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Continuous Data Dependence for an Abstract Volterra Integro-Differential Equation in Hilbert Space with Applications to Viscoelasticity.

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For the Volterra integro-differential equation

$$\boldsymbol{u}_{tt} - \boldsymbol{N}\boldsymbol{u} + \int_0^t \boldsymbol{G}(t-\tau)\,\boldsymbol{u}(\tau)\,d\tau = \mathscr{F}(t)\,,$$

in Hilbert space, with associated homogeneous initial data $\boldsymbol{u}(0) = \boldsymbol{0}$, $\boldsymbol{u}_t(0) = \boldsymbol{0}$, we establish conditions on the operator $\boldsymbol{G}(t)$ which guarantee that solutions which lie in a certain uniformly bounded class must depend continuously on $\mathscr{F}$ in the norm

$$\|\boldsymbol{u}\|_t^2 = \int_0^t \|\boldsymbol{u}(\tau)\|^2\,d\tau\,, \qquad 0 \leqslant t < T\,;$$

this result is then used to prove that $\boldsymbol{u}$ must also depend continuously on perturbations of non-homogeneous initial data, bounded symmetric perturbations of $\boldsymbol{N}$, and perturbations of the initial geometry. Finally, our results are applied to the study of continuous data dependence for solutions to initial boundary value problems in the theory of isothermal linear viscoelasticity.

1. – Introduction.

Let H_+ and H be real Hilbert spaces with inner products $\langle\, ,\, \rangle_+$ and $\langle\, ,\, \rangle$, respectively. We assume that $H_+ \subset H$ algebraically and topologically with

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H_+ dense in H. As in [1] we let H_- denote the dual of H_+ via the inner product $\langle\,,\,\rangle$ of H so that H_- is the completion of H under the norm

$$\|w\|_- = \sup_{v\in H_+} \frac{|\langle v, w\rangle|}{\|v\|_+} .$$

By $\mathscr{L}(H_+, H_-)$ we denote the space of bounded linear operators from H_+ to H_-.

For $0\leqslant t < T$, where $T > 0$ is an arbitrary real number, we consider the initial-value problem

$$u_{tt} - Nu + \int_0^t G(t-\tau)\,u(\tau)\,d\tau = \mathscr{F}(t) \tag{1.1}$$

$$u(\cdot, 0) = f(\cdot), \qquad u_t(\cdot, 0) = g(\cdot) \tag{1.2}$$

where

(i) $N \in \mathscr{L}(H_+, H_-)$ is symmetric, i.e. $\langle Nv, w\rangle = \langle v, Nw\rangle$, $\forall v, w \in H_+$;

(ii) $(\partial^k/\partial t^k)G(t)$ exists a.e. on $[0, T)$, $k\geqslant 4$, and belongs to $\mathscr{L}(H_+, H_-)$ with $G(t)$, $G_t(t)\in L^2\big([0, T);\ \mathscr{L}(H_+, H_-)\big)$;

(iii) $\mathscr{F}(t)\in L^2\big([0, T);\, H_-\big)$ with $\mathscr{F}(0)\neq \mathbf{0}$;

(iv) $f, g\colon J \to H_+$ are continuous, where J, the domain of $u(\cdot, t)$, $\forall t\in[0, T)$, is an arbitrary topological space which is endowed with a positive measure μ.

We are interested in solutions $u\in C^2\big([0, T);\, H_+\big)$ of (1.1), (1.2) for which $u_t \in C^1\big([0, T);\, H_+\big)$ and $u_{tt}\in C\big([0, T);\, H_+\big)$. Our basic aim in this paper will be to establish conditions on $G(t)$ under which solutions of (1.1), (1.2), which lie in a certain uniformly bounded class, are unique and depend continuously on perturbations of the initial data, the initial geometry, and the operator N. Since we make no assumptions of definiteness on N we can not apply the existence and uniqueness results of Dafermos [1] and the problem (1.1), (1.2) is non well-posed. In recent years, however, many non well-posed problems for partial differential equations, including several which arise in continuum mechanics, have been dealt with via logarithmic convexity and related techniques, e.g., [2], [3], [4], [5], and [6]; we mention, in particular, the recent applications by Beevers [7], [8], of the method of Murray and Protter [9] to the study of uniqueness and Hölder stability for solutions to a class of initial-boundary value problems in viscoelasticity (1). Our

(1) See also [13], [14], [15] where convexity techniques yield stability and growth estimates (but not continuous data dependence) theorems for viscoelastic materials.

approach here is modeled, essentially, after a convexity argument which has been used by Knops and Payne [10] to treat the problem of continuous data dependence for classical solutions of initial-boundary value problems in linear elastodynamics. In § 4 we specialize our results in such a way as to deal with the problem of uniqueness and continuous dependence on initial data, for solutions to certain initial-boundary value problems in isothermal linear viscoelasticity; as we make no assumptions concerning the definiteness of the initial value of the relaxation tensor, these problems are all non well-posed.

2. – A basic theorem.

For any $\boldsymbol{v} \in C([0, T); H_+)$ we set

$$\|\boldsymbol{v}\|_t^2 = \int_0^t \|\boldsymbol{v}(\tau)\|^2 \, d\tau, \qquad 0 \leqslant t < T \tag{2.1}$$

and define

$$\mathscr{M}(N) = \{\boldsymbol{v} \in C^2([0, T); H_+) | \boldsymbol{v}_{tt}(0) \neq \boldsymbol{0} \text{ and } \|\boldsymbol{v}\|_T^2 \leqslant N^2,$$
$$\text{for some real number } N\}.$$

We want to show that solutions of (1.1), (1.2), with $\boldsymbol{f} = \boldsymbol{g} = \boldsymbol{0}$, which lie in a certain subclass of $\mathscr{M}(N)$, must depend continuously on $\mathscr{F}$, in the $\|(\cdot)\|_t$ norm, whenever $\boldsymbol{G}(0)$ satisfies

$$-\langle \boldsymbol{v}, \boldsymbol{G}(0)\boldsymbol{v}\rangle \geqslant \varkappa \|\boldsymbol{v}\|_+^2, \qquad \forall \boldsymbol{v} \in H_+ \tag{2.2}$$

with $\varkappa > 0$ sufficiently large. We first prepare some material which shall be needed in the proof of our main theorem.

Let $\boldsymbol{v} \in \mathscr{M}(N)$ and satisfy $\boldsymbol{v}(0) = \boldsymbol{v}_t(0) = \boldsymbol{0}$. For $0 \leqslant t < T$, we define

$$K_{\boldsymbol{v}}(t) \equiv \frac{t \int_0^t \|\boldsymbol{v}(\tau)\|_+^2 \, d\tau}{\int_0^t \int_0^\eta \|\boldsymbol{v}(\tau)\|_+^2 \, d\tau \, d\eta}, \qquad \boldsymbol{v} \neq \boldsymbol{0}. \tag{2.3}$$

If we set $\boldsymbol{g}_{\boldsymbol{v}}(t) = \int_0^t \|\boldsymbol{v}(\tau)\|_+^2 \, d\tau$ then we may write (2.3) in the form

$$K_{\boldsymbol{v}}(t) = t\mathfrak{g}_{\boldsymbol{v}}(t) \Big/ \int_0^t \mathfrak{g}_{\boldsymbol{v}}(\tau) \, d\tau. \tag{2.4}$$

By virtue of the monotonicity of $\mathfrak{g}_v$ on $[0, T)$, and the mean-value theorem for integrals, $K_v(t) \geqslant 1$, $\forall t \in [0, T)$ and any non-zero $v \in \mathcal{M}(N)$. We also have the following

LEMMA 1. If $v \in \mathcal{M}(N)$, with $v(0) = v_t(0) = 0$, then $\sup K_v(t) < \infty$.

PROOF. As $t\mathfrak{g}_v(t) \leqslant TN^2$, $0 \leqslant t < T$, we need only be concerned with the behavior of $K_v(t)$ as $t \to 0^+$. In other words, we have to show that for any $v \in \mathcal{M}(N)$ with $v(0) = v_t(0) = 0$,

$$\lim_{t\to 0^+}\left[t\mathfrak{g}_v(t)\Big/\int_0^t \mathfrak{g}_v(\tau)\,d\tau\right] < \infty . \tag{2.5}$$

But successive differentiation, coupled with the theorem of L'Hôspital, shows that this limit is the same as

$$\lim_{t\to 0^+}\left[t\mathfrak{g}_v^{(k)}(t)/\mathfrak{g}_v^{(k-1)}(t)\right] + 1 \tag{2.6}$$

for any positive integer $k \geqslant 1$. Direct computation using the definition of $\mathfrak{g}_v$ and the hypotheses of the lemma shows, however, that $\mathfrak{g}_v^{(k)}(0) = 0$, $k = 1, 2, \ldots, 5$ but $\mathfrak{g}_v^{(6)}(0) = 6\|v_{tt}(0)\|_+^2$. It then follows that $\lim_{t\to 0^+} K_v(t) = 1$ and that $\sup_{[0,T)} K_v(t) \equiv K_v < \infty$ *Q.E.D.*

Now let $v \in \mathcal{M}(N)$, with $v(0) = 0$, $v_t(0) = 0$. We define $\lambda(v) = \sup_{[0,T)} \|v(t)\|_+^2$ and set $K_v(\lambda) = \lim_{\lambda(v)\to 0} K_v$. Then for any $K > 0$ we may define a subset $\mathcal{P}_K \subset \mathcal{M}(N)$ as follows:

$$\mathcal{P}_K = \{v \in \mathcal{M}(N) | v(0) = 0,\ v_t(0) = 0, \text{ and } K_v \leqslant K_v(\lambda) < 2K\} . \tag{2.7}$$

REMARK. It is a simple matter to exhibit a sequence $\{v_n\} \subset \mathcal{M}(N)$ such that $v_n(0) = 0$, $(v_n)_t(0) = 0$, $\lambda(v_n) \to 0$ as $n \to \infty$, but $\lim_{n\to\infty} K_{v_n}(t) < \infty$, $\forall t \in [0, T)$. For example, let $v \neq 0$ be any element of H_+ and define

$$v_n(t) = \left(\exp[t/n] - \frac{1}{n}t - 1\right)v . \tag{2.8}$$

Clearly, $v_n \in C^\infty([0, T); H_+)$ for each $n = 1, 2, \ldots$ while

$$v_n(0) = 0 , \quad (v_n)_t(0) = 0 \quad \text{and} \quad (v_n)_{tt}(0) = \frac{1}{n^2} \neq 0 .$$

Also,

$$\|\boldsymbol{v}_n(t)\|_+^2 = \left(\exp[t/n] - \frac{1}{n}t - 1\right)^2 \|\boldsymbol{v}\|_+^2, \qquad 0 \leqslant t < T,$$

so that

$$\sup_{[0,T)} \|\boldsymbol{v}_n(t)\|_+^2 \to 0 \qquad \text{as } n \to \infty .$$

In addition,

$$\|\boldsymbol{v}_n\|_T^2 = \left\{\int_0^T \left(\exp\left[\frac{\tau}{n}\right] - \frac{1}{n}\tau - 1\right)^2 d\tau\right\} \|\boldsymbol{v}\|_+^2 \leqslant N^2,$$

for n sufficiently large. Thus, for n sufficiently large, $\{\boldsymbol{v}_n\} \subset \mathscr{M}(N)$ and $\boldsymbol{v}_n(0) = (\boldsymbol{v}_n)_t(0) = \mathbf{0}$. Moreover, a simple computation establishes the fact that

$$\lim_{n\to\infty} K_{\boldsymbol{v}_n}(t) < \infty, \qquad \forall t \in [0, T).$$

