omega_{\tau^0};$$

$$\Psi_k \equiv 0 \quad \text{on } \gamma_k \cap \partial\Omega_{\tau^0}, \qquad k = 1, 2, 3 .$$

*Then inequality* (12) *holds, where*

$$\phi_l(\tau_l) \equiv \int_0^{\tau_l} g_l(\tau_l)\left[\frac{c_l(\tau_l)}{a_l(\tau_l)}\right]^{\frac{1}{2}} d\tau_l , \qquad g_l(\tau_l) = \min_{x \in S_{\tau_l}} h_l(\tau_l, x) .$$

If $c_l(\tau_l)$, $a_l(\tau_l)$, $g_l(\tau_l)$ *are independent of $l$, then* (12) *may be written in the form*

$$\int_{\Omega_0} Q(u, u)\, dx \leqslant \exp\left\{-\int_0^s \left(\frac{c_1(s)}{a_1(s)}\right)^{\frac{1}{2}} g_1(s)\, ds\right\} \int_{\Omega_\tau} Q(u, u)\, dx$$

*where* $\tau = (\tau_1, \dots, \tau_N)$, $\tau_j = \tau_k = s$, $k, j = 1, \dots, N$.

In order to prove this theorem it is sufficient to observe that under the above assumptions $\lambda_l(\tau_l) \geqslant c_l(\tau_l) g_l^2(\tau_l)[a_l(\tau_l)]^{-1}$, $\mu_l(\tau_l) = 0$, $l = 1, \dots, N$, and therefore, in theorem 1, one can take

$$A_l(\tau_l) = [a_l(\tau_l)]^{\frac{1}{2}}[c_l(\tau_l) g_l^2(\tau_l)]^{-\frac{1}{2}} .$$

Making use of theorem 1 we shall now prove certain uniqueness theorems for solutions of the boundary value problems in unbounded domains in classes of growing functions.

In the following theorems it is assumed that the domain $\Omega$ is unbounded, the parameter $\tau$ ranges over the set

$$G = \{\tau\colon 0 \leqslant \tau_l < \infty,\ l = 1, \dots, N\}\,, \qquad \phi_l(\tau_l) \equiv \int_0^{\tau_l} [A_l(\eta)]^{-1}\, d\eta \to \infty$$

as $\tau_l \to \infty$, $l = 1, \dots, N$ and that for any positive $M$ the set $\Omega \setminus \Omega_\tau$ belongs to the domain $\{x\colon |x| > M\}$ if all $\tau_l$, $l = 1, \dots, N$, are sufficiently large.

A function $u(x)$ will be called a weak solution of problem (1), (2) in $\Omega$ for $\Psi_k \equiv 0$, $k = 1, 2, 3$, if in any bounded subdomain $B$ of $\Omega$ the function $u(x)$ is a weak solution of problem (1), (8) with

$$\Gamma_k = \gamma_k \cap \partial B\,, \qquad \Psi_2 \equiv 0 \quad \text{on } \Gamma_2\,, \qquad k = 1, 2, 3\,.$$

In the next theorem sufficient conditions are given for the uniqueness of a solution of the Neuman problem to within an additive constant.

THEOREM 4. *Let $u(x)$ be a weak solution of equation* (1) *in $\Omega$ satisfying boundary conditions* (2) *on $\partial\Omega$, and suppose that $\gamma_1 = \emptyset$, $\gamma_3 = \emptyset$; $\Psi_2 \equiv 0$ on $\partial\Omega$; $F \equiv 0$ in $\Omega$; $b^k \equiv 0$, $k = 1, \dots, n$, $c \equiv 0$ in $\Omega$;*

$$a_0(x)|\xi|^2 \leqslant \sum_{k,j=1}^{n} a^{kj}(x)\xi_k\xi_j \leqslant a_1(x)|\xi|^2\,, \qquad a_0(x) > 0\,, \qquad a_1(x) > 0\,;$$

$$u(x) \in C^1(\bar\Omega) \cap C^2(\Omega \setminus \bar\Omega_0)\,;$$

$$\int_{S_{\tau_l}} \sum_{k,j=1}^{n} a^{kj} u_{x_k} \nu_j\, dS \to 0 \qquad \text{as } \tau_l \to \infty,\ l = 1, \dots, N\,. \tag{25}$$

*If for a certain sequence $R_k \to \infty$*

$$\int_{\Omega_{\varphi(R_k)}} \sum_{k,j=1}^{n} a^{kj} u_{x_k} u_{x_j}\, dx \leqslant \alpha(R_k) \exp\,[R_k]\,, \tag{26}$$

*and $\alpha(R_k) \to 0$ as $R_k \to \infty$, then $u \equiv \text{const}$ in $\Omega$.*

PROOF. According to our assumptions equation (1) is uniformly elliptic. Therefore, $\lambda_l(\tau_l)$ is not smaller than $K_l(\tau_l)\Lambda_l(\tau_l)$ and $K_l(\tau_l) > 0$ depends only on the coefficients of (1) and the function $h_l(\tau_l, x)$; $\Lambda_l(\tau_l)$ is equal to the smallest positive eigenvalue of the Neuman problem for the Laplace equation on $S_{\tau_l}$. Hence, $\lambda_l(\tau_l) \neq 0$, $l = 1, \dots, N$, and, therefore, theorem 1 is valid for $u(x)$ provided that condition (11) is satisfied. It is easy to verify that (25) implies (11). Indeed, for

$$\tau' = (\tau_1, \ldots, \tau_{l-1}, \tau_l', \tau_{l+1}, \ldots, \tau_N), \qquad \tau_l' > \tau_l$$

we have

$$0 = \int_{\Omega_{\tau'} \setminus \Omega_\tau} L(u)\,dx = \int_{S_{\tau_l'}} \sum_{k,j=1}^{n} a^{kj} u_{x_j} \nu_k\, dS - \int_{S_{\tau_l}} \sum_{k,j=1}^{n} a^{kj} u_{x_j} \nu_k\, dS . \tag{27}$$

