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The Sieve of Eratosthenes-Legendre.

HENRYK IWANIEC (*)

By the sieve of Eratosthenes-Legendre we mean that described by Halberstam and Richert in their beautiful book [1], Chapter 1.

Let be given a finite sequence $\mathcal{A}$ of integers and a set $\mathcal{P}$ of primes. For each real number $z \geq 2$ let

$$P(z) = \prod_{p<z,\, p\in\mathcal{P}} p$$

and

$$S(\mathcal{A}, \mathcal{P}, z) = \sum_{\substack{a\in\mathcal{A} \\ (a,P(z))=1}} 1 \,,$$

the number of elements in the sequence $\mathcal{A}$ which are not divisible by any prime number $p < z$ from $\mathcal{P}$. The sieve method deals with estimates of $S(\mathcal{A}, \mathcal{P}, z)$ by linear forms in

(1) $$|\mathcal{A}_d| = \sum_{\substack{a\in\mathcal{A} \\ a\equiv 0 \,(\mathrm{mod}\, d)}} 1 \,,$$

(the number of elements in $\mathcal{A}$ which are divisible by $d|P(z)$)

(2) $$\sum_{d|P(z)} \lambda_d^- |\mathcal{A}_d| \leq S(\mathcal{A}, \mathcal{P}, z) \leq \sum_{d|P(z)} \lambda_d^+ |\mathcal{A}_d| \,.$$

The multipliers $\lambda_d^{\pm}$, usually called « weights », are real numbers satisfying the following conditions

(3) $$\lambda_1^- = \lambda_1^+ = 1 \,,$$

(4) $$\sum_{d|D} \lambda_d^- \leq 0 \leq \sum_{d|D} \lambda_d^+ \qquad \text{for all } D > 1,\ D|P(z) \,.$$

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Next, for each squarefree integer d we choose $\omega(d)$ so that $(\omega(d)/d)X$, where X is a suitable number, approximates $|\mathcal{A}_d|$, and we write the remainder as

$$r_d = |\mathcal{A}_d| - \frac{\omega(d)}{d} X .$$

Inserting this into (2) we get

$$S(\mathcal{A}, \mathcal{P}, z) \leqslant X \sum_{d|P(z)} \lambda_d^+ \frac{\omega(d)}{d} + \sum_{d|P(z)} |\lambda_d^+ r_d| , \tag{5}$$

$$S(\mathcal{A}, \mathcal{P}, z) \geqslant X \sum_{d|P(z)} \lambda_d^- \frac{\omega(d)}{d} - \sum_{d|P(z)} |\lambda_d^- r_d| . \tag{6}$$

We want to make these estimates optimal. This requirement determines the parameter X and the function $\omega(d)$ almost uniquely. It appears in practice that $\omega(d)$ is multiplicative and for some $\varkappa \geqslant 0$ satisfies the condition

$$-L \leqslant \sum_{w \leqslant p < z} \frac{\omega(p) - \varkappa}{p} \log p \leqslant A_2 , \tag{7}$$

for all $2 \leqslant w < z$, where L and A_2 are some constants $\geqslant 1$. The parameter $\varkappa$ is called the « dimension » of the sieve.

The method of Eratosthenes-Legendre rests on the use of Möbius function $\mu(d)$ as common value of the weights

$$\lambda_d^- = \lambda_d^+ = \mu(d) .$$

It turns formula (2) into the Legendre identity

$$S(\mathcal{A}, \mathcal{P}, z) = \sum_{d|P(z)} \mu(d) |\mathcal{A}_d|$$

and this identity usually leads to a bad result because, unless z is very small, the remainder sum

$$\sum_{d|P(z)} |r_d|$$

has too many terms. It was Viggo Brun who first showed how to construct the sieving weights $\lambda_d^{\pm}$ more effectively. For details see [1], Chapter 1.

The aim of this paper is to show that the Eratosthenes-Legendre sieve yields an asymptotic formula for the sifting function $S(\mathcal{A}, \mathcal{P}, z)$ in the case of the dimension $\varkappa < \frac{1}{2}$ and the sequence $\mathcal{A}$ with elements not too large:

$$x = \max_{a \in \mathcal{A}} |a| \leqslant A_3 X \tag{8}$$

for some $A_3 \geqslant 1$. It is assumed that the remainders r_d are also not too large:

$$|r_d| \leqslant A_4 \omega(d) \tag{9}$$

for some $A_4 \geqslant 1$.

