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Smoothness and Analyticity of Free Boundaries in Variational Inequalities.

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dedicated to Hans Lewy

The study of free boundary problems consists basically of two parts; the study of the topological structure and the study of the differentiable structure.

The techniques for the study of the differentiable structure were laid down by H. Lewy in a series of papers [8], [9], [10] and applied later by H. Lewy and G. Stampacchia [11] in the linear case and by D. Kinderlehrer [6] in the minimal surface case. In brief, such techniques were based on the analytic continuation of a conformal mapping, requiring some a priori topological knowledge of the free boundary and analyticity of the data. To obtain the necessary topological structure conditions of geometric type, such as convexity of the boundary and concavity of the obstacle, were imposed.

In this paper we localize the analysis of the free boundary and remove the artifically imposed, geometric conditions. Moreover a more flexible use of quasi-conformal extensions of conformal mappings yields regularity of the free boundary according to the smoothness of the data. The use of quasi-conformal extensions was already considered by D. Kinderlehrer [7], where he obtains the $C^{1,\alpha}$ character of the free boundary when assuming a priori topological behaviour.

The paper is organized as follows. In the first paragraph we prove a series of lemmas which allows the local study of the free boundary.

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In the second paragraph we study specifically the two dimensional free boundary problem arising from the variational inequality that solves the minimal energy problem (Laplace's equation) above a given obstacle.

In the third paragraph we study the nonlinear case as considered in [1], and we extend the regularity results of the linear case; in particular we obtain the regularity of the free boundary of a minimal surface above a given obstacle.

In the fourth paragraph we study the nowhere dense part of the set of coincidence considered in the previous paragraphs.

Finally, in the last paragraph we give a brief application of the techniques of the first paragraph to the Stefan problem and a number of examples are constructed to exhibit the topological behavior of the free boundary of the problems considered in paragraphs 2 and 3.

1. – Density properties of the free boundary.

In this paragraph we show some elementary properties of a free boundary. Although the lemmas of this section are valid for quasilinear elliptic operators of second order, as we will see in Section 3, in this first paragraph we will limit ourselves to the Laplace equation.

Lemma 1.1 shows a density property of the set of non-coincidence at its boundary points, Lemma 1.2 estimates the area of a sequence of surfaces approximating the free boundary, and Lemma 1.3 shows that the solution u can be extended in a C^2 way across the non-dense parts of the coincidence set. Finally, Lemma 1.4 shows how a local construction of a mapping considered by Lewy and Stampacchia [9], preserves, in some weak form, the property of mapping the non-coincidence set into the coincidence set.

LEMMA 1.1. Let Ω be a bounded, connected, open set of R^n and u a real valued function defined in $\bar{\Omega}$ satisfying:

(i) $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$;

(ii) $\Delta u(x) = g(x) > \lambda_0 > 0$, for $x \in \Omega$.

Consider an open set, O, and assume that $u(x) = 0$, $\nabla u(x) = 0$ when $x \in \partial\Omega \cap O$. Then for each $x \in O \cap \partial\Omega$ there exists $y = y(x) \in \partial O \cap \bar{\Omega}$ such that

$$u(y) \geqslant \frac{\lambda_0}{2n} |x - y|^2 .$$

PROOF. For $x \in O \cap \partial\Omega$ set $\varphi_x(z) = (\lambda_0/2n)|x - z|^2 - u(z)$. Clearly $\varphi_x \in C^2(\Omega) \cap \cap C^1(\bar{\Omega})$, it is superharmonic in Ω and $\varphi_x(x) = 0$, $\nabla\varphi_x(x) = 0$.

Assume first that there exists a ball, B, $B \subset \Omega$ and such that $x \in \partial B$ (such x's are dense in $\partial\Omega$ since Ω is open); then, in virtue of the strict maximum principle, $\varphi_x(y)$ must be negative at some point y, $y \in \partial(\Omega \cap O)$. On the other hand observe that $\varphi_x(z) \geqslant 0$ when $z \in O \cap \partial\Omega$. Hence $\varphi_x(y) < 0$ for some $y \in \partial O \cap \bar{\Omega}$ and the lemma follows for such x.

When x is not in the boundary of a ball contained in Ω, there exists a sequence, $\{x_m\}$, for which the property holds and $x_m \to x$ as $m \to \infty$. Since φ_{x_m} converges uniformly in $\bar{\Omega}$ to φ_x, $\varphi_x(z)$ can not be strictly positive in $\partial O \cap \bar{\Omega}$ and the lemma follows

LEMMA 1.2. *Under the assumptions of Lemma* 1.1, *if further* $u \in C^{1,1}(\bar{\Omega})$; *then:*

(i) *Given* $x \in \partial\Omega \cap O$ *and a ball,* $B(x, \varrho)$, $B(x, \varrho) \subset O$; *there exists* δ (δ *fixed*) *and* $y = y(x) \in \partial B(x, \varrho)$ *such that* $B(y, \delta\varrho) \cap O \subset \Omega$.

(ii) *If* Q *is a cube of side* l *contained in* O, *there exist constants* M *and* α; M *an integer,* $0 < \alpha < 1$; *such that if we decompose* Q *into* M^{kn} cubes, $\{Q_i\}$, *of side* lM^{-k} *where* k *is an arbitrary positive integer, and form*

$$U_k = \cup Q_i\,, \qquad Q_i \cap \partial\Omega \neq \emptyset\,,$$

then U_k *has at most* $(M\alpha)^{nk}$ *cubes. In particular the area of* ∂U_k *verifies*

$$A(\partial U_k) \leqslant \sum_i A(\partial Q_i) \leqslant (M\alpha)^k m(Q)$$

and

$$m(U_k) \leqslant \alpha^k m(Q)\,.$$

PROOF. To prove (i) we apply Lemma 1.1 to $B(x, \varrho)$ instead of O. Hence, there exists $y \in \partial B(x, \varrho) \cap \bar{\Omega}$ such that $u(y) \geqslant (\lambda_0/2n)|x - y|^2$; therefore, since ∇u is Lipchitz continuous in $\bar{\Omega}$, there exists δ, depending on the Lipchitz constant only, such that $u(z)$ and $\nabla u(z)$ can not vanish simultaneously when $z \in B(y, \delta\varrho)$, that is $B(y, \delta\varrho) \subset \Omega$.

Given a cube, Q, we partition it into M^n subcubes, $\{Q_i\}$, of side l/M; to prove part (ii) it suffices to show that for M large, but independent of Q, at least one of the cubes of the partition does not intersect $\partial\Omega$. The result would then follow by induction with respect to k.

Let Q_0 be one of the central cubes of the partition and assume that $x_0 \in Q_0 \cap \partial\Omega$. In virtue of (i) there exists $y(x_0) \in \partial B(x_1, l/2)$ such that $B(y, \delta, l/2) \subset \Omega$. If M is chosen large enough, there must exist $Q_i \subset$ $\subset B(y, \delta, l/2) \subset \Omega$, and the lemma follows.

LEMMA 1.3. *Under the hypothesis of Lemma* 1.2 *assume further that* $\Delta u = g$, *with* g *Hölder continuous in* $\bar{\Omega} \cap O$. *Then* $\Delta u = g$ *in* $(\bar{\Omega})^0 \cap O$.

PROOF. It suffices to show that, given $x_0 \in (\bar{\Omega})^0 \cap \complement(\Omega)$, $\Delta u = g$ in the sense of distributions in a neighborhood of x_0.

