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JULIO E. BOUILLET

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Dirichlet Problem for Parabolic Equations with Continuous Coefficients. (*)

JULIO E. BOUILLET (**)

Summary. — L^p -boundary value problems for strongly parabolic operators such that the coefficients of the highest order derivatives are only uniformly continuous have been studied by V. A. Solonnikov [10]. In the case of a cylindre $\Omega \times (0, T)$, Ω a smooth spatial domain, and initial data zero, Solonnikov assumes the Dirichlet data to belong to a trace space. More precisely, if $L \equiv \sum_{|\alpha| \leq 2b} a_\alpha(P, t) D_\alpha^x - D_t$ is the strongly parabolic operator, w^k , $k = 0, 1, \dots, b-1$, the Dirichlet data, then the problem $Lu = 0$ in $\Omega \times (0, T)$, $D_N^k u = w^k$ at $\partial\Omega \times (0, T)$, $u = 0$ on Ω for $t = 0$, D_N indicating normal derivative to $\partial\Omega$, admits a unique solution in the space of functions whose spatial derivatives up to order $2b$, and the time derivative, belong in $L^p(\Omega \times (0, T))$, $p > 1$. Solonnikov observed that this implies that w^k must have spatial derivatives up to order $2b-1-k$, and a « fractional » time derivative of order $(2b-1-k)/2k$ in $L^p(\partial\Omega \times (0, T))$. Moreover, a spatial derivative of order $2b-1-k$ of w^k will have a « fractional » derivative in the spatial direction of order $1/p'$ and in the time direction of order $1/2bp'$. With this information let us denote, for right now, the space of w^k by $\dot{L}_{(2b-1-k+1/p'), (2b-1-k+1/p')/2b}^p(\partial\Omega \times (0, T))$; in the present work we find a class of existence and uniqueness to the problem above with the assumption that $w^k \in \dot{L}_{(b-1-k+\epsilon), (b-1-k+\epsilon)/2b}^p(\partial\Omega \times (0, T))$. Here $\epsilon < 1$ is an arbitrary but fixed positive number. This means that we have reduced the smoothness requirements on the data w^k by at least b derivatives in the space direction, and $b/2b = \frac{1}{2}$ in the time direction. In a subsequent paper we shall discuss the non-zero initial value case. Outline: the definitions and notation appear in I, §§ 1-5, for the case of a half-space, and in VIII, § 24 for the bounded domain Ω . The problem in a half-space is treated in I-VI, using certain surface and volume potentials (III-IV). Using the half-space results, we obtain an elliptic a priori estimate (VII) in the half-space. The problem in a general domain is studied in VIII for the parabolic case (a priori parabolic estimate: Theorem, § 25; existence and uniqueness theorems: Theorem 1 and Theorem 2, § 27). In IX an a priori estimate for the strongly elliptic case is derived. This work is part of a modified version of our Dissertation, under the direction of Professor Eugene B. Fabes. We wish to thank Professor Fabes for many invaluable talks and advise.

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(**) Departamento de Ciencias Exactas, Universidad Nacional de Salta, Salta (FGMB), Argentina.

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I. – Definitions and notations for the problem in the half-space.

§ 1. – Points in R^n will be denoted by the letters x, z, w , while y, v, η, t, r, s will denote real numbers, the last three referring to time. $R_+^{n+1} = R^n \times (0, \infty)$ will be the spatial domain (half-space), with points denoted by $(x, y), (z, v)$. The differential operators will be defined for functions in the « cylinder » $R_+^{n+1} \times (0, T)$, whose points are (x, y, t) , t denoting boundary of this « cylinder » is $S_T = R^n \times \{0, T\}$.

The following notations are standard: $f * g$ for the convolution of the functions f and g , $\mathcal{F}(f)(\cdot)$, occasionally also $\hat{f}(\cdot)$, the Fourier transform $\int f(\xi) \exp[i\langle \xi, \cdot \rangle] d\xi$, where $\langle \cdot, \cdot \rangle$ denotes scalar product of vectors. We will write, e.g. $\mathcal{F}_z(f)$ to specify the transformation in the variable z .

With $x = (x_1, \dots, x_n)$ and $\alpha = (\alpha_1, \dots, \alpha_n)$, α_i nonnegative integers, we set

$$x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}, \quad |\alpha| = \alpha_1 + \dots + \alpha_n,$$

$$D_x^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \dots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} = D_{x_1}^{\alpha_1} \dots D_{x_n}^{\alpha_n}, \quad \Delta = \Delta_x = \sum_{i=1}^n D_{x_i}^2.$$

when there is no confusion we will use this notation to include $\alpha = (\alpha_1, \dots, \alpha_n, \alpha_{n+1})$, and write $D_{x,y}^\alpha$ for $D_x^{(\alpha_1, \dots, \alpha_n)} D_y^{\alpha_{n+1}}$. This will only apply to space variables.

We will denote by $X_D(\cdot)$ the characteristic function of the set D .

§ 2. – DEFINITION. A parabolic singular integral operator is an operator of the form (cf. [3], [4])

$$Kf(x, t) = \alpha(x, t)f(x, t) + \lim_{\varepsilon \rightarrow 0^+} \int_0^{t-\varepsilon} \int_{R^n} k(x, t; x-z, t-s)f(z, s) dz ds,$$

the limit in $L^p(R^n \times (0, T))$, where $\alpha(x, t)$ is a bounded measurable function on $R^n \times (0, T)$, and the variable kernel $k(x, t; z, s)$ is defined for $t \in (0, T)$, $s > 0$ as

$$k(x, t; z, s) = \frac{\Omega(x, t; z/s^{1/2b})}{s^{1+n/2b}},$$

where

- (1) $\Omega(x, t; \cdot) \in \mathcal{S}(R^n)$ = space of rapidly decreasing functions.
- (2) $\int_{R^n} \Omega(x, t; w) dw = 0$,
- (3) $\sup_{(y,s) \in S_T} \left(\int_{R^n} |w^\alpha D_w^\beta \Omega(y, s; w)|^2 dw \right)^{\frac{1}{2}} < \infty$ (see, e.g. [6]).

If in the above definition $a(x, t)$ is a constant and $k(x, t; z, s) = k(z, s)$ is independent of (x, t) , the operator Kf will be said to be of convolution type.

We define the symbol of K to be the function

$$\sigma(K)(x, t; z, s) = a(x, t) + \int_0^\infty \frac{\mathcal{F}_w(\Omega)(x, t; zr^{1/2b})}{r} \exp[itr] dr.$$

K is known to be a continuous mapping of $L^p(R^n x(0, T))$ for $1 < p < \infty$. Its properties (cf. [2], [3], [4]) will be assumed here.

Following [3], [5] we will denote by $\mathfrak{J}_p(S_T)$, $1 < p < \infty$, the class of operators $J: L^p(S_T) \rightarrow L^p(S_T)$ satisfying for any $a \geq 0$

$$(i) \quad JX_{(a, \infty)} = X_{(a, \infty)} JX_{(a, \infty)},$$

$$(ii) \quad \|X_{(a, a+\varepsilon)} J(X_{(a, a+\varepsilon)} f)\|_{L^p(S_T)} \leq \omega(\varepsilon) \cdot \|X_{(a, a+\varepsilon)} f\|_{L^p(S_T)},$$

where $\omega(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$, uniformly in $a \geq 0$. We set $\mathfrak{J}(S_T) = \bigcap_p \mathfrak{J}_p(S_T)$. $\mathfrak{J}_p(R_+^{n+1} x(0, T))$, $\mathfrak{J}(R_+^{n+1} x(0, T))$ are defined the same way.

§ 3. - We introduce the fundamental solution of the operator $(-1)^b \Delta_x^b + D_t$ on S_T , defined by

$$\begin{aligned} F(x, t) &= \mathcal{F}_z(\exp[-|z|^{2b} t])(x) \quad \text{for } t > 0, \\ &= 0 \quad \text{for } t \leq 0. \end{aligned}$$

DEFINITION. For β real > 0 ,

$$\begin{aligned} (\Lambda^{-\beta} f)(x, t) &= (\Lambda^{-\beta} * f)(x, t) = \frac{1}{\Gamma(\beta/2b)} t^{\beta/2b-1} F(x, t) * f(x, t) = \\ &= \frac{1}{t^{1+(n-\beta)/2b}} \Omega_\beta(x/t^{1/2b}) * f(x, t), \quad \Omega_\beta(\cdot) \in \mathcal{S}. \end{aligned}$$

For $0 < \beta \leq 2b$, $\Lambda^{-\beta}(x, t)$ is a tempered distribution on R^{n+1} , whose Fourier transform is $(|x|^{2b} - it)^{-\beta/2b}$. Also if $f \in L^p(S_T)$, $D_x^\alpha \Lambda^{-\beta} f \in L^p(S_T)$ for $|\alpha| \leq [\beta] = \text{largest integer} \leq \beta$, and $D_t \Lambda^{-2b+\beta} f \in L^p(S_T)$. We set $\Lambda^0 f = f$ and proceed to define Λ^β .

DEFINITION. For $0 < \beta \leq 2b$,

$$\Lambda^\beta f = ((-1)^b \Delta_x^b + D_t) \Lambda^{-2b+\beta} f$$

Λ^β is well defined on rapidly decreasing functions f , which vanish for $t \leq 0$. As tempered distributions, $\mathcal{F}(\Lambda^\beta f) = (|x|^{2b} - it)^{\beta/2b} \cdot \hat{f}$. For β integral, $(-1)^b \Lambda^b \Lambda^{-2b+\beta} f$ can also be written [3] as $\sum_{|\alpha|=\beta} K^\alpha D_x^\alpha f$, with K^α a parabolic singular integral operator with symbol

$$\sigma(K_\alpha) = i^{2b-\beta} \frac{P^\alpha(x)}{(|x|^{2b} - it)^{1-\beta/2b}},$$

$P_\alpha(x)$ defined by $|x|^{2b} = \sum_{|\alpha|=\beta} P_\alpha(x) x^\alpha$: considering $(-1)^b \Lambda^b \Lambda^{-2b+\beta}$ as a tempered distribution on R^{n+1} this decomposition is clear if we recall that

$$\mathcal{F}(\Lambda^{-2b+\beta}) = \frac{1}{(|x|^{2b} - it)^{1-\beta/2b}}.$$

If $f = f(z, \eta, s; x, y, v, t)$, z, η, s, y , and v being parameters, we introduce the notation $\Lambda^\beta f(z, \eta, s; x, y, v, t)$ to mean $[\Lambda^\beta f(z, \eta, s; \cdot, y, v, \cdot)](x, t)$.

§ 4. — For $\delta \geq 0$, $L_{2b,1}^p(R^n \times (\delta, \infty) \times (0, T))$ is the space of functions whose derivatives $D_{x,v}^\alpha u$, for $|\alpha| \leq 2b$, and $D_t u$ in the sense of distributions are given by functions belonging to $L^p(R^n \times (\delta, \infty) \times (0, T))$. $L_{2b,1}^p$ is a Banach space with the norm

$$\|u\|_{L_{2b,1}^p(R^n \times (\delta, \infty) \times (0, T))} = \sum_{|\alpha| \leq 2b} \|D_{x,v}^\alpha u\|_{L^p(R^n \times (\delta, \infty) \times (0, T))} + \|D_t u\|_{L^p(R^n \times (\delta, \infty) \times (0, T))}.$$

For $\delta = 0$ we write $L_{2b,1}^p(R_+^{n+1} \times (0, T))$.

$\hat{L}_{2b,1}^p(R^n \times (\delta, \infty) \times (0, T))$ denotes the space of functions $u \in L_{2b,1}^p(R^n \times (\delta, \infty) \times (0, T))$ which are limits in $L_{2b,1}^p(R^n \times (\delta, \infty) \times (0, T))$ of functions $\in C_0^\infty(R^{n+1} \times (0, \infty))$.

§ 5. — A linear differential operator

$$L \equiv \sum_{|\alpha| \leq 2b} a_\alpha(x, y, t) D_{x,v}^\alpha - D_t$$

is said to be parabolic in the sense of Petrovski if

$$\operatorname{Re} \left(\sum_{|\alpha|=2b} a_\alpha(x, y, t) (i\xi)^\alpha \right) < -\pi |\xi|^{2b}, \quad \text{for } 0 \neq \xi \in R^{n+1},$$

$\pi > 0$ independent of (x, y, t) in $R_+^{n+1} \times (0, T)$. Each $a_\alpha(x, y, t)$ is assumed to be measurable, bounded, and for $|\alpha| = 2b$, uniformly continuous in $R_+^{n+1} \times (0, T)$.

We will introduce a distance function: for given $\gamma, 0 < \gamma < 1$, and $1 < p < \infty$, let γ_p be a number such that

$$1 - 1/p \leq \gamma_p < \frac{2b}{2b - \gamma} (1 - 1/p).$$

We define $d_p(y, t) = \min(y, t^{p/2b})$.

Throughout this work C will denote a constant, not necessarily the same at each occurrence. The connection between C and other parameters (eg. parameter of parabolicity, dimension, etc.) will be made explicit when relevant.

We will also let $\psi(r)$ denote any function of the form: constant $\cdot \exp[-\text{constant} \cdot r]$, the constants and r being real and positive. When related to a solution of the operator L , these constants will depend only on the parameter of parabolicity π and on the $\max_{|\alpha| \leq 2b} \sup |a_\alpha(x, y, t)|$.

II. - The parametrix.

§ 6. - We will construct a kernel for a generalized volume potential following [5]. We consider first a differential operator with constant coefficients

$$L_0 \equiv \sum_{|\alpha|=2b} a_\alpha D_{xy}^\alpha - D_t.$$

Let $F(x, y, t) = \mathcal{F}_\xi \left(\exp \sum_{|\alpha|=2b} a_\alpha (i\xi)^\alpha t \right) (x, y)$, $\xi \in R^{n+1}$, be the fundamental solution of L_0 . We construct a function $G_0(x, y, v, t)$, $y, v > 0$, satisfying as a function of (x, y, t) ,

- (i) $G_0(\cdot, \cdot, v, \cdot) \in \dot{L}_{2b,1}^p(R^n \times (\delta, \infty) \times (0, T))$ for every $\delta > 0$, $1 < p < \infty$,
- (ii) $L_0 G_0(x, y, v, t) = 0$ for $y > 0$,
- (iii) for $k = 0, \dots, b-1$,

$$\lim_{y \rightarrow 0^+} \Lambda^{-k} (D_y^k G_0)(x, y, v, t) = [\Lambda^{-k} D_y^k F(\cdot, -v, \cdot)](x, t),$$

the limit taken in $L^p(S_T)$.

It is known [3], that G_0 can be written

$$G_0(x, y, v, t) = \sum_{k,j=0}^{b-1} \int_0^t \int_{R^n} \Lambda^{2b-1-k} D_y^k F(x-z, y, t-s) \cdot T_{k,j}[\Lambda^{-1}(D_y^j F)(\cdot, -v, \cdot)](x, t)](z, s) dz ds,$$

where $(T_{k,j})$ is a $b \times b$ matrix of parabolic singular integral operators (of convolution type).