Suppose that

$$0 \leqslant \frac{\omega(p)}{p} \leqslant 1 - \frac{1}{A_1} \tag{10}$$

for a'l $p \in \mathcal{P}$ and put

$$W(z) = \prod_{p|P(z)} \left(1 - \frac{\omega(p)}{p}\right).$$

The result reads as follows

THEOREM. *Under the assumptions* (7)-(10) *we have*

$$S(\mathcal{A}, \mathcal{P}, z) = \frac{e^{\varkappa\gamma}}{\Gamma(1-\varkappa)} W(z) X \left\{ f(s) + O\left(\frac{(s+1)L}{1-2\varkappa} (\log z)^{2\varkappa-1}\right)\right\},$$

where $\gamma = 0.577 \ldots$ *is the Euler constant,* $s = \log x / \log z$. *The function* $f(s)$ *is defined in Section* 2. *For* $0 < s \leqslant 1$ *we have*

$$f(s) = s^{-\varkappa}.$$

The constant in the symbol 0 *depends only on* A_1, A_2, A_3 *and* A_4.

In his thesis, Sullivan proved the asymptotic formula

$$S(\mathcal{A}, \mathcal{P}, z) \sim W(z) X$$

under the condition

$$\sum_{p<x} \frac{\omega(p)}{p} \log p = o(\log x)$$

instead of (7). His method is based on Halberstam-Richert's Fundamental Lemma [2] (oral communication).

A comparision of our method with Brun's method will be given elsewhere.

Keeping the notations introduced above we have

$$S(\mathcal{A}, \mathcal{P}, z) = \sum_{d|P(z)} \mu(d) |\mathcal{A}_d| = \sum_{d|P(z),\, d\leqslant x} \mu(d) |\mathcal{A}_d| = X \sum_{\substack{d \leqslant x \\ d|P(z)}} \mu(d) \frac{\omega(d)}{d} + \sum_{\substack{d \leqslant x \\ d|P(z)}} \mu(d) r_d.$$

1. – An estimate of the remainder sum.

From (9) we get

$$\sum_{\substack{d\leqslant x\\ d|P(z)}} |r_d| \leqslant A_4 \sum_{\substack{d\leqslant x\\ d|P(z)}} \omega(d)\,. \tag{12}$$

With ω we associate the generalized Mangoldt's function λ as usual by Dirichlet's convolution

$$\omega(d)\log d = \sum_{n|d} \omega(n)\,\lambda(d/n)\,.$$

Since ω is multiplicative, the support of λ is contained in the set of powers of primes. It is easily seen that $\lambda(p) = \omega(p)\log p$, so

$$\begin{aligned}\sum_{\substack{d\leqslant x\\ d|P(z)}} \omega(d)\log d &= \sum_{\substack{n\leqslant x\\ n|P(z)}} \omega(n) \sum_{\substack{m\leqslant x/n\\ mn|P(z)}} \lambda(m)\\ &\leqslant \sum_{\substack{n\leqslant x\\ n|P(z)}} \omega(n) \sum_{p\leqslant x/n} \omega(p)\log p\,.\end{aligned} \tag{13}$$

Using partial summation, from the upper bound (7) we get

$$\sum_{p\leqslant z} \omega(p)\log p \ll z$$

and

$$\sum_{p\leqslant z} \omega(p)/p < \varkappa \log\log z + O(1)\,.$$

Hence

$$\sum_{\substack{d\leqslant x\\ d|P(z)}} \omega(d)\log d \ll x \sum_{\substack{d\leqslant x\\ d|P(z)}} \frac{\omega(n)}{n} \leqslant x \prod_{p\leqslant x}\left(1+\frac{\omega(p)}{p}\right) < x\exp\left(\sum_{p\leqslant x}\omega(p)/p\right) \ll$$
$$\ll x(\log x)^{\varkappa}\,.$$

Using partial summation again we get the final result

$$\sum_{d|P(z),\,d\leqslant x} \omega(d) \ll x(\log x)^{\varkappa-1}$$

and thus we have the same estimate for the remainder sum (12).

2. – The function $f(s)$.