Letting $\tau_l'$ tend to infinity in (27) and taking into account (25) we obtain, that for any $\tau_l > 0$

$$\int_{S_{\tau_l}} \sum_{k,j=1}^{n} a^{kj} u_{x_k} \nu_k\, dS = 0 , \qquad l = 1, \ldots, N .$$

Thus according to theorem 1 for $u(x)$ inequality (18) holds. From (18) and (26) it follows that

$$\int_{\Omega_0} \sum_{k,j=1}^{n} a^{kj} u_{x_k} u_{x_j}\, dS \leqslant \alpha(R_k) .$$

Making $R_k$ tend to infinity in the last inequality yields

$$\int_{\Omega_0} \sum_{k,j=1}^{n} a^{kj} u_{x_k} u_{x_j}\, dx = 0 .$$

Hence, $u \equiv \text{const}$ in $\Omega_0$. Since for any $\tau$ the domain $\Omega_\tau$ may be considered as $\Omega_0$, we can conclude that $u \equiv \text{const}$ in $\Omega$.

Consider now some special cases of theorem 4.

For the cylinder $\Omega = \omega \times \{x_n : 0 < x_n < \infty\}$ considered in example 1), condition (26), specifying the class of uniqueness for the Neuman problem, is reduced to

$$\int_{\Omega_{\tau_1^k}} \sum_{l,j=1}^{n} a^{lj} u_{x_l} u_{x_j}\, dx \leqslant \alpha(\tau_1^k) \exp[\Lambda \tau_1^k] \tag{28}$$

where $\alpha(\tau_1^k) \to 0$ as $\tau_1^k \to \infty$, $\Lambda$ is the constant from inequality (20).

Let the domain $\Omega$ belong to the half-space $\{x : x_n > 0\}$ and let the intersection of $\Omega$ with the plane $\{x : x_n = \sigma + \tau_1\}$, $\sigma = \text{const} > 0$, which we denote by $S_{\tau_1}$, be a ball of radius $f(\tau_1)$ for every $\tau_1 > 0$. Suppose that the domain $\Omega \cap \{x : x_n < \sigma\}$ is finite and that condition (19) of the uniform ellipticity is satisfied. Then the class of uniqueness for the Neuman problem may be specified by the condition

$$\int_{\Omega_{R_k}} \sum_{l,j=1}^{n} a^{lj} u_{x_l} u_{x_j}\, dx \leqslant \alpha(R_k) \exp\left\{\left(\frac{a_0 K_1}{a_1}\right)^{\frac{1}{2}} \int_0^{R_k} \frac{1}{f(\tau_1)}\, d\tau_1\right\},$$

where $\alpha(R_k) \to 0$ as $R_k \to \infty$.

The constant $K_1$ is equal to the smallest positive eigenvalue of problem (22) for the unit ball of $R_{x'}^{n-1}$, $\Omega_{\tau_1} = \Omega \cap \{x\colon x_n < \tau_1 + \sigma\}$. If the domain $\Omega$ coincides for $x_n > \sigma > 0$ with a cone having a vertex at the origin and such that its intersection with the plane $\{x\colon x_n = \tau_1 + \sigma\}$, $\sigma = \text{const} > 0$, is a ball of radius $M_1(\sigma + \tau_1)$, then $\lambda(\tau_1)$ entering (23) equals $K_1[M_1(\tau_1 + \sigma)]^{-2}$, the constant $K_1$ being the first positive eigenvalue of problem (22) for the unit ball. Thus, for $\Omega$ of this kind, inequality (26) takes the form

$$\int_{\Omega_{R_k}} \sum_{l,j=1}^{n} a^{lj} u_{x_l} u_{x_j}\, dx \leqslant \alpha(R_k) \left(\frac{R_k}{\sigma} + 1\right)^{\theta}, \tag{29}$$

where $\alpha(R_k) \to 0$ as $R_k \to \infty$, $\theta = (K_1 a_0 / a_1 M_1^2)^{\frac{1}{2}}$.

3) Consider a domain $\Omega$ such that for some $\sigma > 0$ $\Omega \cap \{x\colon |x| > \sigma\} = \bigcup_{j=1}^{N} \Omega^j$. Suppose that $\Omega^j$ for every $j$ is a domain such that its intersection with the plane, orthogonal to some smooth curve $\mathfrak{L}_j$ at the point $P(\tau_j) \in \mathfrak{L}_j$, forms around that point the domain $S_{\tau_j}$; the parameter $\tau_j$ being equal to the length of $\mathfrak{L}_j$ measured from the surface $\{x\colon |x| = \sigma\}$ to the point $P(\tau_j)$. Suppose that $S_{\tau_j}$ is similar to a domain $S^j$, and that the similarity coefficient is equal to $f_j(\tau_j)$. Denote by $\Omega_{\tau_j}$ the subdomain of $\Omega$ bounded by $S_{\tau_j}$, $\partial\Omega$ and $S_{\tau_j}$ for $\tau_j = 0$. We assume that for $\Omega_\tau$ bounded by $\partial\Omega$ and $\bigcup_{j=1}^{N} S_{\tau_j}$ and containing $\{x\colon |x| \leqslant \sigma\} \cap \Omega$, equality (3) holds with

$$h_l(x, \tau_l) \geqslant T_l > 0 \quad \text{for } x \in S_{\tau_l},$$

where $T_l = \text{const}$, $l = 1, \ldots, N$.