We are now in a position to state the basic theorem, from which all our other continuous data dependence results will follow, i.e., we have

THEOREM I. Let $\boldsymbol{u}$ be any solution of (1.1) which lies in $\mathscr{P}_K$ for some $K < 1$. If $\boldsymbol{G}(0)$ satisfies (2.2) with

$$\varkappa \geqslant K\left(\sup_{[0,T)} \|\boldsymbol{G}(t)\| + 2T \sup_{[0,T)} \|\boldsymbol{G}_t(t)\|\right) \tag{2.9}$$ (2)

then there exist non-negative constants P and Q such that for all t, $0 \leqslant t < T$

$$\|\boldsymbol{u}\|_t^2 \leqslant P Q^{2\delta} \|\mathscr{F}\|_T^{2(1-\delta)}, \qquad \delta = t/T. \tag{2.10}$$

PROOF. For $0 \leqslant t < T$, we consider the real-valued function $F(t)$ which is defined by

$$F(t) = \|\boldsymbol{u}\|_t^2 + T^4 \|\mathscr{F}\|_T^2, \qquad 0 \leqslant t < T. \tag{2.11}$$

(2) $|||\boldsymbol{G}||| \equiv \sup_{[0,T)} \|\boldsymbol{G}(t)\| + 2T \sup_{[0,T)} \|\boldsymbol{G}_t(t)\|$ is, of course a norm on

$$L^2([0, T); \mathscr{L}(H_+, H_-))$$

but is not the natural norm associated with any inner product on this space; such a requirement, however, is not needed in the computations which follow as we do not consider topological properties of the space.

Direct computation then yields

$$F'(t) = \|\boldsymbol{u}(t)\|^2 = 2\int_0^t \langle \boldsymbol{u}(\eta), \boldsymbol{u}_\eta(\eta)\rangle\, d\eta \tag{2.12}$$

as $\boldsymbol{u}(0) = \boldsymbol{0}$, $\boldsymbol{u}_t(0) = \boldsymbol{0}$, and

$$F''(t) = 2\langle \boldsymbol{u}(t), \boldsymbol{u}_t(t)\rangle = 2\int_0^t \{\langle \boldsymbol{u}_\eta(\eta), \boldsymbol{u}_\eta(\eta)\rangle + \langle \boldsymbol{u}(\eta), \boldsymbol{u}_{\eta\eta}(\eta)\rangle\}\, d\eta\,. \tag{2.13}$$

By making use of (1.1) we may rewrite (2.13) in the form

$$\begin{aligned} F''(t) = {} & 2\int_0^t \|\boldsymbol{u}_\eta\|^2\, d\eta + 2\int_0^t \langle \boldsymbol{u}(\eta), \boldsymbol{N}\boldsymbol{u}(\eta)\rangle\, d\eta \\ & + 2\int_0^t \langle \boldsymbol{u}(\eta), \mathscr{F}(\eta)\rangle\, d\eta - 2\int_0^t \langle \boldsymbol{u}(\eta), \int_0^\eta \boldsymbol{G}(\eta-\tau)\boldsymbol{u}(\tau)\, d\tau\rangle\, d\eta\,. \end{aligned} \tag{2.14}$$

We now apply the Schwarz inequality to (2.12), twice in succession, so as to obtain

$$[F'(t)]^2 \leqslant 4\|\boldsymbol{u}\|_t^2 \int_0^t \|\boldsymbol{u}_\eta\|^2\, d\eta\,. \tag{2.15}$$

Therefore,

$$\begin{aligned} FF'' - F'^2 \geqslant {} & 2F\int_0^t \|\boldsymbol{u}_\eta\|^2\, d\eta + 2F\int_0^t \langle \boldsymbol{u}(\eta), \boldsymbol{N}\boldsymbol{u}(\eta)\rangle\, d\eta \\ & + 2F\int_0^t \langle \boldsymbol{u}(\eta), \mathscr{F}(\eta)\rangle\, d\eta \\ & - 2F\int_0^t \langle \boldsymbol{u}(\eta), \int_0^\eta \boldsymbol{G}(\eta-\tau)\boldsymbol{u}(\tau)\, d\tau\rangle\, d\eta - 4\|\boldsymbol{u}\|_t^2 \int_0^t \|\boldsymbol{u}_\eta\|^2\, d\eta \\ = {} & -2F\left\{\int_0^t \|\boldsymbol{u}_\eta\|^2\, d\eta - \int_0^t \langle \boldsymbol{u}(\eta), \boldsymbol{N}\boldsymbol{u}(\eta)\rangle\, d\eta\right\} \\ & + 2F\int_0^t \langle \boldsymbol{u}(\eta), \mathscr{F}(\eta)\rangle\, d\eta \\ & - 2F\int_0^t \langle \boldsymbol{u}(\eta), \int_0^\eta \boldsymbol{G}(\eta-\tau)\boldsymbol{u}(\tau)\, d\tau\rangle\, d\eta + 4T^4\|\mathscr{F}\|_T^2 \int_0^t \|\boldsymbol{u}_\eta\|^2\, d\eta\,, \end{aligned} \tag{2.16}$$

where we have made use of (2.11). To deal with the expression in brackets on the right-hand side of (2.16_2) we take the inner product of (1.1) with $\boldsymbol{u}_\tau$, and use the symmetry of $\boldsymbol{N}$, to get

$$\frac{d}{d\tau}\langle \boldsymbol{u}_\tau, \boldsymbol{u}_\tau\rangle - \frac{d}{d\tau}\langle \boldsymbol{u}, \boldsymbol{N}\boldsymbol{u}\rangle = 2\langle \boldsymbol{u}_\tau, \mathscr{F}(\tau)\rangle - 2\left\langle \boldsymbol{u}_\tau, \int_0^\tau \boldsymbol{G}(\tau-\lambda)\boldsymbol{u}(\lambda)\,d\lambda\right\rangle . \tag{2.17}$$

If we now integrate this last equation, first with respect to τ, over $[0, \eta]$, and then with respect to η, over $[0, t]$, and make use of the fact that $\boldsymbol{u}$ satisfies homogeneous initial conditions, we arrive at the identity,

$$\begin{aligned} \int_0^t \|\boldsymbol{u}_\eta\|^2\,d\eta - \int_0^t \langle \boldsymbol{u}(\eta), \boldsymbol{N}\boldsymbol{u}(\eta)\rangle\,d\eta &= 2\int_0^t (t-\eta)\langle \boldsymbol{u}_\eta, \mathscr{F}(\eta)\rangle\,d\eta \\ &\quad - 2\int_0^t \int_0^n \langle \boldsymbol{u}_\tau, \int_0^\tau \boldsymbol{G}(\tau-\lambda)\boldsymbol{u}(\lambda)\,d\lambda\rangle\,d\tau\,d\eta . \end{aligned} \tag{2.18}$$

Substitution from (2.18) into (2.16_2) now yields

$$\begin{aligned} FF'' - F'^2 \geqslant & -4F\int_0^t (t-\eta)\langle \boldsymbol{u}_\eta, \mathscr{F}(\eta)\rangle\,d\eta \\ & + 2F\int_0^t \langle \boldsymbol{u}(\eta), \mathscr{F}(\eta)\rangle\,d\eta \\ & + 4T^4\|\mathscr{F}\|_T^2\int_0^t \|\boldsymbol{u}_\eta\|^2\,d\eta \\ & - 2F\int_0^t \langle \boldsymbol{u}(\eta), \int_0^n \boldsymbol{G}(n-\tau)\boldsymbol{u}(\tau)\,d\tau\rangle\,d\eta \\ & + 4F\int_0^t \int_0^n \langle \boldsymbol{u}_\tau, \int_0^\tau \boldsymbol{G}(\tau-\lambda)\boldsymbol{u}(\lambda)\,d\lambda\rangle\,d\tau\,d\eta . \end{aligned} \tag{2.19}$$

To deal with the first three expressions on the right-hand side of (2.19) we need.

LEMMA 2. If $F(t)$ is defined by (2.11) then

$$2F\int_0^t \langle \boldsymbol{u}(\eta), \mathcal{F}(\eta)\rangle\, d\eta \geq -T^{-2}F^2(t) \tag{2.20}$$

$$-4F\int_0^t (t-\eta)\langle \boldsymbol{u}_\eta, \mathcal{F}(\eta)\rangle\, d\eta \geq -T^{-2}F^2(t) - 4T^4\|\mathcal{F}\|_T^2\int_0^t \|\boldsymbol{u}_\eta\|^2\, d\eta\,. \tag{2.21}$$

The proof of the validity of the inequalities (2.20) and (2.21) follows that of the analogous results stated in [10] and involves only simple applications of the Schwarz and arithmetic-geometric mean inequalities.

If we make use of (2.20) and (2.21) then we easily reduce (2.19) to

$$\begin{aligned} FF''-F'^2 \geq & -T^{-2}F^2 \\ & -2F\int_0^t \Big\langle \boldsymbol{u}(\eta), \int_0^n \boldsymbol{G}(\eta-\tau)\boldsymbol{u}(\tau)\,d\tau\Big\rangle\, d\eta \\ & + 4F\int_0^t\int_0^n \Big\langle \boldsymbol{u}_\tau, \int_0^\tau \boldsymbol{G}(\tau-\lambda)\boldsymbol{u}(\lambda)\,d\lambda\Big\rangle\, d\tau\, d\eta\,. \end{aligned} \tag{2.22}$$

However,

$$\begin{aligned} \Big\langle \boldsymbol{u}_\tau, \int_0^\tau \boldsymbol{G}(\tau-\lambda)\boldsymbol{u}(\lambda)\,d\lambda\Big\rangle & = \frac{d}{d\tau}\Big\langle \boldsymbol{u}(\tau), \int_0^\tau \boldsymbol{G}(\tau-\lambda)u(\lambda)\,d\lambda\Big\rangle \\ & -\Big\langle \boldsymbol{u}(\tau), \int_0^\tau \boldsymbol{G}_\tau(\tau-\lambda)\boldsymbol{u}(\lambda)\,d\lambda\Big\rangle \\ & -\langle \boldsymbol{u}(\tau), \boldsymbol{G}(0)\boldsymbol{u}(\tau)\rangle\,, \end{aligned}$$

so that

$$\begin{aligned} FF''-F'^2 \geq & -T^{-2}F^2 - 4F\int_0^t\int_0^n \langle \boldsymbol{u}(\tau), \boldsymbol{G}(0)\boldsymbol{u}(\tau)\rangle\, d\tau\, d\eta \\ & -4F\int_0^t\int_0^n \Big\langle \boldsymbol{u}(\tau), \int_0^\tau \boldsymbol{G}_\tau(\tau-\lambda)\boldsymbol{u}(\lambda)\,d\lambda\Big\rangle\, d\tau\, d\eta \\ & + 2F\int_0^t \Big\langle \boldsymbol{u}(\eta), \int_0^n \boldsymbol{G}(\eta-\tau)\boldsymbol{u}(\tau)\,d\tau\Big\rangle\, d\eta\,. \end{aligned} \tag{2.23}$$

