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On the Hölder Continuity of Weak Solutions of Quasilinear Elliptic Systems of Second Order.

STEFAN HILDEBRANDT (*) - KJELL-OVE WIDMAN (**)

dedicated to Hans Lewy

In this paper we shall study bounded weak solutions of a class of elliptic systems of quasilinear partial differential equations,

$$(1) \qquad -D_\beta[A^{\alpha\beta}(x, u, \nabla u) D_\alpha u^l] = f^l(x, u, \nabla u), \qquad 1 \leqslant l \leqslant N,$$

the characteristic properties of which are that the principal part consists of a uniformly elliptic operator times the identity matrix and that the right hand side grows at most quadratically in the derivatives ∇u. Despite the rather special form of the principal part a study of such systems seems worthwhile due to the possible applications in other fields. The harmonic mappings between two Riemannian manifolds form an important example (cf., for example, [5]).

For a discussion of various aspects including possible applications we refer to [3] and [6].

Let $u \in H_2^1 \cap L^\infty$ be a weak solution of (1) such that there are numbers $\lambda > 0$, $a \geqslant 0$, $b \geqslant 0$ satisfying

$$(2) \qquad \lambda|\xi|^2 \leqslant A^{\alpha\beta}(x, u, \nabla u)\xi_\alpha \xi_\beta \quad \text{for all } \xi \in \boldsymbol{R}^n$$

and

$$(3) \qquad |f(x, u, \nabla u)| \leqslant a|\nabla u|^2 + b.$$

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In [3] we have conjectured that the Hölder continuity of u follows from a condition

$$|u|_{L^\infty} < \theta \cdot \lambda / a \tag{4}$$

and that the optimal value of θ is one. We were able to verify this conjecture in the case of two independent variables, while, in general, we had to assume that $\theta < \frac{1}{2}$, and this condition could be improved to $\theta < 2(\sqrt{2}-1) \approx \approx 0.828\,427$ by essentially the same technique [3]. Finally, in three remarkable papers [14]-[16], Wiegner has extended our methods and combined with the technique of Ladyženskaya-Ural'ceva [6] to derive regularity of u for the optimal value $\theta = 1$ as well as an a priori bound for $|u|_{C^\alpha}$, for some $\alpha \in (0, 1)$ depending only on the parameters of the system (1).

However, as we shall show in this paper, his proof can be considerably simplified. Furthermore, the condition (4) yields only a very crude picture of the existing connections between the structure of the right hand side of (1) and the regularity behaviour of its weak solutions. The conjecture (1.8) in [3] may serve as an illustration. To start the investigation in this direction, we have proved various continuity results for the weak solutions of (1) depending on the fine structure of f. In particular, we mention the partial regularity theorem stated as Theorem 3.1, and the somewhat surprising Theorem 3.2, which follow from one-sided conditions on f, and Theorem 4.1 which is an improvement of Wiegner's theorem. Finally, the last section contains a natural extension of the well known result by Ladyženskaya-Ural'ceva for one equation ($N = 1$) to systems which can be derived from none of the previously known theorems. Here, we may allow operators in diagonal form with different principal parts in the diagonal entries.

We end this introduction by noting that by [3], or even by [6], there follows Hölder continuity of solutions, once we have proved continuity. Likewise, given an a priori bound for the modulus of continuity there follows an a priori bound for the Hölder norm. Thus we shall be content to derive continuity for solutions. In case that the coefficients $A^{\alpha\beta}$ of (1) do not depend on the derivatives ∇u, the technique of [6] yields also higher regularity of the solutions.

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1. – Notations.

Ω will always be a bounded open set in $\boldsymbol{R}^n$, and the open ball in $\boldsymbol{R}^n$ with center y and radius R is denoted by $B_R(y)$ while $T_{2R}(y)$ will stand for $B_{2R}(y) - B_R(y)$. Points of $\boldsymbol{R}^n$ are denoted by $x = (x^1, x^2, \ldots, x^n) = (x^\alpha)_{1 \leqslant \alpha \leqslant n}$, and vector-valued functions by $u = u(x) = (u^1, u^2, \ldots, u^N) = (u^i)_{1 \leqslant i \leqslant N}$. Repeated Latin indices $i, k, \ldots$ are to be summed from 1 to N and Greek indices $\alpha, \beta, \ldots$ from 1 to n, and $|\cdot|$ denotes the Euclidean norm. Since we shall not exploit the possibly smooth dependence on u and ∇u of the coefficients $A^{\alpha\beta}(x, u, \nabla u)$ of the introduction, we write

$$a^{\alpha\beta}(x) = A^{\alpha\beta}\big(x, u(x), \nabla u(x)\big)$$

and consider only operators

$$L = - D_\beta\{a^{\alpha\beta}(x) D_\alpha\}$$

which satisfy

$$(1.1) \qquad a^{\alpha\beta} \in L^\infty(\Omega) \quad \text{with} \quad \max_{1 \leqslant \alpha, \beta \leqslant n} |a^{\alpha\beta}| \leqslant \mu$$

$$(1.2) \qquad a^{\alpha\beta}(x)\xi_\alpha\xi_\beta \geqslant \lambda|\xi|^2 \quad \text{for all } \xi \in \boldsymbol{R}^n, \text{ all } x \in \Omega, \text{ and some } \lambda > 0\,.$$

An $\boldsymbol{R}^N$-valued, measurable function $f(x, u, p)$ on $\bar{\Omega} \times \boldsymbol{R}^N \times \boldsymbol{R}^{nN}$ is said to be of class $\mathfrak{Q}(a, b)$ if $f\big(x, u(x), \nabla u(x)\big)$ is measurable for all $u \in H^1_{2,\mathrm{loc}}(\Omega, \boldsymbol{R}^N)$, and if, for any number $M > 0$, there is a number $a(M) \geqslant 0$ and a function $b(\cdot, M) \geqslant 0$ of class $L^q(\Omega)$, $q > n/2$, such that

$$(1.3) \qquad |f(x, u, p)| \leqslant a(M)|p|^2 + b(x, M)$$

for all $(x, u, p) \in \bar{\Omega} \times \boldsymbol{R}^N \times \boldsymbol{R}^{nN}$ with $|u| \leqslant M$.

A function f is said to be of class $\mathfrak{F}(\lambda^*, b^*)$ if it is of type $\mathfrak{Q}$ and, in addition, for any number $M > 0$, there is a number $\lambda^*(M) \geqslant 0$ and a function $b^*(\cdot, M) \in L_q(\Omega)$, $q > n/2$, such that

$$(1.4) \qquad u \cdot f(x, u, p) \leqslant \lambda^*(M)|p|^2 + b^*(x, M)$$

for all $(x, u, p) \in \bar{\Omega} \times \boldsymbol{R}^N \times \boldsymbol{R}^{nN}$ with $|u| \leqslant M$.

Note that $f \in \mathfrak{Q}(a, b)$ is also in $f \in \mathfrak{F}(\lambda^*, b^*)$ with $\lambda^*(M) = Ma(M)$, $b^*(x, M) = Mb(x, M)$. The usefulness of the notion $f \in \mathfrak{F}(\lambda^*, b^*)$ is based on the possibility that there is a number $\lambda^*(M)$ less than $Ma(M)$. For instance, if $f = - u \cdot |p|^2$, we can take $\lambda^* = 0$ while $Ma(M) = M^2$.

For $f \in \mathfrak{Q}$, we call a function $u = u(x) = (u^1(x), \ldots, u^N(x))$ a *weak solution* of the equation

$$Lu = f(x, u, \nabla u) \quad \text{in } \Omega,$$

abbreviated: $u \in W(L, f)$, if $u \in H_2^1 \cap L^\infty(\Omega, \boldsymbol{R}^N)$, and if

$$\int_\Omega a^{\alpha\beta} D_\alpha u \cdot D_\beta \varphi \, dx = \int_\Omega f(x, u, \nabla u) \cdot \varphi \, dx \tag{1.5}$$

for all $\varphi \in \mathring{H}_2^1 \cap L^\infty(\Omega, \boldsymbol{R}^N)$.

The Green function $G(\cdot, y)$ for $L^* = -D_\alpha\{a^{\alpha\beta} D_\beta\}$ and its mollification

$$G^\sigma(x, y) = \fint_{B_\sigma(y)} G(x, z) \, dz \tag{1.6}$$

defined by

$$\int_\Omega a^{\alpha\beta} D_\alpha \varphi D_\beta G^\sigma(x, y) dx = \fint_{B_\sigma(y)} \varphi \, dx \tag{1.7}$$

for all $\varphi \in \mathring{H}_2^1 \cap L^\infty(\Omega, \boldsymbol{R})$, will be used extensively. We have enumerated the needed properties of these functions in the appendix, section 6.

As in [3], the «friend» on $B_{2R}(y)$ is a function $\eta \in C_c^\infty(B_{2R}(y), \boldsymbol{R})$ with $0 \leqslant \eta \leqslant 1$, $|\nabla \eta| < k/R$, k independent of R, and

$$\eta(x) = \begin{cases} 1 & \text{for } |x - y| \leqslant 5R/4, \\ 0 & \text{for } |x - y| \geqslant 7R/4. \end{cases}$$

For the sake of brevity, we shall throughout assume that $N \geqslant 3$. The case $N = 2$ can be treated in an analogous way. In [3], the reader will find various other information about this case.

2. – Basic properties of weak solutions.

It is well known that the class of systems under consideration is invariant with respect to diffeomorphic changes of the independent variables. Our first theorem states that a similar statement holds for diffeomorphic changes of the dependent variables.

2.1. THEOREM. *Let $\hat{F}$ be a C^2-diffeomorphism of $\bar{B}_M^N(0)$ into $\boldsymbol{R}^N$. Then, for every $f \in \mathfrak{Q}$, there is an $\hat{f} \in \mathfrak{Q}$ such that the following holds: If $u \in W(L, f)$ and $|u|_{L^\infty(\Omega)} \leqslant M$, then $\hat{u} = \hat{F}(u) \in W(L, \hat{f})$.*

PROOF. Clearly, we have $\hat{u} \in H_2^1 \cap L^\infty(\Omega, R^N)$ and $u = F(\hat{u})$, if F denotes the inverse of $\hat{F}$.