Let Q be a small cube centered at x, $Q \subset O$ and φ a $C_0^\infty(R^m)$ function with support contained in Q. Following the notation of Lemma 1.2

$$\int_Q u(x)\,\Delta\varphi(x)\,dx = \int_{Q\cap\complement(U_k)} u(x)\,\Delta\varphi(x)\,dx + \int_{U_k} u(x)\,\Delta\varphi(x)\,dx = I + II\,.$$

In virtue of the fact that $m(U_k) \to 0$ as $k \to \infty$, (II) tends to zero.

On the other hand

$$I = \int_{Q\cap\complement(U_k)} u(x)\,\Delta\varphi(x)\,dx = \int_{\partial U_k} \big(u(x)\,\partial_\nu\varphi(x) - \varphi(x)\,\partial_\nu u(x)\big)\,dH_{n-1} + \\ + \int_{Q\cap\complement(U_k)} \Delta u(x)\,\varphi(x)\,dx = P + Q\,.$$

Since, for $x \in U_k$, $|u(x)| \leqslant CM^{-2k}$, $|\nabla u(x)| \leqslant CM^{-k}$ and $A(\partial U_k) \leqslant M^k\alpha^k m(Q)$, it follows that P tends to zero as k tends to infinity.

Also

$$\int_{Q\cap\complement(U_k)} \Delta u(x)\,\varphi(x)\,dx \quad \text{tends to} \quad \int_Q g(x)\,\varphi(x)\,dx$$

and the lemma follows.

Finally, in this first paragraph we want to study a «reflexion» function that will provide us with the local topological behavior needed to obtain the smoothness of the boundary. Assume the hypothesis of Lemma 1.3, and let the coordinates be fixed at a point x_0 of $\big(\partial(\bar{\Omega})\big) \cap O$.

Suppose also that g has a Lipschitz extension $\bar{g}$, to a neighborhood of x_0.

Under such assumptions, using standard arguments, we may construct a function, γ, solution of the equation $\Delta\gamma = -\bar{g}$ in a neighborhood of x_0, and such that

$$\gamma(x) = -\frac{1}{2n}\bar{g}(x_0)\varrho^2 + 0(\varrho^{2+\alpha})\,.$$

with $\varrho = |x - x_0|$ and $0 < \alpha < 1$.

In a small neighborhood, U_0, of x_0, $\nabla\gamma$ is a diffeomorphism of the form

$$\nabla\gamma(x) = \frac{1}{2n}\bar{g}(x_0)(x - x_0) + 0\,(\varrho^{1+\alpha}).$$

Hence, there exists a well defined function $\xi(x)$, $\xi\colon U_0 \cap \Omega \to R^n$, verifying

$$\nabla\gamma\big(\xi(x)\big) = \nabla(\gamma + u)(x)\,,$$

when $x \in U_0$.

Clearly, if $x \in \partial\bar{\Omega}$, $\xi(x) = x$. We want to prove the following

LEMMA 1.4. *If $x \in U_0 \cap \bar{\Omega}$, and there exists a ball B, $B \subset \mathrm{C}(\bar{\Omega})$, $x \in \partial B$, then there exists a sequence $\{x_n\}$, $x_n \in \Omega$, $x_n \to x$, such that $\xi(x_n) \in \mathrm{C}(\bar{\Omega})$.*

PROOF. Assume that the coordinate axes have been chosen so that in a small neighborhood of x,

$$\gamma(y) = l(y) + \sum \alpha_i(x^i - y^i)^2 + O(|x - y|^{2+\alpha})$$

with $\sum \alpha_i = \frac{1}{2}\Delta\gamma(x)$ and l a linear function.

We may assume that U_0 is so small that $\alpha_i < 0$ for every i.

Let us consider the ellipsoid

$$E(\varrho) = \{y\colon \sum(-\alpha_i)(y^i - x^i)^2 < \varrho^2\}\,.$$

We can choose $\varepsilon(\varrho) \to 0$ with $\varrho \to 0$, such that, by the argument of Lemma 1.1, the function $h(y) = u(y) + (1 - \varepsilon(\varrho)) \sum \alpha_i(y^i - x^i)^2$ is subharmonic in $E(\varrho) \cap \Omega$ and hence has a positive maximum at a point $z \in \partial E(\varrho)$, $\varrho < \varrho_0$.

The gradient of h verifies, by the strict maximum principle,

$$(*) \qquad \begin{aligned}\nabla h(z) &= \nabla u(z) + 2(1 - \varepsilon(\varrho))(\alpha_1(z^1 - x^1), \ldots, \alpha_n(z^n - x^n)) \\ &= -\lambda(\alpha_1(z^1 - x^1), \ldots, \alpha_n(z^n - x^n))\end{aligned}$$

with $\lambda > 0$ ($\nabla h(z)$ is normal to $\partial E(\varrho)$ at z).

We recall that by the Lipschitz character of ∇u, there is a ball $B^*(z, C\varrho) \subset \Omega$.

Since B^* cannot intersect the ball B_1, lying in $C(\bar{\Omega})$ and tangent to x, the point z must lay in a cone C, with vertex at x and whose axis is the prolongation of the radius of B_1 passing through x ($C = \{y\colon (y - x, \nu) \geqslant \beta|y - x|\}$, where ν is the exterior normal to B_1 at x).

On the other hand, by definition

$$\nabla\gamma(\xi(z)) = (\nabla u + \nabla\gamma)(z)\,.$$

By the above formula, we get

$$\nabla\gamma(\xi(z)) - \nabla\gamma(z) = -(2 + \lambda - 2\varepsilon(\varrho))(\alpha_1(z^1 - x^1), \ldots, \alpha_n(z^n - x^n))\,.$$

That is

$$2(\alpha_1(\xi^1 - z^1), \ldots, \alpha_n(\xi^n - z^n)) =$$

$$= -2(1 + \lambda)(\alpha_1(z^1 - x^1), \ldots, \alpha_n(z^n - x^n)) + O(|z - x|^{1+\alpha})\,.$$

To complete the proof we will show that λ remains larger than a fixed number $\lambda_0 > 0$ as $\varrho \to 0$. The conclusion will then follow since z lies in the cone C mentioned above.

LEMMA 1.5. *With the hypothesis of the preceeding lemma, let λ be defined as before* (See $(*)$).
Then, λ remains larger than a fixed positive constant λ_0 as ϱ tends to zero.

PROOF. The ellipsoid $E(\varrho)$ contains a ball $B\varrho'$ tangent from inside to $E(\varrho)$ at the point z, with $\varrho' = C\varrho$ and satisfying that $B\varrho' \subset \Omega$.

In order to prove our lemma we make the following claim

$$\inf_{w \in \partial B\varrho} [h(w) - h(z)] < - C\varrho^2 ,$$

with C positive independent of ϱ.

We postpone the proof of the claim and proceed with the lemma.

Observe that the Lipschitz character of the derivatives of u imply that $h(w) - h(z)$ is less than $-\frac{1}{2}C\varrho^2$, as a function of w, over a subset of $\partial B\varrho'$ proportional to its total area. This can be shown noticing that, if $h(w_0) = \inf_{w \in \partial B\varrho'} h(w)$ the tangent derivatives of h vanish at w_0 and therefore the property follows integrating the derivatives of h along meridians passing through w_0.

Finally, observe that the harmonic function $g(w)$ whose value on $\partial B\varrho'$ is $h(w) - h(z)$, bounds $h(w) - h(z)$ from above in $(B\varrho)^0$. But, since $g(w)$ is less than or equal to $- C\varrho^2$ on a proportional part of $\partial(B\varrho')$, $\partial_\nu g(z) < - C\varrho$ ($\partial\nu$ denotes the normal derivative) and the lemma follows.