Considered as a function of x, v, t for $y > 0$, $G_0(x, y, v, t)$ is also a solution of the boundary value problem

$$(i') \quad G_0(\cdot, y, \cdot) \in \tilde{L}_{2b,1}^p(R^n \times (\delta, \infty) \times (0, T)) \text{ for every } \delta > 0, 1 < p < \infty, \\ y > 0,$$

$$(ii') \quad \left(\sum_{|\alpha| \leq 2b} a_\alpha D_x^{\alpha_1 \dots \alpha_n} (-D_v)^{\alpha_{n+1}} - D_t \right) G_0(x, y, v, t) = 0 \text{ for } v > 0,$$

$$(iii') \text{ for } l = 0, \dots, b-1,$$

$$\lim_{v \rightarrow 0^+} \text{ in } L^p(S_T) A^{-1} D_v^l G_0(x, y, v, t) = (-1)^l A^{-1} D_y^l F(x, y, t).$$

Since for $y > 0 \lim_{v \rightarrow 0^+} D_v^l G_0$ exists in $L^p(S_T)$, (iii') implies that for $l = 0, \dots, b-1$,

$$D_v^l G_0(x, y, 0, t) = (-1)^l D_y^l F(x, y, t) \quad \text{for every } y > 0.$$

We now introduce the function

$$K(x, y, v, t) = F(x, y - v, t) - G_0(x, y, v, t).$$

For $l = 0, \dots, b-1$,

$$D_v^l K(x, 0, v, t) = 0, \quad v > 0, \quad \text{and} \quad D_y^l K(x, y, 0, t) = 0, \quad y > 0$$

§ 7. – The proof of the following theorem is long and technical in nature, and will not be included here.

THEOREM. For $y > 0$, $v > 0$, $y \neq v$, and $B > 0$

$$(i) \quad |D_x^\alpha D_y^j D_v^h A^B F(x, y - v, t)| \leq \frac{\psi(|x|/t^{1/2b}) \psi(|y - v|/t^{1/2b})}{t^{(n+1+|\alpha|+j+h+B)/2b}},$$

$$(ii) \quad |D_x^\alpha D_y^j D_v^h A^B G_0(x, y, v, t)| \leq \frac{\psi(|x|/t^{1/2b}) \psi(y/t^{1/2b}) \psi(v/t^{1/2b})}{t^{(n+1+|\alpha|+j+h+B)/2b}},$$

$$(iii) \quad |D_x^\alpha D_y^j D_v^h A_\Delta^B K(x, y, v, t)| \leq \frac{\psi(|x|/t^{1/2b}) \psi(|y - v|/t^{1/2b})}{t^{(n+1+|\alpha|+j+h+B)/2b}}.$$

Clearly, (iii) follows from (i) and (ii).

We shall consider the operator with constant coefficients L_{0zrs} ,

$$(L_{0zrs} u)(x, y, t) \equiv \sum_{|\alpha| = 2b} a_\alpha(z, r, s) D_{xy}^\alpha u(x, y, t) - D_t u(x, y, t).$$

Let $F(z, r, s; x, y, t)$, $G_0(z, r, s; x, y, v, t)$, and $K(z, r, s; x, y, v, t)$ denote the fundamental solution F , and the functions G_0 and K (introduced in § 6) which are associated with the operator L_{0zrs} , $z \in R_n$, r, s , and v being real parameters. Clearly, these functions are solutions of the equation $L_{0zrs}u = 0$.

III. — Estimates for some surface potentials.

§ 8. — For $\Phi(x, t) \in L^p(S_T)$, $1 < p < \infty$, $j = 0, 1, \dots, b-1$, $0 < \mu < 1$, we introduce the following potentials

$$T_{\mu j} \Phi(x, y, t) = \int_0^t \int_{R^n} A^{b-j-\mu} D_y^j F(z, 0, s; x-z, y, t-s) \Phi(z, s) dz ds.$$

LEMMA 1. For $\Phi \in L^p(S_T)$, $1 < p < \infty$,

- (i) $\|D_{x,y}^\alpha A^\mu T_{\mu j} \Phi(\cdot, y, \cdot)\|_{L^p(S_T)} \leq C \cdot T^{(b-1-|\alpha|)/2b} \|\Phi\|_{L^p(S_T)}$ for $|\alpha| < b-1$,
- (ii) $\|A^{b-1+\mu-k} D_y^k T_{\mu j} \Phi(\cdot, y, \cdot)\|_{L^p(S_T)} + \sum_{|\alpha|=b-1} \|D_{x,y}^\alpha A^\mu T_{\mu j} \Phi(\cdot, y, \cdot)\|_{L^p(S_T)} \leq C \cdot \|\Phi\|_{L^p(S_T)}$,
- (iii) $y^{|\alpha|-(b-1)} \|D_{x,y}^\alpha A^\mu T_{\mu j} \Phi(\cdot, y, \cdot)\|_{L^p(S_T)} \leq C \cdot \|\Phi\|_{L^p(S_T)}$ for $|\alpha| > b-1$.
- (iv) For $|\alpha| \leq b-1$,

$$\|D_{x,y}^\alpha A^\mu T_{\mu j} \Phi\|_{L^p(R_+^{n+1} \times (0, T))} \leq C \cdot T^{(b-1-|\alpha|+1/p)/2b} \cdot \|\Phi\|_{L^p(S_T)}.$$

PROOF. By the estimates on F , § 7, and Young's inequality in $dx dt$, we have

$$\|D_{x,y}^\alpha A^\mu T_{\mu j} \Phi(\cdot, y, \cdot)\|_{(L^p S_T)} \leq C \cdot \left\{ y^{b-1-|\alpha|} \int_0^{T/y^{2b}} \frac{\Psi(1/s^{1/2b})}{s^{(b+1+|\alpha|)/2b}} ds \right\} \|\Phi\|_{L^p S_T}.$$

When $|\alpha| < b-1$ the expression in brackets is bounded by $C \cdot T^{(b-1-|\alpha|)/2b}$. When $|\alpha| > b-1$, the integral in ds is finite and bounded independently of y . In both case (i) and case (iii), the constants C depend on $|\alpha|$, the parameter of parabolicity π , and the $\max_{|\alpha|=2b} \sup |a_\alpha(x, y, t)|$.

The discussion above hints that in the case (ii), for $|\alpha| = b-1$ we will find the singularity of a parabolic singular operator. In fact, (ii) is a consequence of the theory of parabolic singular integrals with variable kernel (see [3]). (iv) is obtained by taking L^p -norms in dy on both sides of the estimate above.

The proof of the following Lemma is straightforward

LEMMA 2. If $\Phi \in L^p(S_T)$, $1 < p < \infty$, and $\gamma - 1/p < \mu$, then

$$\|y^{b+1-\gamma} L(T_{\mu}; \Phi)(x, y, t)\|_{L^p(R_+^{n+1} \times (0, T))} \leq \omega(T) \cdot \|\Phi\|_{L^p(S_T)},$$

where $\omega(T) \rightarrow 0$ as $T \rightarrow 0^+$.

IV. — Estimates for the volume potential.

§ 9. — We shall study a volume potential whose kernel is the function $K(z, \eta, s; x, y, v, t)$ (cf. § 6).

DEFINITION. For $f \in L^p(R_+^{n+1} \times (0, T))$, $1 < p < \infty$,

$$V_f(x, y, t) = \int_0^t \int_{R_+^{n+1}} K(z, v, s; x - z, y, v, t - s) f(z, y, s) dz dv ds.$$

THEOREM. If $f \in L^p(R_+^{n+1} \times (0, T))$, $1 < p < \infty$, $0 \leq \beta < \gamma - 1/p$, $\gamma < 1$, then

- (i) for $k = 0, \dots, b-1$, $\|A^{b-1+\beta-k} D_y^k V_f(\cdot, y, \cdot)\|_{L^p(S_T)} \rightarrow 0$ as $y \rightarrow 0^+$;
- (ii)
$$\begin{aligned} & \sum_{k=0}^{b-1} \|A^{b-1-k+\beta} D_y^k V_f(\cdot, y, \cdot)\|_{L^p(S_T)} + \sum_{|\alpha| \leq b-1} \|D_{x,y}^\alpha A^\beta V_f(\cdot, y, \cdot)\|_{L^p(S_T)} + \\ & + \sum_{\substack{b \leq |\alpha| \leq 2b-1}} \|d_p(y, \cdot)^{|\alpha|-(b-1)} D_{x,y}^\alpha A^\beta V_f(\cdot, y, \cdot)\|_{L^p(S_T)} \leq \\ & \leq C \cdot T^{\gamma'/2b} \cdot \|d_p(\cdot, \cdot)^{b+1-\gamma} f(\cdot, \cdot, \cdot)\|_{L^p(R_+^{n+1} \times (0, T))}, \end{aligned}$$

the constant depending on π , $\max_{|\alpha|=2b} \sup |a_\alpha|$, and p , and $\gamma'' > 0$ depending on γ, γ_p, b and a number $\gamma' < \gamma - \beta - 1/p$.

$d_p(y, t) = \min(y, t^{\gamma'/2b})$ is the distance function introduced in § 5.

PROOF. We recall that $D_v^l K(z, \eta, s; x, 0, v, t) = 0$ for $l \leq b-1$ (§ 6). We may therefore write, for $0 \leq l \leq b-1$

$$K(z, \eta, s; x, y, v, t) = \frac{y^{l+1}}{l!} \int_0^1 (1-\lambda)^l D_v^{l+1} K(z, \eta, s; x, \lambda y, v, t) d\lambda.$$

A similar remark applies to $D_v^l K$.

Part (i) is a direct consequence of the techniques used below, applied to

$$A^{b-1+\beta-k} D_y^k V_f = \frac{y^{b-k}}{(b-k-1)!} \int_0^t \int_{R_+^{n+1}} \int_0^1 A^{b-1+\beta-k} D_y^b \cdot$$

$$\cdot K(z, v, s; x - z, \lambda y, v, t - s) (1 - \lambda)^{b-k-1} d\lambda f(z, v, s) dz dv ds.$$

PROOF OF (ii). We prove this in three Lemmas. In Lemmas *A* and *B* (§ 10) we prove the estimate for $d_p = y$. In Lemma *C* (§ 11) we show the estimate for $d_p = t^{\gamma p/2b}$. For the general case we set

$$f(x, y, t) = X_{(y > t^{\gamma p/2b})} \cdot f(x, y, t) + X_{(y \geq t^{\gamma p/2b})} \cdot f(x, y, t) = f_1 + f_2$$

and apply to each potential V_{f_1} , V_{f_2} the corresponding estimate.

§ 10. - LEMMA A, (i). For $k = 0, \dots, b-1$, $\beta + \gamma' < \gamma - 1/p$, $\gamma' > 0$,

$$\|A^{b-1+\beta-k} D_y^k V_f(\cdot, y, \cdot)\|_{L^p(S_T)} \leq C \cdot T^{\gamma'/2b} \cdot \min(1, y^{\gamma-\beta-\gamma'-1/p}) \cdot \|v^{b+1-\gamma} f(z, v, s)\|_{L^p(R_+^{n+1} \times (0, T))}.$$

(ii) For $b-1 < |\alpha| \leq 2b-1$,

$$y^{|\alpha|-(b-1)} \|D_{x,y}^\alpha A^\beta V_f(\cdot, y, \cdot)\|_{L^p(S_T)} \leq C \cdot T^{\gamma'/2b} \cdot \min(1, y^{\gamma-\beta-\gamma'-1/p}) \cdot \|v^{b+1-\gamma} f(z, v, s)\|_{L^p(R_+^{n+1} \times (0, T))}.$$

PROOF. Clearly,

$$A^{b-1+\beta-k} D_y^k V_f(x, y, t) \leq \left(\int_0^{y/2} + \int_{y/2}^\infty \right) \int_0^t \int_{R^n} |A^{b-1+\beta-k} D_y^k \cdot K(z, v, s; x-z, y, v, t-s) \cdot f(z, v, s)| dz dv ds.$$

We estimate first the term $\int_0^{y/2}$. It can be written

$$\int_0^{y/2} \int_0^t \int_{R^n} \left| A^{b-1+\beta-k} D_y^k \int_0^1 \frac{(1-\lambda)^b}{(b-1)!} D_v^b K(z, v, s; x-z, y, \lambda v, t-s) d\lambda \cdot v^b \cdot f(z, v, s) \right| dz dv ds.$$

Applying the estimates for K (§ 7) and Young's inequality, and observing that $|y - \lambda v| \geq y/2$, we get

$$\left\| \int_0^{y/2} \right\|_{L^p(S_T)} \leq C \int_0^{y/2} \int_0^T \frac{\psi(y/s^{1/2b})}{s^{1+\beta/2b}} \left(\frac{T}{s}\right)^{\gamma'/2b} ds \cdot v^{b+1-\gamma} \|f(\cdot, v, \cdot)\|_{L^p(S_T)} \frac{dv}{v^{1-\gamma}},$$

where we have introduced the factor $(T/s)^{\gamma'/2b} > 1$, $\beta + \gamma' \leq \gamma - 1/p$. Applying now Hölder inequality in dv we obtain the desired estimate for $\int_0^{y/2}$, with right-hand side

$$C \cdot T^{\gamma'/2b} \cdot y^{\gamma-\beta-\gamma'-1/p} \cdot \int_0^\infty \frac{\psi(1/s^{1/2b})}{s^{1+(\beta+\gamma')/2b}} ds \cdot \|v^{b+1-\gamma} f(z, v, s)\|_{(L^p(R_+^{n+1} \times (0, T)))}.$$

A similar argument gives the estimate (ii) for the corresponding $\int_0^{y/2}$.

For the term $\int_{y/2}^{\infty}$ in (i), we observe that

$$D_y^k K = \int_0^1 (1-\mu)^{k-1} v^k D_v^k \int_0^1 (1-\lambda)^{b-k-1} D_y^b K(\dots; x-z, \lambda y, \mu v, t-s) d\lambda d\mu.$$

Therefore,

$$\begin{aligned} \left\| \int_{y/2}^{\infty} \right\|_{L^p(S_T)} &\leq C \cdot y^{b-k} \int_{y/2}^{\infty} dv \int_0^1 d\mu \int_0^1 d\lambda \int_0^T \frac{\psi(|\lambda y - \mu v|/s^{1/2b})}{s^{1+\beta/2b}} \left(\frac{T}{s}\right)^{\gamma'/2b} ds \cdot \\ &\quad \cdot \frac{v^k y^{b-k+1-\gamma}}{y^{b-k+1-\gamma}} \cdot \|f(\cdot, v, \cdot)\|_{(L^p S_T)}. \end{aligned}$$

Applying Hölder inequality in $dv d\mu d\lambda$ we get

$$\begin{aligned} \left\| \int_{y/2}^{\infty} \right\|_{L^p(S_T)} &\leq C \cdot T^{\gamma'/2b} \cdot y^{b-k} \cdot \left(\int_{y/2}^{\infty} \int_0^1 \int_0^1 \frac{dv d\lambda d\mu}{\{|\lambda y - \mu v|^{\beta+\gamma'} v^{b-k+1-\gamma}\}^{p/(p-1)}} \right)^{1-1/p} \cdot \\ &\quad \cdot \|v^{b+1-\gamma} f(z, v, s)\|_{L^p(R_+^{n+1} \times (0, T))} \\ &\leq C \cdot T^{\gamma'/2b} \frac{y^{b-k} y^{1-1/p}}{y^{\beta+\gamma'+b-k+1-\gamma}} \cdot \left\{ \int_{\frac{1}{2}}^{\infty} \int_0^1 \int_0^1 \frac{dv d\lambda d\mu}{[|\lambda - \mu v|^{\beta+\gamma'} v^{b-k+1-\gamma}]^{p/(p-1)}} \right\}^{1-1/p} \cdot \\ &\quad \cdot \|v^{b+1-\gamma} f(z, v, s)\|. \end{aligned}$$

For the term $\int_{y/2}^{\infty}$ in (ii) we have

$$\begin{aligned} \left\| \int_{y/2}^{\infty} \right\|_{L^p(S_T)} &\leq C \int_{y/2}^{\infty} \int_0^1 v^{2b-1-|a|} \int_0^T \frac{\psi(|y-\lambda v|/s^{1/2b})}{s^{1+\beta/2b}} \left(\frac{T}{s}\right)^{\gamma'/2b} ds \cdot \frac{v^{b+1-\gamma} \|f(\cdot, v, \cdot)\|_{L^p(S_T)} dv d\lambda}{v^{b+1-\gamma}} \\ &\leq \frac{C \cdot T^{\gamma'/2b} y^{1-1/p}}{y^{|a|-b} y^{2-\gamma+\beta+\gamma'}} \left\{ \int_{\frac{1}{2}}^{\infty} \int_1^0 \frac{dv d\lambda}{[|1-\lambda v|^{\beta+\gamma'} v^{2-\gamma}]^{p/(p-1)}} \right\}^{1-1/p} \cdot \\ &\quad \cdot \|v^{b+1-\gamma} f(z, v, s)\|_{L^p(R_+^{n+1} \times (0, T))}. \end{aligned}$$

For $|a| = 2b-1$, the integral $\int_0^1 d\lambda$ will not be present.

