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A Class of Second-Order Evolution Equations with Double Characteristics.

ANTONIO GILIOLI (*)

Introduction.

Given an abstract Hilbert space H and an unbounded, selfadjoint positive definite operator A on H, which has a bounded inverse A^{-1}, we study in this paper evolution operators of the form

$$(1)\qquad P = \big(\partial_t - a(t, A)A\big)\big(\partial_t - b(t, A)A\big) + c(t, A)A\,,$$

where ∂_t means $\partial/\partial t$ and where the coefficients $a(t, A)$, $b(t, A)$ and $c(t, A)$ are power series with respect to A^{-1}, with coefficients in $C^\infty(J)$, for some open set J on the real line. These power series are assumed to be convergent in $L(H; H)$, as well as each of their t-derivatives, uniformly with respect to t on compact subsets of J. When the leading coefficients $a_0(t)$ and $b_0(t)$ of the power series $a(t, A)$ and $b(t, A)$ vanish simultaneously (we always assume that this happens for $t = 0$), we are in the case of double characteristics.

The study of local solvability and hypoellipticity of such operators was completely made by F. Trèves, in this same abstract set up, under the further condition

$$(2)\qquad a_0(0) = b_0(0) = 0 \quad \text{and} \quad \operatorname{Re} a_0^{(1)}(0)\cdot \operatorname{Re} b_0^{(1)}(0) < 0\,,$$

where $f^{(1)}(t)$ denotes the first derivative of $f(t)$.

The best possible hypoelliptic property of these operators, now in the pseudodifferential form, was studied by F. Trèves and B. de Monvel. For this, they associated to every operator P satisfying (1) and (2) the number

$$(3)\qquad l_P = \frac{c_0(0)}{a_0^{(1)}(0) - b_0^{(1)}(0)} \qquad \big(\text{when } \operatorname{Re} a_0^{(1)}(0) > 0\big)\,,$$

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and it turned out that P only loses one derivative if and only if l_P is *not* an integer $\geqslant 0$.

Here we again study operators in the abstract set up, constructing a scale of Sobolev spaces $\mathcal{H}^s$, but considering the following condition, more general than (2), as well as the corresponding definition of l_P and hypoelliptic property:

(4) for some odd positive integer k, $\operatorname{Re} a_0^{(k)}(0)\cdot\operatorname{Re} b_0^{(k)}(0)<0$, $\operatorname{Re} a_0^{(p)}(0)=$ $=\operatorname{Re} b_0^{(p)}(0)=0$ for all $p\leqslant k-1$, and $\operatorname{Re} c_0^{(p)}(0)=0$ for all $p\leqslant k-2$.

(5)
$$l_P=\frac{k\cdot c_0^{(k-1)}(0)}{a_0^{(k)}(0)-b_0^{(k)}(0)}\qquad \left(\text{when } \operatorname{Re} a_0^{(k)}(0)>0\right).$$

(6) there is an open set J containing 0 such that, given any real number s, any open subset J' of J and any distribution u in J',

$$Pu\in\mathcal{H}^s_{\mathrm{loc}}(J')\quad\text{implies}\quad u\in\mathcal{H}^{s+2/(k+1)}_{\mathrm{loc}}(J').$$

The main result of this article is the following

THEOREM. *Let P satisfy* (1), (4) *and* (5), *and let P^* be the adjoint of P. The following conditions are equivalent*:

(7) *P satisfies* (6);

(8) *P^* satisfies* (6);

(9) *whatever the integer $m\geqslant 0$, $l_P\neq m(k+1)$ and $l_P\neq m(k+1)+1$*

A more precise statement is given in chapter 7, using the spaces ${}^k\mathcal{H}^{s,m}$ constructed in chapter 4

The novelty in this article is the kind of simple concatenations that we use, which incidentally precludes the use of Hermite operators, employed by Monvel-Trèves in their work

0. – Notations.

Throughout this article, we will closely follow the notations of [3] Like there, A will denote a linear operator, densely defined in a Hilbert space H, which is unbounded, but is selfadjoint, positive definite and has a bounded inverse A^{-1}. (We may think of A as being, for instance, $(1-\Delta_x)^s$ on R^n, or a selfadjoint extension of $|D_x|$).

We will consider differential operators on the real line (where the variable is denoted by t and is usually referred to as the time), of the following kind:

$$(0.1) \qquad P = \sum_{r+j\leqslant m} c_{r,j}(t, A) A^r \partial_t^j,$$

where the r's are real numbers $\geqslant 0$, the j's are integers $\geqslant 0$, and the sum is a finite one. The coefficients $c_{r,j}(t, A)$ belong to the ring $\mathcal{Q}_A(J)$ defined as follows: J is a given open subset of the real line; the elements of $\mathcal{Q}_A(J)$ are the series in the nonnegative powers of A^{-1}, with coefficients in $C^\infty(J)$, which converge in $L(H; H)$ (the Banach space of bounded linear operators on H), as well as each one of their t-derivatives, uniformly with respect to t on compact subsets of J.

The operators of the kind (0.1) form a ring which we denote by $\mathcal{T}_A(J)$. The operator given in (0.1) is said to be of order less than or equal to the nonnegative real number m.

We will use the scale of « Sobolev spaces » H^s (for $s \in R$) « on the variables x », defined by A: if $s \geqslant 0$, H^s is the space of elements u of H such that $A^s u \in H$, equipped with the norm $\|u\|_s = \|A^s u\|_0$, where $\| \, \|_0$ denotes the norm in $H = H^0$; if $s < 0$, H^s is the completion of H for the norm $\|u\|_s = \|A^s u\|_0$. The inner product in H^s will be denoted by $(\, , \,)_s$. Whatever $s \in R$, $m \in R$, A^m is an isomorphism (for the Hilbert space structures) of H^s onto H^{s-m}.

By H^∞ we denote the *intersection* of the spaces H^s, equipped with the projective limit topology, and by $H^{-\infty}$ their *union*, with the inductive limit topology. Since, for each $s \in R$, H^s and H^{-s} can be regarded as the dual of each other, so can H^∞ and $H^{-\infty}$: with their topologies, they are the strong dual of each other.

Let J be an open subset of the real line. We denote by $C^\infty(J; H^\infty)$ the space of C^∞ functions in J valued in H^∞. It is the intersection of the spaces $C^j(J; H^k)$ (of the j-continuously differentiable functions defined in J and valued in H^k) as the nonnegative integers j, k tend to $+\infty$. We equip $C^\infty(J; H^\infty)$ with its natural C^∞ topology. If K is any compact subset of J, we denote by $C_c^\infty(K; H^\infty)$ the subspace of $C^\infty(J; H^\infty)$ consisting of the functions which vanish identically outside K. It is a closed linear subspace of $C^\infty(J; H^\infty)$, hence a Fréchet space, and we denote by $C_c^\infty(J; H^\infty)$ the inductive limit of $C_c^\infty(K; H^\infty)$ as K ranges over all compact subsets of J.

We will denote by $\mathcal{D}'(J; H^{-\infty})$ the *dual* of $C_c^\infty(J; H^\infty)$, and refer to it as the space of *distributions in J valued in $H^{-\infty}$*. The « local structure theorem » is valid in $\mathcal{D}'(J; H^{-\infty})$: if u is any distribution in J valued in $H^{-\infty}$ and if J' is any relatively compact open subset of J, we can find a finite set $\{f_{j,k}\}$

($j+k\leqslant M$) of continuous functions in J, valued in H, such that

$$u=\sum_{j+k\leqslant M} A^j \partial_t^k f_{j,k} \quad \text{in } J' . \tag{0.2}$$

Observe that the differential operators $P\in \mathcal{T}_A(J)$ define continuous linear mappings of $C^\infty(J;H^\infty)$ (resp. $C_c^\infty(J;H^\infty)$, resp. $\mathcal{D}'(J;H^{-\infty})$) into itself.

For the sake of completeness, we recall some definitions and results stated in [3].

DEFINITION 0.1. Let t_0 be any point of the open set J. We say that P is *locally solvable at* t_0 if there is an open neighbourhood J' of t_0, contained in J, such that, to every $f\in C_c^\infty(J';H^\infty)$, there is $u\in\mathcal{D}'(J';H^{-\infty})$ satisfying $Pu=f$ in J'. We say that P is locally solvable in a subset S of J if P is locally solvable at every point of S.

DEFINITION 0.2. We say that P is hypoelliptic in J if given any open subset J' of J and any distribution $u\in\mathcal{D}'(J;H^{-\infty})$, the following condition is verified:

$$Pu\in C^\infty(J';H^\infty) \quad \text{implies} \quad u\in C^\infty(J';H^\infty) . \tag{0.3}$$

DEFINITION 0.3. We denote by P^* the formal adjoint of $P\in\mathcal{T}_A(J)$, *i.e.*, the operator defined by

$$\int (P^*u, v)_0\,dt=\int (u, Pv)_0\,dt, \qquad \forall u, v\in C_c^\infty(J;H^\infty) . \tag{0.4}$$

PROPOSITION 0.1. *If P is hypoelliptic in the open subset J_0 of J, then P^* is locally solvable in J_0.*

PROPOSITION 0.2. *Let X be the differential operator $\partial_t - a(t, A)A$, where $a(t, A)\in\mathcal{Q}_A(J)$ (thus, we may write $a(t, A)=\sum_{i=0}^{\infty} a_i(t)A^{-i}$), and suppose that there is an integer $p\geqslant 0$ such that*

$$\operatorname{Re} a_0^{(p)}(0)\neq 0, \qquad \operatorname{Re} a_0^{(q)}(0)=0 \quad \text{if } q<p . \tag{0.5}$$

Then, X is hypoelliptic at $t=0$ if and only if the following condition is satisfied:

(0.6) *either p is even or, if p is odd,* $\operatorname{Re} a_0^{(p)}(0)>0$.

PROPOSITION 0.3. *Let X be as in Prop.* 0.2, *satisfying also* (0.5). *Then, X is locally solvable at $t=0$ if and only if the following condition is satisfied*:

$$\text{(0.7)} \qquad \text{either } p \text{ is even or, if } p \text{ is odd, } \operatorname{Re} a_0^{(p)}(0) < 0.$$

Propositions 0.1, 0.2, *and* 0.3 *are respectively Corollaries* I.1.2, I.3.1 *and* I.2.1 *of* [3].

1. – The « Sobolev » spaces $\mathcal{K}^s$.

We will denote by $\mathcal{S}(R; H^\infty)$ the space of all functions $u \in C^\infty(R; H^\infty)$ such that, for all pairs of polynomials P and Q in the variable t, and with complex coefficients, $P(t)Q(\partial_t)u(t)$ remains in a bounded subset of H^∞ as t varies over R, i.e., such that

$$\text{(1.1)} \qquad \forall s \in R, \qquad \sup_{t \in R} \|P(t)Q(\partial_t)u(t)\|_s < \infty .$$

We equip $\mathcal{S}(R; H^\infty)$ with its natural topology (*i.e.*, we take as a basis of continuous seminorms the expressions in (1.1)).

We define the integral of a continuous function valued in a locally convex vector space as the limit of Riemann sums. Then, if $u \in \mathcal{S}(R; H^\infty)$, we may form its Fourier transform $\hat{u}$ by

$$\text{(1.2)} \qquad \hat{u}(\tau) = \int_R \exp[-it\tau] u(t)\, dt, \qquad \forall \tau \in R .$$

It can be checked at once that $\hat{u}(\tau) \in H^\infty$ for every $\tau \in R$; and that $\hat{u} \in \mathcal{S}(R; H^\infty)$. Moreover, the Fourier transform is a continuous linear map from $\mathcal{S}(R; H^\infty)$ into itself, and it can be verified that its inverse is given by the usual formula:

$$\text{(1.3)} \qquad u(t) = (2\pi)^{-1} \int \exp[it\tau] \hat{u}(\tau)\, d\tau, \qquad \forall t \in R ,$$

which shows that the Fourier transform is an isomorphism from $\mathcal{S}(R; H^\infty)$ onto itself.