Set

$$F_l^i = F_{\hat{u}^l}^i, \qquad F_{lm}^i = F_{\hat{u}^l \hat{u}^m}^i, \qquad G_{kl} = F_k^i F_l^i,$$

and let (G^{kl}) be the inverse of the positive definite matrix function (G_{kl}). Finally, define $\hat{f} = (\hat{f}^1, \hat{f}^2, \ldots, \hat{f}^N)$ for

$$(x, \hat{u}, q) \in \bar{\Omega} \times \boldsymbol{R}^N \times \boldsymbol{R}^{nN} \quad \text{by} \quad f(x, \hat{u}, q) = 0 \qquad \text{if } \hat{u} \notin \hat{F}(\bar{B}_M^N(0)),$$

and

$$\hat{f}^s(x, \hat{u}, q) = \{f^i(x, F(\hat{u}), F_{\hat{u}}(\hat{u})q) F_l^i(\hat{u}) - a^{\alpha\beta}(x) F_k^i(\hat{u}) F_{lm}^i(\hat{u}) q_\alpha^k q_\beta^m\} G^{ls}(\hat{u}) - a^{\alpha\beta}(x) G_{kl}(\hat{u}) G_{\hat{u}^t}^{ls}(\hat{u}) q_\alpha^k q_\beta^t$$

for $\hat{u} \in \hat{F}(\bar{B}_M^N(0))$.

If we choose $\psi \in \overset{\circ}{H}{}_2^1 \cap L^\infty(\Omega, \boldsymbol{R}^N)$ and insert the test vector φ with

$$\varphi^i(x) = F_l^i(\hat{u}(x)) G^{ls}(\hat{u}(x)) \psi^s(x), \qquad 1 \leqslant i \leqslant N,$$

into (1.5), we find after a straight-forward computation the desired relation

$$\int_\Omega a^{\alpha\beta} D_\alpha \hat{u} \cdot D_\beta \psi \, dx = \int_\Omega \hat{f}(x, \hat{u}, \nabla \hat{u}) \cdot \psi \, dx.$$

The proof of the following result is essentially contained in [3], pp. 79-80.

2.2 THEOREM. *Let f be of class $\mathfrak{Q}(a, b)$, and denote by M and ω nonnegative numbers with $\omega < \lambda/a(M)$. Then, there is a number $\alpha \in (0, 1)$ depending only on λ, μ, M, ω, a, and b such that the following holds:*

If $u \in W(L, f)$, $|u|_{L^\infty(\Omega)} \leqslant M$, and $\operatorname{osc}_{\Omega'} u \leqslant \omega$ for some open subset Ω' of Ω, then $u \in C^\alpha(\Omega', \boldsymbol{R}^N)$. Moreover, for any $\Omega'' \subset\subset \Omega'$, there exists a number $K \geqslant 0$ such that

$$|u|_{C^\alpha(\bar{\Omega}'')} \leqslant K(\Omega'', \alpha, \lambda, \mu, M, \omega, a, b)$$

where K depends only on the quantities in parentheses but not on u.

2.3. THEOREM. *Let u be in $W(L, f)$ with $f \in \mathfrak{Q}(a, b)$ and*

$$\int_\Omega |\nabla u(x)|^2 |x - y|^{2-n} \, dx < \infty \tag{2.1}$$

for every $y \in \Omega$. *Then, the limit*

$$\bar{u}(x) = \lim_{R \to 0} \fint_{B_R(x)} u(y)\, dy \tag{2.2}$$

exists for all $x \in \Omega$. *Hence, we have a representation of the Sobolev space element* u *by a bounded measurable function* $\bar{u}$ *of class* ACL (1) *which satisfies*

$$\bar{u}(x) = \lim_{R \to 0} \fint_{B_R(x)} \bar{u}(y)\, dy \quad \text{for all } x \in \Omega\,. \tag{2.2'}$$

For this reason, we shall not distinguish anymore between $u \in H_2^1 \cap L^\infty$ *and its distinguished representation* $\bar{u}$.

Furthermore, we have for all $x \in \Omega$

$$\lim_{R \to 0} \fint_{B_R(x)} |u(x) - u(y)|^2\, dy = 0 \tag{2.3}$$

and

$$u(x) = v(x) + \int_\Omega G(y, x) f(y, u(y), \nabla u(y))\, dy \tag{2.4}$$

where v *is in* $W(L, 0)$ *and satisfies* $u - v \in \mathring{H}_2^1(\Omega, \boldsymbol{R}^N)$.

PROOF. Since $G^\sigma(\cdot, y) \in \mathring{H}_2^1 \cap L^\infty(\Omega, \boldsymbol{R})$ we can take φ with $\varphi^i = G^\sigma(\cdot, y)$ for all i in (1.5), subtract the equation

$$\int_\Omega a^{\alpha\beta} D_\alpha v D_\beta G^\sigma(\cdot, y)\, dx = 0$$

and find

$$\fint_{B_\sigma(y)} u\, dx = \fint_{B_\sigma(y)} v\, dx + \int_\Omega G^\sigma(x, y) f(x, u, \nabla u)\, dx\,, \tag{2.5}$$

on account of (1.7).

It is well known that v is Hölder continuous in Ω (cf., for instance, [9]). This fact, together with (2.1), (6.11), and Lebesgue's theorem on dominated

(1) Cf. Morrey [8], Theorem 3.1.8, p. 66.

convergence, implies that the right hand side of (2.5) has the limit

$$v(y) + \int_{\Omega} G(x, y) f(x, u, \nabla u)\, dx$$

as σ tends to zero. This proves (2.2) and (2.4).

We infer from Lebesgue's differentiation theorem that, for any other representation of u by a bounded measurable function $\tilde{u}$, we have

$$\bar{u}(x) = \tilde{u}(x) \quad \text{a.e. on } \Omega$$

whence we obtain (2.2′).

Finally, to prove (2.3) we put

$$\bar{u}_R = \fint_{B_R(y)} u(x)\, dx$$

and apply Poincaré's inequality:

$$\begin{aligned}\fint_{B_R(y)} |u(x) - u(y)|^2\, dx &\leqslant 2 \fint_{B_R(y)} |u(x) - \bar{u}_R|^2\, dx + 2|u(y) - \bar{u}_R|^2 \\ &\leqslant K R^{2-n} \int_{B_R(y)} |\nabla u|^2\, dx + 2|u(y) - \bar{u}_R|^2 .\end{aligned}$$

Assumption (2.1) implies that

$$\lim_{R \to 0} R^{2-n} \fint_{B_R(y)} |\nabla u|^2\, dx = 0 ,$$

and (2.2′) yields that

$$\lim_{R \to 0} |u(y) - \bar{u}_R| = 0$$

whence we get (2.3). The theorem is proved.

2.4. THEOREM. *Let u be in $W(L, f)$ with $f \in \mathfrak{Q}(a, b)$, and suppose that, for any $\Omega' \subset\subset \Omega$,*

$$\Phi(y, R) = \int_{B_R(y)} |\nabla u(x)|^2 |x - y|^{2-n}\, dx \tag{2.6}$$

tends to zero as $R \to 0$, uniformly in y for $y \in \Omega' \subset\subset \Omega$. That is, there exists a function $\hat{\varepsilon} = \hat{\varepsilon}(R, \Omega') > 0$ such that $\lim_{R \to 0} \hat{\varepsilon}(R, \Omega') = 0$ and

$$\Phi(y, R) \leqslant \hat{\varepsilon}(R, \Omega') \quad \textit{for all } y \in \Omega' . \tag{2.7}$$

Then u is in $C^\alpha(\Omega, \boldsymbol{R}^N)$ for some $\alpha \in (0,1)$, and, for any $\Omega' \subset\subset \Omega$, one can estimate $|u|_{C^\alpha(\Omega')}$ in terms of M and $\hat{\varepsilon}$, where $\sup_\Omega |u| \leqslant M$.

PROOF. Let $\Omega' \subset\subset \Omega'' \subset\subset \Omega$, and choose numbers ϱ and R such that $0 < \varrho < R/2$, $B_{3R/2}(y) \subset \Omega$, and $B_\varrho(y) \subset \Omega''$ for every $y \in \Omega'$.

Fix some $y \in \Omega'$, and let $z_1, z_2 \in B_\varrho(y)$, $x \in \Omega - B_R(y)$. Then

$$|y - z_i| < \varrho < R/2 \leqslant \tfrac{1}{2}|x - y|, \quad i = 1, 2,$$

whence, for some $\beta \in (0,1)$,

$$\begin{aligned} |G(x, z_1) - G(x, z_2)| &\leqslant |G(x, z_1) - G(x, y)| + |G(x, z_2) - G(x, y)| \\ &\leqslant K_3 |x - y|^{2-n-\beta} \sum_{i=1}^{2} |y - z_i|^\beta \\ &\leqslant 2K_3 \varrho^\beta R^{2-n-\beta} \end{aligned}$$

on account of (6.3). Therefore, by (2.4),

$$\begin{aligned} \operatorname*{osc}_{B_\varrho(y)} u \leqslant \operatorname*{osc}_{B_\varrho(y)} v + 2 \sup_{z \in B_\varrho(y)} \int_{B_R(y)} G(x, z) |f(x, u, \nabla u)| dx + \\ + 2K_3 \varrho^\beta R^{2-n-\beta} \int_{\Omega - B_R(y)} \{a(M) |\nabla u|^2 + b(x, M)\} dx \end{aligned}$$

which implies that

$$\operatorname*{osc}_{B_\varrho(y)} u \leqslant \operatorname*{osc}_{B_\varrho(y)} v + 2 \sup_{z \in \Omega''} \int_{B_{R+\varrho}(z)} G(x, z) |f(x, u, \nabla u)| dx + K \varrho^\beta R^{2-n-\beta} (1 + |\nabla u|^2_{L_2(\Omega)})$$

where K depends on β, λ, μ, n, N, M, $a(M)$, and $b(\cdot, M)$. By (6.1) and assumption (2.7), we arrive at

$$\operatorname*{osc}_{B_\varrho(y)} u \leqslant \operatorname*{osc}_{B_\varrho(y)} v + K\{\hat{\varepsilon}(3R/2, \Omega'') + R^\delta + \varrho^\beta R^{2-n-\beta}[1 + |\nabla u|^2_{L_2(\Omega)}]\}. \tag{2.9}$$

Now, the theorem follows from this estimate if we take also **2.2** and the continuity of v into consideration.

2.5. THEOREM. *Suppose that the assumptions of* **2.3** *are satisfied, and that* $\sup_\Omega |u| \leqslant M$. *Then u is Hölder continuous in Ω provided that*

$$\varlimsup_{R \to 0} \sup_{y \in \Omega'} \int_{B_R(y)} G(x, y) |f(x, u, \nabla u)| dx < \lambda / 2a(M) \quad \text{for every } \Omega' \subset\subset \Omega. \tag{2.10}$$

PROOF. This assertion follows immediately from (2.8) and Theorem **2.2**.

2.6. THEOREM. *Suppose that* $f \in \mathfrak{Q}$, $u \in W(L, f)$, *and that*

$$\int_\Omega |\nabla u(x)|^2 |x-y|^{2-n}\, dx < \infty \quad \textit{for all } y \in \Omega. \tag{2.1}$$

Then u *is continuous on a dense open subset of* Ω.