Thus theorem 1 is valid if we take

$$A_j(\tau_j) = \frac{f_j(\tau_j)}{T_j} \left[\frac{a_{j0}(\tau_j) K_j}{a_{j1}(\tau_j)}\right]^{-\frac{1}{2}},$$

where $a_{j0}(\tau_j)|\xi|^2 \leqslant \sum_{k,j=1}^{n} a^{kj}(x) \xi_k \xi_j \leqslant a_{j1}(\tau_j)|\xi|^2$ for $x \in S_{\tau_j}$.

Therefore, the uniqueness class for the Neuman problem can be specified by the inequalities

$$\int_{\Omega_{\tau_j^k}} Q(u, u)\,dx \leqslant \alpha(\tau_j^k) \exp\left\{ \int_0^{\tau_j^k} \left(\frac{a_{j0}(s)}{a_{j1}(s)} K_j\right)^{\frac{1}{2}} \frac{T_j}{f_j(s)}\,ds \right\}, \qquad j = 1, 2, \ldots, N,$$

where $\alpha(\tau_j^k) \to 0$ as $\tau_j^k \to \infty$ and $K_j$ is the first positive eigenvalue of problem (22) on $S^j$.

It is interesting to note that for domains $\Omega$ of the above type the uniqueness class depends on the behavior of the curves $\mathfrak{L}_j$ and, in particular, on the length of the piece of $\mathfrak{L}_j$ enclosed in the ball of radius $R$.

THEOREM 5. (Uniqueness for the Dirichlet problem). *Suppose that* $\partial\Omega = \gamma_1$, $\Psi_1 \equiv 0$ *on* $\partial\Omega$, $F \equiv 0$ *in* $\Omega$,

$$a_1(x)|\xi|^2 \leqslant \sum_{k,j=1}^{n} a^{kj}(x)\xi_k\xi_j \leqslant a_2(x)|\xi|^2,$$

$$a_1(x) > 0, \qquad a_2(x) > 0.$$

*If a weak solution* $u(x)$ *of problem* (1), (2) *in* $\Omega$ *belongs to* $C^1(\overline{\Omega}\setminus\Omega_0)$ *and for a certain sequence* $R_k \to \infty$ *satisfies the condition*

$$\int_{\Omega_{\varphi(R_k)}} Q(u, u)\,dx \leqslant \alpha(R_k) \exp[R_k], \tag{30}$$

*where* $\alpha(R_k) \to 0$ *as* $R_k \to \infty$, $\varphi(R) = (\varphi_1(R), \ldots, \varphi_N(R))$, $\varphi_j(R)$ *is defined in theorem* 1, *then* $u \equiv 0$ *in* $\Omega$.

Theorem 5 follows directly from theorem 1.

Consider now some special cases of theorem 5.

Let $\Omega$ be the domain considered in example 3), but instead of the similarity of $S_{\tau_j}$ to $S^j$ with the coefficient of similarity $f_j(\tau_j)$ we assume here only that $S_{\tau_j}$ can be enclosed in a parallelepiped with the smallest edge equal to $f_j(\tau_j)$ and belonging to the same hyperplane as $S_{\tau_j}$. Then the conditions which guarantee the uniqueness of a weak solution of the Dirichlet problem in $\Omega$ may be written in the form

$$\int_{\Omega_{\tau_j^k}} Q(u, u)\,dx \leqslant \alpha_j(\tau_j^k) \exp\left\{ \int_0^{\tau_j^k} \left(\frac{a_{j0}(\tau_j)}{a_{j1}(\tau_j)}\right)^{\frac{1}{2}} \frac{\pi T_j}{f_j(\tau_j)}\,d\tau_j \right\} \tag{31}$$

where $\alpha_j(\tau_j^k) \to 0$ as $\tau_j^k \to \infty$, $j = 1, \ldots, N$, $k = 1, 2, \ldots$.

It is easy to see that inequality (30) follows from relations (31) because $\lambda_j(\tau_j) \geqslant T_j^2(a_{j0}/a_{j1})\Lambda_j(\tau_j)$ where $\Lambda_j(\tau_j)$ is the smallest eigenvalue of the Dirichlet problem for the Laplace equation on $S_{\tau_j}$, and it is well-known that $\Lambda_j(\tau_j) \geqslant \pi^2(f_j(\tau_j))^{-2}$.

Inequalities (31) show that the admissible growth of solutions in each $\Omega^j$ depends on the functions $f_j$, $a_{j0}$, $a_{j1}$ and the length of the piece of $\mathfrak{L}_j$ enclosed in the ball of radius $R$ as $R$ tends to infinity. Thus, there exist domains of the above type such that the uniqueness class for the corresponding Dirichlet problems includes functions of any preassigned growth in each $\Omega^j$. In a particular case, when $\Omega$ is a cylinder or a cone one can obtain from (31) relations similar to (28), (29). Theorem 5 may be considered as a generalization of the Phragmen-Lindelöf theorem for harmonic functions.