As usually, except for a multiplicative constant, the Fourier transform can be extended as an isometry of $L^2(R; H)$ onto itself. We have precisely:

$$\text{(1.4)} \qquad \int \|\hat{u}(\tau)\|_0^2\, d\tau = 2\pi \int \|u(t)\|_0^2\, dt .$$

We denote by $\mathcal{S}'(R; H^{-\infty})$ the dual of $\mathcal{S}(R; H^{\infty})$, and we refer to it as the space of *tempered distributions on* R, *valued in* $H^{-\infty}$. Since $C_c^{\infty}(R; H^{\infty})$ is dense in $\mathcal{S}(R; H^{\infty})$, we can identify $\mathcal{S}'(R; H^{-\infty})$ (as a set) with a subspace of $\mathcal{D}'(R; H^{-\infty})$. The transposition of the Fourier transform gives an isomorphism from $\mathcal{S}'(R; H^{-\infty})$ onto itself, which extends the initial one, and will be also called Fourier transform.

The operator $1+\tau^2+A^2$, defined in a subspace of $L^2(R; H)$ with values in $L^2(R; H)$, which assigns to each v in its domain of definition the element w in $L^2(R; H)$ given by $w(\tau) = v(\tau) + \tau^2 v(\tau) + A^2(v(\tau))$, is obviously densely defined in $L^2(R; H)$, selfadjoint and positive definite, and its inverse is continuous with norm $\leqslant 1$. Hence, we may consider its powers $(1+\tau^2+A^2)^s$ for any $s \in R$.

We should remark that sometimes A is considered as a function from a subspace of H into H, sometimes as a function from a subspace of $L^2(R; H)$ into itself, as in the expression $1+\tau^2+A^2$. This will never produce any confusion, since it will be clear from the context what is meant by A.

Let us consider a spectral resolution of the operator A (considered in H):

$$A = \int \lambda \, dE(\lambda) . \tag{1.5}$$

Then, for every $v \in L^2(R; H)$, we have both

$$\big((1+\tau^2+A^2)^{s/2} v\big)(\tau) = \int (1+\tau^2+\lambda^2)^{s/2} dE(\lambda) v(\tau) , \tag{1.6}$$

and

$$\|(1+\tau^2+A^2)^{s/2} v\|^2_{L^2(R;H)} = \iint |1+\tau^2+\lambda^2|^s d\|E(\lambda) v(\tau)\|^2 d\tau . \tag{1.7}$$

We will use the scale of « Sobolev » spaces $\mathcal{H}^s$, not to be confused with H^s, for $s \in R$, « on all the variables t and x », defined by $(1-\partial_t^2+A^2)^{\frac{1}{2}}$ (or, in the Fourier transform side, by $(1+\tau^2+A^2)^{\frac{1}{2}}$): if $s \geqslant 0$, $\mathcal{H}^s$ is the space of elements $u \in L^2(R; H)$ such that $(1+\tau^2+A^2)^{s/2} \hat{u} \in L^2(R; H)$, equipped with the norm $|||u|||_s^2 = (2\pi)^{-1} \|(1+\tau^2+A^2)^{s/2} \hat{u}\|^2_{L^2(R;H)}$. Remark that due to (1.6), we have $\mathcal{H}^0 = L^2(R; H)$, with $|||\ |||_0 = \|\ \|_{L^2(R;H)}$. If $s < 0$, $\mathcal{H}^s$ is the completion of $L^2(R; H)$ for the norm $|||u|||_s^2 = (2\pi)^{-1} \|(1+\tau^2+A^2)^{s/2} \hat{u}\|^2_{L^2(R;H)}$. The inner product in $\mathcal{H}^s$ will be denoted by $(\!(\ ,\ )\!)_s$, in order to distinguish it from the inner product in H^s. Whatever s, $m \in R$, $(1-\partial_t^2+A^2)^{m/2}$ is an isomorphism (for the Hilbert space structures) of $\mathcal{H}^s$ onto $\mathcal{H}^{s-m}$.

As in section 0, we may define the spaces $\mathcal{H}^{\infty}$ and $\mathcal{H}^{-\infty}$, and since, for each $s \in R$, $\mathcal{H}^s$ and $\mathcal{H}^{-s}$ can be regarded as the dual of each other, so can $\mathcal{H}^{\infty}$ and $\mathcal{H}^{-\infty}$.

REMARK 1.1. An equivalent definition of the spaces $\mathcal{H}^s$, $s \in R$, is the following: $\mathcal{H}^s$ is the space of tempered distributions u on R, valued in $H^{-\infty}$, such that its Fourier transform $\hat{u}$ is a measurable function and

$$\int \|(1+\tau^2+A^2)^{s/2}\hat{u}(\tau)\|_0^2 d\tau < \infty .$$

DEFINITION 1.1. Let K be a compact subset of R. Then we call $\mathcal{H}_c^s(K) = \{u \in \mathcal{H}^s \,|\, \operatorname{supp} u$ is a subset of $K\}$, with the topology induced by $\mathcal{H}^s$. By $\operatorname{supp} u$ we will always mean the support of u.

DEFINITION 1.2. Let J be an open subset of R. Then we call $\mathcal{H}_c^s(J)$ the inductive limit of the spaces $\mathcal{H}_c^s(K)$, as K ranges over all compact subsets of J.

DEFINITION 1.3. Let J be an open subset of R. Then we call $\mathcal{H}_{\text{loc}}^s(J) = \{u \in \mathcal{D}'(J; H^{-\infty}) \,|\, \forall \varphi \in C_c^\infty(J)$, we have $\varphi u \in \mathcal{H}^s\}$, with the coarsest locally convex topology which renders all the maps $u \to \varphi u$ from $\mathcal{H}_{\text{loc}}^s(J)$ into $\mathcal{H}^s$ continuous.

We remark that $C_c^\infty(J; H^\infty)$ is dense in $\mathcal{H}_c^s(J)$ and $\mathcal{H}_{\text{loc}}^s(J)$ and that $\mathcal{H}_{\text{loc}}^s(J)$ is a Fréchet space which is the dual of $\mathcal{H}_c^s(J)$.

These spaces have the usual properties of the Sobolev spaces. The norm of $\mathcal{H}^s$ is equivalent to the following one: $|||u|||_s' = \|(\tau^2+A^2)^{s/2}\hat{u}\|_{L^2(R;H)}$, which it is easy to check, for $s \geqslant 0$, to be also equivalent to

$$|||u|||_s'' = \|(|\tau|^s + A^s)\hat{u}\|_{L^2(R;H)} .$$

Another fact is:

$$\text{(1.8)} \qquad \forall s, r \in R, \quad r \geqslant 0, \qquad \text{if } A^r u \in \mathcal{H}^s, \text{ then } u \in \mathcal{H}^s ;$$

from which it follows immediately:

$$\text{(1.8')} \qquad \forall s, r \in R, \quad r \geqslant 0, \qquad \text{if } A^r u \in \mathcal{H}_{\text{loc}}^s(J), \text{ then } u \in \mathcal{H}_{\text{loc}}^s(J) .$$

Moreover, due to the way the spaces H^s were defined, we have:

$$\text{(1.9)} \qquad \forall u \in \mathcal{D}'(R; H^{-\infty}), \quad \forall r \in R, \quad \operatorname{supp} u = \operatorname{supp} A^r u .$$

This in connection with (1.8) gives:

$$\text{(1.8'')} \qquad \forall s, r \in R, \quad r \geqslant 0, \qquad \text{if } A^r u \in \mathcal{H}_c^s(J), \text{ then } u \in \mathcal{H}_c^s(J) .$$

We give below a list of important properties of these spaces and operators, leaving their proofs to the reader. We just mention that (1.12) should be first verified for $\mathcal{H}^s$ and then apply (1.9) and (1.8′).

(1.10) $\forall s, r \in R,\ r \geqslant 0$, we have continuous injections

$$\mathcal{H}^{s+r} \to \mathcal{H}^s, \qquad \mathcal{H}^{s+r}_{\mathrm{loc}}(J) \to \mathcal{H}^s_{\mathrm{loc}}(J) \quad \text{and} \quad \mathcal{H}^{s+r}_c(J) \to \mathcal{H}^s_c(J)\,.$$

(1.11) $\forall s, r \in R,\ r \geqslant 0,\ \forall m \in Z_+,\ A^r \partial_t^m$ is a continuous operator from $\mathcal{H}^s$ into $\mathcal{H}^{s-r-m}$, from $\mathcal{H}^s_{\mathrm{loc}}(J)$ into $\mathcal{H}^{s-r-m}_{\mathrm{loc}}(J)$ and from $\mathcal{H}^s_c(J)$ into $\mathcal{H}^{s-r-m}_c(J)$.

(1.12) $\forall s \in R,\ \forall m \in Z_+$,

$u \in \mathcal{H}^s$	iff both $\partial_t^m u$ and $A^m u$ belong to		$\mathcal{H}^{s-m}$,
$u \in \mathcal{H}^s_c(J)$	»	»	$\mathcal{H}^{s-m}_c(J)$,
$u \in \mathcal{H}^s_{\mathrm{loc}}(J)$	»	»	$\mathcal{H}^{s-m}_{\mathrm{loc}}(J)$.

Moreover, the norm of $\mathcal{H}^s$ is equivalent to

$$|||\partial_t^m u|||_{s-m} + |||A^m u|||_{s-m}\,.$$

(1.13) $\forall s \in R,\ \forall m \in Z_+,\ u \in \mathcal{H}^s$ iff there are v, w in $\mathcal{H}^{s+m}$ such that $u = \partial_t^m v + A^m w$. Moreover, there is $C > 0$ (independent of u) such that v and w may be chosen satisfying $|||v|||_{s+m} + |||w|||_{s+m} \leqslant C|||u|||_s$.

(1.14) $\forall s \in R,\ \forall m \in Z_+,\ u \in \mathcal{H}^s(J)$ iff there are v, w in $\mathcal{H}^{s+m}_c(J)$ such that $u = \partial_t^m v + A^m w$. Moreover, if $K = \operatorname{supp} u$ and J' is any open subset of J containing K, we may choose v and w so that both $\operatorname{supp} v$ and $\operatorname{supp} w$ are contained in J'.

We end this section with a proposition which is essential for sections 4 and 5.

PROPOSITION 1.1. *Let a and r be real numbers, $a > 0$ and $r \geqslant 0$, and let k be a positive integer. Then:*

a) If $\partial_t u$ and $|t|^a A^r u$ both belong to $L^2(R; H^s)$, then for every real number b such that $0 \leqslant b \leqslant a$, we have $|t|^b A^{r(b+1)/(a+1)} u \in L^2(R; H^s)$. Moreover, there is a constant $C > 0$ (depending only on a, b and r) such that $\| |t|^b A^{r(b+1)/(a+1)} u \|^2 \leqslant C(\|\partial_t u\|^2 + \| |t|^a A^r u\|^2)$, where $\|\ \|$ means here the norm of $L^2(R; H^s)$.

b) If $\partial_t u$ *and* $t^k Au$ *both belong to* $\mathcal{H}_c^s(R)$, *then*

$$t^j u \in \mathcal{H}_c^{s+(j+1)/(k+1)}(R) \qquad \textit{for } j = 0, 1, \ldots, k .$$

c) Same as b), substituting J *for* R *and* $\mathcal{H}_{\mathrm{loc}}^p$ *for* $\mathcal{H}_c^p$.

Although these are well known facts, we give the proof of this proposition in the Appendix.

2. – Statement of the main theorem.

Let J be an open subset of the real line, containing the origin. We will study a second order evolution operator of the form

$$P = \big(\partial_t - a(t, A)A\big)\big(\partial_t - b(t, A)A\big) + c(t, A)A \tag{2.1}$$

where $a(t, A)$, $b(t, A)$ and $c(t, A)$ are elements of $\mathcal{Q}_A(J)$.

As remarked in [3], there is no gain of generality in considering operators of the form $P + d(t, A)\partial_t$ instead of P.

We will systematically use the notation

$$X = \partial_t - a(t, A)A\,, \qquad Y = \partial_t - b(t, A)A \tag{2.2}$$

$$\delta(t, A) = a(t, A) - b(t, A)\,. \tag{2.3}$$

We will further restrict the class of operators of the kind (2.1) which we propose to study. The restricting conditions will bear on the leading coefficients $a_0(t)$, $b_0(t)$ and $c_0(t)$ of the power series (in A^{-1}) $a(t, A)$, $b(t, A)$ and $c(t, A)$. Let us assume that $a_0(t)$ and $b_0(t)$ do not vanish of infinite order at $t = 0$. Then we can write $a_0(t) = at^m + t^{m+1}f(t)$, $b_0(t) = bt^n + t^{n+1}g(t)$, with $ab \neq 0$. We will further assume that $m = n$, which will be denoted by k, and that $(\operatorname{Re} a)(\operatorname{Re} b) \neq 0$. We will also assume that $c_0(t) = ct^{k-1} + t^k h(t)$, without restrictions on the complex number c.