PROOF. It is an easy exercise in integration theory to show that if the integral

$$\int_\Omega |\nabla u(x)|^2 |x-y|^{2-n}\, dx$$

is finite for all y in some neighborhood of a point $x_0 \in \Omega$ then, to every $\varepsilon > 0$, there exists a ball $B_R(y)$ in this neighborhood such that

$$\int_{B_R(y)} |\nabla u(x)|^2 |x-z|^{2-n}\, dx < \varepsilon \quad \text{for all } z \in B_R(y)\,.$$

Then the assertion follows by an appropriate application of Theorem **2.5**.

3. – One-sided conditions on f.

In this section we assume that $f \in \mathfrak{Q}(a, b) \cap P(\lambda^*, b^*)$ and that $u \in W(L, f)$, $|u|_{L^\infty(\Omega)} \leqslant M$, and

$$\lambda^*(M) < \lambda\,.$$

The inequality of the next theorem is implicitely contained in [3] but we repeat the simple proof.

3.1. THEOREM. *Let* $u \in W(L, f)$, $|u|_{L^\infty(\Omega)} \leqslant M$, $f \in \mathfrak{Q}(a, b) \cap \mathfrak{I}(\lambda^*, b^*)$, *and* $\lambda^*(M) < \lambda$. *Then*

$$\int_\Omega G(x, y) |\nabla u(x)|^2\, dx \leqslant \frac{\operatorname{osc}_\Omega |u|^2}{2\{\lambda - \lambda^*(M)\}} + K[\operatorname{meas} \Omega]^{\delta/n} \tag{3.1}$$

for all $y \in \Omega$, *where* $\delta = 2 - n/q > 0$, *and* K *depends on* λ, μ, M, b^* ($K = 0$, *if* $b^* = 0$). *Hence, the assumptions of Theorem* **2.3** *are fulfilled; thus* (2.2), (2.3), *and* (2.4) *hold, and* u *is continuous on an open, dense subset of* Ω.

Moreover, $|u|^2$ *satisfies*

$$L|u|^2 \leqslant 2b^*(\cdot, M) \quad \text{in } \Omega$$

in the weak sense, is upper semicontinuous, and is represented by

$$|u(x)|^2 = w(x) - 2\int_\Omega G(y, x)\{a^{\alpha\beta} D_\alpha u \cdot D_\beta u - u \cdot f(y, u, \nabla u)\}\, dy \tag{3.2}$$

where $Lw = 0$ *weakly in* Ω, *and* $w - |u|^2 \in \overset{\circ}{H}{}^1_2(\Omega, \mathbf{R})$.

PROOF. In (1.5), we insert the test function $\varphi = uG^\sigma(\cdot, y)$, $y \in \Omega$, where $G^\sigma(\cdot, y)$ is the mollified Green function of L^*. From the resulting equation we subtract

$$\int_\Omega a^{\alpha\beta} D_\alpha w D_\beta G^\sigma(\cdot, y)\, dx = 0$$

whence we get

$$\fint_{B_\sigma(y)} |u|^2\, dx = \fint_{B_\sigma(y)} w\, dx - 2\int_\Omega G^\sigma(x, y)\{a^{\alpha\beta} D_\alpha u \cdot D_\beta u - u \cdot f(x, u, \nabla u)\}\, dx\,. \tag{3.3}$$

By assumption, the quantity $\{\ldots\}$ is greater than or equal to

$$[\lambda - \lambda^*(M)]|\nabla u|^2 - b^*(\cdot, M)\,.$$

Therefore,

$$2[\lambda - \lambda^*(M)]\int_\Omega |\nabla u|^2 G^\sigma(\cdot, y)\, dx \leqslant \fint_{B_\sigma(y)} \{w - |u|^2\} dx + 2\int_\Omega b^* G^\sigma\, dx\,.$$

Letting σ tend to zero, Fatou's lemma yields the inequality

$$\int_\Omega |\nabla u(x)|^2 G(x, y)\, dx \leqslant \frac{w(y) - |u(y)|^2}{2\{\lambda - \lambda^*(M)\}} + K[\operatorname{mes} \Omega]^{2/n - 1/q} \tag{3.4}$$

for almost every $y \in \Omega$.

Now notice that by the maximum principle

$$\sup_\Omega w \leqslant \sup_{\partial\Omega} w \leqslant \sup_{\partial\Omega} |u|^2\,.$$

Thus

$$w(y) - |u(y)|^2 \leqslant w(y) - \sup_{\Omega} w + \sup_{\partial\Omega} |u|^2 - |u(y)|^2$$
$$\leqslant \operatorname*{osc}_{\Omega} |u|^2 .$$

(it is well known how these lines can be given a precise meaning), and (3.1) is proved.

Since

$$G(x, y) \geqslant K_1 |x - y|^{2-n}$$

for x close to y, condition (2.1) is satisfied, and, in virtue of Theorem **2.3**, we need no longer distinguish between u and its special representation $\bar{u}$ which is defined by (2.2). Then we get from (3.3) formula (3.2) by letting $\sigma \to 0$ and invoking Lebesgue's theorem on dominated convergence. This argument proves also that (3.1) holds for all $y \in \Omega$.

From the representation (3.2) it is immediate that $|u|^2$ is upper semicontinuous, since

$$\{a^{\alpha\beta} D_\alpha u \cdot D_\beta u - u \cdot f(x, u, \nabla u)\} \geqslant - b^*(\cdot, M) .$$

Moreover, we have already proved in [3], pp. 74-75, that $L|u|^2 \leqslant 2b^*(\cdot, M)$ holds in Ω in the weak sense.

Finally, Theorem **2.6** implies that u is continuous on an open, dense subset of Ω. For the amusement of the reader, we should like to point out another proof:

The bounded function $|u(x)|^2$ is the pointwise limit of a sequence of continuous functions. This follows either from the upper semicontinuity of $|u|^2$, or also from (2.3) which implies

$$|u(x)|^2 = \lim_{\sigma \to 0} \fint_{B_\sigma(x)} |u(y)|^2 dy ,$$

and

$$\psi_\sigma(x) = \fint_{B_\sigma(x)} |u(y)|^2 dy$$

is clearly a continuous function of x.

A (maybe not so) well-known theorem due to Baire [1] (cf. also [2], p. 178) yields that $|u|^2$ is « pointwise discontinuous » (in German: « punktiert unstetig », cf. [2], p. 143), i.e., continuous on a dense subset of Ω. Taking (3.1) and Theorem **2.4** into account we infer that u is continuous on a dense open subset of Ω. Thus the theorem is proved.

3.2. THEOREM. *Suppose that* $u \in W(L, f)$, $|u|_{L^\infty(\Omega)} \leqslant M$, $f \in \mathfrak{Q}(a, b) \cap \mathfrak{F}(\lambda^*, b^*)$, $\lambda^*(M) < \lambda$, *and*

$$\operatorname{osc}_\Omega |u|^2 < \frac{\lambda\{\lambda - \lambda^*(M)\}}{a^2(M)}. \tag{3.5}$$

Then u *is Hölder continuous in* Ω.

PROOF. This result is an immediate consequence of the theorems **2.5** and **3.1**.

3.3. REMARKS. 1) Let the assumptions of **3.1** be satisfied, and let Ω' be an open subset of Ω. Set

$$\varepsilon_0 = \lambda\{\lambda - \lambda^*(M)\}/a^2(M).$$

Since $|u|^2$ is upper semicontinuous, the set

$$\Omega'_{\varepsilon_0} = \left\{x \in \Omega' : \inf_{\Omega'} |u|^2 \leqslant |u(x)|^2 < \inf_{\Omega'} |u|^2 + \varepsilon_0\right\}$$

is open and, on account of **3.2**, u is Hölder continuous in Ω'_{ε_0}.

This observation yields immediately another proof of the fact that the set of continuity points of u is open and dense in Ω, which does not use Baire's theorem.

2) If the assumptions of **3.1** are satisfied, we have in particular

$$0 \leqslant \inf_\Omega |u| \leqslant \sup_\Omega |u| \leqslant M$$

since $|u|_{L^\infty(\Omega)} \leqslant M$.

If we assume that the stronger estimate

$$\{M^2 - \varepsilon_0\}^{\frac{1}{2}} < \inf_\Omega |u| \leqslant \sup_\Omega |u| \leqslant M$$

is satisfied, then u is continuous in Ω.

Note that this conclusion depends in an essential way on the assumption $\lambda^*(M) < \lambda$. In general, the oscillation of $|u|^2$ may be small, even zero, for $u \in W(L, f)$, while u is discontinuous, as the example $u(x) = x/|x|$ shows where $u \in W(L, f)$, $L = -\Delta$, $f \in \mathfrak{Q}(a, b) \cap \mathfrak{F}(\lambda^*, b^*)$, $M = \lambda = 1$, $n = N \geqslant 3$, $a(1) = \lambda^*(1)$, $b = b^* = 0$, see [3], section 1. For $n = N = 2$, the example

$$u(x) = \left(\sin \log \log \frac{1}{|x|},\ \cos \log \log \frac{1}{|x|}\right)$$

illustrates the same phenomenon.

4. – Two-sided conditions.

M. Wiegner has recently extended our results of [3], showing that any $u \in W(L, f)$ with $f \in \mathfrak{Q}(a, b)$, $|u|_{L^\infty(\Omega)} \leqslant M$, and $Ma(M) < \lambda$, is continuous in Ω. In fact, he has found a priori bounds for the Hölder norm of u.

Here we shall show that Wiegner's proof can be simplified as well as somewhat extended.

4.1. THEOREM. *Let u be in $W(L, f)$ with $f \in \mathfrak{Q}(a, b) \cap \mathfrak{F}(\lambda^*, b^*)$ and $|u|_{L^\infty(\Omega)} \leqslant M$, and suppose that*

$$\lambda^*(M) + Ma(M) < 2\lambda .$$

Then u is Hölder continuous with some exponent $\alpha \in (0, 1)$ which depends only on the parameters of the system $Lu = f$, and, for any $\Omega' \subset\subset \Omega$, we have an estimate

$$|u|_{C^\alpha(\Omega')} \leqslant K'$$

where K' depends only on n, N, M, $\lambda^*(M)$, $a(M)$, λ, $|b(\cdot, M)|_{L_q(\Omega)}$, q, and $\operatorname{dist}(\Omega', \partial\Omega)$.

Moreover, if $\partial\Omega$ and $g = u|_{\partial\Omega}$ are of Lipschitz class then $u \in C^\alpha(\bar{\Omega}, \boldsymbol{R}^N)$, and

$$|u|_{C^\alpha(\bar{\Omega})} \leqslant K$$

where K depends on n, N, $\lambda^*(M)$, $a(M)$, λ, $|b(\cdot, M)|_{L^q(\Omega)}$, q, and the Lipschitz-bound for g and $\partial\Omega$.