The following uniqueness theorem for mixed boundary value problems (1), (2) in unbounded domains is also a consequence of theorem 1.

THEOREM 6. *Suppose that for a weak solution $u(x)$ of problem* (1), (2) *in $\Omega$ the following conditions are satisfied*: $u(x) \in C^1(\bar{\Omega}\setminus\Omega_0)$,

$$\int_{S_{\tau_l}} \sum_{k,j=1}^{n} a^{kj} \nu_j u_{x_k} dS \to 0 \qquad \text{as } \tau_l \to \infty,\ l = p+1, \dots, N,$$

$p \neq 0$, *and suppose that inequalities* (10) *hold for $\Omega_{\tau^0} = \Omega$ and $a^{kj}(x)\xi_k\xi_j > 0$ for $|\xi| \neq 0$ in $\bar{\Omega}$. If $F \equiv 0$ in $\Omega$, $\Psi_k \equiv 0$ on $\gamma_k$, $k = 1, 2, 3$, and for a certain sequence $R_j \to \infty$*

$$\int_{\Omega_{\varphi(R_j)}} Q(u, u)\, dx \leqslant \alpha(R_j) \exp[R_j] \tag{32}$$

*and $\alpha(R_j) \to 0$ as $R_j \to \infty$, then $u \equiv 0$ in $\Omega$.*

REMARK. We specified the classes of functions which ensure the uniqueness of solutions of boundary value problems by imposing restrictions on the growth of the energy integrals $\int_{\Omega_{\varphi(R_j)}} Q(u, u)\, dx$ as $R_j \to \infty$. We shall now show that these classes can be specified by restrictions imposed on the growth of some sequence of integrals of $u^2$.

Let $\Omega' \subset \Omega''$ be arbitrary bounded subdomains of $\Omega$ such that the distance between $\partial\Omega'' \cap \Omega$ and $\partial\Omega' \cap \Omega$ is not smaller than $\varrho = \text{const} \geqslant 1$. Let $\psi \in C^\infty(\bar{\Omega})$, $\psi(x) = 1$ if $x \in \Omega'$, $\psi(x) = 0$ if $x \notin \Omega''$, $0 \leqslant \psi \leqslant 1$ in $\Omega$; $|\psi_{x_j}| \leqslant K\varrho^{-1}$, $j = 1, \dots, n$, where the constant $K$ does not depend on $\varrho$. Taking $v = u\psi^2$ in integral identity (9), we find that

$$\int_{\Omega''} \Big[ \sum_{k,j=1}^{n} a^{kj}(u\psi)_{x_k}(u\psi)_{x_j} + \Big(\tfrac{1}{2}\sum_{k=1}^{n} b^k_{x_k} - c\Big)(u\psi)^2 \Big] dx \leqslant$$
$$\leqslant \int_{\Omega''} \sum_{k,j=1}^{n} a^{kj}\psi_{x_k}\psi_{x_j} u^2\, dx - \int_{\Omega''} \sum_{k=1}^{n} b^k \psi_{x_k} u^2 \psi\, dx .$$

Thus,

$$\int_{\Omega'} Q(u, u)\, dx \leqslant \varrho^{-1} M(\Omega'') \int_{\Omega''} u^2\, dx ,$$

where

$$M(\Omega'') = nK^2 \sup_{\substack{x\in\Omega'' \\ |\xi|=1}} \left\{ \sum_{k,j=1}^{n} a^{kj}\xi_k\xi_j \right\} + K \sum_{i=1}^{n} \sup_{x\in\Omega''} |f^i(x)| .$$

Hence, the uniqueness condition (32) may be replaced by the condition

$$\int_{\Omega_{\tilde{\varphi}(R_k)}} u^2\, dx \leqslant \alpha_1(R_k)[M(\Omega_{\tilde{\varphi}(R_k)})]^{-1} \exp [R_k] ,$$

where $\alpha_1(R_k) \to 0$ as $R_k \to \infty$, the domain $\Omega_{\tilde{\varphi}(R_k)}$ consists of the points of $\Omega$ the distance from which to $\Omega_{\varphi(R_k)}$ does not exceed 1.

Next, we shall prove existence theorems for the above boundary value problems in unbounded domains. From now on we shall assume that at the points of $\Omega$

$$\text{(33)} \qquad \begin{cases} a_0(x)|\xi|^2 \leqslant \sum_{k,j=1}^{n} a^{kj}(x)\xi_k\xi_j \leqslant a_1(x)|\xi|^2 , \qquad a_1(x) > 0,\ a_0(x) > 0 , \\ c - \frac{1}{2}\sum_{k=1}^{n} b^k_{x_k} \leqslant 0 . \end{cases}$$

LEMMA 1. *Let $B$ be a bounded subdomain of $\Omega$ and let*

$$\Gamma_1 = \gamma_1 \cap \partial B , \quad \Gamma_2 = \gamma_2 \cap \partial B , \quad \Gamma_3 = \gamma_3 \cap \partial B , \quad \tilde{\Gamma}_1 = \partial B \setminus (\bar{\Gamma}_2 \cup \bar{\Gamma}_3) .$$

