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On Analyticity in Homogeneous First Order Partial Differential Equations.

HANS LEWY (*)

1. – Let α_1, α_2, s be independent real variables and x, y, u real C^1-functions of these near the origin, satisfying

$$\frac{\partial(x, y, u)}{\partial(\alpha_1, \alpha_2, s)} = 0 \tag{1}$$

and

$$\frac{\partial(x, y)}{\partial(\alpha_1, \alpha_2)} \neq 0 , \qquad \frac{\partial x}{\partial s} \neq 0 . \tag{2}$$

An arbitrary C^1-function $u = f(x, y)$ satisfies (1). We investigate conditions such that f is analytic in x and y.

THEOREM 1. *If x, y, u satisfy* (1) *and* (2) *and*

$$\begin{vmatrix} x_s & y_s & (y_s/x_s)_s \\ x_{\alpha_1} & y_{\alpha_1} & (y_s/x_s)_{\alpha_1} \\ x_{\alpha_2} & y_{\alpha_2} & (y_s/x_s)_{\alpha_2} \end{vmatrix} \neq 0 \tag{3}$$

and if x, y, u can be extended as holomorphic functions of $s + it$, $|t| < t_0$, which remain C^1 in α_1, α_2, s, t, then $u = f(x, y)$ where f is analytic in x and y.

PROOF. We first establish that near the α_1, α_2, s, t-origin the map

$$\alpha_1 \alpha_2, s, t \rightarrow x, y$$

is one-one for $t \neq 0$.

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Put

$$x = x_0 + x_s(s+it) + \tfrac{1}{2}x_{ss}(s+it)^2 + \dots$$
$$y = y_0 + y_s(s+it) + \tfrac{1}{2}y_{ss}(s+it)^2 + \dots$$

where

$$x_0 = x(\alpha_1, \alpha_2, 0)\,, \qquad x_s = \frac{\partial x}{\partial s}(\alpha_1, \alpha_2, 0)\,, \;\dots.$$

An application of Cauchy's integral formula shows the coefficients of the above power series to be C^1-functions of α_1, α_2. The imaginary parts of x and y are of the form $tP(s, t)$ where $P(s, t)$ is a convergent power series in s and t. It follows from (2) that for $t \neq 0$

$$\frac{\operatorname{Im} y}{\operatorname{Im} x} = \frac{y_s + y_{ss}s + \dots}{x_s + x_{ss}s + \dots} = \frac{y_s}{x_s} + s\,\frac{y_{ss}x_s - x_s y_{ss}}{x_s^2} + \dots,$$

a power series in s and t.

Hence near $s = t = \alpha_1 = \alpha_2 = 0$,

$$J = \frac{\partial(\operatorname{Re} x, \operatorname{Re} y, \operatorname{Im} x, \operatorname{Im} y/\operatorname{Im} x)}{\partial(s, t, \alpha_1, \alpha_2)} =$$

$$= \begin{vmatrix} x_s & y_s & 0 & \dfrac{y_{ss}x_s - x_s y_{ss}}{x_s^2} \\ 0 & 0 & x_s & 0 \\ x_{\alpha_1} & y_{\alpha_1} & 0 & \left(\dfrac{y_s}{x_s}\right)_{\alpha_1} \\ x_{\alpha_2} & y_{\alpha_2} & 0 & \left(\dfrac{y_s}{x_s}\right)_{\alpha_2} \end{vmatrix} + \dots = - \begin{vmatrix} x_s & y_s & (y_s/x_s)_s \\ x_{\alpha_1} & y_{\alpha_1} & (y_s/x_s)_{\alpha_1} \\ x_{\alpha_2} & y_{\alpha_2} & (y_s/x_s)_{\alpha_2} \end{vmatrix} \cdot x_s + \dots$$

where the omitted terms are of degree $\geqslant 1$ in s, t.