For the proof of this result we need three lemmata, the first of which is due to Ladyženskaya and Ural'ceva [6]. For a short proof see [4].

4.2. LEMMA. *Let $v \in H_2^1(B_R, \boldsymbol{R})$ be a weak solution of*

$$Lv \leqslant b^* \quad \text{in } \Omega$$

with $b^* \in L^q$, $q > n/2$, and assume that, for some $\sigma \in (0, 1)$ and some $\varrho < R/2$,

$$\operatorname{mes}\Big\{x \in B_\varrho : v(x) - \inf_{B_R} v \leqslant (1 - \sigma) \operatorname*{osc}_{B_R} v\Big\} \geqslant \sigma \operatorname{mes} B_\varrho . \tag{4.1}$$

Then

$$\operatorname*{osc}_{B_\varrho} v \leqslant \gamma(\sigma) \operatorname*{osc}_{B_R} v + K^* R^\tau$$

where $\gamma \in (0, 1)$, $\tau > 0$, *and* $K^* \geqslant 0$ *depend only on* $\sigma, n, \lambda, \mu, q$, *and* $|b^*|_{L^q(\Omega)}$, *and* B_ϱ *and* B_R *denote concentric balls with radius* ϱ *and* R, *respectively.*

The next lemma is essentially due to Wiegner [16].

4.3. LEMMA. *To every* $\varepsilon \in (0, M^2)$ *there is a number* $\sigma = \sigma(\varepsilon) \in (0, 1)$ *with the following property:*

If v *is a weak solution of*

$$Lv \leqslant b^* \quad \text{in } B_R$$

with $b^* \in L_q$, $q > n/2$, *and if* $0 \leqslant v \leqslant M^2$, *then*

$$\sup_{B_R} v - \fint_{B_\varrho} v\,dx \geqslant \varepsilon \quad \text{for some } \varrho \in (0, R/2)$$

implies that

$$\operatorname*{osc}_{B_\varrho} v \leqslant \gamma(\sigma) \operatorname*{osc}_{B_R} v + K^* R^\tau$$

where γ, K^*, *and* τ *are the constants of Lemma* **4.2**.

PROOF. In view of **4.2** we need only show that (4.1) holds with $\sigma = \varepsilon/2M^2$. If (4.1) were not true for this choice of σ we would have

$$v(x) > (1-\sigma) \sup_{B_R} v + \sigma \inf_{B_R} v \geqslant (1-\sigma) \sup_{B_R} v$$

for all x in a subset S_ϱ of B_ϱ of measure greater than $(1-\sigma)$ mes B_ϱ. Hence we have $\sup_{B_R} v > 0$. Since $v \geqslant 0$, we get therefore

$$\begin{aligned}\fint_{B_\varrho} v\,dx &\geqslant \frac{1}{\operatorname{mes} B_\varrho} \int_{B_\varrho} v\,dx \geqslant \frac{\operatorname{mes} S_\varrho}{\operatorname{mes} B_\varrho}(1-\sigma) \sup_{B_R} v \\ &\geqslant (1-\sigma)^2 \sup_{B_R} v > (1-2\sigma) \sup_{B_R} v \\ &\geqslant \sup_{B_R} v - 2\sigma M^2 = \sup_{B_R} v - \varepsilon\,.\end{aligned}$$

This contradicts the assumption

$$\sup_{B_R} v - \varepsilon \geqslant \fint_{B_\varrho} v\,dx\,.$$

4.4. LEMMA. *For every* $\varepsilon > 0$, *there exists an integer* $m > 0$ *and a number* $R_0' > 0$ *with the following property:*

For every $k \in (0, 1)$, *every* $d > 0$, *every* R_0 *with* $0 < R_0 \leqslant min\{R_0', d/8\}$, *every* $y \in \Omega$ *with dist* $(y, \partial\Omega) \geqslant d$, *and every* $u \in W(L, f)$ *with* $\sup_{B_{d/4}(y)}|u| \leqslant M$, $f \in \mathfrak{Q}(a, b) \cap \mathfrak{F}(\lambda^*, b^*)$, $\lambda^*(M) < \lambda$, *there is a number* R *with*

$$(k/8)^m R_0 \leqslant 4R \leqslant R_0$$

such that

$$\int_{B_{2R}(y) - B_{kR}(y)} G(x, y)|\nabla u(x)|^2 \, dx < \varepsilon .$$

PROOF. Fix some $y \in \Omega$ with dist $(y, \partial\Omega) \geqslant d$, and let $R_0 \in (0, d/8]$. For $0 < R \leqslant R_0$ and $|y - z| \leqslant R/4$, $|x - y| \leqslant R$, we get, by (6.1) and (6.2), that

$$K_1|x - z|^{2-n} < G_{2R}(x, z) \leqslant K_2|x - z|^{2-n}$$

where G_{2R} denotes Green's function for L^* on $B_{2R}(y)$ and K_1, K_2 are positive numbers independent of R and y. Set $\tilde{K} = 2^{n-2}K_2/K_1$. Then,

$$G_{2R}(x, y) \leqslant \tilde{K} G_{2R}(x, z)$$

for $z \in B_{kR/4}$ and $x \in B_R - B_{kR/2}$, where k is some fixed number in $(0, 1)$, and B_ϱ stands always for the ball $B_\varrho(y)$, for every $\varrho > 0$.

Thus

$$\int_{B_R - B_{kR/2}} G_{2R}(x, y)|\nabla u(x)|^2 \, dx \leqslant \tilde{K} \int_{B_{2R}} G_{2R}(x, z)|\nabla u(x)|^2 \, dx$$

for all $z \in B_{kR/4}$.

The maximum principle yields that

$$G_{2R}(x, z) \leqslant G(x, z) \leqslant K_2|x - z|^{2-n} \quad \text{for } x, z \in B_{2R}(y) .$$

Let w_{2R} be the weak solution of

$$Lw_{2R} = 0 \quad \text{in } B_{2R}, \qquad w_{2R} - |u|^2 \in \mathring{H}_2^1(B_{2R}, \mathbf{R}) .$$

Then, analogously to (3.4), one obtains the estimates

$$\int_{B_{2R}} |\nabla u(x)|^2 G_{2R}(x, z) \, dx \leqslant \frac{w_{2R}(z) - |u(z)|^2}{2\{\lambda - \lambda^*(M)\}} + \frac{K_2}{2\{\lambda - \lambda^*(M)\}} \int_{B_{2R}} b^*(x, M)|x - z|^{2-n} \, dx$$

and

$$w_{2R}(z) - |u(z)|^2 \leqslant \sup_{B_{2R}} |u|^2 - |u(z)|^2 .$$

Set

$$\Psi_R = \int_{B_R - B_{kR/2}} |\nabla u(x)|^2 G_{2R}(x, y)\, dx .$$

Then we have

$$\Psi_R \leqslant K' \Big\{ \sup_{B_{2R}} |u|^2 - |u(z)|^2 \Big\} + K'' R^\delta$$

for all $z \in B_{kR/4}$ where the numbers K' and K'' do not depend on R and y. Averaging both sides of this inequality over $z \in B_{kR/4}$, we arrive at

$$\Psi_R \leqslant K' \Big\{ \sup_{B_{2R}} |u|^2 - \fint_{B_{kR/2}} |u(z)|^2 dz \Big\} + K'' R^\delta . \tag{4.2}$$

We note that, by Theorem 3.1,

$$L|u|^2 \leqslant 2b^*(\cdot, M) \quad \text{in } B_{2R_0} .$$

Thus we can apply **4.2** and **4.3** to $v = |u|^2$.

Let K^*, τ, σ, and $\gamma(\sigma)$ be the quantities appearing in the previous lemmas. Fix an arbitrary number $\varepsilon > 0$, and let $\sigma = \sigma(\varepsilon/2K')$.

We choose now $R_0 > 0$, in addition, less than a number $R_0' > 0$ which satisfies

$$K'' R_0'^\delta < \varepsilon/2 \quad \text{and} \quad (1-\gamma)^{-1} K' K^* R_0'^\tau < \varepsilon/4 .$$

Then, we determine m as the smallest integer > 0 such that

$$M^2 K' \gamma^m < \varepsilon/4 ,$$

and set

$$R_\nu = \left(\frac{k}{8}\right)^\nu R_0 \quad \text{for } 0 \leqslant \nu \leqslant m ,$$

that is, $R_{\nu+1} = (k/8) R_\nu$.

We claim that

$$\Psi_{R_\nu/2} < \varepsilon \quad \text{for some } \nu \in \{0, 1, 2, \ldots, m\} . \tag{$*$}$$

To this end, we shall prove that

$$\varepsilon \leqslant \Psi_{R_\nu/2} \quad \text{for all } \nu \text{ with } 0 \leqslant \nu \leqslant m-1 \tag{$**$}$$

implies that

$$\Psi_{R_m/2} < \varepsilon .$$

In fact, the estimate (4.2) and (**) yield that

$$\frac{\varepsilon}{2K'} \leqslant \sup_{B_{R_\nu}} |u|^2 - \fint_{B_{kR_\nu/8}} |u|^2 dz\,, \quad 0 \leqslant \nu \leqslant m-1\,,$$

whence, by Lemma **4.3**,

$$\operatorname*{osc}_{B_{R_{\nu+1}}} |u|^2 \leqslant \gamma \operatorname*{osc}_{B_{R_\nu}} |u|^2 + K^* R_\nu^\tau\,, \qquad 0 \leqslant \nu \leqslant m-1\,.$$

Iterating these inequalities, we obtain

$$\begin{aligned}\operatorname*{osc}_{B_{R_m}} |u|^2 &\leqslant \gamma^m \operatorname*{osc}_{B_{R_0}} |u|^2 + (1-\gamma)^{-1} K^* R_0^\tau \\ &\leqslant \gamma^m M^2 + (1-\gamma)^{-1} K^* R_0^\tau \\ &< \varepsilon/4K' + \varepsilon/4K'\,.\end{aligned}$$

That is,

$$\operatorname*{osc}_{B_{R_m}} |u|^2 < \varepsilon/2K'\,.$$

Then it follows from (4.2) that

$$\begin{aligned}\Psi_{R_m/2} &\leqslant K' \Big\{ \sup_{B_{R_m}} |u|^2 - \fint_{B_{R_{m+1}}} |u(z)|^2 dz \Big\} + K'' R_m^\delta \\ &\leqslant K' \fint_{B_{R_{m+1}}} \Big\{ \sup_{B_{R_m}} |u|^2 - |u(z)|^2 \Big\} dz + K'' R_0^\delta \\ &< K' \cdot (\varepsilon/2K') + \varepsilon/2 = \varepsilon\,.\end{aligned}$$

Thus, the relation (*) is verified.