If we assume that k is even, then it can be proved that the operator P is both locally solvable and hypoelliptic at $t = 0$.

Let us assume that k is odd, and $(\operatorname{Re} a)(\operatorname{Re} b) > 0$: if $\operatorname{Re} a > 0$ and $\operatorname{Re} b > 0$, then both X and Y are hypoelliptic but not locally solvable at $t = 0$, and it can be proved that the same happens to P. If $\operatorname{Re} a < 0$ and $\operatorname{Re} b < 0$, then both X and Y are locally solvable but not hypoelliptic at $t = 0$, and it can be proved that the same happens to P.

Moreover, in all the cases above, the degree of regularity of P can be studied, whenever P is hypoelliptic.

We are then left with the case: k odd, $(\mathrm{Re}\, a)(\mathrm{Re}\, b) < 0$. We may even assume that $\mathrm{Re}\, a > 0$, $\mathrm{Re}\, b < 0$, since there is no gain by also considering the case $\mathrm{Re}\, a < 0$, $\mathrm{Re}\, b > 0$, which can be reduced to the previous one by commutation of X and Y and a convenient change of $c(t, A)$.

We will then assume that the operator P given by (2.1) satisfies the following conditions:

(2.4) $a_0(t) = at^k + t^{k+1}f(t)$, $b_0(t) = bt^k + t^{k+1}g(t)$, $c_0(t) = ct^{k-1} + t^k h(t)$, where $f, g, h \in C^\infty(J)$ and a, b, c are complex numbers.

(2.5) $\mathrm{Re}\, a > 0$, $\mathrm{Re}\, b < 0$, and k is an odd integer.

Case (2.5) corresponds to the case when X and Y have « conflicting influences », when (cf. Prop. 0.2 and 0.3):

(2.6) X is not locally solvable but is hypoelliptic at $t = 0$, Y is locally solvable but not hypoelliptic at $t = 0$.

It is then immediate that

(2.7) If $P = XY$, then P is neither locally solvable nor hypoelliptic at $t = 0$.

We will also use the following notations (the one on l_P, by analogy with [2]):

$$\delta = a - b \qquad (\text{hence, } \mathrm{Re}\, \delta > 0) \tag{2.8}$$

$$l_P = c/\delta\,. \tag{2.9}$$

From now on J_1 will be the greatest open interval containing the origin and contained in J, such that $\mathrm{Re}\, \{a_0(t)/t^k\}$ and $\mathrm{Re}\{b_0(t)/t^k\}$ never vanish in J_1.

We are interested in the validity of the following hypoellipticity property:

(2.10) Given any real number s, any open subset J' of J_1 and any distribution u in J',

$$Pu \in \mathcal{H}^s_{\mathrm{loc}}(J') \quad \text{implies} \quad u \in \mathcal{H}^{s+2/(k+1)}_{\mathrm{loc}}(J')\,.$$

The following is the main theorem of this paper:

THEOREM 2.1. *Let P be an operator in J satisfying* (2.1), (2.4) *and* (2.5). *The following are then equivalent:*

$$P \text{ satisfies } (2.10), \tag{2.11}$$

$$P^* \text{ satisfies } (2.10), \tag{2.12}$$

(2.13) *whatever the integer* $m \geqslant 0$, $l_P \neq m(k+1)$ *and* $l_P \neq m(k+1)+1$.

It is well known that property (2.10) has various implications, among them, that P is hypoelliptic and P^* is locally solvable in J_1.

REMARK 2.1. There is an easy generalization of Th. 2.1, obtained replacing in its statement condition (2.4) by

(2.4′) there are real numbers $\alpha_0, \alpha_1, \ldots, \alpha_{k-1}$ such that $c_0(t) = ct^{k-1} + t^k h(t)$; $a_0(t) = i\sum_{j=0}^{k-1} \alpha_j t^j + at^k + t^{k+1} f(t)$; $b_0(t) = i\sum_{j=0}^{k-1} \alpha_j t^j + bt^k + t^{k+1} g(t)$; where $f, g, h \in C^\infty(J)$ and a, b, c are complex numbers.

Let P satisfy (2.1), (2.4′) and (2.5) and set

$$\beta(t) = i\int_0^t \left(\sum_{j=0}^{k-1} \alpha_j t^j\right) dt, \tag{2.14}$$

$$U(t) = \exp\{\beta(t)A\}. \tag{2.15}$$

It is immediately checked that $u \to U(t)u$ defines an automorphism of $\mathcal{D}'(J; H^{-\infty})$, (resp. of $C^\infty(J; H^\infty)$; resp. of $C_c^\infty(J; H^\infty)$). This is, in fact, an isometric automorphism of $L^2(R; H)$, and it can be checked that it is an automorphism of $\mathcal{K}^s_{\text{loc}}(J)$, whatever the real number s and the open subset J of the real line.

Since we have, with the obvious notation, $PU = U\hat{P}$, where $\hat{a}_0(t) = at^k + t^{k+1} f(t)$, $\hat{b}_0(t) = bt^k + t^{k+1} g(t)$, $\hat{c}_0(t) = c_0(t)$, it is evident that, the theorem being true for $\hat{P}$, it must be also true for the more general operator P.

3. – A subelliptic estimate.

We keep the notations of section 2.

THEOREM 3.1. *Let P be a differential operator defined on J, satisfying* (2.1), (2.4) *and* (2.5). *Assume also that P satisfies the following condition:*

$$\operatorname{Re} l_P \leqslant -k/2. \tag{3.1}$$

Then, for suitable constants $C_0, C_1 > 0$ *and all* $\varphi \in C_c^\infty(J; H^\infty)$, *we have*:

$$\int (\|\varphi_t\|_0^2 + \|t^k A\varphi\|_0^2)\, dt \leqslant C_0 \left|\int (P\varphi, \varphi)_0\, dt\right| + C_1 \int |t| (\|\varphi_t\|_0^2 + \|t^k A\varphi\|_0^2)\, dt, \tag{3.2}$$

provided only that J *be bounded.*

PROOF. Throughout the proof, we will omit the subscript 0 in the notation for the norm and the inner product of H. We will set:

$$\tilde{X} = \partial_t - a t^k A, \quad \tilde{Y} = \partial_t - b t^k A, \quad \tilde{P} = \tilde{X}\tilde{Y} + c t^{k-1} A, \tag{3.3}$$

with the constants a, b, c given by (2.4). Observe that we have

$$P - \tilde{P} = t^{k+1} f_1(t, A) A \partial_t + t^{2k+1} g_1(t, A) A^2 + f_0(t, A)\partial_t + t^k g_0(t, A) A + h(t, A),$$

where $f_j(t, A)$, $g_j(t, A)$ $(j = 0, 1)$ and $h(t, A)$ all belong to $\mathfrak{Q}_A(J)$. Therefore:

$$\begin{aligned} &\left|\int (\{P - \tilde{P}\}\varphi, \varphi)\, dt\right| \leqslant \\ &\leqslant C_2 \left\{\int |t| (\|\varphi_t\|\, \|t^k A\varphi\| + \|t^k A\varphi\|^2)\, dt + \right. \\ &\left. + \int (\|\varphi_t\|\, \|\varphi\| + \|t^k A\varphi\|\, \|\varphi\| + \|\varphi\|^2)\, dt \right\} \leqslant \\ &\leqslant 2C_2 \int |t| (\|\varphi_t\|^2 + \|t^k A\varphi\|^2)\, dt + \\ &+ \varepsilon \int (\|\varphi_t\|^2 + \|t^k A\varphi\|^2)\, dt + (C_3 + C_4 \varepsilon^{-1}) \int \|\varphi\|^2\, dt, \end{aligned} \tag{3.4}$$

where ε is a positive number which we are soon going to choose. We note that

$$\int \|\varphi\|^2\, dt = -2\, \mathrm{Re} \int (t\varphi_t, \varphi)\, dt, \quad \forall \varphi \in C_c^\infty(J; H^\infty), \tag{3.5}$$

whence

$$\int \|\varphi\|^2\, dt \leqslant 4 \int t^2 \|\varphi_t\|^2\, dt, \quad \forall \varphi \in C_c^\infty(J; H^\infty), \tag{3.6}$$

hence, if J is bounded (with $|t| \leqslant M$, $\forall t \in J$), we have:

$$\int \|\varphi\|^2\, dt \leqslant 4M \int |t|\, \|\varphi_t\|^2\, dt, \quad \forall \varphi \in C_c^\infty(J; H^\infty). \tag{3.7}$$

If we take (3.7) into account, (3.4) yields:

$$\text{(3.8)}\qquad \left|\int(\{P-\tilde{P}\}\varphi,\varphi)\,dt\right| \leqslant$$
$$\leqslant \varepsilon\int(\|\varphi_t\|^2+\|t^kA\varphi\|^2)\,dt+$$
$$+(C_3'+C_4'\varepsilon^{-1})\int|t|(\|\varphi_t\|^2+\|t^kA\varphi\|^2)\,dt,$$

which has the following implication: it suffices to prove (3.2) with $\tilde{P}$ substituted for P and then choose $\varepsilon=(2C_0)^{-1}$. This yields at once (3.2) for P itself, after some increasing of C_0 and C_1. In other words, we may assume that

$$\text{(3.9)}\qquad P=XY+ct^{k-1}A\,,\qquad X=\partial_t-at^kA\,,\qquad Y=\partial_t-bt^kA\,,$$

remembering that, by (2.5), $\operatorname{Re}a>0$, $\operatorname{Re}b<0$ (hence $\operatorname{Re}\delta>0$). We will set:

$$\text{(3.10)}\qquad X^+=\partial_t+\bar{a}t^kA\,.$$

Note that $-X^+=X^*$, the adjoint of X. We have:

$$\text{(3.11)}\qquad \operatorname{Re}\left\{\delta\int(P\varphi,\varphi)\,dt\right\}=-\operatorname{Re}\left\{\delta\int(Y\varphi,X^+\varphi)\,dt\right\}+\operatorname{Re}(c\delta)\int t^{k-1}(A\varphi,\varphi)\,dt\,,$$

$$\text{(3.12)}\qquad \int(Y\varphi,X^+\varphi)\,dt=\int(\|\varphi_t\|^2-ab\|t^kA\varphi\|^2)\,dt+$$
$$+\int\{a(\varphi_t,t^kA\varphi)-b(t^kA\varphi,\varphi_t)\}\,dt\,,$$

whence:

$$\operatorname{Re}\left\{\bar{\delta}\int(Y\varphi,X^+\varphi)\,dt\right\}=(\operatorname{Re}\delta)\int\|\varphi_t\|^2\,dt-\operatorname{Re}(ab\bar{\delta})\int\|t^kA\varphi\|^2\,dt+$$
$$+|\delta|^2\int\operatorname{Re}(\varphi_t,t^kA\varphi)\,dt+\operatorname{Re}(i\bar{\delta}\{a+b\})\int\operatorname{Im}(\varphi_t,t^kA\varphi)\,dt=$$
$$=(\operatorname{Re}\delta)\int\|\varphi_t\|^2\,dt-\operatorname{Re}(ab\bar{\delta})\int\|t^kA\varphi\|^2\,dt-$$
$$-\{(k|\delta|^2)/2\}\int t^{k-1}(A\varphi,\varphi)\,dt-\operatorname{Im}(\bar{\delta}\{a+b\})\int\operatorname{Im}(\varphi_t,t^kA\varphi)\,dt\,.$$

If we combine this with (3.11), we get:

$$\text{(3.13)}\qquad (\operatorname{Re}(c\bar{\delta})+k|\delta|^2/2)\int t^{k-1}(A\varphi,\varphi)\,dt-\operatorname{Re}\left\{\bar{\delta}\int(P\varphi,\varphi)\,dt\right\}=$$
$$=\int\{(\operatorname{Re}\delta)\|\varphi_t\|^2-2\operatorname{Im}\{\bar{\delta}(a+b)/2\}\operatorname{Im}(\varphi_t,t^kA\varphi)-$$
$$-\operatorname{Re}(ab\bar{\delta})\|t^kA\varphi\|^2\}\,dt\,.$$

Since $\operatorname{Re}\delta > 0$ and, as it was easily proved in [3], we have:

$$(3.14)\qquad -(\operatorname{Re}\delta)\operatorname{Re}(ab\bar\delta) - \{\operatorname{Im}\bar\delta(a+b)/2\}^2 = -(\operatorname{Re}a)(\operatorname{Re}b)|\delta|^2 > 0\,,$$

it follows that, for some constant $C > 0$, we have:

$$(3.15)\qquad C\int(\|\varphi_t\|^2 + \|t^k A\varphi\|^2)\,dt \leqslant (\operatorname{Re}(c\bar\delta) + k|\delta|^2/2)\int t^{k-1}(A\varphi, \varphi)\,dt - \operatorname{Re}\left\{\bar\delta\int(P\varphi, \varphi)\,dt\right\}.$$

Recalling that P satisfies (3.1), i.e., that $\operatorname{Re}(c\bar\delta) + k|\delta|^2/2 \leqslant 0$, we get:

$$(3.16)\qquad C'\int(\|\varphi_t\|^2 + \|t^k A\varphi\|^2)\,dt \leqslant \left|\int(P\varphi, \varphi)\,dt\right|,$$

which finally gives us (3.2).