PROOF OF CLAIM: Notice that for $\varrho < \varrho_0$; $[E(\varrho) \sim E(\varrho/2)] \cap \partial(\bar{\Omega}) \neq \emptyset$, since by hypothesis there exists a ball contained in $C(\bar{\Omega})$ and tangent to x.

An elementary geometric construction provides us with a chain of balls, $\{B_m(\chi_m, \varrho_m)\}_{m=0}^k$, where k depends only on n, and such $B_0 = B\varrho'$, $B_m \subset E(\varrho) \sim E(\varrho/2)$, $x_m \in \partial B_{m-1}$, and $\partial B_k \cap \partial\Omega \neq \emptyset$. Therefore $h(w) - h(z) < - C_0\varrho^2$ for $w \in \partial B_k \cap \partial\Omega$, since $u(w) = 0$, $|w - x| \sim \varrho$, and $h(z) \geqslant 0$.

Proceeding inductively, $h(x_1) - h(z) \leqslant C_k\varrho^2$ and the claim follows.

2. - The topological behaviour of the free boundary.

We now turn our attention to the free boundary problem arising from variational inequalities.

Let D be a bounded connected open set of R^n; φ, the obstacle, a real valued function defined on R^n satisfying

a) $\varphi \in C(\overline{D})$, $\varphi \in C^3(D)$;

b) $\varphi < 0$ on ∂D;

c) $\Delta\varphi$ and $\nabla(\Delta\varphi)$ do not vanish simultaneously.

Let v be the least superharmonic vanishing at ∂D and verifying $v \geqslant \varphi$.
This function can be obtained, for instance, by minimizing

$$\int_D (\nabla u)^2 \, dx$$

on the convex set of $H_0^1(D)$ defined by

$$K = \{u \in H_0^1(D)\, u \geqslant \varphi\}$$

and it is known that v has Lipschitz first derivatives in any compact subset of D (see Frehse [3], Brezis-Kinderlehrer [1]).

The function v, defines two sets

$$\Omega = \{x : v(x) > \varphi(x)\}$$

and

$$\Lambda = D \sim \Omega = \{x : v(x) = \varphi(x)\} .$$

Observe that for $x \in \partial(D \sim \Omega)$, $\nabla v(x) = \nabla\varphi(x)$, and on Ω, $\Delta v = 0$.

LEMMA 2.1. *Under the preceeding conditions* $\complement\Lambda$ *has a finite number of connected components and* $\Delta\varphi < 0$ *on* Λ.

PROOF. Ω contains a neighborhood of ∂D and hence we can chose a compact set K (a finite union of cubes) such that $\Lambda \subset K$, $K \subset D$ and $\complement K$ has finitely many connected components.

On the other hand, since $\Delta\varphi$ and $\nabla(\Delta\varphi)$ do not vanish simultaneously and v is superharmonic we have

a) $V_1 = \{x : \Delta\varphi > 0\}$ and $V_2 = \{x : \Delta\varphi < 0\}$ have a finite number of components that intersect K;

b) $\Lambda \subset V_2$;

c) If C is a component of Ω, $C \not\subset \overline{V}_2 \cap K$.

By b) and c) there are at most as many components of Ω contained in K as those of V_1 and this proves the first part of the lemma.

To obtain the second part, consider $x \in \partial V_1$; in virtue of our assumptions ($\Delta\varphi$ and $\nabla(\Delta\varphi)$ do not vanish simultaneously), the cone

$$C_\varepsilon = \{z_1 \langle z - x, \nabla(\Delta\varphi)(x)\rangle \geqslant \varepsilon|z-x|\} \cap \{z, |z-x| < \delta_\varepsilon\},$$

satisfies $C_\varepsilon \subset V_1$ for small values of ε, $\varepsilon > 0$. On such cone, $v - \varphi$ is superharmonic and strictly positive, and $\nabla(v-\varphi)$ is Lipschitz hence $x \notin \Lambda$.

The preceeding lemma shows that we are in the conditions of the first paragraph of this work. That is, $\Delta\varphi < \varepsilon < 0$ in a neighborhood $\mathcal{U}$, of Λ and hence, if we consider $u = v - \varphi$, we have the following situation

a) $\Delta u = -\Delta\varphi \geqslant \lambda_0 > 0$ on $\Omega \cap \mathcal{U}$.

b) Δu has a Lipschitz extension from $\Omega \cap \mathcal{U}$ to $\bar{\Omega} \cap \mathcal{U}$.

c) u has Lipschitz first derivatives on $\mathcal{U}$.

d) u and ∇u vanish in $(\partial\Omega) \cap \mathcal{U}$.

Lemmas 1.1 to 1.3 show that the free boundary of Ω, $\partial\Omega$, has measure zero and that in $\Omega^* = (\bar{\Omega})^0$, $\Delta v = 0$. Hence it is natural to redefine Ω^* as the non coincidence set, and $\Lambda^* = D \sim \Omega^*$ as the coincidence set (see also § 4).

We recall that H. Brezis and D. Kinderlehrer [1, Theorem 3] show that Δu is locally of bounded variation. They also observe that, if $\Delta\varphi$ is negative then $\Delta u/\Delta\varphi (= 1 - \chi_\Lambda$ a.e.) is of bounded variation. (Λ has finite perimeter in the sense of De Giorgi). The same argument shows in our case that Λ^* has finite perimeter.

In the n-dimensional case each connected component of $(\Lambda^*)^0$ has finite perimeter [2, Lemma IV]. (Notice that $(\partial\Omega^*) \cap D = \partial[(\Lambda^*)^0]$ and its measure is zero).

For the remaining part of this paragraph we will restrict our analysis to the two dimensional case.

In such case, applying Theorem I of [2] it follows that the boundary of each connected component of $(\Lambda^*)^0$ is formed of a finite number of rectifiable Jordan curves.

In order to apply the reflection techniques of H. Lewy, we will prove that, once a finite nunber of points is deleted from each Jordan curve in a neighborhood of any other point of the curve, the only boundary of Ω^* is the curve itself.

Let C be a connected component of $(\Lambda^*)^0$, $x_0 \in \partial C$, and also $B(x_0, r_0)$ a small ball where $\Delta\varphi < 0$. We may assume that $C \cap \complement B(x_0, r_0) \neq \emptyset$. Let $\{C_j\}$ the connected components of $C \cap B(x_0, r_0/2)$ such that $x_0 \in \partial C_j$. Notice that there are a finite number of them (at most, as many as connected components has Ω^*).

Let us fix one of them, C_1, whose boundary is composed of a Jordan curve Γ, passing through x_0. (C_1 is again of finite perimeter and Theorem 1 of [2] applies). Observe that any component of $\Omega^* \cap B(x_0, r_0)$ must intersect $\partial B(x_0, r_0)$.

Let γ and ξ be as in Lemma 1.4. Choose r_1, $\frac{3}{4}r_0 < r_1 < r_0$, such that the Hessian $H(u+\gamma) \neq 0$ ($\nabla(u+\gamma)$ is analytic) in $\partial B(x_0, r_1) \cap \Omega^*$. Let α be small enough such that $\xi(B(x_0, \alpha r_0)) \subset B(x_0, r_0/2)$.

In the next two lemmas we look at the local behaviour of the mapping ξ, in $B(x_0, r_1)$, and show that except for a finite number of exceptional points ξ maps the non-coincidence set into the coincidence set.