*Suppose that* $\frac{1}{2}\sum_{k=1}^{n} b^k \nu_k - a \leqslant 0$ *on* $\Gamma_3$, $\sum_{k=1}^{n} b^k \nu_k \leqslant 0$ *on* $\Gamma_2$ *and that conditions* (33) *hold. Then for any* $F \in L_2(B)$ *there exists a weak solution* $u(x)$ *of equation* (1) *in* $B$ *satisfying the boundary conditions*

$$\text{(34)} \qquad u\Big|_{\tilde{\Gamma}_1} = 0 , \qquad \frac{\partial u}{\partial \beta}\Big|_{\Gamma_2} = 0 , \qquad \left(\frac{\partial u}{\partial \beta} + au\right)\Big|_{\Gamma_3} = 0 ,$$

*and for* $u(x)$ *the following estimate holds*

$$\text{(35)} \qquad \int_B Q(u, u)\, dx \leqslant [\Lambda(B)]^{-1} \int_B F^2\, dx ,$$

*where*

$$\Lambda(B) = \inf_{u \in V(\tilde{\Gamma}_1, B)} \left\{ \int_B Q(u, u)\, dx \Big/ \int_B u^2\, dx \right\} .$$

PROOF. This lemma follows from Lax-Milgram's theorem [5]. Indeed, for any $u$, $v \in V(\tilde{\Gamma}_1, B)$ set

$$K(u, v) = \int_B \Big( \sum_{k,j=1}^n a^{kj} u_{x_k} v_{x_j} - \sum_{k=1}^n b^k u_{x_k} v - cuv \Big) dx + \int_{\Gamma_3} auv \, dS .$$

Then

$$K(v, v) = \int_B \Big( \sum_{k,j=1}^n a^{kj} v_{x_k} v_{x_j} - \Big( c - \tfrac{1}{2} \sum_{k=1}^n b^k_{x_k} \Big) v^2 \Big) dx + \\ + \int_{\Gamma_3} \Big( -\tfrac{1}{2} \sum_{k=1}^n b^k \nu_k + a \Big) v^2 \, dS - \tfrac{1}{2} \int_{\Gamma_2} \sum_{k=1}^n b^k \nu_k v^2 \, dS .$$

It is easy to see that under the above assumptions we have $K(v, v) \geqslant \geqslant M_0 \|v\|^2_{H^1(B)}$ and, therefore, according to the Lax-Milgram theorem there exists a weak solution $u(x)$ of problem (1), (34) in $B$. The estimate (35) follows from the integral identity (9), if we take $v = u$. The lemma is proved.

A priori estimate (12) can also be used to prove the existence of solutions of boundary value problems for equation (1) in unbounded domains.

LEMMA 2. *Suppose that there exists an infinite sequence of bounded subdomains $\Omega_k$, $k = 0, 1, 2, \ldots$, of $\Omega$ such that $\Omega_k \subset \Omega_{k+1}$, $\Omega = \bigcup_{k=0}^{\infty} \Omega_k$. Suppose that for any fixed $i$ $(i = 0, 1, \ldots)$ and any function $w$ which is a weak solution of equation* (1) *in $\Omega_{i+1}$ with $F \equiv 0$ satisfying boundary conditions* (8) *on $\Gamma_j = \gamma_j \cap \partial\Omega_{i+1}$, $j = 1, 2, 3$, with $\Psi_2 = 0$, the following estimate holds*

$$\int_{\Omega_i} Q(w, w) \, dx \leqslant e^{-1} \int_{\Omega_{i+1}} Q(w, w) \, dx . \tag{36}$$

*Suppose that in each region $\Omega_k$ either $\gamma_1 \cap \partial\Omega_k \neq \emptyset$ or $\frac{1}{2} \sum_{k=1}^n b^k_{x_k} - c > 0$. Let $F$ be a function defined in $\Omega$ and let the following condition be imposed on its growth*

$$[\Lambda(\Omega_l)]^{-1} \int_{\Omega_l} F^2 \, dx \leqslant M_1 \exp\{(1 - \varepsilon) l\} , \qquad l = 1, 2, \ldots , \tag{37}$$

*where* $0 < \varepsilon < 1$, $\varepsilon = \text{const}$,

$$\Lambda(\Omega_l) = \inf \Big\{ \int_{\Omega_l} Q(v, v) \, dx \Big/ \int_{\Omega_l} v^2 \, dx, \; v \in V\big(\partial\Omega_l \setminus (\bar{\gamma}_2 \cup \bar{\gamma}_3), \Omega_l\big) \Big\} ,$$

*$M_1$ is a constant independent of $l$. Then there exists a weak solution $u(x)$ of problem* (1), (2) *in $\Omega$ and this solution satisfies the inequalities*

$$\int_{\Omega_l} Q(u, u)\,dx \leqslant M_2 \exp\{(1-\varepsilon)l\}, \qquad l = 1, 2, \ldots, \tag{38}$$

*where $M_2$ is a constant independent of $l$.*

PROOF. Fix any subdomain $\Omega_k$ of the sequence $\Omega_0 \subset \Omega_1 \subset \ldots$ and consider the sequence of subdomains $\Omega_{k+m}$, $m \to \infty$. Denote by $u_m(x)$ a weak solution of problem (1), (34) for $B = \Omega_{k+m}$,

$$\Gamma_2 = \gamma_2 \cap \partial B, \quad \Gamma_3 = \gamma_3 \cap \partial B, \quad \bar{\Gamma}_1 = \partial B \setminus (\bar{\Gamma}_2 \cup \bar{\Gamma}_3).$$