Observe that setting $\gamma' = \gamma - \beta - 1/p$ we have an estimate independent of $y^{\gamma-\beta-\gamma'-1/p}$. The proof is complete.

LEMMA B. For $|\alpha| \leq b-1$,

$$\|D_{x,v}^\alpha A^\beta V_f(\cdot, y, \cdot)\|_{L^p(S_T)} \leq C \cdot T^{\gamma'/2/b} \cdot \min(1, y^{\gamma-\beta-\gamma'-1/p}) \cdot \|v^{b+1-\gamma} f(z, v, s)\|_{L^p(R_+^{n+1} \times (0, T))}.$$

PROOF. Set $D_{x,v}^\alpha A^\beta = D_y^k D_x^\delta A^\beta = (D_x^\delta A^{-(b-1-k)} D_y^k A^{b-1+\beta-k}, |\delta| + k = |\alpha| \leq b-1$. The result follows from Lemma A and the fact that $D_x^\delta A^{-(b-1-k)}$ is an $L^p(S_T)$ operator.

§ 11. - LEMMA C. For $|\alpha| \leq b-1$, γ_p as in § 5, and

$$\gamma_p < (2b - \beta - 1/p)/(2b - \gamma),$$

$$\|D_{x,v}^\alpha A^\beta V_f(\cdot, y, \cdot)\|_{L^p(S_T)} \leq C_1(T) \cdot \|s^{(\gamma_p/2b)(b+1-\gamma)} f(z, v, s)\|_{L^p(R_+^{n+1} \times (0, T))},$$

and for $b \leq |\alpha| \leq 2b-1$,

$$\|t^{(\gamma_p/2b)(|\alpha|-(b-1))} D_{x,v}^\alpha A^\beta V_f(\cdot, y, \cdot)\|_{L^p(S_T)} \leq C_2(T) \cdot \|s^{(\gamma_p/2b)(b+1-\gamma)} f\|_{L^p(R_+^{n+1} \times (0, T))}.$$

PROOF. Case $|\alpha| \leq b-1$.

$$\|D_{x,v}^\alpha A^\beta V_f(\cdot, y, \cdot)\|_{L^p(R^n)} \leq C \int_0^t \frac{1}{(t-s)^{(1+|\alpha|+\beta)/2b}} \int_0^\infty \Psi(|y-v|/(t-s)^{1/2b}) \cdot \|f(\cdot, v, s)\| dv ds.$$

We apply Hölder's inequality in dv and observe that

$$\left(\int_0^\infty \Psi^p(|y-v|/(t-s)^{1/2b}) dv \right)^{1-1/p} \leq C \cdot (t-s)^{(1-1/p)/2b}.$$

Hence

$$\|D_{x,v}^\alpha A^\beta V_f(\cdot, y, t)\|_{L^p(R^n)} \leq C \int_0^\infty \frac{X_{(t,0)}(s) (T/t)^{1-\{|\alpha|+\beta+1/p+\gamma_p(b+1-\gamma)\}/2b}}{(t-s)^{(|\alpha|+\beta+1/p)/2b} s^{(\gamma_p/2b)(b+1-\gamma)}} \cdot (s^{(\gamma_p/2b)(b+1-\gamma)} \|f[\cdot, \cdot, s]\|_{L^p(R_+^{n+1})}) ds.$$

The power of (T/t) is positive, provided $\gamma_p < (b+1-(\beta+1/p))/(b+1-\gamma)$. Now the power of $(t-s)$ is clearly integrable, and $(\gamma_p/2b)(b+1-\gamma) + 1/p < 1$ due to the choice of γ_p . Hence by Hardy-Schur's Lemma (cf. [8], [9]), we obtain the desired inequality.

CASE $|\alpha| \geq b$. We proceed as in previous case, introducing now the factor $t^{(\gamma_p/2b)(|\alpha|-(b-1))}$, to obtain

$$\begin{aligned}
 (*) \quad & t^{(\gamma_p/2b)(|\alpha|-(b-1))} \|D_{xy}^\alpha A^\beta V_f(\cdot, y, t)\|_{L^n(R^n)} \leq \\
 & \leq C \int_0^\infty \frac{X_{(0,t)}(s)(T/t)^{1-\{|\alpha|+\beta+1/p+\gamma_p(2b-\gamma-|\alpha|)\}/2b} t^{(\gamma_p/2b)(|\alpha|-(b-1))}}{(t-s)^{(|\alpha|+\beta+1/p)/2b} s^{(\gamma_p/2b)(b+1-\gamma)}} \cdot \\
 & \quad \cdot (s^{(\gamma_p/2b)(b+1-\gamma)}) \|f(\cdot, \cdot, s)\|_{L^p(R_+^{n+1})} ds.
 \end{aligned}$$

We show that the power of (T/t) is positive by showing that the quantity in brackets is $< 2b$. The ratio $((2b-\beta-1/p)-|\alpha|)/((2b-\gamma)-|\alpha|)$ is an increasing function of $|\alpha|$; therefore its minimum value is attained at $|\alpha|=0$: $1 < (2b-\beta-1/p)/(2b-\gamma)$. As the function $(r-\beta-1/p)/(r-\gamma)$ is decreasing, the condition $\gamma_p < (2b-\beta-1/p)/(2b-\gamma)$ also implies $\gamma_p < (b+1-\beta-1/p)/(b+1-\gamma)$ (see Case $|\alpha| \leq b-1$). It is clear that for $1 < p < \infty$ and $0 < \gamma < 1$, a number γ_p satisfying

$$1-1/p < \gamma_p < \frac{2b}{2b-\gamma} (1-1/p) \quad \text{and} \quad \gamma_p < \frac{2b-\beta-1/p}{2b-\gamma}$$

will suit to our requirements for all $|\alpha| \leq 2b-1$.

Returning to the estimate, we have $\|t^{(\gamma_p/2b)(|\alpha|-(b-1))} D_{xy}^\alpha A^\beta V_f(\cdot, y, \cdot)\|_{L^p(S_T)} \leq$
 $\leq L^p$ -norm in t of the integral in the right hand side of $(*)$. The result follows now from Hardy's Lemma.

V. — The operator J .

§ 12. — We shall study a commutator for the operator

$$(Jf)(x, y, t) = \lim_{\varepsilon \rightarrow 0^+} \int_0^{t-\varepsilon} \iint_{R_+^{n+1}} (L_{xyt} K)(z, v, s; x-z, y, v, t-s) f(z, v, s) dz dv ds,$$

Consider

$$\begin{aligned}
 & [y^{b+1-\gamma} J(f) - J(v^{b+1-\gamma} f(z, v, s))](x, y, t) = \\
 & = \sum_{|\alpha|=2b} T_\alpha^0([a_\alpha(x, y, t) - a_\alpha(z, v, s)]f(z, v, s))(x, y, t) + \\
 & \quad + \sum_{|\alpha| < 2b} a_\alpha(x, y, t) T_\alpha^0(f(z, v, s))(x, y, t),
 \end{aligned}$$

where we have set

$$T_{\alpha}^0(f)(x, y, t) = \int_0^t \iint_{R_+^{n+1}} (y^{b+1-\gamma} - v^{b+1-\gamma}) D_{x,v}^{\alpha} K(z, v, s; x - z, y, v, t - s) \cdot f(z, v, s) dz dv ds.$$

We also set

$$T_{\alpha}^1(f)(x, y, t) = \int_0^t \iint_{R_+^{n+1}} (t^{(\gamma_p/2b)(b+1-\gamma)} - s^{(\gamma_p/2b)(b+1-\gamma)}) (D_{x,v}^{\alpha} K) f(z, v, s) dz dv ds,$$

T_{α}^1 being the corresponding operators for the commutator

$$[t^{(\gamma_p/2b)(b+1-\gamma)} J(f) - J(s^{(\gamma_p/2b)(b+1-\gamma)} f(z, v, s))](x, y, t).$$

LEMMA. Let $f \in L^p(R_+^{n+1} \times (0, T))$. For $0 \leq |\alpha| \leq 2b$, $1/p < \gamma < 1$, and $1 < p < \infty$,

- (i) $\|T_{\alpha}^0 f\|_{L^p(R_+^{n+1} \times (0, T))} \leq C \cdot T^{1-(|\alpha|/2b)} \|y^{b+1-\gamma} f(x, y, t)\|_{L^p(R_+^{n+1} \times (0, T))},$
- (ii) $\|T_{\alpha}^1 f\|_{L^p(R_+^{n+1} \times (0, T))} \leq C \cdot T^{1-(|\alpha|/2b)} \|t^{(\gamma_p/2b)(b+1-\gamma)} f(x, y, t)\|_{L^p(R_+^{n+1} \times (0, T))}.$

PROOF OF (i). We set as usual

$$|(T_{\alpha}^0 f)(x, y, t)| \leq \left(\int_0^{y/2} + \int_{y/2}^{\infty} \right) \int_0^t \int_{R^n} |y^{b+1-\gamma} - v^{b+1-\gamma}| \cdot |D_{x,v}^{\alpha} K| \cdot |f| dz dv ds.$$

To estimate the first term, we use the known properties of K together with Young's inequality in the variables z and s , and the fact that $v \leq y/2$ to obtain

$$\left\| \int_0^{y/2} \right\|_{L^p(S_T)} \leq C \cdot T^{1-(|\alpha|/2b)} \int_0^{\infty} \frac{X_{(0, y/2)}(v)}{y^{\gamma} \cdot v^{1-\gamma}} (v^{b+1-\gamma} \|f(\cdot, v, \cdot)\|_{L^p(S_T)}) dv.$$

The desired inequality follows from this one, applying Hardy's lemma ([8], [9]) and recalling that $\gamma > 1/p$.

For the second term, we apply the known estimate directly to K to get

$$\left\| \int_{y/2}^{2y} \right\|_{L^p(S_T)} \leq C \cdot T^{1-(|\alpha|/2b)} \int_0^{\infty} \frac{X_{(y/2, 2y)}(v) |y^{b+1-\gamma} - v^{b+1-\gamma}|}{v^{b+1-\gamma} |y - v|} \cdot (v^{b+1-\gamma} \|f(\cdot, v, \cdot)\|_{L^p(S_T)}) dv.$$

We now apply Hardy's lemma to obtain the desired inequality.

PROOF OF (ii). It is clear that

$$|T_{\alpha}^1(f)(x, y, t)| \leq \int_0^t \iint_{R_{\alpha}^{n+1}} |t^{(\gamma_p/2b)(b+1-\gamma)} - s^{(\gamma_p/2b)(b+1-\gamma)}| |D_{x,y}^{\alpha} K| |f| dz dv ds.$$

By applying Young's inequality in the variables z, v to the known estimates for K , and observing that $(t-s)^{1-(|\alpha|/2b)} \leq T^{1-(|\alpha|/2b)}$, it is easy to see that

$$\|T_{\alpha}^1 f\|_{L^p(R_+^{n+1} \times (0, T))} \leq C \cdot T^{1-(|\alpha|/2b)} \left\| \int_0^{\infty} k(t, s) \left(s^{(\gamma_p/2b)(b+1-\gamma)} \|f(\cdot, \cdot, s)\|_{L^p(R^{n+1})} \right) ds \right\|_{L^p(0, \infty)},$$

where

$$k(t, s) = \frac{X_{(0,t)}(s)}{s^{(\gamma_p/2b)(b+1-\gamma)}} \frac{t^{(\gamma_p/2b)(b+1-\gamma)} - s^{(\gamma_p/2b)(b+1-\gamma)}}{t-s}.$$

Due to the conditions on γ_p and the integrability of the second factor $k(t, s)$ satisfies the hypotheses in Hardy's lemma, from which part (ii) follows. This completes the proof of the lemma.

§ 13. - LEMMA. For $f \in L^p(R_+^{n+1} \times (0, T))$, $1/p < \gamma < 1$, $1 < p < \infty$,

$$(i) \quad \|y^{b+1-\gamma} J(f) - J(y^{b+1-\gamma} f)\|_{L^p(R_+^{n+1} \times (0, T))} \leq \omega(T) \|y^{b+1-\gamma} f\|_{L^p(R_+^{n+1} \times (0, T))}$$

$$(ii) \quad \|t^{(\gamma_p/2b)(b+1-\gamma)} J(f) - J(s^{(\gamma_p/2b)(b+1-\gamma)} f)\|_{L^p(R_+^{n+1} \times (0, T))} \leq \omega(T) \|t^{(\gamma_p/2b)(b+1-\gamma)} f(x, y, t)\|_{L^p(R_+^{n+1} \times (0, T))},$$

with $\omega(T) \rightarrow 0$ as $T \rightarrow 0^+$, ω depending on the moduli of continuity of the a_{α} , $|\alpha| = 2b$.

PROOF. Let $\varphi \in C_0^{\infty}(R^{n+2})$, $\varphi \geq 0$, with $\iiint_{R^{n+2}} \varphi = 1$ and support φ contained in the set $|x|^2 + y^2 + t^2 < 1$. For $|\alpha| = 2b$, we extend $a_{\alpha}(x, y, t)$ to all R^{n+2} , preserving uniform continuity and define, for $\lambda > 0$,

$$a_{\alpha}^{\lambda}(x, y, t) = (1/\lambda^{n+2}) \iiint_{R^{n+2}} a_{\alpha}(w, u, r) \varphi \left(\frac{x-w}{\lambda}, \frac{y-u}{\lambda}, \frac{t-r}{\lambda} \right) dw du dr.$$

Then $a_{\alpha}^{\lambda} \in C^{\infty}(R^{n+2})$, $|a_{\alpha}(x, y, t) - a_{\alpha}^{\lambda}(x, y, t)| \leq \omega(\lambda) = \text{maximum of moduli of continuity of } a_{\alpha}, |\alpha| = 2b, |Da_{\alpha}(x, y, t)| \leq (C/\lambda)\omega(\lambda)$, D being any derivative, and therefore $|a_{\alpha}^{\lambda}(x, y, t) - a_{\alpha}^{\lambda}(z, v, s)| \leq (C/\lambda)\omega(\lambda)(|x-z| + |y-v| + |t-s|)$.