COROLLARY 3.1. *Same hypothesis as in* Th. 3.1, *in particular* (3.1). *Let us choose* $T_1 > 0$ *such that* $J' =)-T_1, T_1($ *is contained in* J, *and such that* $T_1 \leqslant (2C_1)^{-1}$, *where* C_1 *is the constant in* (3.2). *Then, for every* $\varphi \in C_c^\infty(J'; H^\infty)$, *we have*:

$$(3.17)\qquad \int(\|\varphi_t\|_0^2 + \|t^k A\varphi\|_0^2)\,dt \leqslant 2C_0\left|\int(P\varphi, \varphi)_0\,dt\right|,$$

where C_0 *is the same constant as in* (3.2).

COROLLARY 3.2. *Same hypothesis as in* Cor. 3.1. *There is a constant* $C > 0$ *such that, for every* $\varphi \in C_c^\infty(J'; H^\infty)$ *we have*:

$$(3.18)\qquad \int\Big(\|\varphi_t\|_0^2 + \sum_{j=0}^{k}\|t^j A^{(j+1)/(k+1)}\varphi\|_0^2\Big)\,dt \leqslant C\left|\int(P\varphi, \varphi)_0\,dt\right|.$$

PROOF. Apply Prop. 1.1*a*).

Let $X = \partial_t - a(t, A)A$, as before, with $a_0(t) = at^k + t^{k+1}f(t)$, and $\operatorname{Re}a > 0$, and consider the operator X^+X. If we call $Y = X^+ = \partial_t + \bar a(t, A)A$, we have $X^+X = YX = XY + c(t, A)A$, where $c(t, A) = -a_t(t, A) - \bar a_t(t, A)$, hence $c = -k(a + \bar a)$. In the present situation, $\delta = a + \bar a$, therefore condition (3.1) is automatically verified, and we derive from Corollary 3.2:

COROLLARY 3.3. *Let* $X = \partial_t - a(t, A)A$ *be defined on* J *and suppose that*

(3.19) $a_0(t) = at^k + t^{k+1}f(t)$, *where* $f \in C^\infty(J)$, *k is an odd integer, and* $\operatorname{Re}a > 0$.

Then, there is an open interval J' containing 0 *and contained in J, and a constant $C_0 > 0$ such that, for every $\varphi \in C_c^\infty(J'; H^\infty)$, we have*:

$$\int \Big(\|\varphi_t\|_0^2 + \sum_{j=0}^{k} \|t^j A^{(j+1)/(k+1)} \varphi\|_0^2 \Big)\, dt \leq C_0 \int \|X\varphi\|_0^2\, dt\,. \tag{3.20}$$

REMARK 3.1. In the case of the operator $\tilde{P}$ (as in (3.3)) which is defined on the whole real line, we proved (3.16) under the only restriction that J be bounded. Hence, (3.17) and therefore also (3.18) is automatically verified, for every bounded open set J. A similar remark applies to $\tilde{X}$: (3.20) holds for every bounded open set J.

4. – The spaces ${}^k\mathcal{H}^{s,m}$.

The spaces ${}^k\mathcal{H}^{s,m}_{\text{loc}}(J)$ will be defined in such a way as to have ${}^k\mathcal{H}^{s,0}_{\text{loc}}(J) = \mathcal{H}^s_{\text{loc}}(J)$, and if m is a nonnegative integer, u belongs to ${}^k\mathcal{H}^{s,m}_{\text{loc}}(J)$ if and only if both $\partial_t u$ and $t^k A u$ belong to ${}^k\mathcal{H}^{s-1,m-1}_{\text{loc}}(J)$. Thanks to Prop. 1.1, we can give another definition, easier to handle. We begin by defining the class of operators ${}^k\mathcal{N}^{d,p}$. By Z_+ we will always denote the set of nonnegative integers.

DEFINITION 4.1. Let $d, p \in R$, with $kp \in Z_+$. We will denote by ${}^k\mathcal{N}^{d,p}$ (or ${}^k\mathcal{N}^{d,p}(J)$, if J need to be specified), the linear space of differential operators B of the form

$$B = \sum_{\substack{(\alpha/k)+\beta \leq p \\ \alpha,\beta \in Z_+}} B_{\alpha\beta} t^\alpha \partial_t^\beta\,, \tag{4.1}$$

where $B_{\alpha,\beta}$ is a differential operator of order less than or equal to $d - (kp - \alpha + \beta)/(k+1)$ on J. (Here and in what follows, $B_{\alpha,\beta}$ is understood as being of the form (0.1), which greatly simplifies the exposition. Nevertheless, we will need the results of Prop. 4.1 also in the case $B_{\alpha,\beta}$ is a more general kind of operator, as for instance $(1 - \partial_t^2 + A^2)^q$, with q belonging to R. This, however, will be only needed in the proof of Th. 5.1, parts 3) and 4).)

We give below some of the trivial properties of these spaces:

(4.2) B is a differential operator of order $\leq d$ iff $B \in {}^k\mathcal{N}^{d,0}$.

(4.3) If $B \in {}^k\mathcal{N}^{d,p}$, then the order of B is $\leq d$.

(4.4) ${}^k\mathcal{N}^{d,p}$ is contained in ${}^k\mathcal{N}^{d',p}$ if $d \leq d'$; ${}^k\mathcal{N}^{d,p}$ is contained in ${}^k\mathcal{N}^{d,p-(1/k)}$ if $p \notin Z_+$; ${}^k\mathcal{N}^{d,p+m}$ is contained in ${}^k\mathcal{N}^{d,p}$ if $m \in Z_+$.

(4.5) ${}^k\mathcal{N}^{d,p}$ is contained in ${}^k\mathcal{N}^{d+j/(k+1),p+(j/k)}$, $\forall j \in Z_+$.

(4.6) $t^\alpha \partial_t^\beta \in {}^k\mathcal{N}^{\beta,\alpha/k+\beta}$.

Let $B = B_{\alpha\beta} t^{\alpha} \partial_t^{\beta} \in {}^k\mathcal{N}^{d,p}$, with order of $B_{\alpha\beta}$ less than or equal to $d - (kp - \alpha + \beta)/(k+1)$, and suppose $B_{\alpha\beta} = E\partial_t^m$, where E is independent of ∂_t, i.e., E is a finite sum of terms like $\varphi_r(t, A)A^r$. Then, $B = \sum_{i=0}^{m} Ec_i t^{\alpha-m+i} \partial_t^{\beta+i}$, (where we make the convention that $t^{\alpha-m+i}$ is identically zero if $\alpha - m + i < 0$, and c_i is some constant). If we call $F_i = Ec_i\, t^{\alpha-m+i}\, \partial_t^{\beta+i}$, then $F_i \in {}^k\mathcal{N}^{d,p+i}$, and $F_i = F_{i\alpha\beta} t^{\alpha-m+i} \partial_t^{\beta+i}$, with $F_{i\alpha\beta}$ independent of ∂_t. In general, given any $B \in {}^k\mathcal{N}^{d,p}$, we can find an integer $m \geqslant 0$ and $F_i \in {}^k\mathcal{N}^{d,p+i}$, $i = 0, 1, \dots, m$, such that $B = F_0 + F_1 + \dots + F_m$, and each F_i can be written in the form (4.1), with $F_{i\alpha\beta}$ independent of ∂_t. This remark simplifies the proof of the next proposition.

We will denote by $[B, C]$ the difference $BC - CB$; by Z/k the set of real numbers of the form m/k, where $m \in Z$; and by Z_+/k the set of non-negative numbers belonging to Z/k.

Since the following proposition is not difficult to prove, we will only give a sketch of the proof.

PROPOSITION 4.1. *Let* $B \in {}^k\mathcal{N}^{d,p}$, $B' \in {}^k\mathcal{N}^{d',p'}$, *and* $C = [B, B']$. *Then:*

$$BB' \in {}^k\mathcal{N}^{d+d',p+p'} \text{ and } B^* \in {}^k\mathcal{N}^{d,p}. \tag{4.7}$$

(4.8) *a) If* $p = p' = 0$, *then* $C \in {}^k\mathcal{N}^{d+d'-1,0}$.

b) If $p + p' \neq 0$, $0 \leqslant p < 1$ *and* $0 \leqslant p' < 1$, *then* $C \in {}^k\mathcal{N}^{d+d'-1,p+p'-(1/k)}$.

c) If $pp' = 0$ *and either* $p \geqslant 1$ *or* $p' \geqslant 1$, *then:*

c_1) If $p, p' \in Z_+$, *then* $C \in {}^k\mathcal{N}^{d+d'-1,p+p'-1}$.

c_2) If $p \notin Z$ *or* $p' \notin Z$, *then* $C \in {}^k\mathcal{N}^{d+d'-1+(1/(k+1)),p+p'-1}$. *More precisely,* $C = D + E$, *with*

$$D \in {}^k\mathcal{N}^{d+d'-1,p+p'-1} \quad \text{and} \quad E \in {}^k\mathcal{N}^{d+d'-1,p+p'-(1/k)}.$$

d) If $pp' \neq 0$ *and either* $p \geqslant 1$ *or* $p' \geqslant 1$, *then:*

d_1) If $p \notin Z$ *or* $p' \notin Z$, *then* $C \in {}^k\mathcal{N}^{d+d'-1,p+p'-1-(1/k)}$.

d_2) If $p, p' \in Z$, *then* $C \in {}^k\mathcal{N}^{d+d'-1+(1/(k+1)),p+p'-1}$. *More precisely,* $C = D + E$, *with* $D \in {}^k\mathcal{N}^{d+d'-1,p+p'-1-(1/k)}$ *and* $E \in {}^k\mathcal{N}^{d+d'-1,p+p'-1}$.

SKETCH OF THE PROOF OF (4.8). One should first prove that $[\varphi(t, A) t^{\alpha} \partial_t^{\beta}, \psi(t, A) t^{\gamma} \partial_t^{\delta}]$ is: *a*) 0 if $\beta = \delta = 0$; *b*) an element of ${}^k\mathcal{N}^{\beta+\delta-1,((\alpha+\gamma)/k)+\beta+\delta-1}$ if $\alpha = \gamma = 0$ (or if $\gamma = \delta = 0$ or if $\alpha = \beta = 0$) and $\beta + \delta \geqslant 1$; *c*) an element of ${}^k\mathcal{N}^{\beta+\delta-1,((\alpha+\gamma)/k)+\beta+\delta-1-(1/k)}$ if $\alpha + \gamma \geqslant 1$ (and $\beta + \delta \geqslant 1$).

DEFINITION 4.2. Let $s \in R$ and $m \in Z_+/k$. We will denote by ${}^k\mathcal{H}^{s,m}_{\text{loc}}$ (or ${}^k\mathcal{H}^{s,m}_{\text{loc}}(J)$ if the open set J must be specified) the space of distributions $u \in \mathcal{D}'(J; H^{-\infty})$ having the following property:

(4.9) $\forall B \in {}^k\mathcal{N}^{d,p}(J)$, with $p \leqslant m$, $Bu \in \mathcal{H}^{s-d-\{k(m-p)/(k+1)\}}_{\text{loc}}(J)$.

We give below some of the properties of the spaces ${}^k\mathcal{H}^{s,m}_{\text{loc}}$ (with $m \geqslant 0$), and the proof of (4.12), which is not so easy to verify.

(4.10) ${}^k\mathcal{H}^{s,0}_{\text{loc}} = \mathcal{H}^{s}_{\text{loc}}$.