We have already shown, Lemma 1.4, that ξ crosses the boundary near any point of $\Gamma \cap B(x_0, r_0)$. In the first lemma we will show that the number of components that ξ maps into C_1 is finite. In the second we prove that each of these components attaches to Γ and the exceptional points will be those where two of such components meet.

LEMMA 2.2. *Let $G = \xi^{-1}(C_1) \cap B(x_0, r_1)$, $G = \cup G_i$ (G_i connected components of G); then, the number of components G_i, such that $G_i \cap \partial B(x_0, \alpha r_0) \neq \emptyset$, is finite. Moreover, if $\mathcal{U}$ is a connected component of $\Omega^* \sim \bar{G}$, and $\mathcal{U} \cap B(x_0, \alpha r_0) \neq \emptyset$ then $\mathcal{U} \cap [\partial B(x_0, r_1) \sim \bar{G}] \neq \emptyset$.*

PROOF. Since $\Delta\varphi < 0$, $\Omega^* \cup \complement(\overline{B(x_0, r)})$ is connected, $0 < r < r_0$.

Assume that $\bar{\mathcal{U}} \subset B(x_0, r_1) \cup \bar{G}$. Since ξ is open $\partial\xi(\mathcal{U}) \subset \xi(\partial\mathcal{U})$ and $\xi(\partial\mathcal{U}) \cap [\Omega^* \cup \complement(\overline{B(x_0, r_1)})] = \emptyset$ (since $\partial\mathcal{U} \subset (\partial G) \cup (\partial\Omega^*)$).

On the other hand,

$$\xi(\mathcal{U}) \cap [\Omega^* \cup \complement(\overline{B(x_0, r_1)})] \neq \emptyset . \tag{1}$$

In fact, if $x \in \mathcal{U} \cap B(x_0, \alpha r_0)$ and $\xi(x) \in C_2$ (C_2 a component of $\Lambda^* \cap B(x_0, r_0/2)$) we can construct a bundle of disjoint paths in Ω^*, joing x with a fixed point of G. The images of this paths join $\xi(x)$ with a point of C; since, $\bar{C}_2 \cap \bar{C}_1$ has at most a finite number of points (see [2]) and ξ restricted to any compact subset of Ω has finite multiplicity, there exists a path δ such that $\xi(\delta)$ intersects ∂C_2 strictly before than ∂C_1. Being ξ open, (1) follows, but then $\xi(\mathcal{U}) \supset \Omega^* \cup \complement(\overline{B(x_0, r_1)})$ and that is impossible.

Let us see now that there are finitely many sets G_i.

If G_i is compactly contained in $B(x_0, r_1)$, in virtue of the fact that ξ is open, $\xi(G_i) \supset C$ moreover, since ξ is Lipschitz $\mu(G_i) \geqslant \alpha\mu(C)$ ($\mu(C)$ denotes the Lebesgue measure of C). Therefore, there are only a finite number of G_i compactly contained in $B(x_0, r_1)$.

On the other hand if infinitely many components G_i reach the boundary of $B(x_0, r_1)$, let y be a limit point of them. Clearly $y \notin \partial\Omega^*$ since $\xi(G) \cap B(x_0, r_0/2) \neq \emptyset$ and therefore it would be $\mu(G_i) > \alpha_0$ for infinitely many i.

Hence the Jacobian of ξ does not vanish in a neighborhood $B(y, \varrho)$ of y, and $\xi/B(y, \varrho)$ is a diffeomorphism that maps G onto $C_1 \cap \xi(B)$. Observe that $\xi^{-1}(\Gamma)$ intersects infinitely many times $\partial B(y, \varrho/4)$ and $\partial B(y, \varrho/2)$ alternatively, contradicting the fact that ∂C_1 is a Jordan Curve (C_1 has finite perimeter and hence its boundary is a rectifiable Jordan Curve.)

We recall that, by Lemma 1.4, any point y of Γ, not belonging to $\partial B(x_0, r_0/2)$, verifies $y \in \overline{G}$.

Let $\Gamma^*(t)$, $(a \leqslant t \leqslant b)$, be a portion of Γ verifying $x_0 \in \Gamma^*$, $\Gamma^* \cap \partial B(x_0, \alpha r_0) = \emptyset$. If $x(t_1)$, $x(t_2) \in \Gamma^*$, they divide Γ in two open arcs, Γ_1 and Γ_2.

Let $\Gamma_1^* = \Gamma_1 \cap \Gamma^*$, $\Gamma_2^* = \Gamma_2 \cap \Gamma^*$.

LEMMA 2.3. *If $x(t_1)$, $x(t_2) \in G_0$ then for Γ_1^* or Γ_2^* it follows that: for any $x \in \Gamma_i^*$, there exists $B(x, \varrho(x))$ verifying $\big(B(x, \varrho(x)) \sim \overline{C}_1\big) \subset G_0$.*

PROOF. Let a_n, $b_n \in G_0$, $a_n \to x(t_1)$, $b_n \to x(t_2)$, and Δ_n a poligonal arc, $\Delta_n \subset G_0$, joining a_n with b_n.

If we consider the Jordan curves α_n, formed by Γ_1, Δ_n and two small segments; and β_n, formed by Γ_2, Δ_n and two small segments, then either any compact subarc of Γ_1 lies in the interior of β_n for infinitely many n's, or any compact subarc of Γ_2 lies in the interior of α_n for infinitely many n's.

It suffices to consider either possibility.

Assume the first possibility holds and $x \in \Gamma_1^*$. We will argue by contradiction. If the lemma were to fail we may find a sequence $x_n \to x$, $x_n \notin G_0$ and moreover $x_n \in \Omega^*$. Each x_n can be joined to a point of $(\Omega^* \sim G) \cap \cap B(x_0, \alpha r_0)$ by an arc π_n. By a similar argument to that in the preceeding lemma we may construct π_n so that $\pi_n \subset \complement(\overline{C} \cup \overline{G}_0)$.

It is, then, clear that we can extend π_n to an arc π_n^* joining x_n with a point of $\partial B(x_0, r_1)$ with $\pi_n^* \subset \complement(\overline{C} \cup \overline{G}_0)$.

Choosing adequate subsequences we may assume that x_n lies in the interior of β_n and, since π_n^* can intersect β_m only in the small segments near $x(t_1)$, $x(t_2)$ we may therefore also assume that x_n does not lie in the interior of β_{n+k} $(k \geqslant 1)$ and that π_n^* intersects β_n for the first time always in the small segment near $x(t_1)$.

Let us remove two small balls $B(x(t_1))$ and $B(x)$ and analyze the situation.

Between the portion of π_n^* and π_{n+1}^* from $B(x)$ to $B(x(t_1))$ there is an arc of Δ_n, hence $G_0 \sim \big(B(x) \cup B(x(t_1))\big)$ has infinitely many connected components, G_n^*, each one containing an arc Δ_n^* joining $B(x)$ with $B(x(t_1))$.

To complete the lemma it suffices to show that $\xi(G_u^*)$ contains a fixed component of $C \sim \xi([B(x) \cup B(x(t_1)] \cap \Omega)$ which is a contradiction since then $\mu(G_n^*) > \varepsilon$, for every n.

To show this, we consider a « diameter » separating x from $x(t_1)$ (where by a « diameter » we mean the image of a diameter by a conformal mapping from the disk). The arcs of Δ_n above mentioned intersect this « diameter » if $B(x)$ and $B(x(t_1))$ have been chosen small enough and that completes the proof.