Let $\nu \in \{0, 1, \ldots, m\}$ be an index such that

$$\Psi_{R_\nu/2} < \varepsilon\,.$$

Then

$$\int_{B_{2R} - B_{kR}} G_{4R}(x, y) |\nabla u(x)|^2 dx < \varepsilon$$

for $R = 4^{-1} R_\nu$, $R_\nu = (k/8)^\nu R_0$.

Finally, it follows from (6.1) and (6.2) that there is a number K independent of R such that

$$G(x, y) \leqslant K G_{4R}(x, y) \quad \text{for } |x-y| \leqslant 2R\,.$$

Thus, replacing ε by ε/K, the lemma is proved.

We proceed now with the PROOF OF THEOREM **4.1**.

PROOF OF INTERIOR REGULARITY:

(i) Set

$$a = a(M)\,, \qquad \lambda^* = \lambda^*(M)\,, \qquad b = b(\cdot, M) + b^*(\cdot, M)$$

Since

$$\lambda^* + aM < 2\lambda$$

we can in addition, assume that

$$\lambda^* < \lambda\,.$$

(Otherwise, we should have $aM < \lambda$, and then we can redefine λ^* to be $\lambda^* = aM$ whence $\lambda^* < \lambda$).

Next, we fix some $d > 0$, and some $\varepsilon'' > 0$ with

$$\varepsilon'' < (\lambda/a)^2\,.$$

Set

$$h_0 = \min\{1, (\lambda - \lambda^*)/aM\}\,, \qquad 0 < h_0 \leqslant 1\,.$$

Then there exists a number $\varepsilon' \in (0, 1)$ such that

$$h' = \lambda - a\{(1-h_0)^2 M^2 + \varepsilon'\}^{\frac{1}{2}} > 0\,. \tag{4.3}$$

Let

$$h = \min\{1, h'/aM, h_0\}\,, \qquad 0 < h \leqslant h_0 \leqslant 1\,.$$

and

$$\varepsilon_0 = 2^{-1}h(2-h)\min\{\varepsilon', \varepsilon''/8\}\,, \qquad \varepsilon = \varepsilon_0/4K_{\varepsilon_0}$$

where K_{ε_0} is a constant which is determined in (ii).

Define Q as smallest positive integer such that

$$(1-h)^{2Q} < \varepsilon''/8M^2\,,$$

and let R_0' and m be the numbers appearing in Lemma **4.4** which are associated with ε .

We shall determine a number $\varrho > 0$ depending on d' and the parameters as chosen before but not on u, such that

$$\operatorname*{osc}_{B_\varrho(y)} u < \lambda/a \quad \text{for all } y \in \Omega \quad \text{with dist } (y, \partial\Omega) \geqslant d.$$

Then, in virtue of Theorem **2.2**, the first assertion of Theorem **4.1** is proved.

(ii) Fix now some $y \in \Omega$, and denote in the following with B_R the open ball in $\boldsymbol{R}^n$ with radius R and center y, for any $R > 0$, and $T_{2R} = B_{2R} - B_R$.

Suppose that dist $(y, \partial\Omega) \geqslant d$. Then we shall prove the following

AUXILIARY THEOREM:

(α) *There are numbers* ϱ_0, R_0, R_0^* *depending on the parameters of* (i) *but not on* u *such that* $0 < \varrho_0 < R_0 < R_0^* < \min\{R_0', d/8\}$ *and*

$$\sup_{z \in B_{\varrho_0}} |u(z) - h_0 \bar{u}_{R_0}|^2 \leqslant (1 - h_0)^2 \sup_{B_{R^*}} |u|^2 + \varepsilon_0$$

where

$$\bar{u}_{R_0} = \fint_{T_{2R_0}} u(x)\, dx .$$

(β) *Let* ω *be a constant vector in* $\boldsymbol{R}^N$, *and* $v = u - \omega$. *Suppose that, for some* $R^* \in (0, \min\{R_0', d/8\})$,

$$(*) \qquad \sup_{B_{R^*}} |v|^2 \leqslant (1 - h_0)^2 M^2 + \varepsilon' .$$

Then there are numbers $\varrho = \varrho(R^*, \varepsilon_0, m)$ *and* R *with* $0 < \varrho < R < R^*$ *such that*

$$(+) \qquad \sup_{B_\varrho} |v - h\bar{v}_R|^2 \leqslant (1 - h)^2 \sup_{B_{R^*}} |v|^2 + \varepsilon_0$$

where

$$\bar{v}_R = \fint_{T_{2R}} v(x)\, dx .$$

We shall prove (α) and (β) all at once. Note that $v = u - \omega$ satisfies

$$\int_\Omega a^{\alpha\beta} D_\alpha v \cdot D_\beta \varphi\, dx = \int_\Omega f(x, u, \nabla v) \cdot \varphi\, dx$$

for all $\varphi \in \overset{\circ}{H}{}_2^1 \cap L^\infty(\Omega, \boldsymbol{R}^N)$ where

$$|f(x, u, \nabla v)| \leqslant a|\nabla v|^2 + b(x) .$$

We insert the test vector

$$\varphi = (v - t\bar{v}_R)\, G^\sigma(\cdot, z)\, \eta_R^2$$

where $t = h_0$ or h if $v = u$ or $v \neq u$, respectively, $z \in B_{R/4}$, and $\eta = \eta_R$ is our friend on B_{2R}.

The number $R \in (0, R^*/2)$ will be fixed later on.

Then we obtain

$$\lambda\int|\nabla v|^2 G^\sigma(x,z)\eta_R^2\,dx+\tfrac{1}{2}\int a^{\alpha\beta}D_\alpha[|v-t\bar{v}_R|^2\eta_R^2]D_\beta G^\sigma(x,z)\,dx\leqslant \mathrm{I}+\mathrm{II}+\mathrm{III}\,,$$

where

$$\begin{aligned}\mathrm{I}\ &=-\int a^{\alpha\beta}D_\alpha v D_\beta\eta_R^2\{v-t\bar{v}_R\}\,G^\sigma(x,z)\,dx\,,\\ \mathrm{II}\ &=\tfrac{1}{2}\int a^{\alpha\beta}D_\alpha\eta_R^2 D_\beta G^\sigma(x,z)|v-t\bar{v}_R|^2\,dx\,,\\ \mathrm{III}&=\int\{v-t\bar{v}_R\}\,f(x,u,\nabla v)\,G^\sigma(x,z)\eta_R^2\,dx\,.\end{aligned}$$

The integrals are extended over B_{2R} but note that I and II are only extended over $T_{2R}^*=B_{7R/4}-B_{5R/4}$ since $\nabla\eta_R=0$ outside of T_{2R}^*, and $T_{2R}^*\subset\subset T_{2R}$:

By Young's inequality, (6.1), and (6.12), we obtain

$$|\mathrm{I}|\leqslant KR^{-1}\int\limits_{T_{2R}}|\nabla v|G^\sigma(x,z)\,dx\leqslant\tilde{\varepsilon}R^{-2}\int\limits_{T_{2R}}G^\sigma(x,z)\,dx+K\tilde{\varepsilon}^{-1}\int\limits_{T_{2R}}|\nabla v|^2G^\sigma(x,z)\,dx\,.$$

Choosing $\tilde{\varepsilon}$ appropriately, we find

$$\mathrm{I}\leqslant\frac{\varepsilon_0}{16}+K_{\varepsilon_0}\int\limits_{T_{2R}}|\nabla v|^2G^\sigma(x,z)\,dx\,.$$

On account of (6.10), we see that

$$\begin{aligned}\mathrm{II}&=\tfrac{1}{2}(1-t)^2\,|\bar{v}_R|^2+(1-t)\bar{v}_R\cdot\int\limits_{T_{2R}^*}\{v-\bar{v}_R\}\,a^{\alpha\beta}D_\alpha\eta_R^2D_\beta G^\sigma(x,z)\,dx\\ &\quad+\tfrac{1}{2}\int\limits_{T_{2R}^*}a^{\alpha\beta}D_\alpha\eta_R^2D_\beta G^\sigma(x,z)|v-\bar{v}_R|^2\,dx\\ &\leqslant\tfrac{1}{2}(1-t)^2\sup_{B_R^*}|v|^2+(1-t)|\bar{v}_R|KR^{-1}\int\limits_{T_{2R}^*}|v-\bar{v}_R||\nabla G^\sigma(x,z)|\,dx\\ &\quad+K\sup_{B_R^*}|v-\bar{v}_R|R^{-1}\int\limits_{T_{2R}^*}|\nabla G^\sigma(x,z)||v-\bar{v}_R|\,dx\\ &\leqslant\tfrac{1}{2}(1-t)^2\sup_{B_R^*}|v|^2+KR^{-1}\int\limits_{T_{2R}^*}|v-\bar{v}_R||\nabla G^\sigma(x,z)|\,dx\,.\end{aligned}$$

Young's inequality implies that

$$R^{-1}\int\limits_{T_{2R}^*}|v-\bar{v}_R||\nabla G^\sigma(x,z)|\,dx\leqslant\tilde{\varepsilon}^{-1}R^{2-n}R^{-2}\int\limits_{T_{2R}}|v-\bar{v}_R|^2\,dx+\tilde{\varepsilon}R^{n-2}\int\limits_{T_{2R}^*}|\nabla G^\sigma(x,z)|^2\,dx\,.$$

In virtue of Poincaré's inequality, we get

$$R^{-2}\int_{T_{2R}} |v-\bar{v}_R|^2\,dx \leqslant K\int_{T_{2R}} |\nabla v|^2\,dx\,,$$

and a well-known estimate due to Moser [10] yields that

$$\int_{T^*_{2R}} |\nabla G^\sigma(x,z)|^2\,dx \leqslant KR^{-2}\int_{T_{2R}} |G^\sigma(x,z)|^2\,dx \leqslant KR^{2-n}\,,$$

cf. (6.1), (6.12), (6.14).