(4.11) Let $u \in \mathcal{D}'(J; H^{-\infty})$. Then, $u \in {}^k\mathcal{H}^{s,m}_{\text{loc}}$ if and only if, for all $\alpha, \beta \in Z_+$, with $(\alpha/k) + \beta \leqslant m$, $t^\alpha \partial_t^\beta u \in \mathcal{H}^{s-(km-\alpha+\beta)/(k+1)}_{\text{loc}}$.

(4.12) Let $u \in \mathcal{D}'(J; H^{-\infty})$ and $m \in Z_+$. Then, $u \in {}^k\mathcal{H}^{s,m}_{\text{loc}}$ if and only if $(t^k A)^\alpha \partial_t^\beta u \in \mathcal{H}^{s-(km+\alpha+\beta)/(k+1)}_{\text{loc}}$, for all $\alpha, \beta \in Z_+$ with $\alpha + \beta \leqslant m$.

(4.13) Let $u \in \mathcal{D}'(J; H^{-\infty})$ and $m \in Z_+$. Then, $u \in {}^k\mathcal{H}^{s,m}_{\text{loc}}$ if and only if $t^{k\alpha} \partial_t^\beta u \in \mathcal{H}^{s-\{k(m-\alpha)+\beta\}/(k+1)}_{\text{loc}}$, for all $\alpha, \beta \in Z_+$ with $\alpha + \beta \leqslant m$.

PROOF OF (4.12). The «only if» part is true by (4.11). To prove the «if» part, we have, by (4.11) to prove that

(4.14) $t^{k\alpha+j} \partial_t^\beta u \in \mathcal{H}^{s-(km-k\alpha-j+\beta)/(k+1)}_{\text{loc}}$, if $\alpha, j, \beta \in Z_+$, $\alpha + (j/k) + \beta \leqslant m$ and $0 \leqslant j \leqslant k$.

We will prove (4.14) by induction on α. Let first $\alpha = 0$, so that $(j/k) + \beta \leqslant m$. If $j = 0$, there is nothing to prove. If $j \neq 0$, then we must have $\beta + 1 \leqslant m$, hence by hypothesis both $t^k A \partial_t^\beta u$ and $\partial_t^{\beta+1} u$ belong to $\mathcal{H}^{s-(km+1+\beta)/(k+1)}_{\text{loc}}$. Then, by Prop. 1.1c), (4.14) is satisfied when $\alpha = 0$, $0 \leqslant j \leqslant k$ and $\beta \in Z_+$ with $\alpha + (j/k) + \beta \leqslant m$.

Suppose that (4.14) has already been proved for all $\alpha \leqslant n-1$, $0 \leqslant j \leqslant k$, and $\beta \in Z_+$, with $\alpha + (j/k) + \beta \leqslant m$, and let us prove the same when $\alpha = n$ (if $n \leqslant m$). Let then $n + (j/k) + \beta \leqslant m$. If $j = 0$, instead of $\alpha = n$, $j = 0$ (and that fixed β), we may take $\alpha = n-1$ and $j = k$: this case was already inductively proved. If $j \neq 0$, then we must have $n + 1 + \beta \leqslant m$. We have already inductively proved (cases $(n, k, \beta+1)$ and $(n-1, k-1, \beta)$) that both $t^{kn} \partial_t^{\beta+1} u$ and $t^{kn-1} \partial_t^\beta u$ belong to $\mathcal{H}^{s-(km-kn+1+\beta)/(k+1)}_{\text{loc}}$, hence

$$(4.15) \qquad \partial_t(t^{kn} \partial_t^\beta u) = t^{kn} \partial_t^{\beta+1} u + kn t^{kn-1} \partial_t^\beta u \in \mathcal{H}^{s-(km-kn+1+\beta)/(k+1)}_{\text{loc}}.$$

Consider now the property

$$(4.16) \qquad t^{k(n+1)} A^{n+1-r} \partial_t^\beta u \in \mathcal{H}^{s-(km+n+1+\beta-r(k+1))/(k+1)}_{\text{loc}}, \quad \text{with } r \in Z_+,\ r \leqslant n.$$

If (4.16) is true when $r = n$, then by (4.15) and Prop. 1.1c), we will have proved (4.14) for $\alpha = n$, $0 \leqslant j \leqslant k$, $\beta \in Z_+$ with $\alpha + (j/k) + \beta \leqslant m$, and the proof of (4.12) will be ended. The proof of (4.16) is by induction on r. We know by hypothesis that it is true for $r = 0$. Suppose it is true for $r = p$, i.e., that $t^k A(t^{kn} A^{n-p} \partial_t^\beta u) \in \mathcal{H}_{\mathrm{loc}}^{s-(km+n+1+\beta-p(k+1))/(k+1)}$ and let us prove it for $r = p + 1$ (if $p + 1 \leqslant n$). By (4.15) we get

$$\partial_t(t^{kn} A^{n-p} \partial_t^\beta u) \in \mathcal{H}_{\mathrm{loc}}^{s-(km+n+1+\beta-p(k+1))/(k+1)},$$

hence, by Prop. 1.1c), we must have (4.16) for $r = p + 1$. Thus (4.12) is proved.

The space ${}^k\mathcal{H}_{\mathrm{loc}}^{s,m}(J)$ carries a natural topology: the coarsest locally convex one which renders all the mappings $u \to Bu$ into $\mathcal{H}_{\mathrm{loc}}^{s-d-k(m-p)/(k+1)}(J)$, with B as in (4.9), continuous. It coincides with the coarsest locally convex one which renders all the mappings $u \to t^\alpha \partial_t^\beta u$ into $\mathcal{H}_{\mathrm{loc}}^{s-(km-\alpha+\beta)/(k+1)}(J)$ continuous.

With this topology, ${}^k\mathcal{H}_{\mathrm{loc}}^{s,m}(J)$ is a Fréchet space. If then K is an arbitrary compact subset of J, we denote by ${}^k\mathcal{H}_c^{s,m}(K)$ the closed linear subspace of ${}^k\mathcal{H}_{\mathrm{loc}}^{s,m}(J)$ consisting of the distributions vanishing outside K, and by ${}^k\mathcal{H}_c^{s,m}(J)$ the inductive limit of the Fréchet spaces ${}^k\mathcal{H}_c^{s,m}(K)$, as K ranges over all compact subsets of J. Remark that $C_c^\infty(J; H^\infty)$ is dense in both ${}^k\mathcal{H}_{\mathrm{loc}}^{s,m}$ and ${}^k\mathcal{H}_c^{s,m}$, the inclusion map being continuous. This enables us to identify (as sets) their respective duals with subspaces of $\mathcal{D}'(J; H^{-\infty})$.

DEFINITION 4.3. Let s and m be as in Def. 4.2. We denote by ${}^k\mathcal{H}_{\mathrm{loc}}^{s,-m}(J)$ the dual of ${}^k\mathcal{H}_c^{-s,m}(J)$.

We can of course construct as before ${}^k\mathcal{H}_c^{s,-m}(K)$ and ${}^k\mathcal{H}_c^{s,-m}(J)$. All the spaces ${}^k\mathcal{H}_{\mathrm{loc}}^{s,m}$ and ${}^k\mathcal{H}_c^{s,m}$ contain $C_c^\infty(J; H^\infty)$ as a dense subset, and they are all reflexive, for all real s and all $m \in Z/k$.

PROPOSITION 4.2. *Let $B \in {}^k\mathcal{N}^{d,p}$. Then: a) For every real s, B defines a continuous linear map from ${}^k\mathcal{H}_{\mathrm{loc}}^{s,m}$ into ${}^k\mathcal{H}_{\mathrm{loc}}^{s-d,m-p}$, provided that either:* i) $m - p \geqslant 0$ *or* ii) $m \leqslant 0$ *or* iii) $m \in Z$; *b) For every real s and $\alpha \in Z_+$ (and $m \in Z/k$), the operator « multiplication by t^α » defines a continuous linear map from ${}^k\mathcal{H}_{\mathrm{loc}}^{s,m}$ into ${}^k\mathcal{H}_{\mathrm{loc}}^{s,m-(\alpha/k)}$.*

PROOF: The proof of a)i) is easy, applying (4.7) and the definition of the spaces ${}^k\mathcal{H}_{\mathrm{loc}}^{s,m}$. The case a)ii) is got by duality of a)i) (which is true also for ${}^k\mathcal{H}_c^{s,m}$). Remark that a) i) and a)ii) is all that we need to prove Prop. 4.3, hence we may now assume that Prop. 4.3 was already proved.

b) Is a consequence of *a*)i) and *a*)ii) if either $m-(\alpha/k)\geqslant 0$ or $m\leqslant 0$. Suppose that $m>0$ and $m-(\alpha/k)<0$, and call $\beta=km$: then $\beta\in Z_+$, $0<\beta<\alpha$, hence the operator t^α is the composition of $t^{\alpha-\beta}$ with t^β. Since $t^\beta\colon {}^k\mathcal{H}^{s,m}_{\mathrm{loc}}\to {}^k\mathcal{H}^{s,0}_{\mathrm{loc}}$ and $t^{\alpha-\beta}\colon {}^k\mathcal{H}^{s,0}_{\mathrm{loc}}\to {}^k\mathcal{H}^{s,m-(\alpha/k)}_{\mathrm{loc}}$ are continuous, so also is their composition.

Let us go back to *a*)iii). By ii), we may assume $m\in Z_+$ and $m\geqslant 1$. The easy proof is by induction on m. Since we proved already *a*)iii) to be true when $m=0$, only the inductive step must be proved. Suppose *a*)iii) is true for $m=n-1$, and let us prove it for $m=n$ (where $n\in Z_+$, $n\geqslant 1$). We apply a decomposition similar to the case *b*), remarking that, if $B\in {}^k\mathcal{N}^{d,p}$ with $p\geqslant 1$ (in the case $p<1$ we have $n-p>0$, which case is covered by *a*)i)), then we may write $B=B_1\partial_t+B_2t^k+\sum_{j=0}^{k-1}E_jt^j$, where $B_1\in {}^k\mathcal{N}^{d,p-1}$, $B_2\in {}^k\mathcal{N}^{d,p-1}$, $E_j\in {}^k\mathcal{N}^{d-(kp-j)/(k+1),0}$. By the inductive hypothesis, $B_1\colon {}^k\mathcal{H}^{s-1,n-1}_{\mathrm{loc}}\to {}^k\mathcal{H}^{s-d,n-p}_{\mathrm{loc}}$ and $B_2\colon {}^k\mathcal{H}^{s,n-1}_{\mathrm{loc}}\to {}^k\mathcal{H}^{s-d,n-p}_{\mathrm{loc}}$ are continuous, and so are $\partial_t\colon {}^k\mathcal{H}^{s,n}_{\mathrm{loc}}\to {}^k\mathcal{H}^{s-1,n-1}_{\mathrm{loc}}$ and $t^k\colon {}^k\mathcal{H}^{s,n}_{\mathrm{loc}}\to {}^k\mathcal{H}^{s,n-1}_{\mathrm{loc}}$. Hence, $B_1\partial_t$ and B_2t^k are both continuous maps from ${}^k\mathcal{H}^{s,n}_{\mathrm{loc}}$ into ${}^k\mathcal{H}^{s-d,n-p}_{\mathrm{loc}}$. Finally, $t^j\colon {}^k\mathcal{H}^{s,n}_{\mathrm{loc}}\to {}^k\mathcal{H}^{s,n-(j/k)}_{\mathrm{loc}}$ and $E_j\colon {}^k\mathcal{H}^{s,n-(j/k)}_{\mathrm{loc}}\to {}^k\mathcal{H}^{s-d+(kp-j)/(k+1),n-(j/k)}_{\mathrm{loc}}$ are continuous, as well as (by Prop. 4.3) the inclusion maps from ${}^k\mathcal{H}^{s-d+(kp-j)/(k+1),n-(j/k)}_{\mathrm{loc}}$ into ${}^k\mathcal{H}^{s-d,n-p}_{\mathrm{loc}}$; hence, for each j, $E_jt^j\colon {}^k\mathcal{H}^{s,n}_{\mathrm{loc}}\to {}^k\mathcal{H}^{s-d,n-p}_{\mathrm{loc}}$ is continuous.

PROPOSITION 4.3. *Let* $s, s'\in R$; $m, m'\in Z/k$ *satisfy*

$$s'\leqslant s\,;\qquad s'-km'/(k+1)\leqslant s-km/(k+1)\,. \tag{4.18}$$

Then, we have a continuous injection ${}^k\mathcal{H}^{s,m}_{\mathrm{loc}}\to {}^k\mathcal{H}^{s',m'}_{\mathrm{loc}}$.