According to the decomposition and the estimates in previous §, it will be sufficient to consider the case $|\alpha| = 2b$, $j = 0, 1$. We have

$$\begin{aligned} T_\alpha^j([a_\alpha(x, y, t) - a_\alpha(z, v, s)]f) &= T_\alpha^j([a_\alpha(x, y, t) - a_\alpha^\lambda(x, y, t)]f) + \\ &+ T_\alpha^j([a_\alpha^\lambda(z, v, s) - a_\alpha(z, v, s)]f) + T_\alpha^j([a_\alpha^\lambda(x, y, t) - a_\alpha^\lambda(z, v, s)]f). \end{aligned}$$

The first two terms are in absolute value less than or equal to $\omega(\lambda)$ times (bounds for $|T_\alpha^j f|$ in proof (i) and (ii) of lemma, § 12). For the third, we set $|y^{b+1-\gamma} - v^{b+1-\gamma}| = E^0$, $|t^{(\gamma_p/2b)(b+1-\gamma)} - s^{(\gamma_p/2b)(b+1-\gamma)}| = E^1$ and observe that it is in absolute value

$$\leq \int_0^t \int_0^\infty \int_{R^n} |D_x^\alpha K| \cdot E^j \cdot (C/\lambda) \omega(\lambda) (|x-z| + |y-v| + |t-s|) \cdot |f(z, v, s)| dz dv ds.$$

The usual procedure applies to each term above. For the third, $t-s < T$, so we set $\lambda = T^{1/2b}$ and obtain with a new $\omega(T)$ that tends to zero as $T \rightarrow 0+$, the same type of bound for $|T_\alpha^j f|$ used in Lemma, § 12. The proofs of § 12 apply again.

§ 14. - LEMMA. Let $f \in L^p(R_+^{n+1} \times (0, T))$. Then the operator (Jf) maps $L^p(R_+^{n+1} \times (0, T))$ continuously into itself, and belongs to $\mathfrak{F}(R_+^{n+1} \times (0, T))$ (cf. § 2).

PROOF. Condition (i), § 2 is clear. To prove condition (ii), we set $J = J_1 - J_2$, where

$$\begin{aligned} J_1 f(x, y, t) &= \lim_{\varepsilon \rightarrow 0^+} \int_0^{t-\varepsilon} \int_{R_+^{n+1}} L_{xyt} F(z, v, s; x-z, y-v, t-s) f(z, v, s) dz dv ds, \\ J_2 f(x, y, t) &= \int_0^t \int_{R_+^{n+1}} L_{xyt} G_0(z, v, s; x-z, y, v, t-s) f(z, v, s) dz dv ds. \end{aligned}$$

J_1 is known to belong in $\mathfrak{F}$ (cf. [5]). Furthermore,

$$J_1 = \sum_{|\alpha|=2b} (a_\alpha \bar{K}_\alpha - \bar{K}_\alpha a_\alpha) + \sum_{|\alpha|<2b} a_\alpha J_\alpha,$$

$\bar{K}_\alpha$ being a variable kernel, and the J_α , L^p -operators in $\mathfrak{F}$. For J_2 , we see

that it can be decomposed in two sums, setting

$$N_{\alpha}(f) = \int_0^t \iint_{R_+^{n+1}} D_{x,y}^{\alpha} G_0(z, v, s; x - y, v, t - s) f(z, v, s) dz dv ds$$

$$J_2 f = \sum_{|\alpha|=2b} N_{\alpha}([a_{\alpha}(x, y, t) - a_{\alpha}(z, v, s)]f) + \sum_{|\alpha|<2b} a_{\alpha} N_{\alpha}(f).$$

Consider first the case $|\alpha| = 2b$. Replacing f by $X_{(a,a+\varepsilon)}(s)f(z, v, s)$ we see that each term in the first sum above can be written as

$$N_{\alpha}([a_{\alpha}(x, y, t) - a_{\alpha}^{\lambda}(x, y, t)]X_{(a,a+\varepsilon)}f) + N_{\alpha}([a_{\alpha}^{\lambda}(z, v, s) - a_{\alpha}(z, v, s)]Xf) + \\ + N_{\alpha}([a_{\alpha}^{\lambda}(x, y, t) - a_{\alpha}^{\lambda}(z, v, s)]X_{(a,a+\varepsilon)}f),$$

where each term is in absolute value

$$\leq \omega(\varepsilon^{1/2b}) \int_0^t \iint_{R_+^{n+1}} (\text{bounds for } |D_{x,y}^{\alpha} G_0| \text{ in § 7}) \cdot X_{(a,a+\varepsilon)}(s) |f(z, v, s)| dz dv ds,$$

if $\lambda = \varepsilon^{1/2b}$. (We observe that for $a \leq t \leq a + \varepsilon$ and $a \leq s \leq t$, $t - s \leq \varepsilon$ and $X_{(0,\varepsilon)}(t - s) = 1$).

By Young's inequality in $dx dt$, the $L^p(R^n \times (a, a + \varepsilon))$ -norm of the above expression is bounded by

$$C\omega(\varepsilon^{1/2b}) \int_0^{\infty} \frac{1}{|y + v|} \int_0^{\varepsilon/|y+v|^{2b}} \frac{\Psi(1/s^{1/2b})}{s^{1+1/2b}} ds \|f(\cdot, v, \cdot)\|_{L^p(S_{(a,a+\varepsilon)})} dv.$$

(Here we have set $\Psi(y/s^{1/2b})\Psi(v/s^{1/2b}) \leq \Psi(|y + v|/s^{1/2b})$, the Ψ being of exponential type (cf. § 5). The integral in ds is bounded independently of y, v).

We now apply Hardy's lemma to obtain the inequality

$$\sum_{|\alpha|=2b} \|N_{\alpha}([a_{\alpha}(x, y, t) - a_{\alpha}(z, v, s)]X_{(a,a+\varepsilon)}f)\|_{L^p(R_+^{n+1} \times (a,a+\varepsilon))} \leq \\ \leq C\omega(\varepsilon^{1/2b}) \|f\|_{L^p(R_+^{n+1} \times (a,a+\varepsilon))}.$$

For $|\alpha| < 2b$, using the fact that $\Psi(y)\Psi(v) \leq \Psi(|y - v|)$, $y, v > 0$, and applying the known estimates for G_0 and the remarks on $X_{(0,\varepsilon)}(t - s)$, we easily get

$$\|N_{\alpha}(X_{(a,a+\varepsilon)}f)\|_{L^p(R_+^{n+1} \times (a,a+\varepsilon))} \leq C\varepsilon^{1-(|\alpha|/2b)} \|f\|_{L^p(R_+^{n+1} \times (a,a+\varepsilon))},$$

C depending on $\max_{|z| < 2b} \sup |a_\alpha|$, completing the proof of the lemma.

REMARK. We have essentially shown that

$$L_{xvt} \left(\int_0^t \iint_{R_+^{n+1}} G_0(z, v, s; x - z, y, v, t - s) f(z, v, s) dz dv ds \right) = J_2 f$$

belongs to $L^p(R_+^{n+1} \times (0, T))$ provided $f \in L^p(R_+^{n+1} \times (0, T))$, and its norm is $\leq C \|f\|_{L^p(R_+^{n+1} \times (0, T))}$. We observe that the boundary term in the computation of the D_t is zero, due to the estimates for G_0 (§ 7); we also see that

$$\int_0^t \iint_{R_+^{n+1}} G_0(z, v, s; x - z, y, v, t - s) f(z, v, s) dz dv ds \in \dot{L}_{2b,1}^p(R_+^{n+1} \times (0, T)).$$

Estimate for J . As a consequence of the results above, we have the following

LEMMA. For $f \in L^p(R_+^{n+1} \times (0, T))$, $1/p < \gamma < 1$, $1 < p < \infty$,

$$\|d_p(y, t)^{b+1-\gamma} Jf(x, y, t)\|_{L^p(R_+^{n+1} \times (0, T))} \leq \omega(T) \|d_p(y, t)^{b+1-\gamma} f(x, y, t)\|_{L^p(R_+^{n+1} \times (0, T))},$$

where $\omega \rightarrow 0$ as $T \rightarrow 0^+$.

§ 15. — The operator J having small norm as an operator on $L^p(R_+^{n+1} \times (a, a + \varepsilon))$ for ε suitably small, it follows that $I - J$ is invertible over $L^p(R_+^{n+1} \times (0, T))$. In fact, choosing m large enough,

$$\|J(X_{(a, a+(T/m))} g)\|_{L^p(R_+^{n+1} \times (a, a+(T/m)))} \leq (\tfrac{1}{2}) \|g\|_{L^p(R_+^{n+1} \times (a, a+(T/m)))},$$

provided $a + (T/m) \leq T$. Let g_1 be a function with support in $R_+^{n+1} \times (0, T/m)$ such that $(I - J)g_1 = f$ on $R_+^{n+1} \times (0, T/m)$, and in general let g_k have support in $R_+^{n+1} \times ((k-1)T/m, k(T/m))$, and satisfy $(I - J)g_k = f - \sum_{h=1}^{k-1} (I - J)g_h$ on $R_+^{n+1} \times ((k-1)T/m, k(T/m))$. Set $g = \sum_{k=1}^n g_k$: $g \in L^p(R_+^{n+1} \times (0, T))$, being a sum of functions in that space, and $(I - J)g = f$.

With the construction above it is easy to prove the following

LEMMA. If $f \in L^p(R_+^{n+1} \times (0, T))$, $1 < p < \infty$, $1/p < \gamma < 1$, then

$$\|d_p(y, t)^{b+1-\gamma} (I - J)^{-1} f\|_{L^p(R_+^{n+1} \times (0, T))} \leq C \|d_p(y, t)^{b+1-\gamma} f\|_{L^p(R_+^{n+1} \times (0, T))}.$$

($d_p(y, t)$ can be replaced by y or $t^{(\gamma_p/2b)}$).

REMARK. If the function f and the coefficients of L_{xyt} are differentiable, so is the function $(I - J)^{-1}f = g$. In fact, the only possible points of discontinuity of g or its derivatives with respect to time are those in the partition kT/m , $k \leq m$. To see that g is smooth at those points, we only have to construct $(I - J)^{-1}f$ with a different partition. The uniqueness of $(I - J)^{-1}f$ shows the differentiability of g at the points kT/m .

The differentiability of $(I - J)^{-1} = \sum J^k$ for small t can be seen from the fact that $DJ(f) = J^\sim(f) + J(Df)$, D denoting any derivative, and $J^\sim$ being a $\mathfrak{J}$ -operator that depends on the derivatives of the coefficients. From this identity we derive the recursion inequality

$$\|DJ^n(f)\| \leq \omega(r)(\|J^{n-1}(f)\| + \|DJ^{n-1}(f)\|), \quad \omega \rightarrow 0 \text{ as } r \rightarrow 0^+,$$

which shows convergence in norm of the series $\sum DJ^n(f)$ for r sufficiently small.

VI. — The main results in the half-space.

§ 16. — THEOREM. Let $f \in L^p(\mathbb{R}_+^{n+1} \times (0, T))$ for some p , $1 < p < \infty$. Set

$$\begin{aligned} V(x, y, t) &= - \int_0^t \int_{\mathbb{R}_+^{n+1}} K(z, v, s; x - z, y, v, t - s) (I - J)^{-1} f(z, v, s) dz dv ds \\ &= - V_{(I-J)^{-1}f}(x, y, t). \end{aligned}$$

Then for $0 \leq \beta < \gamma - 1/p$,

(i) $V(x, y, t) \in \dot{L}_{2b,1}^p(\mathbb{R}_+^{n+1} \times (0, T))$, and $LV = f$ for $y > 0$,

(ii) for $k = 0, \dots, b-1$,

$$\lim_{y \rightarrow 0^+} \text{in } L^p(S_T) A^{b-1+\beta-k} D_y^k V(\cdot, y, \cdot) = 0.$$

$$\begin{aligned} \text{(iii)} \quad & \sum_{k=0}^{b-1} \|A^{b-1-k+\beta} D_y^k V(\cdot, y, \cdot)\|_{L^p(S_T)} \\ & + \sum_{|\alpha| \leq b-1} \|D_{x,y}^\alpha A^\beta V(\cdot, y, \cdot)\|_{L^p(S_T)} \\ & + \sum_{b \leq |\alpha| \leq 2b-1} \|d_a(y, \cdot)^{|\alpha|-(b-1)} D_{x,y}^\alpha A^\beta V(\cdot, y, \cdot)\|_{L^p(S_T)} \leq C(T) \|d_p^{b+1-\gamma} f\|_{L^p}, \end{aligned}$$

where $C(T) \rightarrow 0$ as $T \rightarrow 0^+$.

(A similar estimate holds with d_p replaced by y , and with d_p replaced by $t^{\gamma_p/2b}$).

PROOF (i). Recall that

$$K(z, v, x; y, v, t) = F(z, v, s; x, y - v, t) - G_0(z, v, s; x, y, v, t),$$

and that $(I - J)^{-1}f \in L^p(R_+^{n+1} \times (0, T))$. Set $V = -V_1 + V_2$, where

$$V_1(x, y, t) = \int_0^t \int_{R_+^{n+1}} F(z, v, s; x - z, y - v, t - s) (I - J)^{-1}f(z, v, s) dz dv ds,$$

$$V_2(x, y, t) = \int_0^t \int_{R_+^{n+1}} G_0(z, v, s; x - z, y, v, t - s) (I - J)^{-1}f(z, v, s) dz dv ds.$$

We showed (cf. Remark to Lemma, § 14) that $V_2 \in \dot{L}_{2b,1}^p(R_+^{n+1} \times (0, T))$. It is known [2] that $V_1 \in \dot{L}_{2b,1}^p(R_+^{n+1} \times (0, T))$, hence $V \in \dot{L}_{2b,1}^p(R_+^{n+1} \times (0, T))$. Also

$$LV_1 = -(I - J)^{-1}f + J_1((I - J)^{-1}f),$$

and by same Remark, $LV_2 = J_2((I - J)^{-1}f)$. Therefore

$$\begin{aligned} LV &= (I - J)^{-1}f - (J_1 - J_2)(I - J)^{-1}f \\ &= (I - J)^{-1}f - J(I - J)^{-1}f = f. \end{aligned}$$

Part (ii) is an immediate consequence of Theorem § 9, (i), and the fact that $(I - J)^{-1}f$ belongs in $L^p(R_+^{n+1} \times (0, T))$.

Part (iii) follows from the estimates for the volume potential (Theorem, § 9, cf. lemmas A, B, C, §§ 10-11), and from estimate in Lemma, § 15.

§ 17. - For $j = 0, \dots, b - 1$, we introduce the functions

$$\begin{aligned} u_j(x, y, t) &= (T_{\mu_j} \Phi)(x, y, t) + \\ &+ \int_0^t \int_{R_+^{n+1}} K(z, v, s; x - z, y, v, t - s) (I - J)^{-1}(LT_{\mu_j} \Phi)(z, v, s) dz dv ds (*). \end{aligned}$$

(cf. definitions in § 6(K), § 8(T_{μ_j}), and in § 9 (volume potential)).

THEOREM. If $\Phi \in L^p(S_T)$, then for $j = 0, \dots, b - 1$,

- (i) $u_j(x, y, t) \in \dot{L}_{2b,1}^p(R^n \times (\delta, \infty) \times (0, T))$ for every $\delta > 0$,
- (ii) $L\mu_j = 0$ in $R_+^{n+1} \times (0, T)$.

PROOF. Is included in §§ 17-19. We introduce the following notation.