We may now bound the last two expressions on the right-hand side of (2.23) as follows:

$$\begin{aligned}\int_0^t\left\langle \boldsymbol{u}(\eta), \int_0^n \boldsymbol{G}(\eta-\tau)\boldsymbol{u}(\tau)\,d\tau\right\rangle d\eta &\leqslant \int_0^t\left|\left\langle \boldsymbol{u}(\eta), \int_0^n \boldsymbol{G}(\eta-\tau)\boldsymbol{u}(\tau)\,d\tau\right\rangle\right| d\eta\\ &\leqslant \sup_{[0,T)}\|\boldsymbol{G}(t)\|\int_0^t\|\boldsymbol{u}(\eta)\|_+\left(\int_0^\eta\|\boldsymbol{u}(\tau)\|_+\,d\tau\right)d\eta\\ &\leqslant \sup_{[0,T)}\|\boldsymbol{G}(t)\|\left(\int_0^t\|\boldsymbol{u}(\eta)\|_+\,d\eta\right)^2\\ &\leqslant\left(\sup_{[0,T)}\|\boldsymbol{G}(t)\|\right)t\int_0^t\|\boldsymbol{u}(\eta)\|_+^2\,d\eta\end{aligned}$$

for all t, $0\leqslant t<T$, where we have used the assumption that $\boldsymbol{G}(t)\in\mathscr{L}(H_+, H_-)$ for each $t\in[0, T)$. We therefore have the lower bound

$$2F(t)\int_0^t\langle \boldsymbol{u}(\eta), \int_0^n \boldsymbol{G}(\eta-\tau)\boldsymbol{u}(\tau)\,d\tau\rangle\,d\eta \geqslant -2\alpha tF(t)\int_0^t\|\boldsymbol{u}(\eta)\|_+^2\,d\eta \tag{2.24}$$

where $\alpha\equiv\sup_{[0,T)}\|\boldsymbol{G}(t)\|$. In a similar manner we can easily establish the estimate

$$-4F(t)\int_0^t\int_0^n\langle \boldsymbol{u}(\tau), \int_0^\tau \boldsymbol{G}_\tau(\tau-\lambda)\,\boldsymbol{u}(\lambda)\,d\lambda\rangle\,d\tau\,d\eta \geqslant -4\beta tF(t)\int_0^t\|\boldsymbol{u}(\eta)\|_+^2\,d\eta \tag{2.25}$$

where $\beta\equiv T\sup_{[0,T)}\|\boldsymbol{G}_\tau(t)\|$. Therefore,

$$\begin{aligned}FF''-F'^2 &\geqslant -T^{-2}F^2\\ &\quad -2(\alpha+2\beta)\,tF\int_0^t\|\boldsymbol{u}(\tau)\|_+^2\,d\eta\\ &\quad -4F\int_0^t\int_0^\eta\langle \boldsymbol{u}(\tau), \boldsymbol{G}(0)\,\boldsymbol{u}(\tau)\rangle\,d\tau\,d\eta\\ &\geqslant -T^{-2}F^2\end{aligned} \tag{2.26}$$

by virtue of (2.2), (2.9), and the fact that $\boldsymbol{u} \in \mathscr{P}_K$, $K < 1$. However, this last result may be rewritten in the form

$$(2.27) \qquad \frac{d^2}{dt^2}\left(\log\left[\exp\,[t^2/T^2]F(t)\right]\right) \geqslant 0\,, \qquad 0 \leqslant t < T\,,$$

which shows that $E(t) = \exp\,[t^2/T^2]F(t)$ has a convex logarithm on $[0, T)$; thus, we may apply Jensen's inequality to deduce that

$$(2.28) \qquad F(t) \leqslant \exp\,[\delta(1-\delta)][F(T)]^{\delta}[F(0)]^{(1-\delta)}$$

for $0 \leqslant t < T$, where $\delta(t) \equiv t/T$. The desired result, i.e. (2.10), now follows directly from (2.28) if we recall the definition of $F(t)$ and the fact that $\boldsymbol{u} \in \mathscr{P}_K$ implies that $\|\boldsymbol{u}\|_T^2 \leqslant N^2$. *Q.E.D.*

3. – Some continuous data dependence theorems.

The results which we are about to prove all depend on the theorem of the previous section; in rather loose terms, they state that suitably restricted solutions of (1.1), (1.2) are unique and depend continuously on perturbations of the initial data, the initial geometry, and the operator $\boldsymbol{N}$ whenever $\boldsymbol{G}(0)$ satisfies (2.2) with $\varkappa$ sufficiently large. We begin with

A. Continuous dependence on initial data.

Let $\mathscr{R} = \{\boldsymbol{v} \in C^2([0, T); H_+) \,|\, \|\boldsymbol{v}\|_T^2 \leqslant R^2\}$ for some real number R. We consider solutions $\boldsymbol{u} \in \mathscr{R}$ of the system

$$(3.1) \qquad \boldsymbol{u}_{tt} - \boldsymbol{N}\boldsymbol{u} + \int_0^t \boldsymbol{G}(t-\tau)\boldsymbol{u}(\tau)\,d\tau = \boldsymbol{0}$$

$$(3.2) \qquad \boldsymbol{u}(0) = \boldsymbol{f}\,, \qquad \boldsymbol{u}_t(0) = \boldsymbol{g}$$

where it is assumed that $\boldsymbol{N}\boldsymbol{f} \neq \boldsymbol{0}$ so that $\boldsymbol{u}_{tt}(0) \neq \boldsymbol{0}$. For any t, $0 \leqslant t < T$, we define a function $\hat{\boldsymbol{u}} \in C^2([0, T); H_+)$ by

$$(3.3) \qquad \hat{\boldsymbol{u}}(\cdot, t) = \boldsymbol{u}(\cdot, t) - t\boldsymbol{u}_t(\cdot, 0) - \boldsymbol{u}(\cdot, 0)\,.$$

Clearly, $\hat{\boldsymbol{u}}(0) = \boldsymbol{0}$, $\hat{\boldsymbol{u}}_t(0) = \boldsymbol{0}$, and $\hat{\boldsymbol{u}}_{tt}(0) = \boldsymbol{N}\boldsymbol{f} \neq \boldsymbol{0}$. Note also that

$$\begin{aligned} \|\hat{\boldsymbol{u}}\|_T^2 &\leqslant 2\|\boldsymbol{u}\|_T^2 + 2\|t\boldsymbol{u}_t(0) + \boldsymbol{u}(0)\|_T^2 \\ &\leqslant 2\|\boldsymbol{u}\|_T^2 + 2k(T)\max\{\|\boldsymbol{f}\|^2, \|\boldsymbol{g}\|^2\} \\ &\leqslant 2R^2 + 2k(T)\max\{\|\boldsymbol{f}\|^2, \|\boldsymbol{g}\|^2\}\ (^3). \end{aligned}$$

(3) $k(T) \equiv T^3/3 + T^2 + T$.

for any $\boldsymbol{u} \in \mathscr{R}$. Therefore, if we choose N so that

$$N^2 \geqslant 2R^2 + 2k(T)\max\{\|\boldsymbol{f}\|^2, \|\boldsymbol{g}\|^2\} \tag{3.4}$$

and define the class $\mathscr{M}(N)$ accordingly, every function $\hat{\boldsymbol{u}}$ of the form (3.3) lies in $\mathscr{M}(N)$ when $\boldsymbol{u} \in \mathscr{R}$ (4). Furthermore, it is easy to verify that if $\boldsymbol{u}(\cdot, t)$ is any solution of (3.1), (3.2), then $\hat{\boldsymbol{u}}$ satisfies (1.1) with $\mathscr{F}$ replaced by

$$\hat{\mathscr{F}}(t) = \left[t\boldsymbol{N} - \int_0^t \tau \boldsymbol{G}(t-\tau)\,d\tau\right]\boldsymbol{g} + \left[\boldsymbol{N} - \int_0^t \boldsymbol{G}(t-\tau)\,d\tau\right]\boldsymbol{f}. \tag{3.5}$$

For the sake of convenience we now define operators

$$\boldsymbol{A}(t), \boldsymbol{B}(t) \in L^2\big([0, T);\ \mathscr{L}(H_+, H_-)\big)$$

by

$$\boldsymbol{A}(t) = t\boldsymbol{N} - \int_0^t \tau \boldsymbol{G}(t-\tau)\,d\tau \tag{3.6}$$

$$\boldsymbol{B}(t) = \boldsymbol{N} - \int_0^t \boldsymbol{G}(t-\tau)\,d\tau \tag{3.7}$$

so that $\hat{\mathscr{F}} = \boldsymbol{A}(t)\boldsymbol{g} + \boldsymbol{B}(t)\boldsymbol{f}$. If $\hat{\boldsymbol{u}} \in \mathscr{P}_K$, for some $K < 1$, and $\boldsymbol{G}(0)$ satisfies (2.2) with $\varkappa \geqslant K\left(\sup_{[0,T)} \|\boldsymbol{G}(t)\| + 2T \sup_{[0,T)} \|\boldsymbol{G}_t(t)\|\right)$ then we may apply the result of Theorem I to conclude that for all t, $0 \leqslant t < T$

$$\|\hat{\boldsymbol{u}}\|_t^2 \leqslant PQ^{2\delta}\|\boldsymbol{A}(t)\boldsymbol{g} + \boldsymbol{B}(t)\boldsymbol{f}\|_T^{2(1-\delta)}. \tag{3.8}$$

However,

$$\begin{aligned}\|\boldsymbol{u}\|_t &\leqslant \|\hat{\boldsymbol{u}}\|_t + \|t\boldsymbol{u}_t(0) + \boldsymbol{u}(0)\|_T \\ &\leqslant \|\hat{\boldsymbol{u}}\|_t + \sqrt{k(T)\max\{\|\boldsymbol{f}\|^2, \|\boldsymbol{g}\|^2\}},\end{aligned}$$

for $0 \leqslant t < T$, and thus we have the estimate

$$\|\boldsymbol{u}\|_t \leqslant \sqrt{P}Q^\delta\|\boldsymbol{A}(t)\boldsymbol{g} + \boldsymbol{B}(t)\boldsymbol{f}\|_T^{1-\delta} + \sqrt{k(T)\max(\|\boldsymbol{f}\|^2, \|\boldsymbol{g}\|^2)} \tag{3.9}$$

(4) We shall consider the situation where $\max\{\|\boldsymbol{f}\|, \|\boldsymbol{g}\|\} \to 0$ and thus the estimate (3.4) will remain valid for all corresponding $\boldsymbol{u}$ which lie in $\mathscr{R}$.

for all t, $0 \leqslant t < T$. A simple computation yields

$$\|\boldsymbol{A}(t)\boldsymbol{g}+\boldsymbol{B}(t)\boldsymbol{f}\|_T^2 \leqslant 2T\Big\{\sup_{[0,T)}\|\boldsymbol{A}(t)\|\,\|\boldsymbol{g}\|_+^2+\sup_{[0,T)}\|\boldsymbol{B}(t)\|\,\|\boldsymbol{f}\|_+^2\Big\}. \tag{3.10}$$