Therefore, by appropriate choice of $\tilde{\varepsilon}$,

$$\mathrm{II} \leqslant \tfrac{1}{2}(1-t)^2 \sup_{B_R^*} |v|^2 + \frac{\varepsilon_0}{16} + K_{\varepsilon_0}\int_{T_{2R}} G(x,z)|\nabla v|^2\,dx\,.$$

Thirdly,

$$\mathrm{III} = \int_{B_{2R}-B_{R/2}} \ldots + \int_{B_{R/2}} \ldots = \mathrm{III}' + \mathrm{III}''\,,$$

and

$$\mathrm{III}' \leqslant K\int_{B_{2R}-B_{R/2}} G^\sigma(x,z)|\nabla v(x)|^2\,dx + KR^\delta\,.$$

The integral III'' is estimated as follows:

If $v=u$, $t=h_0$, then

$$\mathrm{III}'' \leqslant (\lambda^* + h_0 aM)\int_{B_{R/2}} |\nabla v|^2 G^\sigma(x,z)\,dx + KR^\delta,$$

and for $v \neq u$, $t=h$, we get

$$\mathrm{III}'' \leqslant (1+h)\,a \sup_{B_R^*} |v| \int_{B_{R/2}} |\nabla v|^2 G^\sigma(x,z)\,dx + KR^\delta,$$

since $|v_R| \leqslant \sup_{B_R^*} |v|$.

Collecting these estimates, and noting, that by (6.10)

$$\tfrac{1}{2}\int a^{\alpha\beta} D_\alpha\big[|v-t\bar{v}_R|^2\eta_R^2\big]\,D_\beta G^\sigma(x,z)\,dx = \tfrac{1}{2}\fint_{B_\sigma(z)} |v-t\bar{v}_R|^2\,dx$$

for sufficiently small $\sigma > 0$, we obtain for $\sigma \to 0$ the inequality

$$|v(z) - t\bar{v}_R|^2 + 2\begin{cases}(\lambda - \lambda^* - h_0 aM) & \text{if} \quad v = u \\ (\lambda - a \sup\limits_{B_{R^*}} |v| - haM) & \text{if} \quad v \neq u\end{cases} \int\limits_{B_{R/2}} |\nabla v|^2 G(x, z)\, dx \leqslant$$

$$\leqslant (1-t)^2 \sup_{B_{R^*}} |v|^2 + \varepsilon_0/4 + KR^\delta + K_{\varepsilon_0} \int\limits_{B_{2R} - B_{R/2}} G(x, z) |\nabla v|^2\, dx\,.$$

Suppose that $R^* > 0$ is chosen so small that

$$KR^\delta < \varepsilon_0/4 \qquad \text{for } 0 < R \leqslant R^*\,.$$

(Note that K stands always for a number depending on the parameters of the system but not on R, ω and u).

For $x \in B_{2R} - B_{R/2}$ and $z \in B_\varrho$, $\varrho < R/4$, we have

$$|y - z| < R/4 \leqslant (\tfrac{1}{2}) \cdot |x - y|$$

whence, by (6.3),

$$\begin{aligned} G(x, z) &\leqslant G(x, y) + |G(x, z) - G(x, y)| \\ &\leqslant G(x, y) + K_3 |z - y|^\beta |x - y|^{2-n-\beta} \\ &\leqslant G(x, y) + K\varrho^\beta (R/2)^{2-n-\beta}\,, \qquad 0 < \beta < 1\,. \end{aligned}$$

Therefore, by **3.1**,

$$\begin{aligned} &K_{\varepsilon_0} \int\limits_{B_{2R} - R_{R/2}} G(x, z) |\nabla v|^2\, dx \\ &\leqslant K_{\varepsilon_0} \int\limits_{B_{2R} - B_{R/2}} G(x, y) |\nabla v|^2\, dx + K^* K_{\varepsilon_0} \varrho^\beta (R/2)^{2-n-\beta}\,. \end{aligned}$$

Since $\lambda^* < \lambda$, we can apply Lemma **4.4** to $v = u$, and, because of assumption $(*)$, we have

$$(**) \qquad 0 \leqslant haM \leqslant h' \leqslant \lambda - a \sup_{B_{R^*}} |v|$$

in the case (β), so that we can apply **4.4** also to a situation where u is replaced by v, Ω by B_{4R^*}, d by $4R^*$, and f by $f^*(x, \nabla v) = f(x, u(x), \nabla v)$ since $v \in W(L, f^*)$ while y and ε remain unaltered. Choose $k = \frac{1}{2}$. Then there exists a number R with

$$8^{-1} 16^{-m} R^* \leqslant R \leqslant R^*/8$$

such that

$$\int_{B_{2R}-B_{R/2}} G(x, y)|\nabla v(x)|^2\, dx < \varepsilon = \varepsilon_0/4K_{\varepsilon_0}\,.$$

Having fixed such a number R, we define $\varrho = \varrho(R^*, \varepsilon_0, m)$ by

$$\varrho = \min\{[(\varepsilon_0/4)\cdot(16^{m+1}R^{*-1})^{2-n-\beta}K^{*-1}K_{\varepsilon_0}^{-1}]^{1/\beta},\ 2^{-1}\cdot 16^{-m-1}\cdot R^*\}.$$

Then

$$K_{\varepsilon_0}\int_{B_{2R}-B_{R/2}} G(x, z)|\nabla v|^2\, dx \leqslant \varepsilon_0/4 + \varepsilon_0/4 = \varepsilon_0/2\,.$$

for $z \in B_\varrho$: Combining this inequality with the previous estimate, we obtain

$$|v(z) - t\bar{v}_R|^2 \leqslant (1-t)^2 \sup_{B_R^*} |v|^2 + \varepsilon_0 \qquad \text{for all } z \in B_\varrho$$

taking $(**)$ for $v \neq u$, $t = h$, and $\lambda^* + aM < 2\lambda$ for $v = u$, $t = h_0$ into account.

Thus, we have proved the AUXILIARY THEOREM.

(iii) Now we define v_i, ϱ_i, R_i^* for $i = 0, 1, \ldots, Q$ by the following iteration procedure:

Let ϱ_0, R_0, R_0^* be defined as in (α) of the Auxiliary Theorem, and set

$$v_0 = u - h_0\bar{u}_{R_0}\,,$$

whence

$$\begin{aligned}(4.4)\qquad \sup_{B_{\varrho_0}} |v_0|^2 &\leqslant (1-h_0)^2 \sup_{B_{R'}} |u|^2 + \varepsilon_0 \\ &\leqslant (1-h_0)^2 M^2 + \varepsilon_0\,, \qquad \varepsilon_0 < \varepsilon'\,.\end{aligned}$$

Suppose now that $v_i = u - \omega_i$, $\omega_i = \text{const} \in \boldsymbol{R}^N$, ϱ_i, R_i^* are already determined for $0 \leqslant i \leqslant j-1$ such that

$$(4.5, \text{i})\qquad \sup_{B_{\varrho_i}} |v_i|^2 \leqslant (1-h)^2 \sup_{B_{\varrho_{i-1}}} |v_{i-1}|^2 + \varepsilon_0$$

and

$$(4.6, \text{i})\qquad \sup_{B_{\varrho_i}} |v_i|^2 \leqslant (1-h_0)^2 M^2 + \varepsilon'\,,$$

where $v_{-1} = u$, $\varrho_{-1} = R_0'$.

In the j-th step we choose $R^* = \varrho_{j-1}$ and $v = v_{j-1}$. Since, by induction assumption, $(*)$ is satisfied, there exist numbers $\varrho = \varrho(R^*, \varepsilon_0, m)$ and R with $0 < \varrho < R < R^*$ such that

$$\sup_{B_\varrho} |v - h\bar{v}_R|^2 \leqslant (1-h)^2 \sup_{B_{R^*}} |v|^2 + \varepsilon_0 .$$

Set $R_j = R$, $R_j^* = R^*$, $\varrho_j = \varrho = \varrho(R_j^*, \varepsilon_0, m)$, and $v_j = v_{j-1} - h\bar{v}_{j-1,R_j}$. Then we obtain $(4.5, j)$. Iterating $(4.5, i)$ for $i = 0, 1, \ldots, j$, we arrive at

$$\sup_{B_{\varrho_j}} |v_j|^2 \leqslant (1-h)^{2j} \sup_{B_{R_0^*}} |v_0|^2 + \frac{\varepsilon_0}{1-(1-h)^2} .$$

Since

$$\frac{1}{1-(1-h)^2} + (1-h)^{2j} \leqslant \frac{2}{h(2-h)} ,$$

(4.4) implies that

$$\sup_{B_{\varrho_j}} |v_j|^2 \leqslant (1-h)^{2j}(1-h_0)^2 M^2 + \min_{B_{\varrho_j}} \{\varepsilon', \varepsilon''/8\} .$$

Hence $(4.6, j)$ is satisfied, and the induction procedure can be carried out. Secondly, we get

$$\Big[\operatorname*{osc}_{B_{\varrho_j}} u\Big]^2 \leqslant 4 \sup_{B_{\varrho_j}} |v_j|^2 \leqslant 4(1-h)^{2j} M^2 + \varepsilon''/2 , \tag{4.7}$$

whence

$$\operatorname*{osc}_{B_{\varrho_Q}} u \leqslant \{4(1-h)^{2Q} M^2 + \varepsilon''/2\}^{\frac{1}{2}} \leqslant \sqrt{\varepsilon''} < \lambda/a$$

where $\varrho_Q > 0$ can be calculated and does depend only on the parameters of (i) but not on the particular solution $u \in W(L, f)$.

Thus the proof of interior regularity is complete.

PROOF OF GLOBAL REGULARITY:

Let us assume that the boundary values $g\colon \partial\Omega \to \boldsymbol{R}^N$ are extended to a mapping $g\colon \bar{\Omega} \to \boldsymbol{R}^N$ of class $\mathrm{Lip}(\bar{\Omega}, \boldsymbol{R}^N)$. In view of Theorem 4.3 in [3] it suffices to prove that, for every $\varepsilon > 0$, there is a number $\varrho(\varepsilon) > 0$ independent of u such that

$$|u(x) - g(x)| \leqslant \varepsilon \quad \text{if dist } (x, \partial\Omega) < \varrho(\varepsilon),\ x \in \Omega . \tag{4.8}$$

The interior regularity proof can be modified so as to yield boundary regularity as well. However, we prefer to give a much shorter proof which does not use the Lemmata **4.2**-**4.4**.

Again, we set

$$a = a(M), \quad \lambda^* = \lambda^*(M), \quad b = b(\cdot, M) + b^*(\cdot, M).$$

We have

$$\lambda^* + aM < 2\lambda,$$

and, without loss of generality, we may also assume that

$$\lambda^* < \lambda.$$

Set

$$h_0 = \min\{1, (\lambda - \lambda^*)/aM\}, \quad 0 < h_0 \leqslant 1.$$

There is an $\varepsilon' > 0$ such that

$$h' = \lambda - a\{(1 - h_0)^2 M^2 + \varepsilon'\}^{\frac{1}{2}} > 0$$

Then we choose an integer $Q > 0$ such that

$$h = (1 - h_0)/Q < h'/aM$$

thus dividing the interval $(h_0, 1)$ into Q equal parts of length h, and set

$$h_j = h_0 + jh, \quad j = 0, 1, \ldots, Q,$$

that is, $h_Q = 1$.