For $|\alpha| = 2b$, let $a_\alpha^\lambda(z, v, s)$ denote the regularization of the coefficient $a_\alpha(z, v, s)$ in L (cf. § 13). We will denote by $F^\lambda(z, v, s; x, y, t)$ the $\mathcal{F}_\xi[\exp \sum_{|\alpha|=2b} a_\alpha^\lambda(z, v, s)(i\xi)^\alpha t](x, y)$, by $T_{\mu j}^\lambda \Phi$ the surface potential

$$(T_{\mu j}^\lambda \Phi)(x, y, t) = \int_0^t \int_{R^n} \Lambda^{b-j-\mu} D_y^j F^\lambda(z, 0, s; x-z, y, t-s) \Phi(z, s) dz ds,$$

and by $u_j^\lambda(x, y, t)$ the corresponding functions (*).

If $\Phi \in C_0^\infty(R^n \times (0, \infty))$, it can be seen that $u_j^\lambda(x, y, t) \in \dot{L}_{2b,1}^p(R_+^{n+1} \times (0, T))$, and that $Lu_j^\lambda = 0$ for $y > 0$. The second statement will follow from the theorem in § 16; the first is a consequence of the same theorem in § 16 and of the fact that $(I - J)^{-1}$ is a L^p -mapping, if we observe that $T_{\mu j}^\lambda \Phi \in \dot{L}_{2b,1}^p(R_+^{n+1} \times (0, T))$.

When the coefficients a_α , $|\alpha| = 2b$, are only bounded and uniformly continuous, and $\Phi \in C_0^\infty(R^n \times (0, \infty))$, we will show that the expression in (*), § 17, belongs in $\dot{L}_{2b,1}^p(R_E \times (\delta, \infty) \times (0, T))$ for every $\delta > 0$. We first state two lemmas.

LEMMA 1. Let $\lambda, \lambda' > 0$. For $j = 0, \dots, b-1$,

$$\begin{aligned} \|(T_{\mu j}^\lambda - T_{\mu j}^{\lambda'}) \Phi\|_{L^p(R_+^{n+1} \times (0, T))} &\leq \\ &\leq C \cdot T^{(\mu+1/p)/2b} \max_{|\alpha|=2b} \|a_\alpha^\lambda(z, 0, s) - a_\alpha^{\lambda'}(z, 0, s)\|_{L^\infty(S_T)} \|\Phi\|_{L^p(S_T)}. \end{aligned}$$

LEMMA 2. If $\Phi \in L^p(S_T)$, $1 < p < \infty$, $\gamma - 1/p < \mu$, then

$$\|y^{b+1-\gamma} L(T_{\mu j}^\lambda - T_{\mu j}^{\lambda'}) \Phi(x, y, t)\|_{L^p(R_+^{n+1} \times (0, T))} \leq C \omega(T) \max_{|\alpha|=2b} \|a_\alpha^\lambda - a_\alpha^{\lambda'}\|_{L^\infty(S_T)} \|\Phi\|_{L^p(S_T)}.$$

These lemmas are L^p versions of similar results in Pogorzelski [12], and can be proved by similar arguments.

If now $\theta(y) \in C^\infty(0, \infty)$, $\theta = 0$ for $y < \delta$, $\theta = 1$ for $y \geq 2\delta$, and we set $\theta^{(l)}(y) = D_y^l \theta(y)$, we have the

COROLLARY. For $|\alpha| + l \leq 2b$,

$$\|\theta^{(l)}(y) D_{xy}^\alpha (T_{\mu j}^\lambda - T_{\mu j}^{\lambda'}) \Phi(x, y, t)\|_{L^p(R_+^{n+1} \times (0, T))} \leq C_\delta \omega(T) \max_{|\alpha|=2b} \|a_\alpha^\lambda - a_\alpha^{\lambda'}\|_{L^\infty(S_T)} \|\Phi\|_{L^p(S_T)}.$$

We now continue with the proof of the Theorem, § 17.

§ 18. — We recall that for $\Phi \in C_0^\infty(R^n \times (0, \infty))$, $u_j^\lambda(x, y, t) = (T_{\mu_j}^\lambda \Phi) + V_{(\Phi)}^\lambda$ belongs to $\dot{L}_{2b,1}^p(R_+^{n+1} \times (0, T))$ (we have set $V_{(\Phi)}^\lambda = V_{(I-J)^{-1}(LT_{\mu_j}^\lambda \Phi)}$).

The product $\theta(y)(u_j^\lambda - u_j^{\lambda'})$ vanishes near $y = 0$ and $t = 0$. It is known [4] that

$$\begin{aligned} \|\theta(u_j^\lambda - u_j^{\lambda'})\|_{L_{2b,1}^p(R_+^{n+1} \times (0, T))} &\leq C \cdot \|L(\theta(u_j^\lambda - u_j^{\lambda'}))\|_{L^p(R_+^{n+1} \times (0, T))} \leq \\ &\leq C \sum_{|\alpha|+l \leq 2b} \|\theta^{(l)} D_{x,y}^\alpha (T_{\mu_j}^\lambda - T_{\mu_j}^{\lambda'}) \Phi\|_{L^p(R_+^{n+1} \times (0, T))} + \\ &+ C \|\theta L(T_{\mu_j}^\lambda - T_{\mu_j}^{\lambda'}) \Phi\|_{L^p(R_+^{n+1} \times (0, T))} + \\ &+ C \sum_{l \geq 0} \|\theta^{(l)}(y) D_{x,y}^\alpha V_{(\Phi)}^{\lambda, \lambda'}(x, y, t)\|_{L^p(R_+^{n+1} \times (0, T))}, \end{aligned}$$

where the meaning of $V_{(\Phi)}^{\lambda, \lambda'}(x, y, t)$ is clear.

Now

$$\begin{aligned} &\left\| \sum_{l \geq 0} \theta^{(l)} D_{x,y}^\alpha V_{(\Phi)}^{\lambda, \lambda'} \right\|_{L^p(R_+^{n+1} \times (0, T))} \\ &\leq \sum_{l \geq 0} \|\theta^{(l)}(y)\| D_{x,y}^\alpha V_{(\Phi)}^{\lambda, \lambda'}(\cdot, y, \cdot)\|_{L^p(S_T)} \|_{L^p(0, \infty)} \\ &\leq \sum_{l \geq 0} \left\| \theta^{(l)}(y) \cdot \frac{1}{\delta^{|\alpha|}} \cdot y^{|\alpha|} \right\| \|D_{x,y}^\alpha V_{(\Phi)}^{\lambda, \lambda'}(\cdot, y, \cdot)\|_{L^p(S_T)} \|_{L^p(0, \infty)} \\ &\leq C(T) \sum_{l \geq 0} \left\| \frac{\theta^{(l)}(y)}{\delta^{|\alpha|}} \right\|_{L^\infty(0, \infty)} \|y^{b+1-\gamma} (I-J)^{-1} L(T_{\mu_j}^\lambda - T_{\mu_j}^{\lambda'}) \Phi(x, y, t)\|_{(L^p(R_+^{n+1} \times (0, T)))} \\ &\leq C_\delta(T) \|y^{b+1-\gamma} L(T_{\mu_j}^\lambda - T_{\mu_j}^{\lambda'}) \Phi(x, y, t)\|_{L^p(R_+^{n+1} \times (0, T))}. \end{aligned}$$

(We have used the estimates for the volume potential (§ 9, and lemmas A and B, § 10) with $\beta = 0$, and the Lemma in § 15)

By Lemmas 1 and 2, § 17, we conclude that

$$\|\theta(u_j^\lambda - u_j^{\lambda'})\|_{L_{2b,1}^p(R_+^{n+1} \times (0, T))} \leq C(T), \quad \max_{|\alpha| \leq 2b} \|a_\alpha^\lambda - a_\alpha^{\lambda'}\|_{L^\infty(S_T)} \cdot \|\Phi\|_{L^p(S_T)}.$$

This shows that $\{u_{j,\lambda}^\lambda\}$ is a Cauchy sequence in $\dot{L}_{2b,1}^p(R^n \times (\delta, \infty) \times (0, T))$ for every $\delta > 0$. Let $\bar{u}_j \in \dot{L}_{2b,1}^p(R^n \times (\delta, \infty) \times (0, T))$ for every $\delta > 0$ be its limit. By Lemmas 1 and 2, with $T_{\mu_j}^{\lambda'}$ replaced by T_{μ_j} , it follows that the potentials $T_{\mu_j}^\lambda \Phi$ and $V_{(\Phi)}^\lambda$ converge to $T_{\mu_j} \Phi$,

$$V_{(\Phi)} = \int_0^t \int_{R_+^{n+1}} K(z, v, s; x - z, y, v, t - s) (I - J)^{-1} (L T_{\mu_j} \Phi)(z, v, s) dz dv ds.$$

Hence $\bar{u}_j$ admits the representation (*) for $\Phi \in C_0^\infty(R^n \times (0, \infty))$, i.e.,

$$\bar{u}_j(x, y, t) = (T_{\mu j} \Phi)(x, y, t) + \int_0^t \int_{R_+^{n+1}} K(z, v, s; x - z, y, v, t - s) (I - J)^{-1} (LT^{\mu j} \Phi)(z, v, s) dz dv ds, \quad (*)$$

and $\bar{u}_j \in \dot{L}_{2b,1}^p(R^n \times (\delta, \infty) \times (0, T))$ for every $\delta > 0$.

Clearly $L\bar{u}_j = 0$ for $y > 0$.

§ 19. – The same representation holds for $\Phi \in L^p(S_T)$. To see this, select $\Phi^\lambda \in C_0^\infty(R^n \times (0, \infty))$, $\|\Phi - \Phi^\lambda\|_{L^p(S_T)} \rightarrow 0$ as $\lambda \rightarrow 0$, and let $\bar{u}_j^\lambda$ be the function (*) above with Φ^λ in place of Φ . Again let $\theta(y) \in C^\infty(0, \infty)$, $\theta(y) = 0$ for $y < \delta$, $\theta(y) = 1$ for $y > 2\delta$. We have

$$\begin{aligned} \|\theta(\bar{u}_j - \bar{u}_j^\lambda)\|_{L_{2b,1}^p(R_+^{n+1} \times (0, T))} &\leq \|L[\theta(\bar{u}_j - \bar{u}_j^\lambda)]\|_{L^p(R_+^{n+1} \times (0, T))} \\ &\leq C \sum_{(|\alpha| + l \leq 2b)} \|\theta^{(l)} D_{\mu j}^\alpha T_{\mu j}(\Phi - \Phi^\lambda)\|_{L^p(R_+^{n+1} \times (0, T))} + \\ &\quad + \|\theta LT_{\mu j}(\Phi - \Phi^\lambda)\|_{L^p(R_+^{n+1} \times (0, T))} + \\ &\quad + C \left\| \sum_{l \geq 0} \theta^{(l)}(y) D_{x,y}^\alpha V_{(\Phi - \Phi^\lambda)(x,y,t)} \right\|_{L^p(R_+^{n+1} \times (0, T))} \end{aligned}$$

where $V_{(\Phi - \Phi^\lambda)}$ denotes the volume potential in (*) corresponding to $\Phi - \Phi^\lambda$.

A slight modification of the arguments in § 18 shows that these terms are majorized by $C\|\Phi - \Phi^\lambda\|_{L^p(S_T)}$. Therefore $\bar{u}_j^\lambda \rightarrow \bar{u}_j$ in $\dot{L}_{2b,1}^p(R^n \times (\delta, \infty) \times (0, T))$ for every $\delta > 0$. Also, by Lemmas 1 and 2, § 8, both

$$\|T_{\mu j}(\Phi - \Phi^\lambda)\|_{L^p(R_+^{n+1} \times (0, T))} \quad \text{and} \quad \|y^{b+1-\gamma} LT_{\mu j}(\Phi - \Phi^\lambda)\|_{L^p(R_+^{n+1} \times (0, T))}$$

tend to zero as $\lambda \rightarrow 0$. This shows that

$$\bar{u}_j \in \dot{L}_{2b,1}^p(R^n \times (\delta, \infty) \times (0, T)) \quad \text{for every } \delta > 0,$$

and admits the representation (*) with $\Phi \in L^p(S_T)$. From this we see that $\bar{u}_j = u_j$ and $Lu_j = 0$ for $y > 0$.

This concludes the proof of Theorem, § 17.

§ 20. – It is known (cf. [3]) that

$$\begin{aligned} \Lambda^{b-1+\mu-k} D_y^k T_{\mu j} \Phi(x, y, t) \\ = \int_0^t \int_{R^n} \Lambda^{2b-1-(k+j)} D_y^{k+j} F(z, 0, s; x - z, y, v, t - s) \Phi(z, s) dz ds \end{aligned}$$

converges in $L^p(S_T)$ as $y \rightarrow 0^+$, to a limit of the form $(S_{k,j} + J_{k,j})\Phi$, where $(S_{k,j})$ is a matrix of parabolic singular integral operators whose matrix of symbols admits an inverse $[\sigma(S_{k,j})(z, s; x, t)]^{-1} = (\sigma(S_{k,j}^*)(z, s; x, t))$ provided $(x, t) \neq (0, 0)$, $S_{k,j}^*$ being a parabolic singular integral operator such that $(S_{k,j}) \cdot (S_{k,j}^*) = I + (J_{k,j}^0)$, with $J_{k,j}^0 \in \mathfrak{J}(S_T)$ (cf. [4]). $J_{k,j}$ belongs to $\mathfrak{J}(S_T)$, and is a limit as $y \rightarrow 0$ of a series of commutators in the variables (x, y, t) which belong to $\mathfrak{J}(S_T)$ uniformly in y [3].

From the estimates for the volume potential (lemma A, § 10) and the estimates for $y^{b+1-\gamma}(I - J)^{-1}(\cdot)$ and $y^{b+1-\gamma}LT_{\mu j}\Phi$ (§ 15 and Lemma 2, § 8), it follows that for $\beta < \mu$, with $\beta < \gamma - 1/p < \mu$,

$$A^{b-1+\beta-k}D_y^k \int_0^t \int_{R_+^{1+u}} K(z, v, s; x - z, y, v, t - s)(I - J)^{-1}(LT_{\mu j}\Phi)(z, v, s) dz dv ds \rightarrow 0$$

in $L^p(S_T)$ as $y \rightarrow 0^+$.

Also, for $f \in L^p(R_+^{n+1} \times (0, T))$ (Theorem § 16, (ii)),

$$A^{b-1+\beta-k}D_y^k \int_0^t \int_{R_+^{n+1}} K(z, v, s; x - z, y, v, t - s)(I - J)^{-1}f(z, v, s) dz dv ds \rightarrow 0$$

in $L^p(S_T)$ as $y \rightarrow 0^+$.

We observe now that $(J_{k,j}) \cdot (S_{k,j}^*)$ (the dot indicates matrix multiplication) is a matrix of $\mathfrak{J}$ -operators, due to the fact that if $J \in \mathfrak{J}(S_T)$, and S is a singular integral operator on S_T , $JS \in \mathfrak{J}(S_T)$.

Therefore $I + [(J_{k,j}^0) + (J_{k,j}) \cdot (S_{k,j}^*)]$ has an inverse.

Set

$$(\mathfrak{U}_{k,j}) = (S_{k,j}^*) \cdot [I + (J_{k,j}^0) + (J_{k,j}) \cdot (S_{k,j}^*)]^{-1}.$$

THEOREM. Let there be given $f \in L^p(R_+^{n+1} \times (0, T))$, and for $k = 0, \dots, b-1$, functions $w^k \in L^p(S_T)$ such that $A^{b-1-k+\mu}w^k \in L^p(S_T)$ for some μ , $0 < \mu < 1$.