LEMMA 1. *Let* $0 \leqslant \varkappa < \frac{1}{2}$ *and* $f(s)$ *be the continuous solution of*

$$\begin{cases} f(s) = s^{-\varkappa} & \text{for } 0 < s \leqslant 1 \\ sf'(s) = \varkappa f(s-1) - \varkappa f(s) & \text{for } s > 1 . \end{cases}$$

Then, for $s \to \infty$ *we have*

$$f(s) = e^{-\varkappa\gamma}\Gamma(1-\varkappa) + O(e^{-s}) .$$

PROOF. The derivative $f'(s)$ satisfies the equation

$$sf'(s) = -\varkappa \int_{s-1}^{s} f'(u)\, du ,$$

so $f'(s) = O(e^{-s})$ and thus

$$f(s) = 1 + \int_{1}^{s} f'(u)\, du = c + O(e^{-s}) .$$

It remains to calculate the constant c. To this end, let us consider the Laplace transform

$$L(s) = \int_{0}^{\infty} e^{-st} f(t)\, dt = \int_{0}^{\infty} e^{-st}\big(c + O(e^{-t})\big)\, dt$$

$$= cs^{-1} + O(1) .$$

It can easily be checked that

$$\frac{d}{ds}\big(sL(s)\big) = \varkappa(1 - e^{-s}) L(s)$$

and thus

$$sL(s) = c \exp\left(\varkappa \int_{0}^{s} (1 - e^{-t}) \frac{dt}{t}\right) .$$

Now, we calculate in two ways the limit $\lim_{s\to\infty} s^{1-\varkappa} L(s)$. We have

$$\lim_{s\to\infty} s^{1-\varkappa} L(s) = c \lim_{s\to\infty} s^{-\varkappa} \exp\left(\varkappa \int_0^s (1-e^{-t}) \frac{dt}{t}\right) = c e^{\varkappa\gamma} .$$

On the other hand

$$\begin{aligned}\lim_{s\to\infty} s^{1-\varkappa} L(s) &= \lim_{s\to\infty} s^{1-\varkappa} \int_0^\infty e^{-st} f(t)\, dt \\ &= \lim_{s\to\infty} s^{1-\varkappa} \int_0^1 e^{-st} t^{-\varkappa}\, dt \\ &= \lim_{s\to\infty} \int_0^s e^{-u} u^{-\varkappa}\, du \\ &= \Gamma(1-\varkappa) .\end{aligned}$$

This completes the proof of the Lemma.

COROLLARY. *The function*

$$F(s) = \int_0^s f(u)\, du , \qquad s > 0$$

is of C^1-class and satisfies the equations

$$\begin{aligned} F(s) &= \frac{1}{1-\varkappa} s^{1-\varkappa} , && \text{for } 0 < s \leqslant 1 , \\ sF'(s) &= (1-\varkappa) F(s) + \varkappa F(s-1) , && \text{for } s > 1 , \\ \frac{d}{ds}\left(\frac{F(s)}{s^{1-\varkappa}}\right) &= \varkappa \frac{F(s-1)}{s^{2-\varkappa}} , && \text{for } s > 1 . \end{aligned}$$

3. – An asymptotic formula for the main term.

Let us put

$$g(d) = \mu(d) \frac{\omega(d)}{d}$$

and define for all $x \geqslant 1$, $z \geqslant 1$

$$G(x, z) = \sum_{\substack{d<x \\ d|P(z)}} g(d) .$$

To get an asymptotic formula for $G(x, z)$ we apply the first step of the Levin-Fainleib's iteration method [4]. This method works effectively for sums of positive multiplicative functions, in particular for the one which appears in the Selberg sieve (see [3]). Although our function $g(d)$ changes sign, it turns out that the method still works in the case $\varkappa < \frac{1}{2}$ considered here. Imitating [3], we shall prove

PROPOSITION 1. *For $x \geqslant 2$, $z \geqslant 2$ we have*

$$G(x, z) = Cf(s)(\log z)^{-\varkappa} + O\left(\frac{(s+1)L}{1-2\varkappa}(\log x)^{\varkappa-1}\right),$$

where

$$s = \frac{\log x}{\log z}, \qquad C = \frac{1}{\Gamma(1-\varkappa)} \prod_p \left(1 - \frac{\omega(p)}{p}\right)\left(1 - \frac{1}{p}\right)^{-\varkappa}$$

and the constant in the symbol O depends only on A_1, A_2.