By (2) and (3), $J \neq 0$ near the origin. Accordingly

$$s, t, \alpha_1, \alpha_2 \to \operatorname{Re} x\,, \quad \operatorname{Re} y\,, \quad \operatorname{Im} x\,, \quad \frac{\operatorname{Im} y}{\operatorname{Im} x}$$

is one-one, if $\operatorname{Im} y/\operatorname{Im} x$ is defined by continuity also for $\operatorname{Im} x = 0$; $\operatorname{Im} x = 0$ coincides with $t = 0$ by (2). Now the map

$$\operatorname{Re} x\,, \quad \operatorname{Re} y\,, \quad \operatorname{Im} x\,, \quad \frac{\operatorname{Im} y}{\operatorname{Im} x} \to \operatorname{Re} x\,, \quad \operatorname{Re} y\,, \quad \operatorname{Im} x\,, \quad \operatorname{Im} y$$

is one-one as long as $\operatorname{Im} x \neq 0$. Hence the part of an open neighborhood of the s, t, α_1, α_2-origin for which $t \neq 0$ is in one-one correspondence with a certain open set of $\operatorname{Re} x$, $\operatorname{Re} y$, $\operatorname{Im} x$, $\operatorname{Im} y$-space with $\operatorname{Im} x \neq 0$; and the Jacobian $\partial(x, y, \bar{x}, \bar{y})/\partial(\alpha_1, \alpha_2, s, t) \neq 0$ there.

Now consider for $t \neq 0$ the form

$$\omega = du \wedge dx \wedge dy \wedge d\bar{y}\,.$$

We have

$$\begin{aligned}
\omega &= \frac{\partial u}{\partial \bar{x}} d\bar{x} \wedge dx \wedge dy \wedge d\bar{y} \\
&= \frac{\partial(u, x, y, \bar{y})}{\partial(\alpha_1, \alpha_2, s, t)} d\alpha_1 \wedge d\alpha_2 \wedge ds \wedge dt \\
&= \left(+ \frac{\partial(u, x, y)}{\partial(\alpha_1, \alpha_2, s)} \frac{\partial \bar{y}}{\partial t} - \frac{\partial(u, x, y)}{\partial(\alpha_1, \alpha_2, t)} \frac{\partial \bar{y}}{\partial s} + \frac{\partial(u, x, y)}{\partial(\alpha_1, s, t)} \frac{\partial \bar{y}}{\partial \alpha_2} - \frac{\partial(u, x, y)}{\partial(\alpha_2, s, t)} \frac{\partial \bar{y}}{\partial \alpha_1} \right) \\
&= 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad d\alpha_1 \wedge d\alpha_2 \wedge ds \wedge dt
\end{aligned}$$

since by assumption (1) holds and u, x, y are all holomorphic in $s + it$, *i.e.* satisfy $\partial u/\partial t = i(\partial u/\partial s), \ldots, \partial y/\partial t = i(\partial y/\partial s)$, and the first derivatives with respect to α_1, α_2 are also holomorphic in $s + it$. As $J \neq 0$ in $t \neq 0$ we conclude

$$\frac{\partial u}{\partial \bar{x}} = 0 \qquad \text{for } t \neq 0\,.$$

Similarly, $\partial u/\partial \bar{y} = 0$ for $t \neq 0$. Hence $u = f(x, y)$ in $t \neq 0$ with f holomorphic for $\operatorname{Im} x \neq 0$. As $\operatorname{Im} x \to 0$ we have $t \to 0$, $\operatorname{Im} y \to 0$ and u, x, y tend to their values for $t = 0$ uniformly on compact sets of α_1, α_2, s. This implies that $f(x, y)$ with x, y real is the limit of $f(x, y)$ as x, y tend from complex to real values. Moreover that part of the neighborhood of the origin of the x, y-space which is image of a neighborhood of the origin of α_1, α_2, s, t certainly contains the Cartesian product of

$$|\operatorname{Re} x| < \varepsilon\,, \quad |\operatorname{Re} y| < \varepsilon$$

with

$$0 \neq |\operatorname{Im} x| < \varepsilon\,, \quad \left| \frac{\operatorname{Im} y}{\operatorname{Im} x} - \frac{y_s}{x_s} \right| < \varepsilon$$

with $\varepsilon > 0$ and small, *i.e.* the products of a square E of the $\operatorname{Re} x$, $\operatorname{Re} y$-plane with an open « cone » W of the $\operatorname{Im} x$, $\operatorname{Im} y$-plane (truncated by $\operatorname{Im} x = \varepsilon$), vertex at $(0, 0)$, and with its negative, $-W$. Therefore we may apply the local version of the edge-of-the-wedge theorem [1] which tells that $f(x, y)$ is holomorphic also for real x, y, q.e.d.