Moreover, we fix an arbitrary $\varepsilon > 0$, and set

$$\varepsilon_0 = \min\{\varepsilon', \varepsilon^2\}.$$

Let $\Omega_0, \Omega_1, \ldots, \Omega_Q$ be open subsets of Ω to be specified later on, but such that

$$\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega_2 \subset\subset \ldots \subset\subset \Omega_Q \subset\subset \Omega.$$

We note that, by **3.1**, there are numbers $L > 0$, $\tilde{L}(\Omega') > 0$ depending only on the usual parameters but not on the particular solution u such that

$$\int_\Omega G(x, y)|\nabla u(x)|^2 dx < L \quad \text{for all } y \in \Omega,$$

and

$$\int_{\Omega'} |\nabla u(x)|^2 \, dx < \tilde{L}(\Omega') \qquad \text{for any } \Omega' \subset\subset \Omega .$$

Now we test (1.5) with

$$\varphi = \{u - h_j g\} G^\sigma(\cdot, y), \qquad y \in \Omega .$$

After letting σ tend to zero, and taking (6.10) into account, we arrive at

$$\begin{aligned}
&\int a^{\alpha\beta} D_\alpha u \cdot D_\beta u \, G(x, y) \, dx + \tfrac{1}{2}|u(y) - h_j g(y)|^2 \\
&\leqslant \tfrac{1}{2} w_j(y) + h_j \int_\Omega a^{\alpha\beta} D_\alpha u \cdot D_\beta g \, G(x, y) \, dx \\
&- h_j \int_\Omega a^{\alpha\beta} D_\alpha g \cdot \{u - h_j g\} D_\beta G(x, y) \, dx \\
&+ \int_\Omega \{u - h_j g\} \cdot f(x, u, \nabla u) G(x, y) \, dx
\end{aligned}$$

where w_j is the weak solution of $Lw_j = 0$ in Ω with

$$w_j - |u - h_j g|^2 \in \mathring{H}_2^1(\Omega, \boldsymbol{R}) .$$

In virtue of the maximum principle, we obtain that

$$\sup_\Omega w_j \leqslant \sup_{\partial\Omega} w_j \leqslant (1 - h_j)^2 M^2 .$$

Set

$$C_j = \begin{cases} \lambda - \lambda^* - h_0 a M & \text{if} \quad j = 0 \\ \lambda - a \sup\limits_{\Omega - \Omega_{j-1}} |u - h_{j-1} g| & \text{if} \quad j > 0 \end{cases} .$$

Then we get, by Young's inequality,

$$\begin{aligned}
&|u(y) - h_j g(y)|^2 + 2C_j \int_{\Omega - \Omega_{j-1}} |\nabla u|^2 G(x, y) \, dx \\
&\leqslant (1 - h_j)^2 M^2 + 2Ma \int_{\Omega_{j-1}} |\nabla u|^2 G(x, y) \, dx + 2M \int_\Omega b(x) G(x, y) \, dx \\
&+ (\varepsilon_0 / 5L) \int_\Omega |\nabla u|^2 G(x, y) \, dx + L\varepsilon_0^{-1} K \int_\Omega G(x, y) \, dx + K^* \int_\Omega |\nabla G(x, y)| \, dx .
\end{aligned}$$

On account of (6.4), (6.5), and (6.7), we may choose an $\Omega_0 \subset\subset \Omega$ such that,

for every $y \in \Omega - \Omega_0$,

$$\int_\Omega b(x)\, G(x, y)\, dx < \varepsilon_0/10M\,,$$

$$\int_\Omega G(x, y)\, dx \qquad < \varepsilon_0^2/5LK\,,$$

$$\int_\Omega |\nabla G(x, y)|\, dx \;\; < \varepsilon_0/5K^*\,.$$

Moreover, by (6.4), there is a $K_4 > 0$ and a $\beta \in (0, 1)$ such that

$$G(x, y) \leqslant K_4 \operatorname{dist}^\beta(y, \partial\Omega)|x - y|^{2-n-\beta}\,.$$

Set

$$\varrho_j = \sup\{\operatorname{dist}(y, \partial\Omega) : y \in \Omega - \Omega_j\}\,,$$

$$r_j = \operatorname{dist}\{\Omega_{j-1}, \partial\Omega_j\}\,, \qquad\qquad j = 0, 1, \ldots, Q\,.$$

Now we choose $\Omega_1, \Omega_2, \ldots, \Omega_Q$ iteratively in such a way that

$$2Ma\tilde{L}(\Omega_{j-1})K_4\, \varrho_j^\beta r_j^{2-n-\beta} < \varepsilon_0/5$$

whence

$$2Ma\int_{\Omega_{j-1}} |\nabla u|^2 G(x, y)\, dx < \varepsilon_0/5 \qquad \text{for } y \in \Omega - \Omega_j\,.$$

Collecting all the estimates, we obtain that

$$|u(y) - h_j g(y)|^2 + 2C_j \int_{\Omega - \Omega_{j-1}} |\nabla u|^2 G(x, y)\, dx$$
$$\leqslant (1 - h_j)^2 M^2 + \varepsilon_0 \qquad \text{for every } y \in \Omega - \Omega_j\,.$$

Since $C_0 \geqslant 0$ we conclude that

$$\sup_{\Omega - \Omega_0} |u - h_0 g|^2 \leqslant (1 - h_0)^2 M^2 + \varepsilon_0\,, \qquad \varepsilon_0 \leqslant \varepsilon'\,,$$

whence $C_1 \geqslant 0$, and therefore

$$\sup_{\Omega - \Omega_1} |u - h_1 g|^2 \leqslant (1 - h_1)^2 M^2 + \varepsilon_0$$

$h_j \geqslant h_0$, $\varepsilon' \geqslant \varepsilon_0$. Thus $C_2 \geqslant 0$.

In this way we can proceed to prove

$$\sup_{\Omega-\Omega_j} |u - h_j g|^2 \leqslant (1 - h_j)^2 M^2 + \varepsilon_0$$

for $j = 1, 2, \ldots, Q$. In particular, since $h_Q = 1$, we have

$$\sup_{\Omega-\Omega_Q} |u - g|^2 \leqslant \varepsilon_0 \leqslant \varepsilon^2 .$$

This concludes the proof.

5. – Generalizations.

In this section we shall present another regularity theorem making more detailed use of the fine structure of the right hand side of

$$Lu = f(x, u, \nabla u) .$$

This result appears to be a natural generalization of the well-known regularity theorem of Ladyženskaya-Ural'ceva [6] for a single equation ($N = 1$). However, it cannot be derived from the previously known results. We may also consider the somewhat more general situation of systems of the type

$$\begin{aligned} L_1 u^1 &= f^1(x, u, \nabla u) \\ L_2 u^2 &= f^2(x, u, \nabla u) \\ &\cdots\cdots\cdots \\ L_N u^N &= f^N(x, u, \nabla u) \end{aligned} \tag{5.1}$$

where the operators

$$L_i = - D_\beta \{a_i^{\alpha\beta}(x) D_\alpha\}$$

satisfy the ellipticity conditions (1.1) and (1.2) with constants λ_i and μ_i.

In the following, we revoke the summation convention with respect to Latin indices.

We assume that the right hand side satisfies the following growth condition:

$$|f^i(x, u, p)| \leqslant \sum_{k=1}^{N} \alpha_k^i |p^k|^2 + \beta^i(x) , \qquad 1 \leqslant i \leqslant N , \tag{5.2}$$

for all $(x, u, p) \in \Omega \times \boldsymbol{R}^N \times \boldsymbol{R}^{nN}$ satisfying $|u^i| \leqslant M_i$, $1 \leqslant i \leqslant N$, $\alpha_k^i \geqslant 0$, and $\beta^i \in L_q(\Omega)$ with $q > n/2$.

Set $u = (u^1, \dots, u^N)$, $\lambda = (\lambda_1, \dots, \lambda_N)$, $\mu = (\mu_1, \dots, \mu_N)$,

$$M = (M_1, \dots, M_N), \qquad A = (\alpha_k^i)_{i,k=1\dots,N},$$

$$A_1 = (\alpha_k^i)_{i \leqslant k}, \qquad A_2 = (\alpha_k^i)_{i>k} = A - A_1.$$

A weak solution u of the system (5.1) is an element of class $H_2^1 \cap L^\infty(\Omega, \boldsymbol{R}^N)$ which fulfills the integrated version of (5.1), analogous to (1.5).

Moreover, there is an obvious generalization of Theorem **2.3**, and also of **2.4-2.6**. We leave formulation and proof of the corresponding results to the reader.

Finally, denote by $\|A\|$ the Euclidean norm of a matrix A.

5.1. THEOREM. *Let L_i and f^i, $1 \leqslant i \leqslant N$, satisfy the conditions above. Then there exists a positive number $\varepsilon(\lambda, \mu, M, A_1)$ with the following property:*

If $\|A_2\| < \varepsilon$, and if $u \in H_2^1 \cap L^\infty(\Omega, \boldsymbol{R}^N)$ is a weak solution of (5.1) *with $|u^i|_{L^\infty(\Omega)} \leqslant M_i$, $1 \leqslant i \leqslant N$, then u is Hölder continuous in Ω, and the Hölder norm on compact subsets of Ω can be estimated in the available parameters.*

PROOF. Fix some index i with $1 \leqslant i \leqslant N$, and set $v = u^i$, $w = (u^1, \dots, u^{i-1})$, $w^* = (u^{i+1}, \dots, u^N)$, that is,

$$u = (w, v, w^*) \quad \text{and} \quad F(x, u, \nabla u) = f^i(x, u, \nabla u).$$

The estimate (5.2, i) can be written as

$$|F(x, u, \nabla u)| \leqslant \alpha |\nabla v|^2 + G(\nabla w) + H(\nabla w^*) + \beta(x), \tag{5.3}$$

where the meaning of α, β, G, H is obvious. Now we test the equation

$$\int_\Omega a_i^{\alpha\beta} D_\alpha v D_\beta \varphi \, dx = \int_\Omega F(x, u, \nabla u) \varphi \, dx$$

with

$$\varphi = (v - \bar{v}) \exp\left[t_i |v - \bar{v}|^2\right] \tilde{G}_i^\sigma(\cdot, y) \eta^2$$

where η is our friend on $B_{2R}(y)$ for some $y \in \Omega$ and some sufficiently small $R > 0$,

$$\bar{v} = \int_{T_{2R}(y)} v(x) \, dx,$$

t_i a positive real parameter, and $\tilde{G}_i^\sigma$ is the mollification of Green's function

for the operator

$$\tilde{L}_i^* = - D_\alpha\{\tilde{a}_i^{\alpha\beta} D_\beta\}\,,$$
$$\tilde{a}_i^{\alpha\beta} = \exp\left[t_i|v-\bar{v}|^2\right] a_i^{\alpha\beta}\,.$$