Then there exists a function $u \in \dot{L}_{2b,1}^p(R^n \times (\delta, \infty) \times (0, T))$ for every $\delta > 0$ such that

$$(i) \quad Lu = f \text{ in } R_+^{n+1} \times (0, T),$$

$$(ii) \quad \|A^{b-1-k+\beta}(D_y^k u(\cdot, y, \cdot) - w^k(\cdot, \cdot))\|_{L^p(S_T)} \rightarrow 0 \text{ as } y \rightarrow 0^+, \text{ for every } \beta, \\ 0 \leq \beta < \mu,$$

(iii) For $0 \leq \beta < \gamma - 1/p < \mu$,

$$\begin{aligned} & \sum_{|\alpha| \leq b-1} \|D_{x,y}^\alpha \Lambda^\beta u(\cdot, y, \cdot)\|_{L^p(S_T)} + \\ & + \sum_{k=0}^{b-1} \|\Lambda^{b-1+\beta-k} D_y^k u(\cdot, y, \cdot)\|_{L^p(S_T)} + \\ & + \sum_{b \leq |\alpha| \leq 2b-1} \|d_p(y, \cdot)^{|\alpha|-(b-1)} D_{x,y}^\alpha \Lambda^\beta u(\cdot, y, \cdot)\|_{L^p(S_T)} \leq \\ & \leq C \sum_{k=0}^{b-1} \|\Lambda^{b-1+\mu-k} w^k\|_{L^p(S_T)} + C(T) \|d_p^{b+1-\gamma} f\|_{L^p(R_+^{n+1} \times (0, T))} \end{aligned}$$

(d_p can be replaced with y or $t^{\gamma p/2b}$).

PROOF. We set

$$g_{\mu l}(x, y, t) = \sum_{j=0}^{b-1} (\mathcal{U}_{ij} \Lambda^{b-1+\mu-j} w)(x, y, t)$$

and define

$$\begin{aligned} u(x, y, t) &= \sum_{l=0}^{b-1} (T_{\mu l} g_{\mu l})(x, y, t) + \\ & + \sum_{l=0}^{b-1} \int_0^t \iint_{R_+^{n+1}} K(z, v, s; x-z, y, v, t-s) \cdot (I-J)^{-1} (LT_{\mu l} g_{\mu l})(z, v, s) dz dv ds - \\ & - \int_0^t \iint_{R_+^{n+1}} K(z, v, s; x-z, y, v, t-s) (I-J)^{-1} f(z, v, s) dz dv ds. \end{aligned}$$

The fact that $u \in \dot{L}_{2b,1}^p(R^n \times (\delta, \infty) \times (0, T))$ for every $\delta > 0$, and Part (i) follow from the theorems, §§ 16 and 17. To prove (ii), it is enough to consider the terms $\sum_{l=0}^{b-1} (T_{\mu l} g_{\mu l})(x, y, t)$ and to recall the definition of $g_{\mu l}$ and the fact that $\Lambda^{\beta-\mu}$ is an integrable function.

§ 21. - THEOREM. Suppose $u \in \dot{L}_{2b,1}^p(R_+^{n+1} \times (0, T))$, $1 < p < \infty$. Then $u(x; y, t)$ admits the representation $u = u_1 + u_2$, where

$$\begin{aligned} u_1(x, y, t) &= \\ &= \sum_{l=0}^{b-1} \{ (T_{\mu l} g_{\mu l})(x, y, t) + \int_0^t \iint_{R_+^{n+1}} K(z, v, s; x-z, y, v, t-s) (I-J)^{-1} (LT_{\mu l} g)_{\mu l} dz dv ds \}, \end{aligned}$$

$$u_2(x, y, t) = - \int_0^t \int_{\mathbb{R}_+^{n+1}} K(z, v, s; x - z, y, v, t - s) (I - J)^{-1} (Lu)(z, v, s) dz dv ds,$$

with

$$g_{\mu l}(x, y, t) = \sum_{j=0}^{b-1} (\mathfrak{U}_{ij} A^{b-1+\mu-j} D_y^j u(\cdot, 0, \cdot))(x, y, t), \quad (\text{cf. } \S 20).$$

Furthermore, for $0 \leq \beta < \gamma - 1/p < \mu$,

$$\begin{aligned} & \sum_{k=0}^{b-1} \|A^{b-1+\beta-k} D_y^k u(\cdot, y, \cdot)\|_{L^p(S_T)} + \sum_{|\alpha| \leq b-1} \|D_{xy}^\alpha A^\beta u(\cdot, y, \cdot)\|_{L(S_T)} + \\ & + \sum_{b \leq |\alpha| \leq 2b-1} \|d_p(y, \cdot)^{|\alpha|-(b-1)} D_{x,y}^\alpha A^\beta u(\cdot, y, \cdot)\|_{L^p(S_T)} \leq \\ & \leq C \sum_{k=0}^{b-1} \|A^{b-1+\mu-k} D_y^k u(\cdot, 0, \cdot)\|_{L^p(S_T)} + C(T) \cdot \|d_p^{b+1-\gamma}(Lu)\|_{L^p(\mathbb{R}_+^{n+1} \times (0, T))}, \end{aligned}$$

where $C(T) \rightarrow 0$ as $T \rightarrow 0$, d_p can be replaced by y or $t^{p/2b}$, and $C, C(T)$ depend only on the parameter of parabolicity π and on the $\max_{|\alpha|=2b} \sup |a_\alpha|$.

PROOF. We assume first $u \in C_0^\infty(\mathbb{R}_+^{n+1} \times (0, \infty))$, and let $u^\lambda = u_1^\lambda + u_2^\lambda$, u_1^λ and u_2^λ being the terms in the decomposition above, with

$$T_{\mu l}^\lambda, g_{\mu l}^\lambda, \mathfrak{U}_{ij}^\lambda = ((S_{ij}^\lambda) \cdot [I + (J_{ij}^{0\lambda}) + (J_{ij}^\lambda) \cdot (S_{ij}^\lambda)]^{-1})_{ij}$$

being the expressions corresponding to $F^\lambda(z, 0, s; x, y, t)$ (cf. §§ 13 and 20).

We observe that

$$g_{\mu l}^\lambda(x, y, t) = \sum_{j=0}^{b-1} \{ \mathfrak{U}_{ij}^\lambda A^{b-1+\mu-j} D_y^j u(\cdot, 0, \cdot) \}(x, y, t)$$

vanishes near $t = 0$ and has all derivatives in $L^p(\mathbb{R}_+^{n+1} \times (0, T))$.

By an argument similar to that in § 17, we see that

$$T_{\mu l}^\lambda g_{\mu l}^\lambda \in \hat{L}_{2b,1}^p(\mathbb{R}_+^{n+1} \times (0, T)).$$

Hence $u_1^\lambda \in \hat{L}_{2b,1}^p(\mathbb{R}_+^{n+1} \times (0, T))$. u_2^λ belongs to $\hat{L}_{2b,1}^p(\mathbb{R}_+^{n+1} \times (0, T))$ by Theorem, § 16 ($Lu \in L^p$).

Therefore $u^\lambda \in \hat{L}_{2b,1}^p(\mathbb{R}_+^{n+1} \times (0, T))$, and it is clear that $Lu^\lambda \equiv Lu$ for $y > 0$.

Also $A^{b-1+\mu-k} D_y^k u^\lambda$ tends to $A^{b-1+\mu-k} D_y^k u(\cdot, 0, \cdot)$ in $L^p(S^T)$ as $y \rightarrow 0^+$. On the other hand, $D_y^k u^\lambda$ and $D_y^k u$ converge in $L^p(S_T)$ as $y \rightarrow 0^+$, being func-

tions in $\dot{L}_{2b,1}^p(R_+^{n+1} \times (0, T))$. Hence we must have $D_y^k u^\lambda(\cdot, 0, \cdot) = D_y^k u(\cdot, 0, \cdot)$, and we see that $u - u^\lambda$ satisfies a homogeneous initial-boundary value problem with homogeneous data. By [3] it follows that

$$u \equiv u^\lambda \equiv u_1^\lambda + u_2^\lambda.$$

Now as $\lambda \rightarrow 0$, $g_{\mu i}^\lambda \rightarrow g_{\mu i}$ in $L^p(S_T)$ (it would be enough to show that $S^{\lambda*} \rightarrow S^*$, $J^\lambda \rightarrow J$ as operators on $L^p(S_T) \times \dots \times L^p(S_T)$, $(b-1)$ times; we observe that

$$\|J_{ih}^\lambda(X_{(a,a+\varepsilon)}f)\|_{L^p(R^n \times (a,a+\varepsilon))} \leq \omega(\varepsilon)\|f\|_{L^p(R^n \times (a,a+\varepsilon))},$$

ω in dependent of a and λ , cf. definition of J_{ih} in [3]).

Thus we have the representation $u = u_1 + u_2$ for $u \in C_0^\infty(R^{n+1} \times (0, \infty))$. Any function in $\dot{L}_{2b,1}^p(R_+^{n+1} \times (0, T))$ being a limit of $C_0^\infty(R^{n+1} \times (0, \infty))$ -functions in the $L_{2b,1}^p$ sense, the general result follows by a density argument, recalling the obvious convergence in L^p of sequences like $(g_{\mu i})_\nu$ and Lu_ν , to their corresponding expressions for $u \in L_{2b,1}^p$, and the fact that

$$\|y^{b+1-\nu}LT_{\mu i}(g_{\mu i})_\nu - g_{\mu i}\|_{L^p(R^n \times (0,T))} \rightarrow 0.$$

The estimate is an immediate consequence.

§ 22. – Using the representation above, we can prove the uniqueness of the solution to the problem (cf. Theorem, § 20): $u \in L_{2b,1}^p(R^n \times (\delta, \infty) \times (0, T))$ for every $\delta > 0$,

$$Lu = f \in L^p(R_+^{n+1} \times (0, T)) \quad \text{for } y > 0,$$

$$\|A^{b-1+\mu-k}(D_y^k u(\cdot, y, \cdot) - w^k(\cdot, \cdot))\|_{L^p(S_T)} \rightarrow 0 \quad \text{as } y \rightarrow 0^+$$

for every β , $0 \leq \beta < \mu < 1$, and $k = 0, \dots, b-1$.

Here $A^{b-1+\mu-k}w^k \in L^p(S_T)$.

THEOREM. If

(i) $u(x, y, t) \in \dot{L}_{2b,1}^p(R^n \times (\delta, \infty) \times (0, T))$ for every $\delta > 0$,

(ii) for $k = 0, \dots, b-1$, and some μ , $0 < \mu < 1$,

$$\|A^{b-1+\mu-k}D_y^k u(\cdot, y, \cdot)\|_{L^p(S_T)} \rightarrow 0 \quad \text{as } y \rightarrow 0^+,$$

and

(iii) $Lu = 0$ for $y > 0$,

Then $u \equiv 0$.

PROOF. We observe that for $\delta > 0$,

$$u(x, y + \delta, t) \in \dot{L}_{2b,1}^p(R_+^{n+1} \times (0, T))$$

is a solution of

$$L^\delta g \equiv \sum_{|\alpha| \leq 2b} a_\alpha(x, y + \delta, t) D_{x,y}^\alpha g + D_t g = 0 \quad \text{for } y > 0.$$

By the estimate of previous theorem, with $\beta = 0$, we have

$$\|u(\cdot, y + \delta, \cdot)\|_{L^p(S_T)} \leq C \sum_{k=0}^{b-1} \|A^{b-1+\mu-k} D_y^k u(\cdot, \delta, \cdot)\|_{L^p(S_T)}.$$

Hence,

$$\|u(\cdot, y, \cdot)\|_{L^p(S_T)} \leq \lim_{\delta \rightarrow 0^+} \|u(\cdot, y + \delta, \cdot)\|_{L^p(S_T)} = 0 \quad \text{for } y > 0,$$

proving the theorem (we recall that the constants in the representation theorem depend only on π and $\max_{|\alpha|=2b} \sup |a_\alpha|$).

VII. — An elliptic estimate.

§ 23. — Let now

$$\mathcal{E} \equiv \sum_{|\alpha| \leq 2b} a_\alpha(x, y) D_{x,y}^\alpha$$

be an operator in R_+^{n+1} , strongly elliptic in the sense that

$$\operatorname{Re} \left(\sum_{|\alpha|=2b} a_\alpha(x, y) (i\xi)^\alpha \right) < -\pi \cdot |\xi|^{2b},$$

$\pi > 0$ and independent of (x, y) , $\xi \in R^{n+1}$, $\xi \neq 0$. We assume each $a_\alpha(x, y)$ to be bounded and measurable, and for $|\alpha| = 2b$, uniformly continuous in $\overline{R_+^{n+1}}$.

For $1 < p < \infty$ we define $\bar{d}_p(y) = \min(y, T^{p/2b})$, T being a constant.

For B any real number, we will denote by $G_{-B}(x)$ the Bessel potential, defined by

$$\mathcal{F}(G_{-B}f)(x) = (1 + |x|^2)^{B/2} \cdot \mathcal{F}(f)(x).$$

Now set $L \equiv \mathfrak{E} - D_t$ and assume $u(x, y) \in L_{2b}^p(R_+^{n+1})$.

Clearly $t \cdot u(x, y) \in \dot{L}_{2b,1}^p(R_+^{n+1} \times (0, T))$ and we have (cf. § 21)

$$\begin{aligned} \sum_{k=0}^{b-1} \|D_y^k A^{b-1+\beta-k}(s \cdot u)(\cdot, y, \cdot)\|_{L^p(S_T)} + \sum_{|\alpha| \leq b-1} \|A^\beta(s \cdot D_{x,y}^\alpha u)(\cdot, y, \cdot)\|_{L^p(S_T)} + \\ + \sum_{b \leq |\alpha| \leq 2b-1} \|(\bar{d}_p(y, \cdot))^{| \alpha | - (b-1)} A^\beta(s \cdot D_{x,y}^\alpha u)(\cdot, y, \cdot)\|_{L^p(S_T)} \\ \leq C \sum_{k=0}^{b-1} \|D_y^k A^{b-1+\mu-k}(s \cdot u)(\cdot, 0, \cdot)\|_{L^p(S_T)} + \\ + C(T) \|(\bar{d}_p(y, t))^{b+1-\gamma} u(x, y)\|_{L^p(R_+^{n+1} \times (0, T))} + \\ + C(T) \|(\bar{d}_p(y, t))^{b+1-\gamma} tLu(x, y)\|_{L^p(R_+^{n+1} \times (0, T))} \quad (*), \end{aligned}$$

where

$$0 \leq \beta < \gamma < 1/p < \mu, \quad \bar{d}_p(y, t) = \min(y, t^{p/2b}) \quad (\S 5),$$

and $C(T) \rightarrow 0$ as $T \rightarrow 0^+$.