According to lemma 1 such a solution $u_m(x)$ exists. From relations (35) and (37) the following is obtained:

$$\int_{\Omega_{k+m}} Q(u_m, u_m)\,dx \leqslant M_1 \exp\{(1-\varepsilon)(k+m)\}, \qquad m = 1, 2, \ldots \tag{39}$$

Set

$$\langle v \rangle_{\Omega_l} = \Big( \int_{\Omega_l} Q(v, v)\,dx \Big)^{\frac{1}{2}}.$$

For any positive integers $m$, $m'$ the function $w = u_m - u_{m+m'}$ is a solution of equation (1) in $\Omega_{k+m}$ with $F \equiv 0$, satisfying boundary conditions (8) on $\Gamma_s = \partial\Omega_{k+m} \cap \gamma_s$, $s = 1, 2, 3$, with $\Psi_2 \equiv 0$. Thus, applying successively estimate (36) in the domains $\Omega_k \subset \ldots \subset \Omega_{k+m}$ and taking into account inequality (39) we deduce that

$$\begin{aligned}
\langle u_m - u_{m+m'} \rangle_{\Omega_k} &\leqslant \exp\Big(-\frac{m}{2}\Big) \langle u_m - u_{m+m'} \rangle_{\Omega_{k+m}} \leqslant \\
&\leqslant \exp\Big(-\frac{m}{2}\Big) \big( \langle u_m \rangle_{\Omega_{k+m}} + \langle u_{m+m'} \rangle_{\Omega_{k+m+m'}} \big) \leqslant \\
&\leqslant 2\sqrt{M_1} \exp\Big\{-\frac{m}{2}\Big\} \exp\Big\{\frac{1}{2}(1-\varepsilon)(k+m+m')\Big\} = \\
&= 2\sqrt{M_1} \exp\Big\{\frac{1}{2}(1-\varepsilon)(k+m')\Big\} \exp\Big\{-\frac{\varepsilon m}{2}\Big\}.
\end{aligned} \tag{40}$$

For any $s > 0$ and $t > 0$ one may conclude from inequalities (40), that

$$(41)\qquad \langle u_s - u_{s+t}\rangle_{\Omega_k} \leqslant \sum_{i=0}^{t-1} \langle u_{s+i} - u_{s+i+1}\rangle_{\Omega_k} \leqslant$$
$$\leqslant 2\sqrt{M_1} \exp\left\{\frac{1}{2}(1-\varepsilon)k\right\} \exp\left\{\frac{(1-\varepsilon)}{2}\right\} \sum_{i=0}^{t-1} \exp\left\{\frac{-\varepsilon(s+i)}{2}\right\} \leqslant$$
$$\leqslant M_3 \exp\left\{\frac{(1-\varepsilon)k}{2}\right\} \exp\left\{-\frac{\varepsilon s}{2}\right\},$$

where the constant $M_3$ does not depend on $s$ and $t$. Hence, for any $t > 0$ we have $\langle u_s - u_{s+t}\rangle_{\Omega_k} \to 0$ as $s \to \infty$. According to our assumptions, either $\bar{\Omega}_k \cap \gamma_1 \neq \emptyset$ or $\frac{1}{2} \sum_{k=1}^{n} b^k_{x_k} - c > 0$ in $\Omega_k$ and, therefore, $\|u_s - u_{s+t}\|^2_{H^1(\Omega_k)} \leqslant \leqslant M \langle u_s - u_{s+t}\rangle^2_{\Omega_k}$, $M = \text{const}$. Thus, the relation $\langle u_s - u_{s+t}\rangle_{\Omega_k} \to 0$ as $s \to \infty$ implies that the sequence $\{u_s\}$ converges with respect to $H^1(\Omega_k)$-norm to a function $u(x) \in H^1(\Omega_k)$ and, owing to the well-known imbedding theorems, it also implies that on the set $\gamma_3 \cap \bar{\Omega}_k$ the functions $u_s(x)$ converge to $u(x)$ with respect to the $L_2(\gamma_3 \cap \bar{\Omega}_k)$ norm.

So it is possible to make $s$ tend to infinity in the integral identity (9) for $B = \Omega_k$ and $u = u_s$. It follows that $u(x)$ is a weak solution of problem (1), (2) in $\Omega$. Setting $s = 1$ in (41), making $t$ tend to infinity and taking into account (39), we find that

$$\langle u\rangle_{\Omega_k} \leqslant M_3 \exp\left\{\frac{(1-\varepsilon)k}{2}\right\} + \langle u_1\rangle_{\Omega_k} \leqslant$$
$$\leqslant M_3 \exp\left\{\frac{(1-\varepsilon)k}{2}\right\} + M_1^{\frac{1}{2}} \exp\left\{\frac{(1-\varepsilon)}{2}(k+1)\right\},$$

which implies (38). If $u$ and $v$ are solutions of the problem (1), (2) in $\Omega$ such that for $u$ and $v$ relation (38) is valid, then it follows from estimates (36) that

$$\int_{\Omega_l} Q(u-v, u-v)\,dx \leqslant \exp[-j] \int_{\Omega_{l+j}} Q(u-v, u-v)\,dx \leqslant$$
$$\leqslant \exp[-j]\, 2\left(\int_{\Omega_{l+j}} Q(u,u)\,dx + \int_{\Omega_{l+j}} Q(v,v)\,dx\right) \leqslant M_4 \exp[-\varepsilon j],$$

where $M_4$ is a constant independent of $j$. Letting $j$ tend to infinity, we find that $u - v = 0$ in $\Omega$. The lemma is proved.