Then,

$$\begin{aligned}(5.4)\qquad & 2t_i\int \tilde{a}_i^{\alpha\beta} D_\alpha v D_\beta v\, |v-\bar{v}|^2 \tilde{G}_i^\sigma \eta^2\, dx + \int \tilde{a}_i^{\alpha\beta} D_\alpha v D_\beta v\, \tilde{G}_i^\sigma \eta^2\, dx \\ & + \tfrac{1}{2}\int \tilde{a}_i^{\alpha\beta} D_\alpha[|v-\bar{v}|^2\eta^2]\, D_\beta \tilde{G}_i^\sigma\, dx \\ & = \tfrac{1}{2}\int \tilde{a}_i^{\alpha\beta} D_\alpha \eta^2 D_\beta \tilde{G}_i^\sigma |v-\bar{v}|^2\, dx - \int \tilde{a}_i^{\alpha\beta} D_\alpha v D_\beta \eta^2 (v-\bar{v})\tilde{G}_i^\sigma\, dx \\ & + \int (v-\bar{v}) F(x, u, \nabla u)\tilde{G}_i^\sigma \eta^2 \exp\left[t_i|v-\bar{v}|^2\right] dx\,.\end{aligned}$$

Since

$$\int_\Omega \tilde{a}_i^{\alpha\beta} D_\alpha\left[|v-\bar{v}|^2\right] D_\beta \tilde{G}_i^\sigma(x, y)\, dx = \fint_{B_\sigma(y)} |v-\bar{v}|^2\, dx \geqslant 0$$

we may discard the third term on the left hand side of (5.4). Furthermore,

$$\alpha|v-\bar{v}||\nabla v|^2 \leqslant \frac{\lambda_i}{2}|\nabla v|^2 + \frac{\alpha^2}{2\lambda_i}(v-\bar{v})^2|\nabla v|^2\,.$$

Combining these estimates with (5.3) and (5.4), it follows that

$$\begin{aligned}(5.5)\qquad & 2t_i\lambda_i\int \exp\left[t_i|v-\bar{v}|^2\right]\tilde{G}_i^\sigma \eta^2 (v-\bar{v})^2|\nabla v|^2\, dx \\ & + \lambda_i\int \exp\left[t_i|v-\bar{v}|^2\right] G_i^\sigma \eta^2 |\nabla v|^2\, dx \\ & \leqslant \mathrm{I} + \mathrm{II} + \frac{\alpha^2}{2\lambda_i}\int \exp\left[t_i|v-\bar{v}|^2\right]\tilde{G}_i^\sigma \eta^2 |v-\bar{v}|^2|\nabla v|^2\, dx + \\ & + \frac{\lambda_i}{2}\int \exp\left[t_i|v-\bar{v}|^2\right]\tilde{G}_i^\sigma \eta^2 |\nabla v|^2\, dx \\ & + \int \{G(\nabla w) + H(\nabla w^*) + \beta(x)\}|v-\bar{v}|\tilde{G}_i^\sigma \eta^2 \exp\left[t_i|v-\bar{v}|^2\right] dx\end{aligned}$$

where I and II denote the first two integrals on the right hand side of (5.4).

Note that

$$G(\nabla w) = \sum_{k<i} \alpha_k^i |\nabla u^k|^2 \leqslant \|A_2\|\,|\nabla w|^2$$
$$H(\nabla w^*) = \sum_{k>i} \alpha_k^i |\nabla u^k|^2 \leqslant \|A_1\|\,|\nabla w^*|^2$$

and that $|v-\bar{v}|\leqslant 2M_i$. Moreover, we choose

$$t_i=\alpha^2/4\lambda_i^2\,,\qquad \alpha=\alpha_i^i\,.$$

Then (5.5) implies, that

$$\begin{aligned}&(\lambda_i/2)\int \tilde{G}_i^\sigma\eta^2|\nabla v|^2\,dx\\ &\leqslant \mathrm{I}+\mathrm{II}+KR^\delta+K\int|\nabla w^*|^2\tilde{G}_i^\sigma\eta^2\,dx\\ &+K\|A_2\|\int|\nabla w|^2\tilde{G}_i^\sigma\eta^2\,dx\,.\end{aligned}$$

Furthermore, we get, as in [3], pp. 79-80,

$$\mathrm{I}+\mathrm{II}\leqslant KR^{2-n}\int_{T_{2R}(y)}|\nabla v|^2\,dx\,.$$

Here, and for the rest of the proof, K denotes a positive number depending on λ, μ, M, and $\|A_1\|$, but not on u, R, and $\|A_2\|$.

Set

$$g^\sigma(x,y)=\fint_{B_\sigma(y)}|x-z|^{2-n}\,dz$$

and let $B_{4R}(y)\subset\Omega$. By (6.1) and (6.2), there are numbers K' and K'' depending on λ and μ but independent of R such that

$$K'\leqslant G_i^\sigma(x,y)/g^\sigma(x,y)\leqslant K''$$

for $1\leqslant i\leqslant N$, $0<\sigma\ll 1$, and $|x-y|<2R$.

Thus, for $B_{2R}=B_{2R}(y)$, $T_{2R}=T_{2R}(y)$, $g^\sigma=g^\sigma(x,y)$,

$$\begin{aligned}&\int_{B_{2R}}|\nabla u^i|^2\,g^\sigma\,\eta^2\,dx\\ &\leqslant K\int_{T_{2R}}|\nabla u^i|^2g^\sigma\,dx+KR^\delta+K\sum_{k>i}\int_{B_{2R}}|\nabla u^k|^2g^\sigma\eta^2\,dx\\ &+K\|A_2\|\cdot\int_{B_{2R}}|\nabla u|^2g^\sigma\eta^2\,dx\,.\end{aligned}$$

Starting with $i=N$, we find successively for $i=N-1,\ N-2,\ldots,$ that

$$\begin{aligned}(5.6)\qquad &\int_{B_{2R}}|\nabla u^i|^2g^\sigma\eta^2dx\leqslant K\int_{T_{2R}}\sum_{j=i}^{N}|\nabla u^j|^2g^\sigma dx+KR^\delta\\ &+K\|A_2\|\int_{B_{2R}}|\nabla u|^2g^\sigma\eta^2dx\end{aligned}$$

using (5.4) for $i+1, i+2, \ldots, N$. Summing (5.6) over i from 1 to N, we get

$$\int_{B_{2R}} |\nabla u|^2 g^\sigma \eta^2 dx \leqslant K \|A_2\| \int_{B_{2R}} |\nabla u|^2 g^\sigma \eta^2 dx + K \int_{T_{2R}} |\nabla u|^2 g^\sigma dx + KR^\delta .$$

Suppose now that

$$K(A_1, \lambda, \mu, M) \cdot \|A_2\| < 1 .$$

Then, letting σ tend to zero, we may apply Fatou's lemma, and obtain

$$\int_{B_R} |\nabla u|^2 |x-y|^{2-n} dx \leqslant K \int_{T_{2R}} |\nabla u|^2 |x-y|^{2-n} dx + KR^\delta \tag{5.7}$$

since $g^\sigma(x, y) \to |x-y|^{2-n}$ as $\sigma \to 0$.

Now, the hole-filling procedure (cf. [13] and [3], p. 80) applied to (5.7) yields the desired result.

REMARK. We are able to estimate $\varepsilon(\alpha, \mu, M, A_1)$ explicitely. However, since our estimate is probably far from best possible, we do not insist on this point. In fact, with a different choice of the test function φ one would get a better estimate, particularly in the case when all L_i are equal.

We also note that if $A_2 = 0$ then the regularity follows by easier means.

6. – Appendix.

In this section we have collected various informations about Green's function which were needed before. They can be found in [7] and [12], or, at least, may be derived from results of the quoted papers. For (6.8) and (6.14), cf. also [3] and [10].

$$0 \leqslant G(x, y) \leqslant K_2 |x-y|^{2-n} . \tag{6.1}$$

$$K_1 |x-y|^{2-n} \leqslant G(x, y) \quad \text{if } |x-y| \leqslant \tfrac{3}{4} \operatorname{dist}(y, \partial\Omega) . \tag{6.2}$$

(6.3) For some $\beta \in (0, 1)$, we have

$$|G(x, y) - G(z, y)| \leqslant K_3 |x-z|^\beta |x-y|^{2-n-\beta} \quad \text{if } |x-z| \leqslant \tfrac{1}{2}|x-y| .$$

$$G(x, y) \leqslant K_4 \operatorname{dist}^\beta(y, \partial\Omega) |x-y|^{2-n-\beta} . \tag{6.4}$$

$$\int_{\Omega} |G(x, y)|^p \, dx \to 0 \quad \text{as } \operatorname{dist}(y, \partial\Omega) \to 0 \quad \text{if } p < n/(n-2) . \tag{6.5}$$

$$\nabla G(\cdot, y) \in L^p \quad \text{for } p < n/(n-1) . \tag{6.6}$$

$$\int_{\Omega} |G(x, y)| \, dx \to 0 \quad \text{as dist } (y, \partial\Omega) \to 0 . \tag{6.7}$$

$$\int_{T_{2R}} |\nabla G(x, y)|^2 \, dx \leqslant K_5 R^{2-n} \quad \text{if } B_{4R}(y) \subset \Omega . \tag{6.8}$$

$$G^\sigma(\cdot, y) \in \mathring{H}_2^1 \cap L^\infty(\Omega, \boldsymbol{R}) , \quad \text{for } y \in \Omega . \tag{6.9}$$

$$\int_{\Omega} a^{\alpha\beta} D_\alpha \varphi D_\beta G^\sigma(x, y) \, dx = \fint_{B_\sigma(y)} \varphi \, dx \quad \text{for all } \varphi \in \mathring{H}_2^1(\Omega, \boldsymbol{R}) . \tag{6.10}$$

$$G^\sigma(x, y) \to G(x, y) \quad \text{as } \sigma \to 0, \text{ for } x \neq y . \tag{6.11}$$

$$G^\sigma(x, y) \leqslant K_6 G(x, y) \quad \text{if } |x - y| < 2^{-1} \text{ dist } (y, \partial\Omega) . \tag{6.12}$$

$$K_7 G(x, y) \leqslant G^\sigma(x, y) \quad \text{if } |x - y| \geqslant 2\sigma \text{ and } B_{2\sigma}(y) \subset \Omega . \tag{6.13}$$

$$\int_{T_{2R}} |\nabla G^\sigma(x, y)|^2 \, dx \leqslant K_8 R^{2-n} . \tag{6.14}$$

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