THEOREM. If $\mathfrak{E}$ is a strongly elliptic operator in R_+^{n+1} , and $(x, y) \in L_{2b}^p(R_+^{n+1})$, then, with $0 \leq \beta < \gamma - 1/p < \mu < 1$,

$$\begin{aligned} \sum_{k=0}^{b-1} \|G_{-(b-1+\beta-k)}(D_y^k u)(\cdot, y)\|_{L^p(R^n)} + \sum_{|\alpha| \leq b-1} \|G_{-\beta}(D_{x,y}^\alpha u)(\cdot, y)\|_{L^p(R^n)} + \\ + \sum_{b \leq |\alpha| \leq 2b-1} \|(\bar{d}_p(y))^{| \alpha | - (b-1)} G_{-\beta}(D_{x,y}^\alpha u)(\cdot, y)\|_{L^p(R^n)} \leq \\ \leq C \sum_{k=0}^{b-1} \|G_{-(b-1+\beta-k)}(D_y^k u)(\cdot, 0)\|_{L^p(R^n)} + C \|(\bar{d}_p(y))^{b+1-\gamma} \cdot \\ \cdot \mathfrak{E}u(x, y)\|_{L^p(R_+^{n+1} \times (0, T))} + C \|u\|_{L^p(R_+^{n+1} \times (0, T))}. \end{aligned}$$

PROOF. Consider first the following inequalities (see [5] for the first two)

$$(i) \quad \frac{1}{C} (\bar{d}(y))^{| \alpha |} T^{1/p} \leq \left(\int_{\tau}^T (\bar{d}(y, t))^{| \alpha | p} dt \right)^{1/p} \leq C (\bar{d}_p(y))^{| \alpha |} T^{1/p}$$

where $\tau = 0, T/2$, and $|\alpha|$ can be replaced with $b + 1 - \gamma$;

$$(ii) \quad \|A^{-k}f\|_{L^p(S_T)} \leq C \cdot \|G_k f\|_{L^p(R^n)};$$

$$(iii) \quad \|(\bar{d}_p(y, t))^{b+1-\gamma} \cdot t \cdot \varepsilon u(x, y)\|_{L^p(R_+^{n+1} \times (0, T))} \leq CT^{1+1/p} \|(\bar{d}_p(y))^{b+1-\gamma} \cdot \varepsilon u(x, y)\|_{L^p(R_+^{n+1})};$$

$$(iv) \quad \|(\bar{d}_p(y, t))^{b+1-\gamma} u(x, y)\|_{L^p(R_+^{n+1} \times (0, T))} \leq CT^{(\gamma/p)(2b)(b+1-\gamma)+1/p} \cdot \|u(x, y)\|_{L^p(R_+^{n+1})};$$

$$(v) \quad \text{for } 0 < B < 2b, \quad \|A^B(sf(z))\|_{L^p(S_T)} \leq C \|G_{-B}f\|_{L^p(R^n)}.$$

(i)-(v) show that the right hand side of the parabolic estimate (*) is majorized by the right hand side of the elliptic one. It is clear that the proof will be completed by the following result, whose proof follows the lines of [5]. Appendix

LEMMA. Let $f(x) \in C_0^\infty(R^n)$. Then for $1 < p < \infty$, $0 \leq B \leq 2b$,

$$\|G_{-B}f\|_{L^p(R^n)} \leq C_T \|A^B(s \cdot f(z))\|_{L^p(R^n \times (T/2, T))}.$$

REMARK. If u is assumed to have support contained in $\{(x, y): |x|^2 + y^2 \leq r^2, y \geq 0\}$, the term $\|u\|_{L^p(R_+^{n+1})}$ in the estimate may be replaced by $\|u\|_{L^1(R_+^{n+1})}$, with a change in the constants.

VIII. — The main results in a general domain.

§ 24. — We now consider the Initial-Dirichlet boundary-value problem with initial data zero, for a parabolic equation where the cylinder is defined as the product of a domain in R^{n+1} with the time interval $(0, T)$. P shall denote a point inside that domain, $D^\alpha = D_P^\alpha$ a spatial derivative of order $|\alpha|$, Q a point on the boundary. The differential operator, L , is assumed to satisfy Petrovski's condition and the conditions on the coefficients stated in § 5 with (x, y, t) replaced by (P, t) .

From here on, Ω will denote a bounded, smooth domain in R^{n+1} . By this we mean

(i) There exists a finite number of functions f_i having continuous and bounded derivatives up to order $2b + 2$, and each mapping the disc $\{(x, 0): |x|^2 < r_i^2, r_i > 0\} \subset R^{n+1}$ into $\partial\Omega$ in a 1-1 manner, such that every point $Q \in \partial\Omega$ can be written $Q = f_i(x, 0)$, $|x|^2 < r_i^2$, for some i ;

(ii) If N_Q denotes the unit inner normal to $\partial\Omega$ at Q , and $\Omega_\delta = \{P \in \Omega: \text{dist}(P, \partial\Omega) > \delta, \delta \geq 0\}$, then there is a number $\delta_0 > 0$ such that for each i

the function

$$(x, y) \rightarrow f_i(x, y) = Q + y \cdot N_Q, \quad Q = f_i(x, 0) \in \partial\Omega$$

maps the set $\{(x, y): |x|^2 < r_i^2, 0 \leq y \leq 4\delta_0\}$ into $\bar{\Omega} - \Omega_{4\delta_0}$ in a 1-1 manner. We assume further that every point $P \in \bar{\Omega} - \Omega_{4\delta_0}$ can be uniquely written as $P = Q + y \cdot N_Q$, $Q \in \partial\Omega$.

We consider the finite covering of the set $\bar{\Omega} - \Omega_{\delta_0}$ by the sets U_i , image of

$$\{(x, y): |x|^2 < r_i^2, 2\delta_0 > y > \kappa, \kappa \text{ a suitable number } < 0\}$$

under $f_i(x, y) = f_i(x, 0) + y \cdot N_{f_i(x, 0)}$. We let $\{\varphi_i\}$ denote a fixed C_0^∞ partition of unity subordinate to $\{U_i\}$, and we denote by $\{\zeta_i\}$ a family of functions such that $\zeta_i \in C_0^\infty(U_i)$ and $\zeta_i \equiv 1$ in a neighborhood of the support of φ_i . We point out here that when $0 < \delta < \delta_0$, for the domain Ω_δ we can associate the sets $U_{\delta i}$, and the families $\{\varphi_{\delta i}\}$, $\{\zeta_{\delta i}\}$ obtained from the ones above by the transformation $P \rightarrow P + \delta N_Q$, where $P = Q + y N_Q \in U_i$. It follows that the derivatives of $\varphi_{\delta i}$, $\zeta_{\delta i}$ can be bounded uniformly on δ , $0 < \delta < \delta_0$.

If $u(P)$ is a function defined in $\bar{\Omega}$ we will set, for simplicity, $u_i^\sim(x, y) = u \circ f_i(x, y)$. For the functions φ_i , ζ_i , we will set, e.g., $(\varphi_i) = \varphi_i^\sim$ when there is no confusion. We will also write $u_i^\sim(x, y, t)$ for $u(f_i(x, y), t)$. We observe that $(D_{N_Q} u_i^\sim)(x, y) = D_y(u_i^\sim)(x, y)$.

The L^p -norm of a function $u(Q, t)$ defined on $\partial\Omega \times (0, T)$ is equivalent to $\sum_i \|\varphi_i u\|$, or $\sum_i \|\zeta_i u\|$, which are computed as integrals over $R^n \times (0, T) = S_T$:

We introduce now the operators A on $\partial\Omega$.

DEFINITION. If $u(Q, t) \in L^p(\partial\Omega \times (0, T))$, $1 < p < \infty$, and $0 \leq \beta \leq 2b$,

$$(A^{-\beta} u)(Q, t) = \sum_i \zeta_i(Q) A^{-\beta}[\varphi_i^\sim \cdot u_i^\sim](f_i^{-1}(Q), t),$$

where $A^{-\beta}[\cdot]$ is the operator already defined on S_T .

DEFINITION. For functions $u(Q, t) \in C^\infty(\partial\Omega \times (0, T))$ which are identically zero for t near zero,

$$(A^\beta u)(Q, t) = \sum_i \zeta_i(Q) A^\beta[\varphi_i^\sim \cdot u_i^\sim](f_i^{-1}(Q), t).$$

It can be shown that $A^\beta A^{-\beta}$ is not in general the identity on C^∞ functions that vanish near $t = 0$, but it is extendible to an invertible operator on $L^p(\partial\Omega \times (0, T))$, $1 < p < \infty$.

The Bessel potentials are defined in a similar way.

DEFINITION. If $u(Q) \in C^\infty(\partial\Omega)$,

$$(G_{-\beta}u)(Q) = \sum_i \zeta_i(Q) G_{-\beta}[\varphi_i^\sim \cdot u_i^\sim](f_i^{-1}(Q), t),$$

where $\mathcal{F}_x(G_{-\beta}[\varphi_i^\sim \cdot u_i^\sim]) = (|x|^2 + 1)^{b/2} \cdot \mathcal{F}_x[\varphi_i^\sim \cdot u_i^\sim]$, $x \in R^n$.

In the next paragraphs we follow the method used in [5], article (4.1), for the corresponding problem with initial data zero.

§ 25. — THEOREM. If $u \in \tilde{L}_{2b,1}^p(\Omega \times (0, T))$, $1 < p < \infty$, $0 \leq \beta < \mu < 1$, and $Lu = 0$ in $\Omega \times (0, T)$, then

$$\begin{aligned} \sup_{\nu > \delta_0} \left\{ \sum_{|\alpha| \leq 2b-1} y^{[|\alpha|-(b-1)]} \|D^\alpha(\Lambda^\beta u)(Q + yN_Q, t)\|_{L^p(\partial\Omega \times (0, T))} + \right. \\ \left. + \sum_{k=0}^{b-1} \|\Lambda^{b-1+\beta-k}[D_{N_Q}^k u](Q + yN_Q, t)\|_{L^p(\partial\Omega \times (0, T))} \right\} \\ \leq C \cdot \sum_{k=0}^{b-1} \|\Lambda^{b-1+\mu-k}[D_{N_Q}^k u](Q, t)\|_{L^p(\partial\Omega \times (0, T))}. \end{aligned}$$

(Here and in the following, the expression $y^{[|\alpha|-(b-1)]}$ is meant to be replaced by 1 for $|\alpha| \leq b-1$).

PROOF. We shall sketch the proof of this result, which proceeds first for small T (this §), and then in the general case (next §).

By applying the definition of Λ^β (§ 24), dropping continuous functions of compact support, and introducing a new constant C_{δ_0} , we can see that the left hand side ($LHS_T^\beta(u; \Omega)$) of the estimate above is less than or equal to

$$\begin{aligned} \|u\|^\sim \equiv C_{\delta_0} \sum_i \sup_{\nu < 2\delta_0} \left\{ \sum_{|\alpha| \leq 2b-1} y^{[|\alpha|-(b-1)]} \|D_{xy}^\alpha \Lambda^\beta(\varphi_i^\sim \cdot u_i^\sim)(\cdot, y, \cdot)\|_{L^p(S_T)} + \right. \\ \left. + \sum_{k=0}^{b-1} \|\Lambda^{b-1+\beta-k} D_y^k(\varphi_i^\sim \cdot u_i^\sim)(\cdot, y, \cdot)\|_{L^p(S_T)} + \right. \\ \left. + \sum_{k=1}^{b-1} \sum_{m=1}^k \|\Lambda^{b-1+\beta-k} (D_y^m \varphi_i^\sim \cdot D_y^{k-m} u_i^\sim)(\cdot, y, \cdot)\|_{L^p(S_T)} \right\}. \end{aligned}$$

We now

(i) Define a parabolic operator $L_i^\sim$ on $R_+^{n+1} \times (0, T)$, with coefficients bounded and measurable, and those of the leading terms, uniformly continuous in $\overline{R_+^{n+1}} \times [0, T]$, that satisfies

$$L_i^\sim(u_i^\sim)(x, y, t) = L(u)(f_i(x, y), t) \quad \text{for } f_i(x, y) \in U_i;$$

- (ii) Apply the estimates for $L_i^\sim$ (§ 20) to the first two terms above;
- (iii) Omit somewhat lengthy considerations to obtain, with $0 < \delta < \delta_0$ and $C(T) \rightarrow 0$ as $T \rightarrow 0^+$

$$\begin{aligned}
LHS_T^\beta(u; \Omega) &\leq \|u\|^\sim \leq C \sum_{k=0}^{b-1} \|A^{b-1+\mu-k}(D_{N_Q}^k u)(Q, t)\|_{L^p(\partial\Omega \times (0, T))} + \\
&+ C_{\delta, \delta_0} C(T) \sup_{\delta < y < 2\delta_0} \sum_{|\alpha| \leq 2b-1} \|(D^\alpha u)(Q + yN_Q, t)\|_{L^p(\partial\Omega \times (0, T))} + \\
&+ C_{\delta_0} C(T) \delta^{1-\gamma+1/p} \sup_{y < \delta} \sum_{|\alpha| \leq 2b-1} y^{[|\alpha|-(b-1)]} \|(D^\alpha u)(Q + yN_Q, t)\|_{L^p(\partial\Omega \times (0, T))} + \\
&+ C_{\delta_0} \sum_i \sup_{y < 2\delta_0} \sum_{k \geq m \geq 0} \|A^{-1+\mu-\beta} A^{b-1+\beta-(k-m)} (D_y^m \varphi_i^\sim D_y^{k-m} u_i^\sim)(\cdot, y, \cdot)\|_{L^p(S_T)};
\end{aligned}$$

(iv) Take δ small enough so we can move the third term in (iii) over to the left hand sides. We modify the constants (introducing C_δ in front of $\|u\|^\sim$) and fix this δ from now on. And finally

(v) CLAIM. There exists a $T_0 > 0$ such that the estimate in the Theorem holds for $T \leq T_0$.

Clearly, the second term in (iii) (modified as in (iv)) can be bounded with

$$C_{\delta, \delta_0} C(T) LHS_T^\beta(u; \Omega) \leq C_\delta C_{\delta, \delta_0} C(T) \cdot \|u\|^\sim.$$

Also, the fourth term in (iii) can be shown to be bounded by

$$C_{\beta, \mu}(T) C_{\delta, \delta_0} C_\delta \|u\|^\sim, \quad \text{with } C_{\beta, \mu}(T) \rightarrow 0 \text{ as } T \rightarrow 0^+.$$

Therefore, if T is selected so that

$$C_{\delta_0} C_{\beta, \mu}(T) + C_{\delta, \delta_0} C(T) < 1,$$

then

$$(\#) \quad LHS_T^\beta(u; \Omega) \leq C_{\delta, \delta_0, \beta, \mu} \sum_{k=1}^{b-1} \|A^{b-1+\mu-k}(D_{N_Q}^k u)(Q, t)\|_{L^p(\partial\Omega \times (0, T))},$$

i.e. there exists a $T_0 > 0$, depending on $\delta, \delta_0, \beta, \mu$ such that the theorem is true for $\partial\Omega \times (0, T)$, $T \leq T_0$.

REMARK. For $T \leq T_0$, (#) is also true for all the domains $\Omega_{\delta_0^1}$, $0 \leq \delta_0^1 < \delta_0$, with the same constant $C_{\delta, \delta_0, \beta, \mu}$ (which, we recall, depends on the families $\{U_i\}$, $\{\varphi_i\}$, $\{\zeta_i\}$, cf. definitions in § 24).

§ 26. — To prove the theorem for any T , we rewrite the estimates in (iii) with third term deleted, see (iv), as

$$LHS_T^\beta(u; \Omega) \leq C_{\delta_0} \left\{ \sum_{k=0}^{b-1} \|A^{b-1+\mu-k}(D_{N_Q}^k u)\|_{L^p(\partial\Omega \times (0, T))} + \|u\| \approx \right\},$$

where we have set

$$\begin{aligned} \|u\| \approx & \sup_{\delta < y < 2\delta_0} \sum_{|\alpha| \leq 2b-1} \|(D^\alpha u)(Q + yN_Q, t)\|_{L^p(\partial\Omega \times (0, T))} + \\ & + \sum_i \sup_{y < 2\delta_0} \sum_{k \geq m \geq 0}^{b-1} \|A^{-1+\mu-\beta} A^{b-1+\beta-(k-m)} (D_y^m \varphi_i \tilde{D}_y^{k-m} u_i)(\cdot, y, \cdot)\|_{L^p(S_T)}, \end{aligned}$$

and observe that the desired result is a consequence of the following lemma, whose proof is also omitted

LEMMA — Given $\varepsilon > 0$, there exists a constant C_ε such that for all $u \in \dot{L}_{2b,1}^p(\Omega \times (0, T))$ that satisfy $Lu = 0$ we have

$$\|u\| \approx \leq \varepsilon LHS_T^\beta(u; \Omega) + C_\varepsilon \sum_{k=0}^{b-1} \|A^{b-1+\mu-k}(D_{N_Q}^k u)\|_{L^p(\partial\Omega \times (0, T))}.$$

REMARK. The a priori estimate in Theorem, § 25, holds for all domains $\Omega_{\delta_0^1}$, with $\delta_0^1 \ll \delta_0$. The constant in the theorem is independent of δ_0^1 (cf. also Remark to § 25).