Theorem will follow from

COROLLARY. *For $x \geqslant 2$, $z \geqslant 2$ we have*

$$G(x, z) = \frac{e^{\gamma\varkappa}}{\Gamma(1-\varkappa)} W(z) \left\{f(s) + O\left(\frac{(s+1)L}{1-2\varkappa}(\log z)^{2\varkappa-1}\right)\right\}.$$

The constant implied in the symbol O depends only on A_1 and A_2.

To derive Corollary from Proposition 1 we have to show

$$W(z) = e^{-\varkappa\gamma}\Gamma(1-\varkappa)\, C(\log z)^{-\varkappa}\left\{1 + O\left(\frac{L}{\log z}\right)\right\}.$$

But this is a simple consequence of the Mertens formula

$$\prod_{p<z}\left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log z}\left(1 + O\left(\frac{1}{\log z}\right)\right)$$

and of the estimate

$$\Gamma(1-\varkappa)\, C = \prod_{p<z}\left(1 - \frac{\omega(p)}{p}\right)\left(1 - \frac{1}{p}\right)^{-\varkappa}\left\{1 + O\left(\frac{L}{\log z}\right)\right\}.$$

For details see [1], Lemma 5.3.

Before we prove Proposition 1 we shall show a few auxiliary lemmas. Let

$$T(x,z)=\int_1^x \frac{G(t,z)}{t}\,dt=\sum_{\substack{d<x\\ d|P(z)}} g(d)\log\frac{x}{d}$$

for $x\geqslant 1$ and $z\geqslant 1$. For $0<x\leqslant 1$ we put $T(x,z)=0$.

LEMMA 2. *For $x\geqslant 2$ and $z\geqslant 2$ we have*

$$G(x,z)\log x=(1-\varkappa)T(x,z)+\varkappa T\left(\frac{x}{z},z\right)+O(L\log^{\varkappa}x)\,. \tag{14}$$

The constant in the symbol O depends only on A_1 and A_2.

PROOF. We start with the definition of generalized Mangoldt's function χ associated with g:

$$g(d)\log d=\sum_{n|d} g(n)\chi(d/n)\,.$$

Since g is multiplicative, the support of χ is contained in the set of powers of primes. It is easily seen that $\chi(p)=-\big(\omega(p)/p\big)\log p$, so

$$\sum_{\substack{d<x\\ d|P(z)}} g(d)\log d=\sum_{\substack{d<x\\ d|P(z)}} g(d)\sum_{\substack{n<x/d\\ dn|P(z)}}\chi(n)=-\sum_{\substack{d<x\\ d|P(z)}} g(d)\sum_{\substack{p<x/d\\ p<z,\,p\nmid d}}\frac{\omega(p)}{p}\log p\,.$$

Since

$$\sum_{p<y}\frac{\omega(p)}{p}\log p=\varkappa\log y+O(L)\,,\qquad \sum_{d<x}|g(d)|<\prod_{p<x}\left(1+\frac{\omega(p)}{p}\right)\ll(\log x)^{\varkappa}\,,$$

$$\sum_{d<x}|g(d)|\sum_{p|d}\frac{\omega(p)}{p}\log p\leqslant\sum_{d<x}\frac{\omega(d)}{d}\sum_{p<x}\left(\frac{\omega(p)}{p}\right)^2\log p\ll\sum_{d<x}\frac{\omega(d)}{d}\ll(\log x)^{\varkappa}$$

we obtain

$$\sum_{\substack{d<x\\ d|P(z)}} g(d)\log d=-\varkappa\sum_{\substack{d<x\\ d|P(z)}} g(d)\log\left(\min\left(z,\frac{x}{d}\right)\right)+O(L\log^{\varkappa}x)\,.$$

If we add the sum $\sum_{d|P(z),\,d<x} g(d)\log x/d$ to both sides, we arrive at (14).

LEMMA 3. *For $x \geqslant y \geqslant 2$ and $z \geqslant 2$ we have*

$$(15)\qquad \frac{T(x,z)}{(\log x)^{1-\varkappa}} = \frac{T(y,z)}{(\log y)^{1-\varkappa}} + \varkappa\int_y^x \frac{T(t/z,z)}{(\log t)^{1-\varkappa}}\, d(\log\log t) + O\left(\frac{L}{1-2\varkappa}\right)(\log y)^{2\varkappa-1}.$$

The constant in the symbol O depends only on A_1 and A_2.