Thus the proof of the existence of weak solutions of problem (1), (2) in $\Omega$ is based on the assumption that estimates (36) hold. Estimates of this kind can be obtained by utilizing theorem 1. However, in theorem 1, the only weak solutions which are considered are those that belong to the class $C^1(\bar{\omega})$ in some subdomains $\omega$ of $\Omega$, which means in general that there are no intersections of $\bar{\gamma}_1$, $\bar{\gamma}_2$, $\bar{\gamma}_3$ on $\partial\omega$. Therefore, in order to obtain the existence of weak solutions on the basis of theorem 1 and lemma 2, one has to require that certain parts of $\partial\Omega$ belong either to $\gamma_1$, or $\gamma_2$, or $\gamma_3$, and also that the coefficients of equation (1) and the domain $\Omega$ be sufficiently smooth. Actually theorem 1 is also valid for weak solutions $u(x)$ of class $H_1^{\text{loc}}(\Omega)$ only, provided that the boundary of $\Omega$ and the coefficients of (1) are sufficiently smooth. The proof of that fact is given in paper [7], which also contains results on the existence of solutions from this class. Estimates which constitute Saint-Venant's Principle for parabolic equations are proved in [8].

Lemma 2 and theorem 1 lead to the following result concerning the existence and the uniqueness of a solution of the boundary value problem (1), (2).

THEOREM 7. *Suppose that the coefficients of equation* (1) *in* $\bar{\Omega}\setminus\Omega_0$, *the function* $a(x)$ *on* $\gamma_3$ *and also* $\partial\Omega\setminus\bar{\Omega}_0$ *are sufficiently smooth. Let*

$$\bar{S}_{\tau_l}\cap\partial\Omega\subset\gamma_1\,,\qquad l=1,\dots,q\,;$$

$$\bar{S}_{\tau_l}\cap\partial\Omega\subset\gamma_3\,,\qquad c-\tfrac{1}{2}\sum_{k=1}^{n}b^k_{x_k}<0\qquad on\ \bar{S}_{\tau_l}\,,\ l=q+1,\dots,p_1\,;$$

$$\bar{S}_{\tau_l}\cap\partial\Omega\subset\gamma_2\,,\qquad c-\tfrac{1}{2}\sum_{k=1}^{n}b^k_{x_k}<0\qquad on\ \bar{S}_{\tau_l}\,,\ l=p_1+1,\dots,N\,;$$

$$0<\tau_l<\infty\,,\qquad l=1,\dots,N\,;\qquad \Psi_k\equiv 0,\ k=1,2,3\,.$$

*Suppose that conditions* (10) *are valid for* $\Omega_{\tau^0}=\Omega$, $p=N$ *and for the function* $F(x)$ *the following inequalities hold*

$$[\Lambda(\Omega_{\varphi(k)})]^{-1}\int_{\Omega_{\varphi(k)}}F^2\,dx\leqslant M_1\exp\{k(1-\varepsilon)\}\,,\qquad k=1,2,\dots,$$

*where* $0<\varepsilon<1$, *the constants* $\varepsilon$ *and* $M_1$ *do not depend on* $k$, $\varphi(R)=$ $=(\varphi_1(R),\dots,\varphi_N(R))$ *is the vector-function defined in theorem* 1,

$$\Lambda(\mathfrak{D})=\inf\left\{\int_{\mathfrak{D}}Q(u,u)\,dx\Big/\int_{\mathfrak{D}}u^2\,dx,\ u\in V(\partial\mathfrak{D}\setminus(\bar{\gamma}_2\cup\bar{\gamma}_3),\mathfrak{D})\right\}.$$

*Then there exists a weak solution* $u(x)$ *of problem* (1), (2), *which satisfies the* *inequalities*

$$\int_{\Omega_{\varphi(k)}} Q(u,u)dx \leqslant M_2 \exp\big(k(1-\varepsilon)\big), \qquad k=1,2,\ldots,$$

*where* $M_2$ *is a constant independent of* $k$. *Such a solution* $u(x)$ *is unique.*

PROOF. Let us set $\Omega_j = \Omega_{\varphi(j)}$ in lemma 2. Inequalities (36) for $w$ follow from theorem 1, since $w \in C^1(\Omega_{\varphi(k+1-\delta)} \diagdown \Omega_{\varphi(k)})$ for any $\delta \in (0,1)$, which is due to our assumptions of the smoothness of $\partial\Omega \diagdown \bar{\Omega}_0$, of the coefficients of (1) and of $a(x)$. (See [3], [4]). Thus, the assertions of theorem 7 follow from lemma 2.

Consider the domain $\Omega$ described in example 3). Let $\partial\Omega = \gamma_1$ and suppose that the functions $f_j(\tau_j)$, $a_{j1}(\tau_j)$, $j=1,\ldots,N$, are uniformly bounded. Then there exists a solution $u(x)$ of the Dirichlet problem (1), (2), if the right hand side of (1) $F(x)$ is such that

$$\int_{\Omega_{\tau_j}} F^2 dx \leqslant M_j \exp\left\{(1-\varepsilon)\int_0^{\tau_j} \left(\frac{a_{j0}(s)}{a_{j1}(s)}\right)^{\frac{1}{2}} \frac{\pi}{f_j(s)}\, ds\right\}, \quad j=1,\ldots,N,$$

where $0<\varepsilon<1$ and the constants $M_j$, $\varepsilon$ are independent of $\tau_j$, $j=1,\ldots,N$.