Also, $\|\boldsymbol{w}\|^2 \leqslant \pi\|\boldsymbol{w}\|_+^2$ for every $\boldsymbol{w} \in H_+$ and some $\pi > 0$, since we have assumed that $H_+ \subset H$ topologically as well as algebraically; using this fact, and the estimate (3.10) with

$$a \equiv \sup_{[0,T)}\|\boldsymbol{A}(t)\|\,, \qquad b \equiv \sup_{[0,T)}\|\boldsymbol{B}(t)\| \tag{3.11}$$

we may replace (3.9) by

$$\|\boldsymbol{u}\|_t \leqslant \gamma(t)[a\|\boldsymbol{g}\|_+^2 + b\|\boldsymbol{f}\|_+^2]^{\frac{1}{2}-\delta/2} [\pi k(T)]^{\frac{1}{2}}[\max(\|\boldsymbol{f}\|_+^2, \|\boldsymbol{g}\|_+^2)]^{\frac{1}{2}} \tag{3.12}$$

where we have set $\gamma(t) \equiv \sqrt{P}Q^{\delta}(2T)^{\frac{1}{2}-\delta/2}$. Our results can be summarized as

THEOREM II. Let $\boldsymbol{u} \in \mathscr{R}$ be any solution of (3.1), (3.2) for which $\hat{\boldsymbol{u}} \equiv (\boldsymbol{u} - t\boldsymbol{g} - \boldsymbol{f}) \in \mathscr{P}_K$ for some $K < 1$. Then if $\boldsymbol{G}(0)$ satisfies (2.2) with $\varkappa \geqslant K\Big(\sup_{[0,T)}\|\boldsymbol{G}(t)\| + 2T\sup_{[0,T)}\|\boldsymbol{G}_t(t)\|\Big)$, there exists a constant $\bar{k}(T)$ and a bounded real-valued function $\bar{\gamma}(t)$, defined on $[0, T)$, such that for all t, $0 \leqslant t < T$,

$$\|\boldsymbol{u}\|_t \leqslant \bar{\gamma}(t)[\max(\|\boldsymbol{f}\|_+^2, \|\boldsymbol{g}\|_+^2)]^{\frac{1}{2}-\delta/2} + \bar{k}(T)[\max(\|\boldsymbol{f}\|_+^2, \|\boldsymbol{g}\|_+^2)]\,. \tag{3.13}$$

REMARK. The appropriate forms for $\bar{k}(T)$ and $\bar{\gamma}(t)$ are

$$\bar{k}(T) = \big(\pi k(T)\big)^{\frac{1}{2}} \quad \text{and} \quad \gamma(t) = \sqrt{P}Q^{\delta}(4cT)^{\frac{1}{2}-\delta/2}$$

where

$$c \equiv \max\Big(\sup_{[0,T)}\|\boldsymbol{A}(t)\|,\ \sup_{[0,T)}\|\boldsymbol{B}(t)\|\Big).$$

REMARK (UNIQUENESS OF SOLUTIONS TO (1.1), (1.2)). Suppose that $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ are solutions of (1.1), (1.2) which correspond, respectively, to $\mathscr{F}_1$ and $\mathscr{F}_2$, where $\mathscr{H}(t) = \mathscr{F}_1(t) - \mathscr{F}_2(t)$ vanishes for all t, $0 < t < T$, but

$\mathcal{H}(0) \neq \mathbf{0}$. Then $\boldsymbol{u} = \boldsymbol{u}_1 - \boldsymbol{u}_2$ satisfies

$$\boldsymbol{u}_{tt} - \boldsymbol{N}\boldsymbol{u} + \int_0^t \boldsymbol{G}(t-\tau)\boldsymbol{u}(\tau)\,d\tau = \mathcal{H}(t)$$

$$\boldsymbol{u}(0) = \mathbf{0}, \qquad \boldsymbol{u}_t(0) = \mathbf{0}$$

for $0 \leqslant t < T$. If we assume that $\boldsymbol{u}_i \in C^2([0, T); H_+)$ and $\|\boldsymbol{u}_i\|_T^2 \leqslant \bar{N}^2$, for $i = 1, 2$, then (with N chosen so that $N^2 \geqslant 4\bar{N}^2$) it follows that $\boldsymbol{u} \in \mathcal{M}(N)$. Provided

$$\lim_{\lambda(\boldsymbol{u}_1 - \boldsymbol{u}_2)\to 0} \frac{1}{2} \sup_{[0,T)} \frac{t\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_t^2}{\int_0^t \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_\tau^2\, d\tau} < 1$$

we have $\boldsymbol{u} \in \mathcal{P}_K$, for some $K < 1$. Therefore, if $\boldsymbol{G}(0)$ satisfies (2.2) with $\varkappa \geqslant K\left(\sup_{[0,T)} \|\boldsymbol{G}(t)\| + 2T \sup_{[0,T)} \|\boldsymbol{G}_t(t)\|\right)$, the result of Theorem I may be applied and we deduce that

$$\|\boldsymbol{u}\|_t^2 \leqslant PQ^{2\delta}\left[\int_0^T \|\mathcal{H}(t)\|^2\, dt\right]^{1-\delta} = 0 \tag{3.14}$$

for all t, $0 \leqslant t < T$. From (3.14) it follows that $\boldsymbol{u}_1 = \boldsymbol{u}_2$ *a.e.* on $[0, T)$.

B. Continuous Dependence on $\delta\boldsymbol{N}$.

We now want to consider the result of perturbing the operator $\boldsymbol{N}$ by a symmetric operator $\delta\boldsymbol{N} \in \mathcal{L}(H_+, H_-)$; in other words, we consider solutions $\boldsymbol{u}(\cdot, t)$ and $\boldsymbol{u}_\delta(\cdot, t)$ of the system (1.1), (1.2), which correspond, respectively, to the operator pairs $\{\boldsymbol{N}, \boldsymbol{G}(t)\}$ and $\{\boldsymbol{N} - \delta\boldsymbol{N}, \boldsymbol{G}(t)\}$. Clearly $\delta\boldsymbol{u} \equiv \boldsymbol{u} - \boldsymbol{u}_\delta$ must satisfy

$$(\delta\boldsymbol{u})_{tt} - \boldsymbol{N}(\delta\boldsymbol{u}) + \int_0^t \boldsymbol{G}(t-\tau)\,\delta\boldsymbol{u}(\tau)\,d\tau = \delta\boldsymbol{N}(\boldsymbol{u}_\delta) \tag{3.15}$$

$$\delta\boldsymbol{u}(0) = \mathbf{0}, \qquad (\delta\boldsymbol{u})_t(0) = \mathbf{0} \tag{3.16}$$

for all t, $0 \leqslant t < T$. We require that $\delta\boldsymbol{N}\boldsymbol{f} \neq \mathbf{0}$ so that $(\delta\boldsymbol{u})_{tt}(0) \neq \mathbf{0}$ and assume that $\boldsymbol{u}, \boldsymbol{u}_\delta \in C^2([0, T); H_+)$ with $\|\boldsymbol{u}\|_T^2 \leqslant \tilde{N}^2$, $\|\boldsymbol{u}_\delta\|_T^2 \leqslant \tilde{N}^2$, for some real number $\tilde{N}$. If we choose N so that $N^2 \geqslant 4\tilde{N}^2$ then, clearly, $\boldsymbol{u} \in \mathcal{M}(N)$. If, in addition,

$$\lim_{\lambda(\delta\boldsymbol{u})\to 0} \frac{1}{2} \sup_{[0,T)} \frac{t\|\delta\boldsymbol{u}\|_t^2}{\int_0^t \|\delta\boldsymbol{u}\|^2\, d\tau} < 1 \tag{3.17}$$

then $\delta \boldsymbol{u} \in \mathscr{P}_{\bar{K}}$ for some $\mathrm{K} < 1$. This leads us to the following result:

THEOREM III. Let $\boldsymbol{u}, \boldsymbol{u}_\delta \in C^2([0, T); H_+)$ be two solutions of (1.1), (1.2), which correspond to the operator pairs $\{\boldsymbol{N}, \boldsymbol{G}(t)\}$ and $\{\boldsymbol{N} - \delta\boldsymbol{N}, \boldsymbol{G}(t)\}$, respectively, where $\delta\boldsymbol{N} \in \mathscr{L}(H_+, H_-)$ is symmetric and $\delta\boldsymbol{N}\boldsymbol{f} \neq \boldsymbol{0}$. Assume that $\delta\boldsymbol{u}(\cdot, t) = \boldsymbol{u}(\cdot, t) - \boldsymbol{u}_\delta(\cdot, t)$ satisfies (3.17), so that $\delta\boldsymbol{u} \in \mathscr{P}_{\bar{K}}$ for some $\bar{K} < 1$, and that $\boldsymbol{G}(0)$ satisfies (2.2) with

$$\varkappa \geqslant \bar{K}\left(\sup_{[0,T)} \|\boldsymbol{G}(t)\| + 2T \sup_{[0,T)} \|\boldsymbol{G}_t(t)\|\right).$$

Then there exist non-negative constants P and Q such that for all t, $0 \leqslant t < T$

$$\|\boldsymbol{u} - \boldsymbol{u}_\delta\|_t^2 \leqslant PQ^{2\delta} \|\delta\boldsymbol{N}(\boldsymbol{u}_\delta)\|_T^{2(1-\delta)}. \tag{3.18}$$

COROLLARY. Provided the conditions of Theorem III are satisfied,

$$\|\boldsymbol{u} - \boldsymbol{u}_\delta\|_t \to 0, \quad 0 \leqslant t < T, \qquad \text{as } \|\delta\boldsymbol{N}\| \to 0.$$

C. Continuous Dependence on Initial Geometry.

The results which we now present for the system (1.1), (1.2), generalize earlier results of Bloom [11] for the special case of (1.1) where $\boldsymbol{G} \equiv \boldsymbol{0}$. Thus, suppose that $\chi: J \to R^+$ is a non-negative continuous function on J such that $\sup_{x \in J} |\chi(x)| < \varepsilon$ for some $\varepsilon > 0$. We consider solutions $\boldsymbol{u}^\chi(\cdot, t)$ of (1.1) for which associated initial data $\boldsymbol{f}, \boldsymbol{g}$ are prescribed on the surface $t = -\chi(\cdot)$, i.e.,

$$\boldsymbol{u}^\chi(\cdot, -\chi(\cdot)) = \boldsymbol{f}(\cdot), \quad \boldsymbol{u}_t^\chi(\cdot, -\chi(\cdot)) = \boldsymbol{g}(\cdot). \tag{3.19}$$

Now let $\boldsymbol{u}(\cdot, t)$ be any solution of (1.1), (1.2) and assume that $\|\boldsymbol{u}\|_T^2 \leqslant M^2$, $\|\boldsymbol{u}^\chi\|_T^2 \leqslant M^2$ for ε sufficiently small. If we set $\boldsymbol{u}^\varepsilon = \boldsymbol{u}^\chi - \boldsymbol{u}$ then, clearly, $\boldsymbol{u}^\varepsilon$ is a solution of (1.1) corresponding to $\mathscr{F} = \boldsymbol{0}$ and $\boldsymbol{u}^\varepsilon \in \mathscr{R}$ when R is chosen so that $R^2 \geqslant 4M^2$. Moreover

$$\begin{aligned} \boldsymbol{u}^\varepsilon(\cdot, 0) &= \boldsymbol{u}^\chi(\cdot, 0) - \boldsymbol{u}(\cdot, 0) \\ &= \boldsymbol{u}^\chi(\cdot, 0) - \boldsymbol{u}^\chi(\cdot, -\chi(\cdot)) \\ &= \int_0^{-\chi(\cdot)} \eta \boldsymbol{u}_{\eta\eta}\, d\eta + \chi(\cdot)\boldsymbol{g}(\cdot) \end{aligned} \tag{3.20}$$

and

$$\boldsymbol{u}_t^\varepsilon(\cdot, 0) = \boldsymbol{u}_t^\chi(\cdot, 0) - \boldsymbol{u}_t(\cdot, 0) \tag{3.21}$$
$$= \boldsymbol{u}_t^\chi(\cdot, 0) - \boldsymbol{u}_t^\chi(\cdot, -\chi(\cdot))$$
$$= \int_{-\chi(\cdot)}^{0} \boldsymbol{u}_{\eta\eta}^\chi \, d\eta \, .$$