PROOF. Let us write (14) with x replaced by t and divide throughout by $t(\log t)^{2-\varkappa}$. Integrating with respect to t from y to x, we obtain

$$\int_y^x \frac{G(t,z)}{t(\log t)^{1-\varkappa}}\, dt = (1-\varkappa)\int_y^x \frac{T(t,z)}{t(\log t)^{2-\varkappa}}\, dt + \varkappa\int_y^x \frac{T(t/z,z)}{t(\log t)^{2-\varkappa}}\, dt + O\left(\frac{L}{1-2\varkappa}(\log y)^{2\varkappa-1}\right).$$

If we integrate the identity

$$\frac{\partial}{\partial t}\left(\frac{T(t,z)}{t(\log t)^{1-\varkappa}}\right) = \frac{G(t,z)}{t(\log t)^{1-\varkappa}} - (1-\varkappa)\frac{T(t,z)}{t(\log t)^{2-\varkappa}}$$

and add the result to the formula above, we arrive at (14).

4. – An asymptotic formula for $T(x, z)$.

LEMMA 4. *For $z \geqslant x \geqslant 2$, we have*

$$T(x,z) = \frac{C}{1-\varkappa}(\log x)^{1-\varkappa} + O\left(\frac{L}{1-2\varkappa}\log^{\varkappa} x\right).$$

The constant in the symbol O depends only on A_1 and A_2.

PROOF. For $z \geqslant t \geqslant 2$ we have

$$G(t,z) = G(t,t) = G(t),$$

$$T(t,z) = T(t,t) = T(t)$$

and by Lemma 2

$$(16)\qquad r(t) = G(t)\log t - (1-\varkappa)\,T(t) \ll L\log^{\varkappa} t.$$

If we divide (16) throughout by $t(\log t)^{2-\varkappa}$ and integrate with respect to t

from 2 to x, we obtain

$$\int_2^x \frac{G(t)}{t(\log t)^{1-\varkappa}}dt-(1-\varkappa)\int_2^x \frac{T(t)}{t(\log t)^{2-\varkappa}}dt=\int_2^x \frac{r(t)}{t(\log t)^{2-\varkappa}}dt=c_1+O\left(\frac{L}{1-2\varkappa}(\log x)^{2\varkappa-1}\right).$$

But

$$\frac{d}{dt}\left(\frac{T(t)}{(\log t)^{1-\varkappa}}\right)=\frac{G(t)}{t(\log t)^{1-\varkappa}}-(1-\varkappa)\frac{T(t)}{t(\log t)^{2-\varkappa}},$$

so integration by parts leads to

$$\frac{T(x)}{(\log x)^{1-\varkappa}}-\frac{T(2)}{(\log 2)^{1-\varkappa}}=c_1+O\left(\frac{L}{1-2\varkappa}(\log x)^{2\varkappa-1}\right)$$

and finally

$$T(x)=c_2(\log x)^{1-\varkappa}+O\left(\frac{L}{1-2\varkappa}\log^{\varkappa} x\right).$$

It remains to calculate the constant $c_2=c_1+\log^{\varkappa}2$.
From (16) we get

$$G(x)=(1-\varkappa)c_2(\log x)^{-\varkappa}+O\left(\frac{L}{1-2\varkappa}(\log x)^{\varkappa-1}\right) \tag{17}$$

for $x\geqslant 2$. Hence, for $s>0$ we have

$$\prod_p\left(1-\frac{\omega(p)}{p^{s+1}}\right)=\sum_{n=1}^{\infty}\frac{g(n)}{n^{s+1}}=s\int_1^{\infty}\frac{G(x)}{x^{s+1}}dx=s\int_1^2+s\int_2^{\infty}=O(s)+(1-\varkappa)c_2 s\cdot$$

$$\cdot\int_2^{\infty}\frac{\log^{-\varkappa}x}{x^{s+1}}dx+O\left(\frac{Ls}{1-2\varkappa}\int_2^{\infty}\frac{(\log x)^{\varkappa-1}}{x^{s+1}}dx\right)=(1-\varkappa)c_2 s^{\varkappa}\Gamma(1-\varkappa)+O\left(\frac{L}{1-2\varkappa}s^{1-\varkappa}\right).$$

Since $\lim\limits_{s\to+0}s\zeta(s+1)=1$, by the Euler product formula we obtain

$$(1-\varkappa)c_2\Gamma(1-\varkappa)=\lim_{s\to+0}s^{-\varkappa}\prod_p\left(1-\frac{\omega(p)}{p^{s+1}}\right)=\prod_p\left(1-\frac{\omega(p)}{p}\right)\left(1-\frac{1}{p}\right)^{-\varkappa}.$$

This completes the proof of Lemma 4.