§ 27. — We are now in a position to prove the main results.

THEOREM 1. Let $w^k \in L^p(\partial\Omega \times (0, T))$, $k = 0, \dots, b-1$, be such that $A^{b-1+\mu-k} w^k \in L^p(\partial\Omega \times (0, T))$ for some μ , $0 < \mu < 1$. Then there exists a solution to the problem

- (i) $u(P, t) \in \dot{L}_{2b,1}^p(\Omega^* \times (0, T))$ for every subdomain Ω^* such that $\bar{\Omega}^* \subset \Omega$,
- (ii) $Lu = 0$ in $\Omega \times (0, T)$, and
- (iii) for every β , $0 \leq \beta < \mu$,

$$\lim_{y \rightarrow 0^+} \|A^{b-1+\beta-k}[D_{N_Q}^k u(Q + yN_Q, t) - w^k(Q, t)]\|_{L^p(\partial\Omega \times (0, T))} = 0.$$

PROOF. Let $w_j^k \in C_0^\infty(\partial\Omega \times (0, \infty))$, such that $w_j^k \rightarrow w^k$ and $A^{b-1+\mu-k} w_j^k \rightarrow A^{b-1+\mu-k} w^k$, in $L^p(\partial\Omega \times (0, T))$ as $j \rightarrow \infty$, $0 \leq 1 \leq b-1$. It is known [10] that we can find $u_j(P, t) \in \dot{L}_{2b,1}^p(\Omega \times (0, T))$ satisfying (ii) and (iii) with w^k replaced by w^k .

Now if $\theta(P) \in C_0^\infty(\Omega)$, $\theta \equiv 1$ in Ω_δ , then

$$\|\theta u_j\|_{L_{2b,1}^p(\Omega \times (0,T))} \leq C_\delta \sum_{\substack{|\beta| > 0 \\ |\beta| + |\alpha| \leq 2b}} \|(D^\beta \theta)(D^\alpha u_j)\|_{L^p(\Omega \times (0,T))}.$$

This, together with the theorem, § 25, imply that if $\bar{\Omega}^* \subset \Omega$,

$$\|u_j - u_l\|_{L_{2b,1}^p(\Omega^* \times (0,T))} \leq C_{\Omega^*} \sum_{k=0}^{b-1} \|A^{b-1+\mu-k}(w_j^k - w_l^k)\|_{L^p(\partial\Omega \times (0,T))}.$$

Hence $\{u_i\}$ is a Cauchy sequence in $L_{2b,1}^p(\Omega^* \times (0, T))$, for every subdomain Ω^* with $\bar{\Omega}^* \subset \Omega$. Let $u(P, t)$ denote the limit of $\{u_i\}$. Clearly $u(P, t)$ satisfies (i) and (ii).

To prove (iii) we observe that

$$\begin{aligned} \sup_{y < \delta_0} \sum_{k=0}^{b-1} \|A^{b-1+\beta-k} D_{N_Q}^k(u_j - u_l)(Q + yN_Q, t)\|_{L^p(\partial\Omega \times (0,T))} &\leq \\ &\leq C \sum_{k=0}^{b-1} \|A^{b-1+\mu-k}[w_j^k - w_l^k]\|_{L^p(\partial\Omega \times (0,T))}, \end{aligned}$$

which implies

$$\begin{aligned} \sup_{y < \delta_0} \sum_{k=0}^{b-1} \|A^{b-1+\beta-k} D_{N_Q}^k(u_j - u)(Q + yN_Q, t)\|_{L^p(\partial\Omega \times (0,T))} &\leq \\ &\leq C \cdot \lim_{j \rightarrow \infty} \sum_{k=0}^{b-1} \|A^{b-1+\mu-k}[w_j^k - w_l^k]\|_{L^p(\partial\Omega \times (0,T))} \rightarrow 0 \quad \text{as } j \rightarrow \infty. \end{aligned}$$

Now we observe that

$$A^{b-1+\beta-k} D_{N_Q}^k u = A^{b-1+\beta-k} D_{N_Q}^k(u - u_j) + A^{b-1+\beta-k} [D_{N_Q}^k u_j],$$

and that

$$\lim_{y \rightarrow 0^+} \|A^{b-1+\beta-k} [D_{N_Q}^k(u - u_j)]\|_{L^p(\partial\Omega \times (0,T))} \leq \sup \|A^{b-1+\beta-k} D_{N_Q}^k(u - u_j)\|_{L^p(\partial\Omega \times (0,T))},$$

and we see that

$$\lim_{y \rightarrow 0^+} A^{b-1+\beta-k} [D_{N_Q}^k u] = \lim_{j \rightarrow \infty} A^{b-1+\beta-k} w_j^k = A^{b-1+\beta-k} w^k \quad \text{in } L^p(\partial\Omega \times (0, T)).$$

THEOREM 2. Let $u(P, t)$ be any function such that

- (i) $u(P, t)$ belongs in $L_{2b,1}^p(\Omega^* \times (0, T))$ for every subdomain Ω^* such that $\bar{\Omega}^* \subset \Omega$,

- (ii) $Lu \equiv 0$ in $\Omega \times (0, T)$,
 (iii) for $k = 0, \dots, b-1$, and for some μ , $0 < \mu < 1$,

$$\lim_{y \rightarrow 0^+} \|A^{b-1+\mu-k}[D_{N_Q}^k u](Q + yN_Q, t)\|_{L^p(\partial\Omega \times (0, T))} < \infty.$$

Then, for $0 \leq \beta < \mu$,

$$\begin{aligned} \sup_{y < \delta_0} \left\{ \sum_{|\alpha| \leq 2b-1} y^{[|\alpha|-(b-1)]} \|D^\alpha(A^\beta u)(Q + yN_Q, t)\|_{L^p(\partial\Omega \times (0, T))} + \right. \\ \left. + \sum_{k=0}^{b-1} \|A^{b-1+\beta-k}[D_{N_Q}^k u](Q + yN_Q, t)\|_{L^p(\partial\Omega \times (0, T))} \right\} \leq \\ \leq C \cdot \sum_{k=0}^{b-1} \lim_{y \rightarrow 0^+} \|A^{b-1+\mu-k}[D_{N_Q}^k u](Q + yN_Q, t)\|_{L^p(\partial\Omega \times (0, T))}. \end{aligned}$$

COROLLARY. The solution to problem (i)-(iii) in Theorem 1 is unique.

PROOF OF THEOREM 2. We recall that $(\cdot)^{[|\alpha|-(b-1)]} \equiv 1$ for $|\alpha| \leq b-1$ (§ 25). We consider any fixed y , $0 < y < \delta_0$, we take $0 < \delta_0^1 \ll y$, and study $\Omega_{\delta_0^1} \times (0, T)$; clearly $u \in \tilde{L}_{2b,1}^p(\Omega_{\delta_0^1} \times (0, T))$ and $Lu = 0$. By the a priori estimate in Theorem, § 25 (cf. also the Remark to § 26), and with $0 \leq \beta < \mu$,

$$\begin{aligned} \sum_{|\alpha| \leq 2b-1} (y - \delta_0^1)^{[|\alpha|-(b-1)]} \|D^\alpha(A^\beta u)(Q + yN_Q, t)\|_{L^p(\partial\Omega \times (0, T))} + \\ + \sum_{k=0}^{b-1} \|A^{b-1+\beta-k}[D_{N_Q}^k u](Q + yN_Q, t)\|_{L^p(\partial\Omega \times (0, T))} \leq \\ \leq C_{\beta, \mu, \delta_0} \cdot \sum_{k=0}^{b-1} \|A^{b-1+\mu-k}[D_{N_Q}^k u](Q + \delta_0^1 N_Q, t)\|_{L^p(\partial\Omega \times (0, T))}. \end{aligned}$$

The theorem follows by taking $\lim$ on both sides and recalling that C_{β, μ, δ_0} does not depend on δ_0^1 (cf. loc. cit.).

IX. - The elliptic estimate.

§ 28. - As in § 23, we let $\mathcal{E} \equiv \sum_{|\alpha| \leq 2b} a_\alpha(P) D^\alpha$ be a strongly elliptic operator on Ω , that is, $\mathcal{E} - D_t$ is parabolic in the sense of Petrovski. Again $\pi > 0$ will denote the parameter of parabolicity of $\mathcal{E} - D_t$ and the a 's are assumed to be bounded and measurable, and for $|\alpha| = 2b$, uniformly continuous in $\bar{\Omega}$. $G_{-\beta}$ shall denote the Bessel potentials defined in § 24.

As done in §§ 25-27, the expression $(\cdot)^{[|\alpha|-(b-1)]}$ is to be replaced by 1 for $|\alpha| \leq b-1$.

THEOREM. If $u \in L^p_{2b}(\Omega)$ and $\varepsilon u = 0$, then for $0 < \beta < \mu$,

$$\begin{aligned} \sup_{y < \delta_0} \left\{ \sum_{|\alpha| \leq 2b-1} y^{[|\alpha|-(b-1)]} \|D_\alpha[G_{-\beta}u](Q + yN_Q)\|_{L^p(\partial\Omega)} + \right. \\ \left. + \sum_{k=0}^{b-1} \|G_{-(b-1+\beta-k)}[D_{N_Q}^k u](Q + yN_Q)\|_{L^p(\partial\Omega)} \right\} \\ \leq C \left\{ \sum_{k=0}^{b-1} \|G_{-(b-1+\mu-k)}[D_{N_Q}^k u](Q)\|_{L^p(\partial\Omega)} + \|u\|_{L^1(\Omega)} \right\}. \end{aligned}$$

The proof of this result follows lines analogous to those in the parabolic estimate of § 25. We shall only sketch them.

By application of the definitions of the Bessel potentials to the left hand side ($LHS^\beta(u)$) of the inequality, a bound is obtained to whose terms the estimates of § 23 and Remark, § 23 apply with elliptic operators $\varepsilon_i^\sim$ defined by

$$\varepsilon_i^\sim(u_i^\sim)(x, y) = \varepsilon u(f_i(x, y)) \quad \text{for } f_i(x, y) \in U_i.$$

Support considerations on the C_0^∞ functions in the definition of $G_{-\beta}$ (§ 24) lead to the estimate

$$\begin{aligned} LHS^\beta(u) \leq C \left(\sum_{k=0}^{b-1} \|G_{-(b-1+\mu-k)}(D_{N_Q}^k u)(Q)\|_{L^p(\partial\Omega)} + \|u\|_{L^1(R_+^{n+1})} \right) + \\ + \|u\|^\infty + C_{\delta_0} \delta^{1-\gamma+1/p} \sup_{y < \delta} \sum_{|\alpha| \leq 2b-1} y^{[|\alpha|-(b-1)]} \|D^\alpha u(Q + yN_Q)\|_{L^p(\partial\Omega)} \\ + C_{\delta, \delta_0} \sup_{\delta < \gamma < 2\delta_0} \sum_{|\alpha| \leq 2b-1} \|D^\alpha u(Q + yN_Q)\|_{L^p(\partial\Omega)}, \end{aligned}$$

where we have set

$$\|u\|^\infty \equiv C_{\delta_0} \sum_i \sup_{y < 2\delta_0} \sum_{k \geq m \geq 0}^{b-1} \|G_{1-\mu+\beta} G_{-(b-1+\beta-(k-m))} (D_y^m \varphi_i^\sim D_y^{k-m} u_i^\sim)(y)\|_{L^p(R^n)}.$$

Fixing δ small enough, the third term in the estimate above can be moved over to the left hand side. The last term can be shown to be $\leq \varepsilon \cdot C \|\varepsilon(\varphi u)\|_{L^p(\Omega)} + C_\varepsilon \|u\|_{L^1(\Omega)}$, where $\varphi \in C_0^\infty(\Omega)$, $\varphi \equiv 1$ on $\Omega' \supset \supset \Omega_\delta$, by using a trace theorem [11], and an estimate in [1], already needed for Remark, § 23.

But $\|\varepsilon(\varphi u)\|_{L^p(\Omega)} \leq C_{\delta\varphi} LHS^\beta(u)$, so choosing ε small enough we even-

tually get

$$LHS^\beta(u) \leq C \left\{ \sum_{k=0}^{b-1} \|G_{-(b-1+\mu-k)}(D_{N_Q}^k u)\|_{L^p(\partial\Omega)} + \|u\|_{L^1(\Omega)} + \|u\|_{\approx} \right\}.$$

The proof can be completed by proving the following

LEMMA. To every $\varepsilon' > 0$ there is a constant $C_{\varepsilon'}$ such that every $u \in L_{2b}^p(\Omega)$ with $\delta u = 0$ satisfies

$$\|u\|_{\approx} \leq \varepsilon' LHS^\beta(u) + C_{\varepsilon'} \|u\|_{L^1(\Omega)}.$$

REFERENCES

- [1] S. AGMON - A. DOUGLIS - L. NIRENBERG, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, I*, Communications on Pure and Applied Mathematics, **12**, no. 4 (1959).
- [2] E. B. FABES, *Singular integrals and partial differential equations of parabolic type*, Studia Mathematica, **28** (1966).
- [3] E. B. FABES - M. JODEIT Jr., *L^p -boundary value problems for parabolic equations*, Bulletin A.M.S., **74**, no. 6 (1968). Also *Singular integrals and boundary value problems for parabolic equations in a half-space*, University of Minnesota.
- [4] E. B. FABES - N. M. RIVIÈRE, *Systems of parabolic equations with uniformly continuous coefficients*, Journal d'Analyse Mathématique, **17** (1966).
- [5] E. B. FABES - N. M. RIVIÈRE, *L^p -estimates near the boundary for solutions of the Dirichlet problem*, Annali della Scuola Normale Superiore, Pisa, **24**, fasc. III (1970).
- [6] E. B. FABES, *Singular integrals and their applications to the Cauchy and Cauchy-Dirichlet problems for parabolic equations*, Notes from a course given at the Istituto Matematico, Università di Ferrara, Italia, 1970.
- [7] A. FRIEDMAN, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1964.
- [8] G. H. HARDY - J. E. LITTLEWOOD - G. PÓLYA, *Inequalities*, Cambridge at the University Press, 1952.
- [9] E. M. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.
- [10] V. A. SOLONNIKOV, *On boundary value problems for linear parabolic systems of differential equations of a general type* (Russian), Trudy Matematicheskogo Instituta Akademii Nauk (Steklov), **83**, Leningrad, 1965.
- [11] E. GAGLIARDO, *Proprietà di alcune classi di funzioni di più variabili*, Ricerche di Matematica, **7** (1958).
- [12] W. POGORZELSKI, *Étude de la matrice des solutions fondamentales du système parabolique d'équations aux dérivées partielles*, Ricerche di Matematica, **7** (2) (1958).