Therefore, $\boldsymbol{u}^\varepsilon \in \mathscr{R}$ is a solution of (3.1), (3.2) with

$$\boldsymbol{f} \to \boldsymbol{f}^\varepsilon \equiv \chi(\cdot)\boldsymbol{g}(\cdot) + \int_0^{-\chi(\cdot)} \eta \boldsymbol{u}_{\eta\eta}^\chi \, d\eta$$

and

$$\boldsymbol{g} \to \boldsymbol{g}^\varepsilon \equiv \int_{-\chi(\cdot)}^{0} \boldsymbol{u}_{\eta\eta}^\chi \, d\eta \, .$$

We assume that $\boldsymbol{N}\boldsymbol{f}^\varepsilon \neq \boldsymbol{0}$, at least for ε sufficiently small, and that

$$\hat{\boldsymbol{u}}^\varepsilon \equiv \boldsymbol{u}^\varepsilon - t\boldsymbol{g}^\varepsilon - \boldsymbol{f}^\varepsilon$$

satisfies

$$\lim_{\lambda(\hat{\boldsymbol{u}}^\varepsilon)\to 0} \frac{1}{2} \sup_{[0,T)} \frac{t\|\hat{\boldsymbol{u}}^\varepsilon\|_t^2}{\int_0^t \|\hat{\boldsymbol{u}}^\varepsilon\|_\tau^2 \, d\tau} < 1 \tag{3.22}$$

if ε is chosen sufficiently small. As (3.22) is satisfied $\boldsymbol{u}^\varepsilon \in \mathscr{P}_{\tilde{K}}$ for some $\tilde{K} < 1$ and the estimate

$$\|\boldsymbol{u}^\varepsilon\|_t \leqslant \bar{\gamma}(t)[\max(\|\boldsymbol{u}^\varepsilon(0)\|_+^2, \|\boldsymbol{u}_t^\varepsilon(0)\|_+^2)]^{\frac{1}{2}-\delta/2} \tag{3.23}$$
$$+ \bar{k}(T)[\max(\|\boldsymbol{u}^\varepsilon(0)\|_+^2, \|\boldsymbol{u}_t^\varepsilon(0)\|_+^2)]^{\frac{1}{2}}$$

will apply, for all $t \in [0, T)$, whenever $\boldsymbol{G}(0)$ satisfies (2.2) with

$$\varkappa \geqslant \tilde{K}\left(\sup_{[0,T)} \|\boldsymbol{G}(t)\| + 2T \sup_{[0,T)} \|\boldsymbol{G}_t(t)\|\right).$$

If we can show that both $\|\boldsymbol{u}^\varepsilon(0)\|_+^2 \to 0$ and $\|\boldsymbol{u}_t^\varepsilon(0)\|_+^2 \to 0$, for each $x \in J$, as $\varepsilon \to 0$, then it will follow directly from (3.23) that $\|\boldsymbol{u}^\varepsilon\|_t \to 0$, $0 \leqslant t < T$, for each $x \in J$, as $\varepsilon \to 0$. We have, in fact, the following result:

THEOREM IV. Suppose that the prerequisite conditions for the validity of (3.23) are satisfied and that

$$\|\chi \boldsymbol{g}\|_+ \geqslant (1+\varepsilon)\|\boldsymbol{u}_t^\varepsilon(0)\|_+ \tag{3.24}$$

at each $x \in J$, for all ε sufficiently small. Then for every $x \in J$ and each t, $0 \leqslant t < T$, $\|\boldsymbol{u}^\varepsilon\|_t \to 0$ as $\varepsilon \to 0$.

PROOF. In what follows we shall assume that $x \in J$ is arbitrary and we will suppress the dependence of all quantities, such as $\|\boldsymbol{u}^\varepsilon(0)\|_+$, on x; an alternative approach (5) would be to introduce a new inner product for elements $\boldsymbol{f}, \boldsymbol{g} \in L^2(J, H_+)$ via

$$\langle \boldsymbol{f}, \boldsymbol{g}\rangle_J \equiv \int_J \langle \boldsymbol{f}(x), \boldsymbol{g}(x)\rangle_+ \, d\mu_x \, .$$

so that $\|\boldsymbol{u}\|_t^2 \equiv \int_0^t \Big(\int_J \langle \boldsymbol{u}(x,\tau), \boldsymbol{u}(x,\tau)\rangle_+ \, d\mu_x\Big)\, d\tau$.

We begin by taking the inner product of (3.20) with $\boldsymbol{u}^\varepsilon(0)$ and in so doing we obtain

$$\|\boldsymbol{u}^\varepsilon(0)\|_+^2 = \chi\langle \boldsymbol{g}, \boldsymbol{u}^\varepsilon(0)\rangle_+ + \int_0^{-\chi(\cdot)} \eta\, \langle \boldsymbol{u}^\varepsilon(0), \boldsymbol{u}^\chi_{\eta\eta}\rangle_+ \, d\eta \, . \tag{3.25}$$

Now for a given $\chi\colon J \to R^+$, and any fixed $\varepsilon > 0$,

$$U(\varepsilon; \chi; \eta) \equiv \langle \boldsymbol{u}^\varepsilon(0), \boldsymbol{u}^\chi_{\eta\eta}\rangle_+ \tag{3.26}$$

is a continuous function of η on $[0, T)$. Therefore, we may apply the standard mean-value theorem to the integral in (3.25) to obtain

$$\begin{aligned} \|\boldsymbol{u}^\varepsilon(0)\|_+^2 &= \chi\langle \boldsymbol{g}, \boldsymbol{u}^\varepsilon(0)\rangle_+ + \eta_\chi \int_0^{-\chi(\cdot)} \langle \boldsymbol{u}^\varepsilon(0), \boldsymbol{u}^\chi_{\eta\eta}\rangle_+ \, d\eta \\ &= \chi\langle \boldsymbol{g}, \boldsymbol{u}^\varepsilon(0)\rangle_+ + \eta_\chi \Big\langle \boldsymbol{u}^\varepsilon(0), \int_0^{-\chi(\cdot)} \boldsymbol{u}^\chi_{\eta\eta}\, d\eta \Big\rangle_+ \\ &= \chi\langle \boldsymbol{g}, \boldsymbol{u}^\varepsilon(0)\rangle_+ - \eta_\chi \langle \boldsymbol{u}^\varepsilon(0), \boldsymbol{u}^\varepsilon_t(0)\rangle_+ \end{aligned} \tag{3.27}$$

where $-\chi < \eta_\chi < 0$ on J and we have made use of (3.21) in going from (3.27_2)

(5) I am indebted to Prof. H. A. LEVINE for this suggestion.

to (3.27_3). As $|\eta_\chi| < |\chi| < \varepsilon$ everywhere on J, (3.27) implies that

$$\|\boldsymbol{u}^\varepsilon(0)\|_+^2 \leqslant \varepsilon\|\boldsymbol{g}\|_+\|\boldsymbol{u}^\varepsilon(0)\|_+ + \varepsilon\|\boldsymbol{u}^\varepsilon(0)\|_+\|\boldsymbol{u}_t^\varepsilon(0)\|_+ \tag{3.28}$$

or

$$\|\boldsymbol{u}^\varepsilon(0)\|_+ \leqslant \varepsilon(\|\boldsymbol{g}\|_+ + \|\boldsymbol{u}_t^\varepsilon(0)\|_+) \tag{3.29}$$

everywhere on J. Now, take the inner product of (3.20) with $\boldsymbol{g}$ so as to get

$$\langle \boldsymbol{g}, \boldsymbol{u}^\varepsilon(0)\rangle_+ = \chi\|\boldsymbol{g}\|_+^2 + \int_0^{-\chi(\cdot)} \eta\langle \boldsymbol{g}, \boldsymbol{u}_{\eta\eta}^\chi\rangle_+ \, d\eta\,. \tag{3.30}$$

However,

$$\bar{U}(\varepsilon;\chi;\eta) \equiv \langle \boldsymbol{g}, \boldsymbol{u}_{\eta\eta}^\chi\rangle_+ \tag{3.31}$$

is a continuous function of η on $[0, T)$, so there exists a function $\bar{\eta}_\chi$, with $-\chi < \eta_\chi < 0$ everywhere on J, such that

$$\begin{aligned}\langle \boldsymbol{g}, \boldsymbol{u}^\varepsilon(0)\rangle_+ &= \chi\|\boldsymbol{g}\|_+^2 + \eta_\chi\langle \boldsymbol{g}, \int_0^{-\chi(\cdot)} \boldsymbol{u}_{\eta\eta}^\chi \, d\eta\rangle_+ \\ &= \chi\|\boldsymbol{g}\|_+^2 - \bar{\eta}_\chi\langle \boldsymbol{g}, \boldsymbol{u}_t^\varepsilon(0)\rangle_+\,.\end{aligned} \tag{3.32}$$

Therefore,

$$\|\boldsymbol{g}\|_+\|\boldsymbol{u}^\varepsilon(0)\|_+ \geqslant |\chi|\,\|\boldsymbol{g}\|_+^2 - \varepsilon\|\boldsymbol{g}\|_+\|\boldsymbol{u}_t^\varepsilon(0)\|_+ \tag{3.33}$$

where we have used the fact that $|\bar{\eta}_\chi| \leqslant |\chi| < \varepsilon$ on J. Clearly, (3.33) implies that

$$\|\boldsymbol{u}^\varepsilon(0)\|_+ \geqslant |\chi|\,\|\boldsymbol{g}\|_+ - \varepsilon\|\boldsymbol{u}_t^\varepsilon(0)\|_+ \tag{3.34}$$

if $\boldsymbol{g} \neq 0$. Using (3.34) and the hypothesis (3.24) we easily find that, for ε sufficiently small,

$$\begin{aligned}\|\boldsymbol{u}^\varepsilon(0)\|_+ &\geqslant (1+\varepsilon)\|\boldsymbol{u}_t^\varepsilon(0)\|_+ - \varepsilon\|\boldsymbol{u}_t^\varepsilon(0)\|_+ \\ &= \|\boldsymbol{u}_t^\varepsilon(0)\|_+\,.\end{aligned} \tag{3.35}$$

From (3.29) we then obtain

$$\|\boldsymbol{u}_t^\varepsilon(0)\|_+ \leqslant \|\boldsymbol{u}^\varepsilon(0)\|_+ \leqslant \varepsilon(\|\boldsymbol{g}\|_+ + \|\boldsymbol{u}_t^\varepsilon(0)\|_+) \tag{3.36}$$

or

$$\|\boldsymbol{u}_t^\varepsilon(0)\|_+ \leqslant \frac{\varepsilon}{1-\varepsilon}\|\boldsymbol{g}\|_+ \tag{3.37}$$

from which it is immediate that $\|\boldsymbol{u}_t^\varepsilon(0)\|_+ \to 0$, as $\varepsilon \to 0$, everywhere on J; using this result, in conjunction with (3.29) again, we see that $\|\boldsymbol{u}^\varepsilon(0)\|_+ \to 0$, as $\varepsilon \to 0$, everywhere on J. The desired result, i.e. $\|\boldsymbol{u}^\varepsilon\|_t \to 0$, $0 \leqslant t < T$, everywhere on J as $\varepsilon \to 0$, now follows directly from the estimate (3.23). *Q.E.D.*

4. – Some continuous dependence results for linear viscoelasticity.