At this point of our work, we have shown that for each connected component C of Λ^*, its boundary, ∂C, is except for a finite number of points, « clean », that is, if $x \in \partial C$ and is not an exceptional point there is a neighborhood H of x that is divided by a Jordan arc $\Gamma \subset \partial C$, into two connected, simply connected domains, D_1 and D_2, $D_1 \subset C$, $D_2 \subset \Omega$. Hence on the domain D_2, we are in the conditions of the theorem 1 of Kinderlehrer [7] and therefore the curve Γ has a $C^{1,\alpha}$ parametrization ($\alpha < 1$). The results in [7] that interest us can be resumed as follows:

Let w be a conformal mapping from a portion, H, of the half plane $R_2^+ = \{(x, y), y > 0\}$ onto D_2 that carries a segment, (a, b) of the x-axis onto $\Gamma \cap H$ in a one to one fashion. Then $w \in C^{1,\alpha}$, $\alpha < 1$, up to (a, b) and at any point $z_0 \in (a, b)$, either $w(z) = a(z - z_0) + O(|z - z_0|^{1+\alpha})$, $a \neq 0$, or $w(z) = b(z - z_0)^2 + O(|z - z_0|^{2+\alpha})$, $b \neq 0$.

In the second case, centering w at z_0 and letting $b = 1$, $w(z) = z^2 + O(|z|^{2+\alpha}) = x^2 - y^2 + i2xy + O((x^2 + y^2)^{(2+\alpha)/2})$ in a neighborhood of $(0, 0)$, that is, $w(z)$ forms a quadratic cusp., D_2 being the exterior of it. But, by the properties of the reflexion mapping considered in Lemma 1.4; identity (1.41); near $(0, 0)$ ξ would not reflect the noncoincidence set D_2 into the coincidence set, property which must hold on D_2. Therefore, Γ is differentiable and has a nonzero tangent at each point.

The following lemma is a refinement of the result of D. Kinderlehrer [7]. We improve the reflexion mapping to obtain as much regularity of the free boundary as that of $\nabla\varphi$.

We will also keep track of the C^k continuity norms in order to prove analyticity of the free boundary. We must point out that a simplier reflexion argument yields the analyticity in the linear case (see [11]). However, the estimates are needed in the nonlinear case where a direct reflexion argument fails.

Let $z_0 \in \Gamma$ and assume that for a fixed r.

a) $\|D^j\varphi\|_\infty = \sup_{B_r(0)} |D^j\varphi| \leqslant C^j j!$, $1 \leqslant j \leqslant k+2$;

b) $\|D^{k+2}\varphi\|_\alpha = \sup_{x,y \in B_r(0)} \left| \dfrac{D^{k+2}\varphi(x) - D^{k+2}\varphi(y)}{|x - y|^\alpha} \right| \leqslant C^{k+2}(k+2)!$;

c) $\Delta\varphi < -\varepsilon$ on $B_r(z_0)$.

Let us fix $r'<r$, depending only on C, r, η, such that $Cr'<\eta$. (η to be chosen later). We will prove:

LEMMA. *Let w be a conformal mapping from the half circle into $D\subset B(z_0, r')\cap\Omega^*$, mapping the segment $(-1,1)$ (boundary diameter of the half circle) onto a segment of Γ (Γ is known to be $C^{1,\alpha}$), $w(0)=z_0$.*

Assume that w is Lipschitz with constant Λ in such domain and that for $t_0\in(-1,1)$

$$w(t)=\sum_0^k a_i(t-t_0)^i+O(1)|t-t_0|^{k+\alpha}) .$$

Then, under hypothesis (a) and (b), $w(t)$ verifies

$$w(t)=\sum_0^{k+1} a_i(t-t_0)^i+O'(1)|t-t_0|^{k+1+\alpha}$$

where

$$|a_{k+1}|\leqslant M\left[\sum_0^k\frac{|a_i|d^i}{d^{k+1}}+r'\Lambda\frac{(1+\lambda)}{d^{k+2}}+\frac{\|D^{k+2}\varphi\|_\alpha}{(k+2)!}\Lambda^{k+2}\frac{2}{\varepsilon}\right],$$

($d=1-|t_0|$, M a universal constant, λ depending only on the second derivatives of φ) .

PROOF. Suppose that $0=w(t_0)$ and define

$$\nabla^*u=u_x-iu_y\,,\qquad \nabla^*\varphi=\varphi_x-i\varphi_y\,.$$

Then $\nabla^*\varphi(x,y)=P^{k+1}(z,\bar z)+O(1)(|z|^{k+1+\alpha})$, $(O(1)\leqslant C)$.

$$P^{k+1}(z,\bar z)=\sum_{i+j\leqslant k+1}\frac{1}{i!j!}(D_z^iD_{\bar z}^j\varphi)z^i\bar z^j\,,$$

since

$$\Delta\varphi=D_{\bar z}\nabla^*\varphi\leqslant-\varepsilon\quad\text{ in } Br;\ Cr'<\eta\,;$$

choosing $\eta=\eta(\varepsilon, C)$ small enough

$$|D_{\bar z}P^{k+1}|\geqslant\varepsilon-\sum\frac{|D_z^iD_z^j\varphi|}{i!j!}r'^{i+j}+O(1)r'^{k+\alpha}\geqslant\varepsilon/2\,.$$

Furthermore, we can choose η so small that the implicit function problem:

$$\nabla^*u(z)=P^{k+1}(z,\xi)+O(1)|z|^{k+1+\alpha}$$

for $z \in \Omega$ has a unique solution $\xi = \xi(z)$ which is Lipschitz continuous with constant λ depending only on the second derivatives of φ.

$$0 = \Delta\varphi(0)\,\xi_{\bar z} + \left(\sum_{i+j\leqslant k+1} j(D_z^i D_{\bar z}^j \varphi)\, z^i \xi^{j-1}\right)\xi_{\bar z} + D_{\bar z}(O(1))\,|z|^{k+1+\alpha}\,.$$

Hence

$$|\xi_{\bar z}| \leqslant \frac{2C}{\varepsilon(k+1)!}\,|z|^{k+\alpha}\,.$$

Now, we proceed to extend the conformal mapping quasi-conformally to the whole disk; set

$$\Omega(t) = \begin{array}{ll} w(t), & \operatorname{Im}(t) \geqslant 0 \\ \bar\xi(w(\bar t)), & \operatorname{Im}(t) \leqslant 0\,. \end{array}$$

(Remember that Ω depends on t_0. To construct ξ we develope $\nabla^*\varphi$ around $O = w(t_0)$).