PROOF. One should first prove that ${}^k\mathcal{H}^{s,m}_{\mathrm{loc}}\to {}^k\mathcal{H}^{s',m}_{\mathrm{loc}}$ if $s'\leqslant s$; ${}^k\mathcal{H}^{s,m}_{\mathrm{loc}}\to {}^k\mathcal{H}^{s,m'}_{\mathrm{loc}}$ if $m\leqslant m'$, and that always ${}^k\mathcal{H}^{s,m}_{\mathrm{loc}}\to {}^k\mathcal{H}^{s-1/(k+1),m-1/k}_{\mathrm{loc}}$. The first two facts are easily checked. As for the last one, it is enough to remark that $I\in {}^k\mathcal{N}^{0,0}$, which is contained in ${}^k\mathcal{N}^{1/(k+1),1/k}$ (cf. (4.5)), and apply Prop. 4.2, cases *a*)i) and *a*)ii), which cover all possible values of m. An iteration of these three facts gives the whole result.

COROLLARY 4.1. *If* $m\geqslant 0$, *then*: $\mathcal{H}^s_{\mathrm{loc}}(J)\to {}^k\mathcal{H}^{s,m}_{\mathrm{loc}}(J)\to \mathcal{H}^{s-km/(k+1)}_{\mathrm{loc}}(J)$. *If* $m\leqslant 0$, *then*: $\mathcal{H}^{s-km/(k+1)}_{\mathrm{loc}}(J)\to {}^k\mathcal{H}^{s,m}_{\mathrm{loc}}(J)\to \mathcal{H}^s_{\mathrm{loc}}(J)$.

Let K be a compact subset of J: the spaces $\mathcal{H}^s_c(K)$, ${}^k\mathcal{H}^{s,m}_c(K)$ are normable (in fact, they can be equipped with a Hilbert space structure). Let us select in each of them a norm, $\| \|_s$ and $\| \|_{s,m}$ (from now on, we will always denote the norm of $\mathcal{H}^s$ by $\| \|_s$, since that of H^s will not appear in the sequel), submitted to the sole requirement that they define the topologies of the spaces. For instance, when $m \geqslant 0$, we may take

$$\|u\|_{s,m}^2 = \sum_{\substack{(\alpha/k)+\beta \leqslant m \\ \alpha,\beta \in Z_+}} \|t^\alpha \partial_t^\beta u\|^2_{s-(km-\alpha+\beta)/(k+1)} .$$

If $m \in Z_+$, we may take $\|u\|_{s,m}^2 = \sum_{\alpha+\beta \leqslant m} \|t^{k\alpha} \partial_t^\beta u\|^2_{s-(km-k\alpha+\beta)/(k+1)}$ or even $\|u\|_{s,m}^2 = \sum_{\alpha+\beta \leqslant m} \|(t^k A)^\alpha \partial_t^\beta u\|^2_{s-(km+\alpha+\beta)/(k+1)}$, as (4.12), (4.13) and an easy application of the open mapping theorem shows.

PROPOSITION 4.3′. *Suppose that*

$$s' < s \; ; \quad s' - km'/(k+1) < s - km/(k+1) . \tag{4.18′}$$

Then, the injection ${}^k\mathcal{H}_c^{s,m}(K) \to {}^k\mathcal{H}_c^{s',m'}(K)$ *is compact and, given any real number* s_1 *(arbitrary close to* $-\infty$*), and any* $\varepsilon > 0$*, there is* $C > 0$ *such that, for all* $u \in {}^k\mathcal{H}_c^{s,m}(K)$*, we have*

$$\|u\|_{s',m'} \leqslant \varepsilon \|u\|_{s,m} + C\|u\|_{s_1} . \tag{4.19}$$

The easiest way to prove the above proposition is by using the analogous properties of the spaces $\mathcal{H}^s$ and the description of ${}^k\mathcal{H}^{s,m}$. We give below other properties of the spaces ${}^k\mathcal{H}^{s,m}$: (4.20), (4.21) and (4.24) are got by duality respectively of (4.11), (4.12) and (4.22); moreover, (4.23) can be easily got from (4.22). We will therefore prove only (4.22).

(4.20) Let $m \in Z_+/k$, and $u \in \mathcal{D}'(J; H^{-\infty})$. Then, $u \in {}^k\mathcal{H}_{\text{loc}}^{s,-m}(J)$ if and only if there are $f_{\alpha\beta} \in \mathcal{H}_{\text{loc}}^{s+(km-\alpha+\beta)/(k+1)}$ (for $\alpha, \beta \in Z_+$ with $(\alpha/k) + \beta \leqslant m$), such that $u = \sum \partial_t^\beta (t^\alpha f_{\alpha\beta})$; $u \in {}^k\mathcal{H}_{\text{loc}}^{s,-m}(J)$ iff there are $g_{\alpha\beta} \in \mathcal{H}_{\text{loc}}^{s+(km-\alpha+\beta)/(k+1)}$, with α, β as before, such that $u = \sum t^\alpha \partial_t^\beta g_{\alpha\beta}$. The summation is in both cases over all $\alpha, \beta \in Z_+$ such that $(\alpha/k) + \beta \leqslant m$.

(4.21) Let $m \in Z_+$, and $u \in \mathcal{D}'(J; H^{-\infty})$. Then, $u \in {}^k\mathcal{H}_{\text{loc}}^{s,-m}(J)$ iff there are $g_{\alpha\beta} \in \mathcal{H}_{\text{loc}}^{s+(km+\alpha+\beta)/(k+1)}$ (for $\alpha, \beta \in Z_+$ with $(\alpha/k) + \beta \leqslant m$) such that $u = \sum_{(\alpha/k)+\beta \leqslant m} (t^k A)^\alpha \partial_t^\beta g_{\alpha\beta}$.

(4.22) Let $m, j \in Z_+$, $1 \leqslant j \leqslant k-1$, and $u \in \mathcal{D}'(J; H^{-\infty})$. Then, $u \in {}^k\mathcal{H}_{\text{loc}}^{s,m+(j/k)}$ iff $u \in {}^k\mathcal{H}_{\text{loc}}^{s-1/(k+1),m+(j-1)/k}$ and $tu \in {}^k\mathcal{H}_{\text{loc}}^{s,m+(j-1)/k}$.

(4.23) Let m, j and u be as in (4.22). Then, $u \in {}^k\mathcal{H}_{\text{loc}}^{s,m+j/k}$ iff $t^r u \in {}^k\mathcal{H}_{\text{loc}}^{s-(j-r)/(k+1),m}$ for $r = 0, 1, \ldots, j$.

(4.24) Let m, j, u be as in (4.22), with $m \geqslant 1$. Then, $u \in {}^k\mathcal{H}_{\text{loc}}^{s,-m+j/k}$ iff there are $v \in {}^k\mathcal{H}_{\text{loc}}^{s,-m+(j+1)/k}$ and $w \in {}^k\mathcal{H}_{\text{loc}}^{s+1/(k+1),-m+(j+1)/k}$ such that $u = tv + w$.

PROOF OF (4.22). The « only if » part is true by Props. 4.2 and 4.3. By (4.11), in order to prove that $u \in {}^k\mathcal{H}^{s,m+j/k}_{\text{loc}}$ it is enough to show that

$$(4.25) \quad t^\alpha \partial_t^\beta u \in \mathcal{H}^{s-(km+j-\alpha+\beta)/(k+1)}_{\text{loc}}, \text{ for all } \alpha, \beta \in Z_+ \text{ such that } (\alpha/k) + \beta \leqslant m + j/k.$$

Since $u \in {}^k\mathcal{H}^{s-1/(k+1),m+(j-1)/k}_{\text{loc}}$, (4.11) implies that (4.25) is true when $(\alpha/k) + \beta \leqslant m + (j-1)/k$, so it remains to show that (4.25) is true when $\alpha/k + \beta = m + j/k$. If this last relation holds, we must have $\alpha \geqslant 1$ (since $1 \leqslant j \leqslant k-1$ and $\beta \in Z_+$). Since $tu \in {}^k\mathcal{H}^{s,m+(j-1)/k}_{\text{loc}}$, we have by (4.11):

$$t^{\alpha-1} \partial_t^\beta (tu) \in \mathcal{H}^{s-(km+j-\alpha+\beta)/(k+1)}_{\text{loc}},$$

because $(\alpha-1)/k + \beta = m + (j-1)/k$. But $t^{\alpha-1}\partial_t^\beta(tu) = t^\alpha \partial_t^\beta u + \beta t^{\alpha-1}\partial_t^{\beta-1} u$, and since we already know that $t^{\alpha-1}\partial_t^{\beta-1} u \in \mathcal{H}^{s-(km+j-\alpha+\beta)/(k+1)}_{\text{loc}}$, it follows that (4.25) is satisfied also in the case $\alpha/k + \beta = m + j/k$.

PROPOSITION 4.4. *Let $m \in Z/k$ be such that either $m \geqslant 1$ or $m \leqslant 0$, and let $u \in \mathcal{D}'(J; H^{-\infty})$. Then:*

i) *$u \in {}^k\mathcal{H}^{s,m}_{\text{loc}}$ iff both $\partial_t u$ and $t^k A u$ belong to ${}^k\mathcal{H}^{s-1,m-1}_{\text{loc}}$. (iff $\partial_t u \in {}^k\mathcal{H}^{s-1,m-1}_{\text{loc}}$ and $t^k u \in {}^k\mathcal{H}^{s,m-1}_{\text{loc}}$).*

ii) *$u \in {}^k\mathcal{H}^{s,-m}_{\text{loc}}$ iff there are v, w both in ${}^k\mathcal{H}^{s+1,-m+1}_{\text{loc}}$, such that $u = \partial_t v + t^k A w$.*

Moreover, the « if » part of i) is true without restrictions on m.

The complete proof of Prop. 4.4 will be only given in the next section. Here we will limit ourselves to the case $m \geqslant 1$, which will be needed in the proof of the general case, and also in some proofs of the next section.

PROOF (for $m \geqslant 1$): We will limit ourselves to the proof of i), since ii) follows by duality. The « only if » part of i) is true by Prop. 4.2 (if $m \geqslant 1$ or $m \leqslant 0$). We have to prove that, if $t^k A u$ and $\partial_t u$ are both in ${}^k\mathcal{H}^{s-1,m-1}_{\text{loc}}$, then $u \in {}^k\mathcal{H}^{s,m}_{\text{loc}}$, if $m \geqslant 1$. We do that first in the case $m \in Z_+$, $m \geqslant 1$: availing ourselves of (4.12), we see that we need to show that

$$(4.26) \quad (t^k A)^\alpha \partial_t^\beta u \in \mathcal{H}^{s-(km+\alpha+\beta)/(k+1)}_{\text{loc}}, \quad \text{for } \alpha, \beta \in Z_+ \text{ with } \alpha + \beta \leqslant m.$$

Since $\partial_t u \in {}^k\mathcal{H}^{s-1,m-1}_{\text{loc}}$, we get (4.26) when $\beta \geqslant 1$ by (4.12). Since $t^k A u \in {}^k\mathcal{H}^{s-1,m-1}_{\text{loc}}$, we get (4.26) when $\beta = 0$ and $\alpha \geqslant 1$, by (4.12). Hence, it only remains to show that (4.26) holds when $\alpha = \beta = 0$. Now, since both $t^k A u$ and $\partial_t u$ are in $\mathcal{H}^{s-(km+1)/(k+1)}_{\text{loc}}$, it follows by Prop. 1.1*c*) that (4.26) is satisfied when $\alpha = \beta = 0$.

In order to finish the proof, we have to show that the « if » part of i) is true when $m = n + j/k$, with $n, j \in Z_+$, $n \geqslant 1$, $0 \leqslant j \leqslant k-1$. The proof is by induction on j. When $j=0$, it was just proved. Suppose proved for $j=p$ and let us prove it for $j=p+1$ (if $p+1\leqslant k-1$). By (4.22), in order to show that $u\in{}^k\mathcal{H}^{s,n+(p+1)/k}_{\rm loc}$, it is enough to prove that: $a)$ $tu\in{}^k\mathcal{H}^{s,n+p/k}_{\rm loc}$ and $b)$ $u\in{}^k\mathcal{H}^{s-1/(k+1),n+p/k}_{\rm loc}$. Since t^kAu and $\partial_t u$ are both in ${}^k\mathcal{H}^{s-1,n-1+(p+1)/k}_{\rm loc}$, we have: $c)$ $t^{k+1}Au$ and $t\partial_t u$ are both in ${}^k\mathcal{H}^{s-1,n-1+p/k}_{\rm loc}$; $d)$ t^kAu and $\partial_t u$ are both in ${}^k\mathcal{H}^{s-1-1/(k+1),n-1+p/k}_{\rm loc}$. By $d)$ and the inductive hypothesis we get $b)$, which implies by Prop. 4.3 that $u\in{}^k\mathcal{H}^{s-1,n-1+p/k}_{\rm loc}$. From this, from $c)$ and the inductive hypothesis, we get $a)$. Q.E.D.