THEOREM 8. (Existence of a solution of the Neuman problem). *Let* $\partial\Omega$ *be a smooth surface of class* $C^2$, $a^{kj} \in C^2(\bar{\Omega})$, *and let* $S_\tau$ *be a connected set (i.e.* $N=1$). *Suppose that* $\Psi_2 \equiv 0$; $b^k \equiv 0$, $k=1,\ldots,n$; $c \equiv 0$, $\partial\Omega = \gamma_2$ *and the function* $F(x)$ *satisfies the conditions*

$$[\Lambda(\Omega_{\varphi(k)})]^{-1} \int_{\Omega_{\varphi(k)}} F^2 dx \leqslant \tilde{M}_1 \exp\{(1-\varepsilon)k\}, \qquad k=1,2,\ldots,$$

*where* $\varphi(R)$ *is defined in theorem* 1, $\tilde{M}_1$, $\varepsilon$ *are constants,* $\varepsilon \in (0,1)$,

$$\Lambda(\Omega_{\varphi(k)}) = \inf\left\{ \int_{\Omega_{\varphi(k)}} Q(u,u)\,dx \Big/ \int_{\Omega_{\varphi(k)}} u^2 dx,\ u \in V(\partial\Omega_{\varphi(k)} \diagdown \gamma_2, \Omega_{\varphi(k)})\right\}.$$

*Then there exists a solution* $u(x)$ *of the Neuman problem* (1), (2) *which satisfies the inequalities*

$$\int_{\Omega_{\varphi(l)}} Q(u,u)\,dx \leqslant \tilde{M}_2 \exp\big(l(1-\varepsilon)\big), \qquad l=1,2,\ldots$$

*where the constant* $\tilde{M}_2$ *is independent of* $l$.

*Such a solution is unique to within an additive constant.*

PROOF. We assumed that $\partial\Omega = \gamma_2$, $N=1$. Therefore, if $w$ is a weak solution of (1) in $\Omega_{\varphi(i+1)}$ for $F \equiv 0$ satisfying boundary conditions (8) for $\Gamma_1 = \Gamma_3 = \emptyset$, $\Gamma_2 = \partial\Omega_{\varphi(i+1)} \cap \partial\Omega$, $\Psi_2 \equiv 0$, then

$$\int_{S_\tau} \sum_{k,j=1}^{n} a^{kj} \nu_j w_{x_k} dS = 0 , \qquad \tau < i+1 .$$

From the results on local smoothness of weak solutions of the boundary value problems ([3], [4], [6]) it follows that $w \in C^1(\overline{\Omega}_{\varphi(i+1-\delta)})$ for any $\delta \in (0,1)$. Thus, theorem 1 yields estimates (36). Fix an arbitrary integer $k$ and consider the sequence of the subdomains $\Omega_{k+m} = \Omega_{\varphi(k+m)}$ as $m \to \infty$. We define the functions $u_m(x)$ and deduce that for any $t>0$ $\langle u_s - u_{s+t}\rangle_{\Omega_k} \to 0$ as $s \to \infty$, in exactly the same manner as in lemma 2. It follows that $\partial u_s/\partial x_j$ $(j=1,\dots,n)$ converges in $L_2(\Omega_k)$-norm to $\partial U^k/\partial x_j$ respectively, and $U^k$ belongs to $H^1(\Omega_k)$. It is easy to see that for $k'>k$ we have $U^{k'} = U^k + c^{k'}$ in $\Omega_k$, where $c^{k'}$ is a constant. Choose the constants $c^{k'}$ in such a way that $U^{k'} = u(x)$ and for any bounded subdomain $\Omega'$ of $\Omega$ $u(x) \in H^1(\Omega')$. It is evident that $u(x)$ is a weak solution of problem (1), (2) in $\Omega$. The uniqueness of $u(x)$ (to within an additive constant) follows from theorem 4.

## BIBLIOGRAPHY


[1] R. A. TOUPIN, *Saint-Venant's principle*, Arch. Ration. Mech. and Anal., Vol. 18, no. 2 (1965).

[2] M. GURTIN, *The linear theory of elasticity*, Handbuch der physik, VI a/2, Springer Verlag, 1973.

[3] C. MIRANDA, *Equazioni alle derivate parziali di tipo ellittico*, Springer Verlag, 1955.

[4] E. I. MOISEEV, *On the Neuman problem is piecewise-smooth domains*, Diff. uravnenia, Vol. 7, no. 9 (1971), pp. 1655-1666.

[5] K. YOSIDA, *Functional analysis*, Springer Verlag, 1965.

[6] G. FICHERA, *Existence theorems in elasticity*, Handbuch der physik, VI a/2, Springer Verlag, 1973.

[7] O. A. OLEINIK - G. A. YOSIFIAN, *Energy estimates for weak solutions of boundary value problems for second order elliptic equations and their applications*, Dokl. Akad. Nauk SSSR, **232**, no. 6 (1977).

[8] O. A. OLEINIK - G. A. YOSIFIAN, *An analogue of Saint-Venant's principle and the uniqueness of solutions of boundary value problems for parabolic equations in unbounded domains*, Uspechi Mat. Nauk, **31**, no. 6 (1976), pp. 142-166.