Set $\tilde\Omega(t) = \Omega(t) - \sum_{i=0}^{k} a_i (t-t_0)^i$, then

$$|\tilde\Omega_{\bar t}(t)| \leqslant \frac{2\Lambda^{k+1+\alpha}}{\varepsilon}\, C^{k+2} |t-t_0|^{k+\alpha}\,.$$

In virtue of Green's identity

$$\frac{\tilde\Omega(t) - \tilde\Omega(t_0)}{(t-t_0)^{k+1}} = -\frac{1}{\pi}\iint\limits_{|s|\leqslant 1} (s-t_0)^{-(k+1)}\,\tilde\Omega_{\bar s}(s)\,\frac{1}{(s-t)}\,ds + {} + \frac{1}{2\pi_i}\int\limits_{|s|=1} (s-t_0)^{-(k+1)}\,\tilde\Omega(s)\,\frac{1}{s-t}\,ds$$

and a_{k+1} is the value of the above integrals for $t = t_0$, we obtain the desired bound for a_{k+1} and the required estimate for

$$\left|\frac{\tilde\Omega(t) - \tilde\Omega(t_0)}{(t-t_0)^{k+1}} - a_{k+1}\right| \leqslant \frac{1}{\pi}\iint \left|\frac{(s-t_0)^{-(k+2)}}{(s-t)}\,\tilde\Omega_{\bar s}(s)(t-t_0)\right| ds + {} + \frac{1}{2\pi}\int\limits_{|s|=1} \left|\frac{(s-t_0)^{-(k+2)}}{(s-t)}\,\tilde\Omega(s)(t-t_0)\right| ds\,. \qquad \text{Q.E.D.}$$

As an immediate consequence of the lemma we have the following:

THEOREM I. *If $\Delta\varphi \in C^{k,\alpha}$ and G is a connected component of Λ^*, then ∂G is composed of a finite number of Jordan arcs with a $C^{k+1,\alpha}$ nondegenerated* *parametrization. Moreover, if $\Delta\varphi$ is real analytic, the Jordan arcs are real analytic.*

Observe that the argument actually shows that there is a virtual extension, across the free boundary, of the solution, u, which is in the class $C^{k+2,\alpha}$.

In the case studied by H. Lewy and G. Stampacchia (D convex and φ concave) where it is known that $(\Lambda^*)^0$ has only one component and Ω^* is mapped into (Λ^*) by ξ, the boundary of $(\Lambda^*)^0$ is a $C^{k+1,\alpha}$ Jordan curve, with no exceptional points.

3. – The non-linear case.

In this section we want to show how the preceeding methods apply also to the non-linear case treated by H. Brezis and D. Kinderlehrer [1], in particular to the minimal surface equation.

In this case we study the regularity of the solution u to a variational problem of the form

$$\int a_j(Du)\,D_j(v-u)\,dx \geqslant 0$$

on the convex set of functions $v \geqslant \varphi$, (φ the obstacle), $v \in H_0^1(D)$. (a_j is locally a strictly coercitive, C^2, vector field, and $\varphi \in C^4(D) \cap C(\bar{D})$, $\varphi < 0$ on ∂D); such that $\nabla S(\varphi)$ and $S\varphi$ do not vanish simultaneously.

As before, we have two sets,

$$\Lambda = \{x\colon u = \varphi\}$$

and

$$\Omega = \{x\colon u > \varphi\}\,.$$

Brezis and Kinderlehrer [1] prove that $u \in C^{1,1}$ in a neighborhood of Λ.

On Ω, $S(u) = \sum_{i,j} D_j\big(a_i(D_u)\big(D_{ij}(u) = 0$, the system $D_j\big(a_i(Du)\big)\xi_i\xi_j$ being uniformly elliptic for bounded Du.

By exactly the same reasoning as in Lemma 2.1, using now, instead of the strict maximum principle, Lemma 2, of MacNabb [12], we prove the following lemma.

LEMMA 3.1.

a) $\Lambda \subset V_1 = \{x\colon S(\varphi) < 0\}$,

b) *Ω has a finite number of connected components.*

To apply the results of Lemmas 1.1 to 1.4 to this case we begin linearizing the problem.

LEMMA 3.2. *For $x \in \Omega$, let g be*

$$g = S_\varphi(u) = \sum D_j a_i(D\varphi) D_{ij} u .$$

If we extend g by $g/\Lambda = 0$, g is Lipschitz in a neighborhood of Λ.

PROOF. The Lipschitz character of g in the interior of Ω follows from the interior Schauder estimates of the second derivatives of u. The Lipschitz constant being fixed whenever the distance between the points is

$$d(x, y) \leqslant \alpha(\min d(x, \partial\Omega), d(y, d\Omega)) \qquad (\alpha < 1) .$$

But near the boundary of Λ

$$|S_\varphi(u)(x) - S(u)(x)| = |\sum [D_j a_i(D\varphi) - D_j a_i(Du)] D_{ij}(u)| \leqslant C d(x, \partial\Omega)$$

and that completes the proof.

For the linearized problem it is easy to obtain the equivalents of Lemmas 1.1 to 1.3.

LEMMA 3.3. *Under the same hypothesis as Lemma* 1.1, *but now, $S_\varphi(u) \geqslant \geqslant \lambda_0 > 0$, instead of $\Delta u \geqslant \lambda_0 > 0$.*

Then, $u(y) \geqslant C|y - x_0|^2$ for some point y on ∂O, the constant C depending on the modulus of ellipticity of S_φ.

LEMMA 3.4. *Under the same hypothesis of Lemma* 1.2, *with S_φ instead of Δ, the same conclusions hold.*

LEMMA 3.5. *Under the same hypothesis of Lemma* 3.4. *Assume further that $S(u) = f$ on Ω, where f has a Lipschitz extension to $\bar{\Omega} \cup 0$ (we keep the notation of Lemma* 1.3). *Then u is a $C^{2,\alpha}$ solution of $S(u) = f$ in $(\bar{\Omega})^0 \cap 0$.*

PROOF. As in Lemma 1.3, we can show that

$$\int \sum a^i(Du) D^i v \, dx = -\int f v \, dx , \qquad v \in C_0^\infty((\bar{\Omega})^0)$$

and therefore the extension of u to $(\bar{\Omega})^0$ follows.

We define Ω^* and Λ^* as in Section 2.

To prove the equivalent of Lemma 1.4 we can reduce our situation pointwise to that of the lemma. However, since in this paper we apply it to the two dimensional case only, by means of a Beltrami transformation we can reduce

$$S_\varphi(u-\varphi)=g$$

(g a Lipschitz function different from 0 in a neighborhood of the free boundary) to $\Delta\big((u-\varphi)\circ w\big)=h$ ($\circ$ denotes composition) where h is again Lipschitz, $h\neq 0$ in a neighborhood of $w^{-1}(\Lambda^*)$ and

$$\nabla[(u-\varphi)\circ w]=0\,, \qquad (u-\varphi)\circ w=0$$

on $\nabla^{-1}(\partial\Lambda^*)$.

The mapping w being a diffeomorphism, the results of Theorem 1 apply to our new Λ^* and hence it remains valid for the general, non-linear case:

THEOREM 3. *If C is a connected component of Λ^*, ∂C is composed by a finite number of $C^{1,\alpha}$ parametrizable arcs, Γ_i, each one of which is locally the only boundary between Λ^* and Ω^*.*

From now on we restrict our attention to one of the $C^{1,\alpha}$ arcs, Γ, separating a portion, of Ω^* from a component, C, of Λ^*.

We recall that, by construction (see the linear case) such arcs have a continuous tangent. Therefore, a conformal mapping, $t(z)$, from D, into a portion of the half plane, $\{(x, y), y>0\}$, carrying Γ into a segment (a, b) of the real line is $C^{1,\alpha}$ on any compact subarc of Γ (understood as a uniform interior estimate (1)). Let us fix, as in the linear case, a point $z_0\in\Gamma$, and $B_r(z_0)$ such that

$$\|D^j\varphi\|_{\infty,B_r}\leqslant Cj!\,, \qquad j\leqslant k+2$$
$$\|D^{k+2}\varphi\|_{\alpha,B_r}\leqslant C(k+2)!$$
$$S(\varphi)\leqslant -\varepsilon, \qquad z\in B_r(z_0)\,.$$

Also, assume that

$$\|D^j a_i(p, q)\|_{\infty,B_r}\leqslant Cj!\,, \qquad j\leqslant k+1\,,$$
$$\|D^{k+1}a_i(p, q)\|_{\alpha,B_r}\leqslant C(k+2)!$$

for

$$|(p, q)-\nabla\varphi(z_0)|<\lambda r$$

(1) See O. D. KELLOG, *On the derivatives of harmonic function on the boundary*, TAMS, **33** (1931), pp. 486-510.