In the light of this section, we may rewrite (3.18) as $\|\varphi\|^2_{1,1}\leqslant C|(P\varphi,\varphi)_{0,0}|$, from which we derive:

$$\|\varphi\|_{1,1}\leqslant C\|P\varphi\|_{-1,-1}, \quad \forall\varphi\in C^\infty_c(J';H^\infty). \tag{4.27}$$

The expression (3.20) can be rewritten as:

$$\|\varphi\|_{1,1}\leqslant \sqrt{C_0}\|X\varphi\|_{0,0}, \quad \forall\varphi\in C^\infty_c(J';H^\infty). \tag{4.28}$$

Hence, we may state the following

REMARK 4.1. Corollaries 3.2 and 3.3, and Remark 3.1 remain true if we substitute everywhere (4.27) for (3.18) and (4.28) for (3.20).

5. – A class of stable estimates.

Property (2.10) will be obtained via an estimate involving the ${}^k\mathcal{H}^{s,m}$ norms. In the present section, we wish to investigate the interrelation between the estimate in question and local existence and regularity results, and also the dependence of such results on the indices s and m.

Throughout the section, Ω will be an open set of R, containing the origin; J will denote an open bounded subset of R, containing the origin and with its closure $\bar{J}$ contained in Ω, and k will be a fixed odd integer.

Let us list the three types of properties we are interested in. They will apply to an operator $P\in{}^k\mathcal{N}^{d,p}(\Omega)$, where $p\in Z_+$, $p\geqslant 1$ and P is of the form (0.1), i.e., it is not of the more general kind mentioned after Def. 4.1 (in sections 2 and 3 we had $p=2$, but for our present purposes this limitation is unnecessary).

We begin by the property which is closest to (2.10):

$$(5.1)_{s,m} \qquad \forall u\in\mathcal{D}'(J;H^{-\infty}), \quad Pu\in{}^k\mathcal{H}^{s,m}_{\rm loc}(J) \text{ implies } u\in{}^k\mathcal{H}^{s+d,m+p}_{\rm loc}(J).$$

Here s is any real number, m any belonging to Z/k. Note that if $(5.1)_{s,m}$ holds for a family of open sets J_i contained in J, which covers J (i.e., with the union of all the J_i equal to J), then it holds for J itself. The converse (that if it holds for J it then holds for all open sets J' contained in J) is not immediately apparent but will result from the forthcoming argument (at least for most of the values of m).

The next property is the estimate we have alluded to:

$(5.2)_{s,m}$ To every $\theta \in C_c^\infty(J)$, to every $s' \in R$, and to every compact subset K of Ω, there is a constant $C > 0$ such that, for all $\varphi \in C_c^\infty(K; H^\infty)$,

$$\|\theta\varphi\|_{s+d,m+p} \leqslant C(\|P\theta\varphi\|_{s,m} + \|\varphi\|_{s'}) . \qquad (5.2')_{s,m}$$

In applying (5.2′) one usually chooses s' close to $-\infty$. It is clear that, if $(5.2)_{s,m}$ holds, it also holds when we replace J by anyone of its open subsets. The converse is not so evident.

We come now to the third property, which is relative to the (local) existence of solutions to the inhomogeneous adjoint equation (solutions modulo arbitrarily regular functions):

$(5.3)_{s,m}$ Let $\theta \in C_c^\infty(J)$, $s'' \in R$ be given arbitrarily. To every $g \in {}^k\mathcal{H}_{\mathrm{loc}}^{-s-d,-m-p}(J)$, there is $f \in {}^k\mathcal{H}_c^{-s,-m}(J)$ such that

$$\theta(P^*f - g) \in \mathcal{H}_c^{s''}(J) . \qquad (5.3')_{s,m}$$

Remark that such property is equivalent to the one we get by giving $g \in {}^k\mathcal{H}_c^{-s-d,-m-p}(J)$ instead of ${}^k\mathcal{H}_{\mathrm{loc}}^{-s-d,-m-p}(J)$.

It will be shown that $(5.3)_{s,m}$ holds for J only if it holds for every open subset of J and that, if there is a family J_i of open subsets of J which covers J, and $(5.3)_{s,m}$ holds for each J_i, then it also holds for J. This will be a consequence of the first part of Th. 5.1.

We will denote by S the set of $m \in Z/k$ such that either $m \geqslant 0$ or $m + p \leqslant 0$ or $m \in Z$. We remember that $P \in {}^k\mathcal{N}^{d,p}$, where p is a *positive integer*.

THEOREM 5.1. *a) If* $(5.j)_{s,m}$ *is true for some* $j \in \{1, 2, 3\}$ *and for some* $(s_0, m_0) \in R \times S$, *it is true for all such j's and all* (s, m_0) *with* $s \in R$. *b) If* $m_0 \in Z$ *and hypothesis of a) is verified, then* $(5.j)_{s,m}$ *is true for all such j's and all* $(s, m) \in R \times S$. *c) Moreover, if hypothesis of b) is verified,* $(5.1)_{s,m}$ *is true for all* $(s, m) \in R \times Z/k$, *such that* $m \geqslant -1$ *or* $m \leqslant -p + 1$. *In the case* $p = 2$, *this covers all the possible values of m.*

PROOF. 1) *For a fixed pair* $(s, m) \in R \times S$, $(5.1)_{s,m}$ *implies* $(5.2)_{s,m}$.

Let us denote momentarily by E the intersection of the spaces $\mathcal{H}_c^{s'}(K)$ and ${}^k\mathcal{H}_{\mathrm{loc}}^{s+d,m+p}(J)$. It is clear that E can be equipped with a natural Fréchet space topology: the coarsest locally convex one which renders continuous the injection in $\mathcal{H}_c^{s'}(K)$ and also all the mappings $f \to \theta f$ from E into ${}^k\mathcal{H}_{\mathrm{loc}}^{s+d,m+p}(J)$, as θ ranges over $C_c^\infty(J)$. We equip now E with a second topology (which we denote by τ_P). Since $m \in S$, we know by Prop. 4.2*a*) that $P({}^k\mathcal{H}_{\mathrm{loc}}^{s+d,m+p}(J))$ is contained in ${}^k\mathcal{H}_{\mathrm{loc}}^{s,m}(J)$, so we may talk about the coarsest topology on E which renders continuous the injection in $\mathcal{H}_c^{s'}(K)$ and also all the mappings $f \to \theta P f$ from E into ${}^k\mathcal{H}_{\mathrm{loc}}^{s,m}(J)$, as θ ranges over $C_c^\infty(J)$. The topology τ_P is metrizable and $(5.1)_{s,m}$ implies that it is complete. Since it is obviously coarser than the natural topology on E, it is identical to it, by the open mapping theorem. We derive at once from this that, to every $\theta_1 \in C_c^\infty(J)$, there is $\theta_2 \in C_c^\infty(J)$ and a constant $C' > 0$ such that, for all $f \in E$,

$$\|\theta_1 f\|_{s+d,m+p} \leqslant C'(\|\theta_2 P f\|_{s,m} + \|f\|_{s'}) . \tag{5.4}$$

Let us take $f = \theta\varphi$, $\varphi \in C_c^\infty(K; H^\infty)$, $\theta \in C_c^\infty(J)$, and choose θ_1 identically 1 in a neighborhood of $\operatorname{supp}\theta$: (5.2′) follows at once.

2) *For a fixed pair* $(s, m) \in R \times Z/k$, $(5.2)_{s,m}$ *is equivalent to* $(5.3)_{s,m}$. Here there is no restriction at all on m.

First we show that $(5.2)_{s,m}$ implies $(5.3)_{s,m}$.

Let θ, s'' and g be given as in (5.3). Let $\theta_1 \in C_c^\infty(J)$ be real and equal to 1 in a neighborhood of $\operatorname{supp}\theta$. We apply (5.2′) with $s' = -s''$ (and where $(\ ,\ )$ is the inner product on $\mathcal{H}^0$):

$$\begin{aligned} |(\theta g, \varphi)| = |(\theta_1 g, \theta_1\bar\theta\varphi)| \leqslant C_1\|\theta_1\bar\theta\varphi\|_{s+d,m+p} \leqslant C(\|P\theta_1\bar\theta\varphi\|_{s,m} + \|\bar\theta\varphi\|_{-s''}) = \\ = C(\|\theta_1 P\bar\theta\varphi\|_{s,m} + \|\bar\theta\varphi\|_{-s''}) , \end{aligned} \tag{5.5}$$

which shows that the antilinear functional $\varphi \to (\theta g, \varphi)$ is continuous (say on $C_c^\infty(J; H^\infty)$) for the seminorm in the last member of (5.5), and therefore, by the Hahn-Banach theorem, it is equal to an antilinear functional $\varphi \to (\theta P^* \bar\theta_1 f_1, \varphi) + (\theta h, \varphi)$, where $f_1 \in {}^k\mathcal{H}_{\mathrm{loc}}^{-s,-m}(J)$ and $h \in \mathcal{H}_{\mathrm{loc}}^{s''}(J)$. If we set $f = \bar\theta_1 f_1$, we get (5.3′).

Next we show that (5.3) implies (5.2) for each (s, m).

Let K be an arbitrary compact subset of Ω, K' another compact subset of Ω, containing the union of K with $\bar J$. Let us denote by F the space $C_c^\infty(K; H^\infty)$ equipped with the single seminorm $\varphi \to (\|P\theta\varphi\|_{s,m} + \|\varphi\|_{-s''})$, where $\theta \in C_c^\infty(J)$ is given. Consider then the sesquilinear functional, defined on ${}^k\mathcal{H}_c^{-s-d,-m-p}(K') \times F$, which sends the pair (g, φ) in the complex number given by the inner product $(g, \theta\varphi)$. It follows at once from $(5.3)_{sm}$ that it is separately continuous; but on the product of a Fréchet space with a metrizable space, any separately continuous sesquilinear functional is continuous, hence, for a suitable constant $C > 0$,

$$|(g, \theta\varphi| \leqslant C\|g\|_{-s-d,-m-p}\left(\|P\theta\varphi\|_{s,m} + \|\varphi\|_{-s''}\right).$$

Taking $s' = -s''$, we get at once $(5.2')$.

Before continuing the proof of the theorem, we need the following

PROPOSITION 5.1. *If $(5.2)_{s_0,m_0}$ is true, it remains true after we have replaced P by $P - R$, where R is an arbitrary element of ${}^k\mathcal{N}^{d-1,p-1}$, provided that either:* i) $m \geqslant 0$ *or* ii) $m + p - 1 \leqslant 0$ *or* iii) $m \in Z$. *In particular, this happens if $m \in S$. A similar statement holds if $R \in {}^k\mathcal{N}^{d-\varepsilon,p}$, with $\varepsilon > 0$.*

PROOF. Use Propositions 4.2 and 4.3′.

3) *If $(5.2)_{s_0,m_0}$ holds for some $(s_0, m_0) \in R \times S$, $(5.2)_{s,m_0}$ holds for all $s \in R$.*

Let t be an arbitrary real number and Q an elliptic operator of order t in Ω (for example, $Q = (1 - \partial_t^2 + A^2)^{t/2}$). Setting $R = P - Q^{-1}PQ$, (4.8)c) implies that $R \in {}^k\mathcal{N}^{d-1,p-1}$, hence we may apply Prop. 5.1. It suffices then to observe that $(5.2)_{s_0,m_0}$ for $Q^{-1}PQ$ is equivalent to $(5.2)_{s_0+t,m_0}$ for P.