Now, we are ready to prove the general result

PROPOSITION 2. *For $x \geqslant 2$ and $z \geqslant 2$ we have*

$$T(x, z) = CF(s)(\log z)^{1-\varkappa} + O\left(\frac{(s+1)L}{1-2\varkappa} \log^{\varkappa} x\right). \tag{18}$$

The constant in the symbol O depends only on A_1 and A_2.

PROOF. The proof will proceed by induction. Putting

$$T(x, z) = CF(s)(\log z)^{1-\varkappa} + R(x, z) \tag{19}$$

we have to prove

$$R(x, z) \ll \frac{(s+1)L}{1-2\varkappa} \log^{\varkappa} x , \tag{20}$$

for all $x \geqslant 2$ and $z \geqslant 2$. This has already been proved in Lemma 4 for all $z \geqslant x \geqslant 2$. In this range $R(x, z) = R(x, x)$.

If we introduce (19) into (15) we find out that the leading terms disappear throughout and we are left with a relation between the remainder terms only, namely

$$\frac{R(x, z)}{(\log x)^{1-\varkappa}} = \frac{R(y, z)}{(\log y)^{1-\varkappa}} + \varkappa \int_y^x \frac{R(t/z, z)}{(\log t)^{1-\varkappa}} d(\log\log t) + O\left(\frac{L}{1-2\varkappa} (\log y)^{2\varkappa-1}\right) \tag{21}$$

for all $x \geqslant y \geqslant 2$ and $z \geqslant 2$.

Let us assume for a moment that $y = z$ and

$$z \leqslant x \leqslant z^2 . \tag{22}$$

Accordingly, we can use (20) to the right-hand side of (21) and in the result we arrive again at (20) but now for the range (22). Therefore, we already have

$$R(x, z) < \frac{BL}{1-2\varkappa} (s+1) \log^{\varkappa} x , \tag{23}$$

for all $z \geqslant 2$ and $2 \leqslant x \leqslant z^2$. The constant B depends only on A_1 and A_2: Now, we shall show by induction that if B is sufficiently large then (23) holds for all $z \geqslant 2$ and $x \geqslant 2$. For that, it is enough to prove the implication: if (23) holds for all $x \leqslant z^u$ then it holds for $x = z^{u+1}$, where u is real number $\geqslant 1$.

After putting

$$y = z^u, \qquad x = z^{u+1}$$

in (21), we get by the inductive assumption

$$\frac{R(x,z)}{(\log x)^{1-\varkappa}} < \frac{BL}{1-2\varkappa}\left\{(u+1)(\log x)^{2\varkappa-1} + \varkappa\int_y^x \frac{\log^\varkappa t}{\log z}\left(\log\frac{t}{z}\right)^\varkappa d(\log\log t)\right\} +$$
$$+ O\left(\frac{L}{1-2\varkappa}(\log x)^{2\varkappa-1}\right) < \frac{BL}{1-2\varkappa}\left\{u+1+\varkappa\left(\frac{u+1}{u}\right)^{1-\varkappa} + \varepsilon\right\}(\log x)^{2\varkappa-1}.$$

If B is sufficiently large then $\varepsilon < 1 - 1/\sqrt{2}$ and thus the term in the bracket is less than $u+2$. This completes the proof of Proposition 2.

5. – Completion of the proof of Proposition 1.

Putting $F(s) = 0$ for all $s \leqslant 0$ we find out from (14) and (18) that

$$G(x,z) = Cs^{-1}\{(1-\varkappa)F(s) + \varkappa F(s-1)\}(\log z)^{-\varkappa} +$$
$$+ O\left\{\frac{(s+1)L}{1-2\varkappa}(\log x)^{\varkappa-1} + L(\log x)^{\varkappa-1}\right\} = CF'(s)(\log z)^{-\varkappa} +$$
$$+ O\left(\frac{(s+1)L}{1-2\varkappa}(\log x)^{\varkappa-1}\right)$$

for all $x \geqslant 2$ and $z \geqslant 2$. This completes the proof of Proposition 1.

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