(λ the Lipschitz constant of ∇u, $\nabla\varphi$ in $B_r(z_0)$. Let us choose a $r' \leqslant \varrho/C$ with ϱ, small, to be chosen later.

Finally let $z(t)$ be a Lipschitz conformal mapping from the half circle $A = \{t: |t| < \alpha, \operatorname{Im} t > 0\}$, into $z(A) \subset \Omega^* \cap B_{r'}(z_0)$, mapping $(-1, 1)$ into Γ, $(z(0) = z_0)$.

We are going to prove, inductively, that if $\nabla\varphi \in C^{k,\alpha}$, and $a_j(p, q) \in C^{k,\alpha}$ then $z(t) \in C^{k,\alpha}$ up to $\operatorname{Im}(t) = 0$, $(k > 1)$. Moreover, we will keep track of the constants in order to obtain analyticity whenever $\nabla\varphi$ and $a_j(p, q)$ are analytic.

Let us first observe that $\Delta u \in C^{k,\alpha}$ for any $\alpha < 1$ up to Γ. To prove it, observe that $S_\varphi(Du)$ is bounded and hence $\Delta(Du \circ w)$ is also bounded. Composing $Du \circ w$ with a $C^{k,\alpha}$ conformal mapping, ζ, we reduce the problem to a half space problem where the Laplacian of the composition is bounded, the composition takes $D\varphi \circ w \circ \zeta$ boundary values on (a, b) and therefore $\nabla u \in C^{k,\alpha}$ up to Γ. Therefore ∇u has a $C^{k,\alpha}$ extension, $\tilde{\nabla} u$, to a neighborhood of any compact subarc of Γ.

Let us assume that on B_r, $\tilde{\nabla} u$ has $C^{1,\alpha}$ norm smaller than C.

LEMMA 3.6. *Assume, inductively, that for* $t_0 \in (-1, 1)$, $t \in A\big(z(t_0) = 0\big)$.

$$z(t) = \sum_1^k a_j (t - t_0)^j + O(1)\big(|t - t_0|^{k+\alpha}\big)$$

and

$$\|D^j u\|_{\infty, z(A_\gamma)} \leqslant (C\beta\gamma)^j j!\,, \qquad j \leqslant k + 1$$

$$\|D^{k+1} u\|_{\alpha, z(A_\gamma)} \leqslant (C\beta\gamma)^{k+1} (k+1)!$$

(*where* β *is a large constant and* $A_\gamma = A \cap \{t: |t| \leqslant 1 - 1/\gamma\}$. *Then*

$$z(t) = \sum_0^{k+1} a_j (t - t_0)^j + O(1)\big(|t - t_0|^{k+1+\alpha}\big)$$

where

$$|a_{k+1}| \leqslant C_0 \left(\sum_0^k \frac{|a_j| d^j}{d^{k+1}} + \frac{\Lambda \lambda r'}{d^{k+2}} + C\beta\Lambda^{k+2} \frac{2}{\varepsilon} \right)$$

where $d = 1 - |t_0|$, Λ *is the Lipschitz constant of* $z(t)$ *on* A *and* C_0 *an absolute constant. Also,*

$$\|D^{k+2} u\|_{\alpha, z(A_\gamma)} \leqslant (C\beta\gamma)^{k+2}\,.$$

PROOF. Since $0 = z(t_0)$, then $(\nabla u)(z) = P(x, y) + O(1)(z - z_0)^{k+\alpha}\big(P, (P_1, P_2)\big)$.

Let

$$a_{ij}(p, q) = a^*_{ij}(p - p_0, q - q_0) + O\big((|p - p_0| + |q - q_0|)^{k+\alpha}\big), \quad \big((p_0, q_0) = \nabla u(0)\big) .$$

Set

$$S^*(u) = \sum a^*_{ij}(P)\, u_{ij} ,$$

if ϱ is chosen small enough depending on C, λ and the modulus of ellipticity of S_{p_0,q_0}; then S^* remains strictly elliptic on $B_{r'}(z_0)$. Let μ be the Beltrami coefficient associated to S^*.

Clearly μ is a real analytic function

$$\mu = \sum (\mu_{ij}) z^i \bar{z}^j$$

$$|\mu_{ij}| \leqslant (C\beta\gamma\tau)^{i+j} = (C_1)^{i+j}$$

(τ an absolute constant depending only on the definition of μ as a rational function, in particular as a series, on a^*_{ij}). Also $|\mu| < 1 - \delta$ with δ depending only on Γ', that is on C, S_{p_0,q_0} and λ. Therefore, there is a unique real analytic Beltrami transformation $f = \sum b_{ij} z^i \bar{z}^j$ verifying $b_{k0} = \delta_{k1}$ and such that if $f^{-1} = \sum b'_{ij} \xi^i \bar{\xi}^j$, b_{ij}, as well as b'_{ij} verify

$$|b_{ij}| , \qquad |b'_{ij}| \leqslant (C' C_1)^{i+j} ,$$

C' again an absolute constant.

On the new coordinate system, $u - \varphi$ and $\nabla(u - \varphi)$ vanish on $f(\Gamma)$. Moreover, around $0 = f(0)$, they verify

$$\Delta(u - \varphi) + \sum c_{ij} D_{\xi_i \xi_j}(f^{-1}) D_{X_i}(u - \varphi) = \frac{S^*(u) - S^*(\varphi)|_{p^{-1}(\xi)}}{T(f)}$$

where c_{ij} and $T(f)$ depend only on the Beltrami transformation, are analytic, and with bounds for its coefficients for the same type as before.

Let us now consider a conformal mapping $\zeta(\eta)$ of the half circle, A onto

$$f(z(A)) , \qquad \zeta((-1, 1)) = f(z(-1, 1)) , \qquad \eta_0 = \eta\Big(\zeta\big(f(z(t_0)) = t_0\Big) .$$

Then $\eta_{\bar{t}} = \tilde{\mu}\eta_t$ with $\tilde{\mu} = \mu(z(t))$. Hence, $\tilde{\mu}$ is of class $C^{k,\alpha}$, with the same type of constants as those of μ and z and therefore $\eta(t)$ is a $C^{k+1,\alpha}$ function on t_0, with an asymptotic development $\eta(t)$

$$\eta(t) = \eta(t_0) + \sum_{j=1}^{k+1} \eta_j (t - t_0)^j + O\big(|t - t_0|^{k+1,\alpha}\big) .$$

(This can be easily checked writing η as the composition of a particular solution with a conformal mapping.) The constants η_j are bounded by

$$|\eta_j| \leqslant (C''' d)^j j!\,.$$

Applying Lemma 3.6, $\zeta(\eta)$ has a $C^{k+1,\alpha}$ asymptotic development, and hence so does $z(\zeta(\eta)(t))$ around t_0. The estimate for $\|D^{k+2}u\|_{\alpha,z(A_\gamma)}$, $\nabla_z u$ as a function of η, provided β is chosen large enough to absorve the constants that appear in standard a priori estimates.

4. – The non-dense part of the free boundary.