We begin by considering the following abstract form of the standard initial-boundary value problem of linear viscoelasticity:

$$\boldsymbol{u}_{tt} - \boldsymbol{N}\boldsymbol{u} + \int_{-\infty}^{t} \boldsymbol{G}(t-\tau)\,\boldsymbol{u}(\tau)\,d\tau = \hat{\mathscr{F}}(t) \tag{4.1}$$

$$\boldsymbol{u}(\tau) = \boldsymbol{U}(\tau), \qquad -\infty < \tau < 0 \tag{4.2}$$

$$\boldsymbol{u}(0) = \boldsymbol{f}, \qquad \boldsymbol{u}_t(0) = \boldsymbol{g} \tag{4.3}$$

where $0 \leqslant t < T$, and

I) $\boldsymbol{U}(t) \in C^1\big((-\infty, 0); H_+\big)$ is a prescribed function (called the past history) which must satisfy

a) $\lim_{t\to 0^-} \boldsymbol{U}(t) = \boldsymbol{f}$, $\lim_{t\to 0^-} \boldsymbol{U}_t(t) = \boldsymbol{g}$;

b) $\lim_{t\to -\infty} \|\boldsymbol{U}(t)\|_+ = 0$;

c) $\int_{-\infty}^{0} \|\boldsymbol{U}(\tau)\|_+^2\,d\tau < \infty$.

II) $\boldsymbol{G}(t) \in L^2\big((-\infty, \infty); \mathscr{L}(H_+, H_-)\big)$ such that

a) $\boldsymbol{G}_t(t) \in L^2\big([0, T); \mathscr{L}(H_+, H_-)\big)$

b) $\partial^k \boldsymbol{G}(t)/\partial t^k$, $k = 2, 3, 4$, exists a.e. on $[0, T)$ and belongs to $\mathscr{L}(H_+, H_-)$.

$$\text{III)} \qquad \int_{-\infty}^{0} \boldsymbol{G}(-\tau)\,\boldsymbol{U}(\tau)\,d\tau - \boldsymbol{N}\boldsymbol{f} \neq \boldsymbol{0}\,. \tag{4.4}$$

For solutions $\boldsymbol{u} \in \mathscr{R}$ of (4.1)-(4.3), we may easily derive an estimate which yields joint continuous dependence on perturbations of the past history $\boldsymbol{U}(\tau)$, $-\infty < \tau < 0$, and the initial data $\boldsymbol{f}$ and $\boldsymbol{g}$, when $\hat{\mathscr{F}} = \mathbf{0}$. If $\hat{\mathscr{F}} \neq \mathbf{0}$ then it will be clear that we can include continuous dependence on perturbations of $\hat{\mathscr{F}}$ if we replace condition III above with

$$\text{III} \qquad \int_{-\infty}^{0} \boldsymbol{G}(-\tau)\,\boldsymbol{U}(\tau)\,d\tau \neq \boldsymbol{N}\boldsymbol{f} + \hat{\mathscr{F}}(0)\,.$$

Thus, suppose that $\boldsymbol{u} \in \mathscr{R}$ is a solution of (4.1)-(4.3) and define $\boldsymbol{v} \in C^2([0, T); H_+) \cap C^1((-\infty, 0); H_+)$ by

$$\boldsymbol{v}(t) = \begin{cases} \boldsymbol{u}(t) - t\boldsymbol{u}_t(0) - \boldsymbol{u}(0)\,, & 0 \leqslant t < T \\ \boldsymbol{U}(t)\,, & -\infty < t < 0\,. \end{cases} \tag{4.5}$$

Then it is a simple matter to show that $\boldsymbol{v}(0) = \mathbf{0}$, $\boldsymbol{v}_t(0) = \mathbf{0}$, and that $\boldsymbol{v}$ satisfies

$$\begin{aligned} \boldsymbol{v}_{tt} - \boldsymbol{N}\boldsymbol{v} + \int_0^t \boldsymbol{G}(t-\tau)\,\boldsymbol{v}(\tau)\,d\tau \\ = \boldsymbol{A}(t)\boldsymbol{g} + \boldsymbol{B}(t)\boldsymbol{f} - \int_{-\infty}^{0} \boldsymbol{G}(t-\tau)\,\boldsymbol{U}(\tau)\,d\tau \end{aligned} \tag{4.6}$$

for each t, $0 \leqslant t < T$, where $\boldsymbol{A}(t)$ and $\boldsymbol{B}(t)$ are given, respectively, by (3.6) and (3.7). Also, by virtue of (4.4), we have $\boldsymbol{v}_{tt}(0) \neq \mathbf{0}$. If we choose $\boldsymbol{N}$ so as to satisfy (3.4) then $\boldsymbol{v}_{|[0,T)} \in \mathscr{M}(N)$. Therefore, if

$$\lim_{\lambda(\boldsymbol{v})\to 0} \frac{1}{2} \sup_{[0,T)} \frac{t\|\boldsymbol{v}\|_t^2}{\int_0^t \|\boldsymbol{v}\|_\tau^2\,d\tau} < 1 \tag{4.7}$$

it will follow that $\boldsymbol{v} \in \mathscr{P}_{\bar{K}}$ for some $\bar{\varkappa} < 1$. We assume that (4.7) is satisfied and that

$$\langle \boldsymbol{w}, \boldsymbol{G}(0)\boldsymbol{w}\rangle \geqslant -\varkappa\|\boldsymbol{w}\|_+^2\,,\ \forall \boldsymbol{w} \in H_+$$

where

$$\varkappa \geqslant \bar{\varkappa}\left[\sup_{[0,T)} \|\boldsymbol{G}(t)\| + 2T \sup_{[0,T)} \|\boldsymbol{G}_t(t)\|\right]. \tag{4.8}$$

Then the result of Theorem I may be used to deduce the stability estimate

$$\|\boldsymbol{u}\|_t \leqslant \sqrt{P}\,Q^{\delta}\|\mathscr{F}_U(\boldsymbol{f},\boldsymbol{g})\|_T^{1-\delta} + \sqrt{k(T)\max(\|\boldsymbol{f}\|^2, \|\boldsymbol{g}\|^2)} \tag{4.9}$$

for $0 \leqslant t < T$, where

$$\mathscr{F}_U(\boldsymbol{f},\boldsymbol{g}) \equiv \boldsymbol{A}(t)\boldsymbol{g} + \boldsymbol{B}(t)\boldsymbol{f} - \int_{-\infty}^{0} \boldsymbol{G}(t-\tau)\,\boldsymbol{U}(\tau)\,d\tau\,.$$

Each of the terms on the right-hand side of (4.9) may be bounded, as follows:

$$\|\mathscr{F}_U(\boldsymbol{f},\boldsymbol{g})\|_T^2 \leqslant 2\|\boldsymbol{A}(t)\boldsymbol{g} + \boldsymbol{B}(t)\boldsymbol{f}\|_T^2 + 2\left\|\int_{-\infty}^{0} \boldsymbol{G}(t-\tau)\,\boldsymbol{U}(\tau)\,d\tau\right\|_T^2. \tag{4.10}$$

Also,

$$\left\|\int_{-\infty}^{0} \boldsymbol{G}(t-\tau)\,\boldsymbol{U}(\tau)\,d\tau\right\|_T^2 \leqslant \int_0^T\left(\int_{-\infty}^{0} \|\boldsymbol{G}(t-\tau)\|\,\|\boldsymbol{U}(\tau)\|_+\,d\tau\right)^2 dt$$
$$\leqslant \int_0^T\left(\int_{-\infty}^{0} \|\boldsymbol{G}(t-\tau)\|^2\,d\tau\right)dt\int_{-\infty}^{0} \|\boldsymbol{U}(\tau)\|_+^2\,d\tau$$

so that

$$\left\|\int_{-\infty}^{0} \boldsymbol{G}(t-\tau)\,\boldsymbol{U}(\tau)\,d\tau\right\|_T^2 \leqslant \left[T\sup_{[0,T)}\int_{-\infty}^{0} \|\boldsymbol{G}(t-\tau)\|^2\,d\tau\right]\left(\int_{-\infty}^{0} \|\boldsymbol{U}(\tau)\|_+^2\,d\tau\right)$$

while for the first expression on the right-hand side of (4.10) we have

$$\|\boldsymbol{A}(t)\boldsymbol{g} + \boldsymbol{B}(t)\boldsymbol{f}\|_T^2 \leqslant 2\|\boldsymbol{A}(t)\boldsymbol{g}\|_T^2 + 2\|\boldsymbol{B}(t)\boldsymbol{f}\|_T^2$$
$$\leqslant 2T\left[\sup_{[0,T)}\|\boldsymbol{A}\|^2\|\boldsymbol{g}\|_+^2 + \sup_{[0,T)}\|\boldsymbol{B}\|^2\|\boldsymbol{f}\|_+^2\right]$$

so that

$$\|\boldsymbol{A}(t)\boldsymbol{g} + \boldsymbol{B}(t)\boldsymbol{f}\|_T^2 \leqslant 4T\max\left(\sup_{[0,T)}\|\boldsymbol{A}\|^2,\ \sup_{[0,T)}\|\boldsymbol{B}\|^2\right)\max(\|\boldsymbol{f}\|_+^2, \|\boldsymbol{g}\|_+^2)\,.$$

Combining these estimates we have

$$(4.11)\qquad \|\boldsymbol{u}\|_t \leqslant \sqrt{P}\,Q^{\delta}\Big[\Big(2T \sup_{[0,T)} \int_{-\infty}^{0} \|\boldsymbol{G}(t-\tau)\|^2 d\tau\Big) \int_{-\infty}^{0} \|U\|_+^2 d\tau + 8T \max\Big(\sup_{[0,T)} \|\boldsymbol{A}\|^2, \sup_{[0,T)} \|\boldsymbol{B}\|^2\Big) \max(\|\boldsymbol{f}\|_+^2, \|\boldsymbol{g}\|_+^2)\Big]^{\frac{1}{2}-\delta/2} + [\pi k(T) \max(\|\boldsymbol{f}\|_+^2, \|\boldsymbol{g}\|_+^2)]^{\frac{1}{2}}.$$

Therefore, if $\boldsymbol{u} \in \mathscr{R}$ is any solution of (4.1)-(4.3), with $\hat{\mathscr{F}} = \boldsymbol{0}$, we have the following result:

THEOREM V. Assume that conditions I-III (above) are satisfied and that $\boldsymbol{u}(t) - t\boldsymbol{g} - \boldsymbol{f} \in \mathscr{P}_{\bar{K}}$ for some $\bar{\varkappa} < 1$. If $\boldsymbol{G}(0)$ satisfies (2.2) with

$$\varkappa \geqslant \bar{\varkappa}\Big(\sup_{[0,T)} \|\boldsymbol{G}(t)\| + 2T \sup_{[0,T)} \|\boldsymbol{G}_t(t)\|\Big)$$

then $\|\boldsymbol{u}\|_t \to 0$, $0 \leqslant t < T$, as

$$\max\Big[\int_{-\infty}^{0} \|\boldsymbol{U}\|_+^2 d\tau, \max(\|\boldsymbol{f}\|_+^2, \|\boldsymbol{g}\|_+^2)\Big] \to 0.$$

In order to apply Theorem V to initial-boundary value problems which arise in isothermal linear viscoelasticity, we consider a bounded domain $\Omega \subset E^3$ with smooth boundary $\partial\Omega$. In the cylinder $\Omega \times (-\infty, T)$ we have the equations

$$(4.12)\qquad \varrho(\boldsymbol{x}) \frac{\partial^2 u_i}{\partial t^2} - \frac{\partial}{\partial x_j}\Big[\int_{-\infty}^{t} g_{ijkl}(\boldsymbol{x}, t-\tau) \frac{\partial u_{k,l}}{\partial \tau} d\tau\Big] = 0$$

while

$$(4.13)\qquad u_i(\boldsymbol{x}, \tau) = U_i(\boldsymbol{x}, \tau), \quad (\boldsymbol{x}, \tau) \in \Omega \times (-\infty, 0)$$

and

$$(4.14)\qquad \boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{0}, \quad (\boldsymbol{x}, t) \in \partial\Omega \times (-\infty, T)$$

$$(4.15)\qquad \boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{f}(\boldsymbol{x}), \quad \frac{\partial \boldsymbol{u}}{\partial t}(\boldsymbol{x}, 0) = \boldsymbol{g}(\boldsymbol{x}) \quad \text{on } \Omega.$$

In (4.12), $\boldsymbol{u}$ is the displacement vector, $\varrho(\boldsymbol{x})$ is the nonhomogeneous density, and the $g_{ijkl}(\boldsymbol{x}, t)$ are the components of the relaxation tensor at time t,

$-\infty < t < T$. Under the usual assumption (6) that $\lim_{t\to-\infty} g_{ijkl}(\boldsymbol{x}, t) = 0$, uniformly in $\boldsymbol{x}$, we may recast (4.12) in the form

$$\varrho(\boldsymbol{x})\frac{\partial^2 u_i}{\partial t^2}(\boldsymbol{x}, t) - \frac{\partial}{\partial x_j}\left[g_{ijkl}(\boldsymbol{x}, 0)\frac{\partial u_k}{\partial x_l}(\boldsymbol{x}, t)\right] \tag{4.16}$$

$$+\frac{\partial}{\partial x_j}\int_0^t \frac{\partial}{\partial \tau} g_{ijkl}(\boldsymbol{x}, t-\tau)\frac{\partial u_k}{\partial x_l}(\boldsymbol{x}, \tau)\,d\tau = -\frac{\partial}{\partial x_j}\int_{-\infty}^0 \frac{\partial}{\partial \tau} g_{ijkl}(\boldsymbol{x}, t-\tau)\frac{\partial U_k}{\partial x_l}(\boldsymbol{x}, \tau)\,d\tau\,,$$

for $(\boldsymbol{x}, t)\in \Omega\times[0, T)$. Following Dafermos [1] we now introduce Hilbert spaces H_+, H, and H_- as follows: Let $C_0^\infty(\Omega)$ denote the set of three-dimensional vector fields with compact support in Ω whose components belong to $C_0^\infty(\Omega)$. Then H is obtained by completing $C_0^\infty(\Omega)$ under the norm induced by the inner product

$$\langle \boldsymbol{v}, \boldsymbol{w}\rangle \equiv \int_\Omega v_i w_i\, d\boldsymbol{x} \tag{4.17}$$

while H_+ is defined to be the completion of $C_0^\infty(\Omega)$ under the norm induced by the inner product

$$\langle v, w\rangle_+ \equiv \int_\Omega \frac{\partial v_i}{\partial x_j}\frac{\partial w_i}{\partial x_j}\, d\boldsymbol{x}\,. \tag{4.18}$$

Finally we define H_- to be completion of $C_0^\infty(\Omega)$ under the norm

$$\|\boldsymbol{v}\|_- \equiv \sup_{\boldsymbol{w}\in H_+}\left[\left|\int_\Omega v_i w_i\, d\boldsymbol{x}\right| \Big/ \left(\int_\Omega \frac{\partial w_i}{\partial x_j}\frac{\partial w_i}{\partial x_j}\, d\boldsymbol{x}\right)^{\frac{1}{2}}\right]. \tag{4.19}$$

An operator $\boldsymbol{S}(t)\in L^2\big((-\infty, T);\ \mathscr{L}(H_+, H_-)\big)$ can now be defined as follows: for any $\boldsymbol{v}\in H_+$ and $t\in(-\infty, T)$

$$[\boldsymbol{S}(t)\boldsymbol{v}]_i = \frac{\partial}{\partial x_j}\left[g_{ijkl}(\boldsymbol{x}, t)\frac{\partial v_k}{\partial x_l}\right] \tag{4.20}$$

or

$$S_{ik}(t) \equiv \frac{\partial}{\partial x_j}\left[g_{ijkl}(\boldsymbol{x}, t)\frac{\partial}{\partial x_l}\right]. \tag{4.21}$$

The system of equations (4.16), taken in conjunction with the homogeneous

(6) We also assume that $\varrho(\boldsymbol{x})$ and the $g_{ijk}(\boldsymbol{x}, t)$, for each $t\in[0, T)$, are Lebesgue measurable and essentially bounded on Ω (with $g_{ijk}(\cdot, t)\in C^1(\Omega)$).

boundary condition (4.13), is then precisely of the form (4.1) provided we define

$$\mathbf{N} = \mathbf{S}(0) \quad \text{and} \quad \mathbf{G}(t-\tau) = \frac{\partial}{\partial \tau}\mathbf{S}(t-\tau)\,. \tag{4.22}$$

From (4.22) we easily obtain

$$\begin{aligned} \mathbf{G}(0) &= \frac{\partial}{\partial \tau}\mathbf{S}(t-\tau)\bigg|_{t=\tau} \\ &= -\frac{\partial}{\partial t}\mathbf{S}(t-\tau)\bigg|_{t=\tau} \\ &= -\dot{\mathbf{S}}(0) \end{aligned}$$

so that the condition (2.2) becomes

$$\langle \boldsymbol{v}, \dot{\mathbf{S}}(0)\boldsymbol{v}\rangle \geqslant \varkappa \|\boldsymbol{v}\|_+^2\,, \qquad \forall \boldsymbol{v} \in H_+\,. \tag{4.23}$$

By using the definitions of H, H_+, and $\mathbf{S}(t)$, and the divergence theorem, it is easy to show that (4.23) assumes the specific form

$$\int_\Omega \dot{g}_{ijkl}(\boldsymbol{x}, 0)\frac{\partial v_i}{\partial x_j}\frac{\partial v_k}{\partial x_l}\,d\boldsymbol{x} \leqslant -\varkappa\int_\Omega \frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial x_j}\,d\boldsymbol{x} \tag{4.24}$$

for all $\boldsymbol{v} \in H_+$. In the one-dimensional homogeneous situation, with $\Omega \equiv \{x | x \in [0, 1]\}$, the equations of motion (4.16) reduce to

$$\varrho\frac{\partial^2 u}{\partial t^2} - g(0)\frac{\partial^2 u}{\partial x^2} + \int_{-\infty}^{0}\frac{\partial}{\partial \tau}g(t-\tau)\frac{\partial^2 u}{\partial x^2}\,dx = 0 \tag{4.25}$$

for $(x, t) \in [0, 1] \times (-\infty, T)$, where $g(t)$ is the relaxation function. Condition (4.24) now reduces to the statement that

$$\dot{g}(0) \leqslant -\varkappa\,, \qquad \text{for some } \varkappa > 0\,. \tag{4.26}$$

In order to apply Theorem V to the system consisting of (4.25) together with

$$\begin{cases} u(x, \tau) = U(x, \tau)\,, & (x, \tau) \in [0, 1] \times (-\infty, 0) \\ u(0, t) = u(1, t) = 0\,, & -\infty < t < T\,, \\ u(x, 0) = f(x)\,, \quad u_t(x, 0) = g(x)\,, & x \in [0, 1]\,, \end{cases} \tag{4.27}$$

the following conditions must be satisfied (in addition to the obvious smoothness requirements which must be imposed on $g(t)$, $U(t)$, and the initial data by virtue of conditions I, II above):

$$(4.28)\qquad (i)\ \int_{-\infty}^{0} \dot{g}(-\tau)\frac{\partial^2 U(x,\tau)}{\partial x^2}\,d\tau \neq g(0)\frac{\partial^2 f}{\partial x^2}$$

(ii) for $v \equiv u - tg - f$

$$(4.29)\qquad \tfrac{1}{2}\lim_{\lambda(v)\to 0}\ \sup_{[0,T)}\left\{t\int_0^t\int_0^1 v^2(x,\tau)\,dx\,d\tau \Big/ \int_0^t (t-\tau)\int_0^1 v^2(x,\tau)\,dx\,d\tau\right\} < 1$$

so that $v \in \mathscr{P}_{\bar{\varkappa}}$ for some $\bar{\varkappa} < 1$.

(iii) $\dot{g}(0)$ satisfies (4.26) with

$$(4.30)\qquad \varkappa \geqslant \bar{\varkappa}\left[\sup_{[0,T)}|\dot{g}(t)| + 2T\sup_{[0,T)}|\ddot{g}(t)|\right].$$

If (i), (ii), (iii) above are satisfied then

$$(4.31)\qquad \int_0^t\int_0^1 u^2(x,\tau)\,dx\,d\tau \to 0\,, \qquad 0 \leqslant t < T\,,$$

as

$$(4.32)\qquad \max\left[\int_{-\infty}^{0}\int_0^1\left(\frac{\partial U(x,\tau)}{\partial x}\right)^2 dx\,d\tau\,,\ \max\left(\int_0^1\left(\frac{\partial f}{\partial x}\right)^2 dx,\int_0^1\left(\frac{\partial g}{\partial x}\right)^2 dx\right)\right] \to 0\,.$$

It has been observed experimentally that the relaxation function of a one-dimensional linear viscoelastic material is both non-negative and monotonically decreasing in time; this latter condition has been analytically established by Day [12] who has given an interesting interpretation in terms of an assertion concerning the work done on the material over certain closed paths in strain space. What (4.30)-(4.32) say about solutions of the system (4.25), (4.27), (whose growth behavior is mild in the sense that it conforms to (4.29)) is that, provided the relaxation function $g(t)$ is decreasing sufficiently fast at $t = 0$, $u(x,t)$ exhibits joint continuous dependence on perturbations of the past history and the initial data (7).

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(7) Similarly, results concerning continuous dependence on perturbations of the initial geometry and the initial value of the relaxation tensor, for the system (4.12)-(4.15), may be obtained from Theorems III and IV.