2. – Theorem 1 contains the special case when $x = \alpha_1$, $y = \alpha_2$ for $s = 0$. We then have $u(\alpha_1, \alpha_2, 0) = f(\alpha_1, \alpha_2)$ with analytic f.

Note that (1) (with the aid of (2)) can be formulated thus: x, y, u are solutions of

$$\frac{\partial v}{\partial s} = A_1 \frac{\partial v}{\partial \alpha_1} + A_2 \frac{\partial v}{\partial \alpha_2} \tag{4}$$

with

$$A_1 = \frac{\partial(x, y)}{\partial(s_1, \alpha_2)} \Big/ \frac{\partial(x, y)}{\partial(\alpha_1, \alpha_2)}, \qquad A_2 = \frac{\partial(x, y)}{\partial(\alpha_1, s)} \Big/ \frac{\partial(x, y)}{\partial(\alpha_1, \alpha_2)}.$$

This suggests the following corollary of Theorem 1.

THEOREM 2. *Let $A_1(\alpha_1, \alpha_2, s)$, $A_2(\alpha_1, \alpha_2, s)$ be real valued analytic functions of α_1, α_2, s, extensible holomorphically as functions of $s + it$, $|t| < t_0$. Let v be a C^1-solution of (4) which can be extended to a C^1-function of $s + it, \alpha_1, \alpha_2$, holomorphic in $s + it$ for $|t| < t_0$. Then v is holomorphic in all three variables α_1, α_2, s, provided $A_1(\partial A_2/\partial s) - A_2(\partial A_1/\partial s) \neq 0$.*

PROOF. There exist by Cauchy-Kovalewski two solutions x, y of (4) which reduce to $x = \alpha_1$, $y = \alpha_2$ for $s = 0$. We find for $s = 0$, if w.l.o.g., $A_1 = \partial x/\partial s \neq 0$,

$$\frac{\partial x}{\partial \alpha_1} = 1, \quad \frac{\partial x}{\partial \alpha_2} = 0, \quad \frac{\partial y}{\partial \alpha_1} = 0, \quad \frac{\partial y}{\partial \alpha_2} = 1, \quad \frac{\partial x}{\partial s} = A_1, \quad \frac{\partial y}{\partial s} = A_2,$$

$$\frac{\partial^2 x}{\partial s^2} = \frac{\partial A_1}{\partial s} + \sum_2^1 A_j \frac{\partial A_1}{\partial \alpha_j}, \qquad \frac{\partial^2 y}{\partial s^2} = \frac{\partial A_2}{\partial s} + \sum_1^2 A_j \frac{\partial A_2}{\partial \alpha_j}$$

so that (3) becomes for $s = 0$

$$\begin{vmatrix} x_s & y_s & \left(\dfrac{y_s}{x_s}\right)_s \\ x_{\alpha_1} & y_{\alpha_1} & \left(\dfrac{y_s}{x_s}\right)_{\alpha_1} \\ x_{\alpha_2} & y_{\alpha_2} & \left(\dfrac{y_s}{x_s}\right)_{\alpha_2} \end{vmatrix} = x_s^{-2} \begin{vmatrix} A_1 & A_2 & y_{ss}A_1 - x_{ss}A_2 \\ 1 & 0 & A_1(A_2)_{\alpha_1} - A_2(A_1)_{\alpha_1} \\ 0 & 1 & A_1(A_2)_{\alpha_2} - A_2(A_1)_{\alpha_2} \end{vmatrix} =$$

$$= x_s^{-2}\left(A_1 \frac{\partial A_2}{\partial s} - A_2 \frac{\partial A_1}{\partial s}\right) \neq 0$$

so that Theorem 1 applies. Hence v is a holomorphic function of x, y which in turn are analytic in α_1, α_2, s as Cauchy-Kowalevski solutions of (4). Thus v is holomorphic in α_1, α_2, s, q.e.d.

Dr. T. KAWAI has kindly communicated to the A. how to deduce Theorem 2 as a special case of a general analyticity Theorem to be found in [2].

REFERENCES

[1] W. RUDIN, *Lectures on the edge-of-the wedge Theorem*, Regional Conference Series in Mathematics 1971, p. 10.

[2] M. SATO - T. KAWAI - N. KASHIWARA, *Hyperfunctions and Pseudodifferential Equations*, Lectures Notes 287 (Springer 1973), p. 265 ff.