We would like now to discuss the remaining part of Λ.

First, we notice that $\Lambda \sim \Lambda^*$ is a false coincidence set in the sense that φ can be replaced by another obstacle $\varphi - h$ $(h \in C_0^\infty(\Omega))$ still satisfying our hypothesis and such that u is still the solution of the variational problem, but $\Lambda(\varphi - h) = \Lambda^*$.

To construct h, we consider a C_0^∞ function g such that

$$g > 0 \qquad \text{on } [\{x\colon S(\varphi) < \varepsilon\} \sim \Lambda^*] \cap H$$

(where H is a suitable neighborhood of $\partial\Lambda$) and $g = 0$ on the rest of Ω. Set $h = \gamma g$. For γ small enough the condition that $S(\varphi - h)$ and $\nabla S(\varphi - h)$ do not vanish simultaneously is still fulfilled.

On the other hand, the fact that $S(u) = 0$ across $\Lambda \sim \Lambda^*$ shows that locally $\Lambda \sim \Lambda^*$ is contained in an arc of a differentiable curve.

If the a_j's are real analytic functions of Du and φ is also real analytic, the set $\Lambda \sim \Lambda^*$ is locally an analytic arc or an isolated point (Notice that if there exists a sequence of points x_n of $\Lambda \sim \Lambda^*$ accumulating on a point x not in Λ^*, they must lay in an analytic arc, Γ, near x; then $(u - \varphi)\circ w$ has to be the solution to the Cauchy problem $\nabla v = 0$ on Γ, $\Delta v = \Delta[(u - \varphi)\circ w]$, once the problem has been linearized by means of the Beltrami transformation, w).

Furthermore, since the ball property of Lemma 1.2 (resp. 3.4) has to hold at both sides of the curve Γ, we have two possibilities: either Γ is a closed analytic curve or its closure, $\bar{\Gamma}$, is a new arc, Γ^*, eventually closed, with its end points (point) laying in Λ^* (Precisely $\Gamma^* \sim \Gamma$ consists of these end points (end point)).

In the linear case, this second possibility can also be excluded.

THEOREM 5. *In the linear case with a real analytic obstacle, $\Lambda \sim \Lambda^*$ is composed of isolated points plus a finite number of closed analytic curves.*

PROOF. The Lewy-Stampacchia [11], reflection function, let us call it w, is defined in a uniform neighborhood of $\partial\Lambda$.

Let us choose a neighborhood H in such a way that, if $w(x) \notin \Lambda$, $w(w(x))$ is again defined, and let x_0 be an end point of Γ, $x_0 \in \Lambda^*$. Then there are two components, S_1 and S_2, of $\Omega \cap B(x_0, \varrho_0)$ that have Γ as common boundary.

If $(\Lambda^*)^0$ has a finite number of connected components in a neighborhood of x_0, by means of the reflection w it follows that ∂S_i is an analytically parametrizable arc, part of which is Γ. Hence, the arc is the same for S_1 and S_2 and that is not possible. Therefore, one of them, let us say S_1 has to have boundary points in common with infinitely many components of $(\Lambda^*)^0$ in any neighborhood of x_0.

On the other hand, near Γ, w maps S_1 into S_2, and $w(w(x)) = x$ ($w \circ w$ is analytic and $w(x) = x$ on Γ). Hence in a subneighborhood of x_0, $w(S_1) \subset S_2$ and $w(w(x)) = x$ which means that $\partial S_1 \subset \partial S_2$ but that is not possible

5. – On the Stefan problem and other remarks.

In this last section we will make a few remarks that we postponed for the purpose of continuity. Let us first mention the Stefan problem as treated by Friedman and Kinderlehrer [4].

There, we have two domains Ω and Λ in $Dx[0, \infty]$. (D a domain in R^n) and a function, u, verifying

a) $u \in C^{1,1}$ in the variable x for fixed t. (Personal communication of David Kinderlehrer.)

b) $H(u) = \Delta u - D_t u = k$ in Ω.

c) $\nabla_x(u) = 0$ at $\partial\Lambda$.

d) $D_t u \geqslant 0$ in Ω.

e) $\mathcal{T}(\Lambda \cap R^n x\{t_0\}) < \infty$, $\forall t_0 > 0$ ($\mathcal{T}$ denotes the De Georgi n-dimensional perimeter).

Hence Lemmas 1.1 and 1.2 apply for each fixed t and we obtain

1) $m(\partial\Lambda \cap [R^n x\{t_0\}]) = 0$, for every $t_0 > 0$.

2) Since u_t is bounded, there is a « parabolic ball condition »: if $(x, t_0) \in \partial\Lambda \cap [R^n x \{t_0\}]$ and $B(x, \varrho)$ is a ball (in R^n) with ϱ small enough, there is a cylinder $C = B(y, C\varrho) x [t_0 - C\varrho^2, t_0]$ verifying $C \subset \Omega$, with $y \in \partial B(y_0, \varrho)$.

It is worth noting that this cylinder goes backwards in time, which in some sense bounds the « velocity » of melting of the ice.

3) As a consequence of the results of [2], for each fixed time, $n = 2$, if we consider $\Lambda^* = (\bar{\Lambda}^0)$, each connected component of Λ^* being simply connected, has a boundary composed of a rectifiable Jordan curve.

Our next remark is about the number and shape of the components of Λ^* in the two dimensional elliptic case. First let us show an example in which the number of components is as large as we please.

Consider the periodic analytic curve

$$\Gamma = \cos\theta + i(\cos^2 n\theta)\sin\theta\,.$$

The Cauchy problem $\Delta u = 1$ with the initial data $u = 0$, $\nabla u = 0$ on Γ can be solved in the exterior side of Γ. To construct our obstacle, we consider a curve Γ' enclosing Γ and such that u is defined and positive between Γ and Γ'. We then solve the Dirichlet problem $\Delta\varphi = -1$, $\varphi/\Gamma' = -u$. Then φ is our obstacle, $v = \varphi + u$ our solution and the interior of Γ is the set of coincidence, $(\Lambda_\varphi)^0$.

If we now consider the obstacle $\varphi - \varepsilon$

$$\Lambda_{\varphi-\varepsilon} \subset (\Lambda_\varphi)^0$$

and $v_{\varphi-\varepsilon} > v - \varepsilon$. Hence for ε small, $\Lambda_{\varphi-\varepsilon}$ has at least n-different non-tangent connected components.

The second example shows how a curve of $\Lambda \sim \Lambda^*$ can appear. Consider a sixth degree polynomial $P(t)$ such that P is symmetric has an absolute maximum at 0 and two relative maxima at 1 and -1.

Also suppose that $\Delta(P(|x|))$ verifies the nondegeneracy condition above imposed $(\Delta P(|x|))$ and $\nabla(\Delta P(|x|))$ don't vanish simultaneously and $P(0) = 1$. If we solve now the minimum energy problem for $\varphi(x) = P(|x|)$ and $D = D(R)$ the disc of radius R, for $R > R_0$ the coincidence set is a circle, for R_0 it is a circle and a circumference $(\Lambda \sim \Lambda^*)$ and for $R < R_0$ it is a circle and a ring.

Finally let us notice that the fact that the boundary of each component C, of $(\Lambda^*)^0$ is formed by a finite number of Jordan curves can also be obtained from Lemma 1.1 under the weaker assumption, $\varphi \in C^{2+\alpha}$.

First we prove that each two points of C can be joined by a curve on C, laying in a square proportional to the distance between the points, and then the result follows using accessibility of the boundary.

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