Appendix.

Condition (2.7) has been imposed to handle the following kind of situation: For each $n = 1, 2, 3, \ldots$ let

$$\boldsymbol{u}_n \in \mathscr{R} \equiv \{\boldsymbol{v} \in C^2([0, T); H_+) \mid \|\boldsymbol{v}\|_T^2 \leqslant R^2\}$$

be a solution of (3.1) corresponding to

$$\boldsymbol{u}_n(0) = \boldsymbol{f}_n, \qquad \frac{\partial u_n}{\partial t}(0) = \boldsymbol{g}_n$$

where for $n = 1, 2, \ldots$, $\boldsymbol{f}_n, \boldsymbol{g}_n \in H_+$ and $\max\{\|\boldsymbol{f}_n\|_+, \|\boldsymbol{g}_n\|_+\} \to 0$, as $n \to \infty$. For each $n = 1, 2, \ldots$ define

$$\hat{\boldsymbol{u}}_n(\cdot, t) = \boldsymbol{u}_n(\cdot, t) - t\boldsymbol{g}_n - \boldsymbol{f}_n .$$

Then $\hat{\boldsymbol{u}}_n$ satisfies (1.1) with $(\hat{\boldsymbol{u}}_n)_t(\cdot, 0) = 0$ and $\hat{\boldsymbol{u}}_n(\cdot, 0) = 0$, for each $n = 1, 2, \ldots$ and

$$\mathscr{F}(t) \to \mathscr{F}_n(t) \equiv \left[t\boldsymbol{N} - \int_0^t \tau \boldsymbol{G}(t-\tau)\, d\tau\right]\boldsymbol{g}_n + \left[\boldsymbol{N} - \int_0^t \boldsymbol{G}(t-\tau)\, d\tau\right]\boldsymbol{f}_n$$

As

$$K\hat{\boldsymbol{u}}_n \equiv \sup_{[0,T)} \frac{t\int_0^t \|\hat{\boldsymbol{u}}_n(\tau)\|_+^2\, d\tau}{\int_0^t \int_0^\eta \|\hat{\boldsymbol{u}}_n(\tau)\|_+^2\, d\tau\, d\eta} < \infty, \qquad n = 1, 2, \ldots$$

(i.e., *Lemma* 1) an estimate analogous to (2.10), i.e.,

$$\text{(A.1)} \qquad \|\hat{\boldsymbol{u}}_n\|_t^2 \leqslant P_n Q_n^{2\delta} \|\mathscr{F}_n(t)\|_T^2, \qquad 0 \leqslant t < T$$

can be pushed through without requiring the imposition of (2.7). However, we now want to examine what happens as $n \to \infty$, i.e., as $\max\{\|\boldsymbol{f}_n\|_+, \|\boldsymbol{g}_n\|_+\} \to 0$. If the problem is well-posed in the sense of Hölder we expect that $\|\boldsymbol{u}_n\|_t \to 0$ as $n \to \infty$ for each t, $0 \leqslant t < T$.

Now (1) implies (note the development from (3.8) to (3.12) and the result (3.13)) that

$$\|\boldsymbol{u}_n\|_t \leq \bar{\gamma}_n(t) \max\left[(\|\boldsymbol{f}_n\|_+^2, \|\boldsymbol{g}_n\|_+^2)\right]^{\frac{1}{2}-\delta/2} + \bar{\varkappa}(T)\left[\max(\|\boldsymbol{f}_n\|_+^2, \|\boldsymbol{g}_n\|_+^2)\right] \tag{A.2}$$

where $\bar{\gamma}_n(t)$ depends on n through both the P_n and Q_n of (A.1) [see the remark following (3.13)] and P_n, Q_n depend on $K\hat{\boldsymbol{u}}_n$ defined above. As $n \to \infty$ the second expression on the right hand side of (A.2) goes to zero as does the factor $\max(\|\boldsymbol{f}_n\|_+^2, \|\boldsymbol{g}_n\|_+^2)]^{\frac{1}{2}-\delta/2}$ in the first expression on the right hand side of (A.2). But without the imposition of (2.7) there is no guarantee that $\{\bar{\gamma}_n(t)\}$ is bounded above as $n \to \infty$ and without such boundedness for $\{\bar{\gamma}_n(t)\}$, $t \in [0, T)$, we lose the desired continuous dependence result $\|\boldsymbol{u}_n\|_t \to 0$ as $n \to \infty$. In effect, therefore, (2.7) characterizes the class of uniformly bounded solutions of (1.1) which depend Hölder continuously on the initial data as being those for which $\sup_n K\hat{\boldsymbol{u}}_n < \infty$.

Because of the uniform boundedness of the functions $\hat{\boldsymbol{u}}_n(t)$ on $[0, T)$, $n = 1, 2, \ldots$ the only way that the quotient

$$\frac{t\int_0^t \|\hat{\boldsymbol{u}}_n(\tau)\|_+^2\, d\tau}{\int_0^t \int_0^\eta \|\hat{\boldsymbol{u}}_n(\tau)\|_+^2\, d\tau\, d\eta} \tag{A.3}$$

can «blow up» for fixed $t \in [0, T)$ is when the denominator goes to zero, i.e., when

$$\int_0^t \|\hat{\boldsymbol{u}}_n(\tau)\|_+^2\, d\tau \to 0$$

so let

$$h_n(t) \equiv \int_0^t \|\hat{\boldsymbol{u}}_n(\tau)\|_+^2\, d\tau$$

so that (A.3) assumes the form

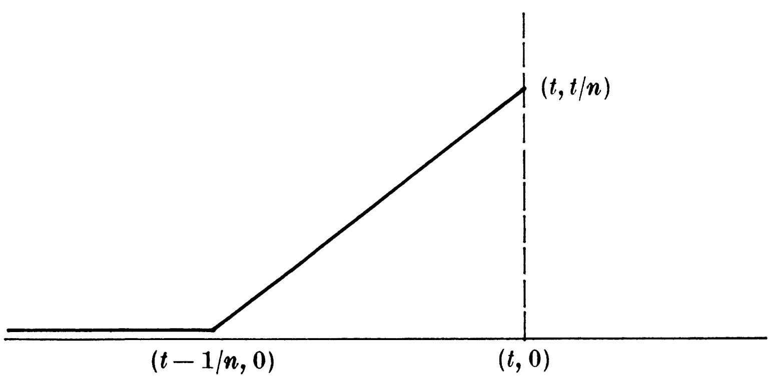
$$\frac{h_n(t)}{1/t\int_0^t h_n(\tau)\, d\tau} \equiv [K\hat{\boldsymbol{u}}_n](t)\,. \tag{A.4}$$

Note that $K\hat{\boldsymbol{u}}_n = \sup_{[0,T)} [K\hat{\boldsymbol{u}}_n](t)$ and that $\{h_n(t)\}$ is a monotonically nondecreasing sequence of differentiable non-negative functions defined on $[0, T)$ so that $\sup_{[0,t)} h_n(s) = h_n(t)$ for all t, $0 \leqslant t < T$. In order for Hölder continuity to follow we must avoid situations where

$$h_n(t) \to 0 \quad \text{as} \quad n \to \infty \quad \textit{and} \quad \frac{h_n(t)}{1/t \int_0^t h_n(\tau)\, d\tau} \to \infty .$$

Clearly examples of such « bad » sequences can be easily constructed, i.e. define, for $t < T$,

$$\text{(A.5)} \qquad \tilde{h}_n(s) = \begin{cases} 0, & 0 \leqslant s < t - \dfrac{1}{n} \\ \left(s - t + \dfrac{1}{n}\right) t, & t - \dfrac{1}{n} \leqslant s < t \end{cases}$$



Clearly (see diagram above) the sequence $\{\tilde{h}_n(s)\}$ satisfies the required monotonicity condition *and* we simply round-off the corner at $(t - 1/n, 0)$ to get the required smoothness. Also

$$\sup_{[0,t)} \tilde{h}_n(s) = \tilde{h}_n(t) = \frac{t}{n}, \qquad n = 1, 2, \ldots$$

so that $\lim_{n\to\infty} \sup_{[0,t)} \tilde{h}_n(s) = 0$. But

$$\frac{h_n(t)}{1/t \int_0^t \tilde{h}_n(\tau)\, d\tau} = \frac{t/n}{1/t[\frac{1}{2}(1/n)(t/n)]} = 2nt \to \infty , \qquad \text{as } n \to \infty$$

for each t, $0 \leqslant t < T$.

Thus if $\boldsymbol{u}_n \in \mathscr{R}$ is a solution of (3.1) corresponding to $\boldsymbol{u}_n(0) = \boldsymbol{f}_n$, $(\partial \boldsymbol{u}_n/\partial t)(0) = \boldsymbol{g}_n$ for which the sequence of functions

$$\left\{\int_0^t \|\boldsymbol{u}_n(\tau) - \tau \boldsymbol{g}_n - \boldsymbol{f}_n\|_+^2 \, d\tau\right\} \tag{A.6}$$

is of the form specified by (A.5) the prerequisite conditions guaranteeing the Hölder continuity of solutions, under perturbations of the initial data, breakdown. It should be noted that there are wide ranges of behavior which the sequence (A.6) may conform to for which

$$\lim_{n\to\infty} \sup_{[0,T)} \int_0^t \|\boldsymbol{u}_n(\tau) - \tau \boldsymbol{g}_n - \boldsymbol{f}_n\|_+^2 \, d\tau = 0$$

and yet Hölder continuity follows. For instance, a simple computation shows that if

$$\int_0^s \|\boldsymbol{u}_n(\tau) - \tau \boldsymbol{g}_n - \boldsymbol{f}_n\|_+^2 \, d\tau = \begin{cases} \dfrac{s}{n^2}, & 0 \leqslant s < t - \dfrac{1}{n} \\ \left(\dfrac{t - 1/n}{n^2}\right) \exp[1 - nt] \exp[ns], & t - \dfrac{1}{n} \leqslant s \leqslant t \end{cases}$$

then the sequence

$$\{\bar{h}_n(s)\} \equiv \left\{\int_0^s \|\boldsymbol{u}_n(\tau) - \tau \boldsymbol{g}_n - \boldsymbol{f}_n\|_+^2 \, d\tau\right\}, \qquad n = 1, 2, \ldots$$

satisfies $\lim\limits_{n\to\infty} \sup\limits_{[0,t)} \bar{h}_n(s) = 0$, with $\{\bar{h}_n(s)\}$ monotonically nondecreasing on $[0, t)$ and yet

$$\lim_{n\to\infty} \frac{\bar{h}_n(t)}{1/t \int_0^t \bar{h}_n(\tau)\, d\tau} = \lim_{n\to\infty} \frac{\exp\left[\dfrac{t - 1/n}{n^2}\right]}{\dfrac{t}{n^2} - \dfrac{2}{n^3} + \dfrac{1}{n^4 t} + \left(\dfrac{t - 1/n}{tn^3}\right)[e - 1]} = e, \quad \text{for all } t,\ 0 \leqslant t < T.$$

It then follows that $\|\boldsymbol{u}_n\|_t \to 0$ as $n \to \infty$ for all t, $0 \leqslant t < T$, if $\max\{\|\boldsymbol{f}_n\|_+, \|\boldsymbol{g}_n\|_+\} \to 0$ as $n \to \infty$.

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