4) *For a fixed $m_0 \in S$, if $(5.2)_{s,m_0}$ holds for all s, so does $(5.1)_{s,m_0}$*

Let η, θ_1 and θ belong to $C_c^\infty(J)$, with θ_1 equal to 1 in a neighbourhood J' of supp η, θ equal to 1 in a neighbourhood J'' of supp θ_1, and let J''' be an open neighbourhood of supp θ, whose closure is compact and contained in J. Let $u \in \mathcal{D}'(J; H^{-\infty})$ be such that $Pu \in {}^k\mathcal{H}_{\text{loc}}^{s,m}(J)$. If the real number σ is sufficiently close to $-\infty$, we have $u \in {}^k\mathcal{H}_{\text{loc}}^{\sigma+d,m+p}(J''')$ (easy consequence of (0.2) and Corollary 4.1). Consider now, for $\varepsilon \geqslant 0$, the operator

$$B_\varepsilon = \eta \exp\left[-\varepsilon(1 - \partial_t^2 + A^2)\right]\theta_1 .$$

When $\varepsilon > 0$, $B_\varepsilon v \in C_c^\infty(J; H^\infty)$, whatever $v \in \mathcal{H}_{\text{loc}}^{-\infty}(J)$; we also have $B_0 = \eta I$, and if $v \in {}^k\mathcal{H}_{\text{loc}}^{s,m}(J)$, then $B_\varepsilon v$ converges to ηv in ${}^k\mathcal{H}_c^{s,m}(J''')$. We will say then that « B_ε converges to ηI » in ${}^k\mathcal{N}^{0,0}$. Now,

$$P(\theta B_\varepsilon u) = B_\varepsilon P(\theta u) - [B_\varepsilon, P](\theta u) + P[\theta, B_\varepsilon]u . \tag{5.6}$$

By the choice of θ_1 and θ, we have $[\theta, B_\varepsilon] = 0$ and $B_\varepsilon P(\theta u) = B_\varepsilon(\theta Pu)$, and since we know that $Pu \in {}^k\mathcal{H}_{\text{loc}}^{s,m}(J)$, it follows that $B_\varepsilon P(\theta u)$ converges to $\eta Pu = \eta P(\theta u)$ in ${}^k\mathcal{H}_c^{s,m}(J''')$. As for $[B_\varepsilon, P](\theta u)$, we observe (cf. (4.8)$c$) that « $[B_\varepsilon, P]$ converges to $[B_0, P]$ in ${}^k\mathcal{N}^{d-1,p-1}$ », and therefore, by Prop. 4.2

(and since $m_0 \in S$), $[B_\varepsilon, P](\theta u)$ converges to $[\eta I, P](\theta u)$ in ${}^k\mathcal{H}_c^{\sigma+1,m+1}(J''')$, which is contained in ${}^k\mathcal{H}_{\mathrm{loc}}^{\sigma+1/(k+1),m}(J''')$.

Hence, we see that $P(\theta B_\varepsilon u)$ converges to $P(\eta u)$ in ${}^k\mathcal{H}_c^{\sigma',m}(J''')$, where $\sigma' = \inf(s, \sigma + 1/(k+1))$. We apply (5.2′) to $\varphi = B_\varepsilon u$, with $K = \overline{J'''}$, and σ' substituted for s (and with s' sufficiently close to $-\infty$). We conclude, taking the limit as $\varepsilon \to 0$, that ηu belongs to ${}^k\mathcal{H}_{\mathrm{loc}}^{\sigma'+d,m+p}(J)$. By iterating this reasoning, we eventually reach the stage where $\sigma' = s$.

REMARK 5.1. This ends the proof of part *a*) of the theorem. Remark that this implies, in particular, that if $(5.j)_{s_0,m_0}$ (with $m_0 \in S$) holds for J, then it holds for every open subset of J, and conversely, if it holds for a family J_i of open subsets of J which covers J, then it also holds for J.

5) *If* $(5.1)_{s,m_0}$ *holds for all* s, *then so does* $(5.1)_{s,m_0+1/k}$, *if* $m_0 + 1/k \notin Z$ *and if either* $m_0 + 1 \geq 0$ *or* $m_0 + p \in Z_+$.

Let $u \in \mathcal{D}'(J; H^{-\infty})$ be such that $Pu \in {}^k\mathcal{H}_{\mathrm{loc}}^{s,m_0+1/k}(J)$: then, $tPu \in {}^k\mathcal{H}_{\mathrm{loc}}^{s,m_0}(J)$ and $Pu \in {}^k\mathcal{H}_{\mathrm{loc}}^{s-1/(k+1),m_0}(J)$. From this last fact we derive, by $(5.1)_{s-1/(k+1),m_0}$, that $u \in {}^k\mathcal{H}_{\mathrm{loc}}^{s+d-1/(k+1),m_0+p}(J)$. On the other hand, by (4.8)*d*), $[P, t] \in {}^k\mathcal{N}^{d-1,p-1}$ hence $[P, t]u \in {}^k\mathcal{H}_{\mathrm{loc}}^{s+k/(k+1),m_0+1}(J)$, which is contained in ${}^k\mathcal{H}_{\mathrm{loc}}^{s,m_0}(J)$, if $m_0 + 1 \geq 0$ or $m_0 + p \leq 0$ or $m_0 \in Z$. Then, $P(tu) = tPu + [P, t]u \in {}^k\mathcal{H}_{\mathrm{loc}}^{s,m_0}(J)$, which yields, by means of $(5.1)_{s,m_0}$, $tu \in {}^k\mathcal{H}_{\mathrm{loc}}^{s+d,m_0+p}(J)$.

We have shown that, if $m_0 + 1 \geq 0$ or $m_0 \in Z$, then $u \in {}^k\mathcal{H}_{\mathrm{loc}}^{s+d-1/(k+1),m_0+p}(J)$ and $tu \in {}^k\mathcal{H}_{\mathrm{loc}}^{s+d,m_0+p}(J)$. If we assume that $m_0 + 1/k \notin Z$ and $m_0 + p \geq 0$, then by (4.22) we get $u \in {}^k\mathcal{H}_{\mathrm{loc}}^{s+d,m_0+p+1/k}(J)$, which ends the proof of 5).

6) *If* $(5.1)_{s,m_0}$ *holds for all* s, *then so does* $(5.1)_{s,m_0+1}$, *if* $m_0 \in S$.

First, we restrict ourselves to the case $m_0 \geq -p$.

Let $u \in \mathcal{D}'(J; H^{-\infty})$ be such that $Pu \in {}^k\mathcal{H}_{\mathrm{loc}}^{s,m_0+1}(J)$. Since ${}^k\mathcal{H}_{\mathrm{loc}}^{s,m_0+1}(J)$ is contained in ${}^k\mathcal{H}_{\mathrm{loc}}^{s-k/(k+1),m_0}(J)$, we derive by $(5.1)_{s-k/(k+1),m_0}$ that $u \in {}^k\mathcal{H}_{\mathrm{loc}}^{s+d-k/(k+1),m_0+p}(J)$. Since $Pu \in {}^k\mathcal{H}_{\mathrm{loc}}^{s,m_0+1}(J)$, we have $\partial_t Pu \in {}^k\mathcal{H}_{\mathrm{loc}}^{s-1,m_0}(J)$, if $m_0 \geq 0$ or $m_0 \leq -1$. On the other hand, $[P, \partial_t] \in {}^k\mathcal{N}^{d+1/(k+1),p}$, by (4.8)*d*), which yields $[P, \partial_t]u \in {}^k\mathcal{H}_{\mathrm{loc}}^{s-1,m_0}(J)$ if $m_0 \in S$. Hence, $P\partial_t u = \partial_t Pu + [P, \partial_t]u \in$ $\in {}^k\mathcal{H}_{\mathrm{loc}}^{s-1,m_0}(J)$, which gives by $(5.1)_{s-1,m_0}$: $\partial_t u \in {}^k\mathcal{H}_{\mathrm{loc}}^{s+d-1,m_0+p}(J)$, if $m_0 \in S$. We can similarly prove that $t^k u \in {}^k\mathcal{H}_{\mathrm{loc}}^{s+d,m_0+p}(J)$, if $m_0 \in S$.

If we also assume that $m_0 \geq -p$, then by Prop. 4.4 (the case already proved) we have $u \in {}^k\mathcal{H}_{\mathrm{loc}}^{s+d,m_0+p+1}(J)$.

In order to remove the restriction $m_0 \geq -p$, we must settle a certain number of particular cases of the general result we are seeking. We state the first of them:

(5.7) If $P = P^*$ and $m_0 \in S$, then $(5.1)_{s,m_0}$ holds for all s if and only if $(5.1)_{s,-m_0-p}$ holds for all s.

PROOF OF (5.7). Let us call $m_1 = -m_0 - p$. Remarking that, if $m_0 \in S$ then $m_1 \in S$ and $m_0 = -m_1 - p$, we see that it is enough to prove (5.7) in one direction: if $(5.1)_{s,m_0}$ holds for all s, then $(5.1)_{s,m_1}$ holds for all s.

We already know that, for $m \in S$, $(5.1)_{s,m}$ and $(5.3)_{s,m}$ are equivalent. In particular, we see that $(5.3)_{s,m_0}$ holds for all s, and that it is enough to prove that $(5.3)_{s,m_1}$ holds for all s.

Let $\theta \in C_c^\infty(J)$, $g \in {}^k\mathcal{H}_c^{-s-d,-m_1-p}(J)$ and $s'' \in R$ be given arbitrarily, with s'' sufficiently close to $+\infty$, and let us take $\theta_1, \theta_2 \in C_c^\infty(J)$, with θ_1 equal to 1 in neighbourhood J' of supp θ, θ_2 equal to 1 in a neighbourhood J'' of supp θ_1. If we take a real number ν such that $\nu \leqslant 0$ and $\nu \leqslant k(m_1 - m_0)/(k+1)$, we have ${}^k\mathcal{H}_c^{-s-d,-m_1-p}(J)$ contained in ${}^k\mathcal{H}_c^{-s-d+\nu,-m_0-p}(J)$, hence, applying $(5.3)_{s-\nu,m_0}$ with θ_2 substituted for θ, we see that there is $f_1 \in {}^k\mathcal{H}_c^{-s+\nu,-m_0}(J)$ such that $\theta_2(Pf_1 - g) \in \mathcal{H}_c^{s''}(J)$ (which implies $\theta(Pf_1 - g) \in \mathcal{H}_c^{s''}(J)$). Since s'' is sufficiently near to $+\infty$, and $g \in {}^k\mathcal{H}_c^{-s-d,-m_1-p}(J)$, we get $\theta_2 Pf_1 \in {}^k\mathcal{H}_c^{-s-d,-m_1-p}(J) = {}^k\mathcal{H}_c^{-s-d,m_0}(J)$, hence $Pf_1 \in {}^k\mathcal{H}_{\mathrm{loc}}^{-s-d,m_0}(J'')$. Since $(5.1)_{-s-d,m_0}$ holds for J, hence for J'' (cf. Remark 5.1), we have $f_1 \in {}^k\mathcal{H}_{\mathrm{loc}}^{-s,m_0+p}(J'') = {}^k\mathcal{H}_{\mathrm{loc}}^{-s,-m_1}(J'')$. Setting $f = \theta_1 f_1$ we have then $f \in {}^k\mathcal{H}_c^{-s,-m_1}(J)$, and remarking that $\theta P f_1 = \theta P(\theta_1 f_1) = \theta P f$, we finally get $\theta(Pf - g) \in \mathcal{H}_c^{s''}(J)$, which proves $(5.3)_{s,m_1}$.

We will apply (5.7) to the operator $\mathcal{T}$ defined as follows: if we call $X = \partial_t - t^k A$ and $X^+ = -X^* = \partial_t + t^k A$, then $\mathcal{T} = X^+ X$. It is clear that $\mathcal{T} = \mathcal{T}^* \in {}^k\mathcal{N}^{2,2}$. Moreover, by the choice of X and $\mathcal{T}$ and Remark 4.1, (4.27) is satisfied for any bounded open set J, i.e., $(5.2)_{-1,-1}$ holds for $\mathcal{T}$ on these same sets.

LEMMA 5.1. *When $P = \mathcal{T}$, $(5.j)_{s,m}$ is true for all $j \in \{1, 2, 3\}$ and all $(s, m) \in R \times S$. Moreover, $(5.1)_{s,m}$ is true for all $(s, m) \in R \times Z/k$.*

PROOF. As we already saw, $(5.2)_{-1,-1}$ holds for $\mathcal{T}$. Hence, by part a), $(5.1)_{s,-1}$ holds for all s. Since in this case $p = 2, -1 \in S$ and $-1 \geqslant -p$, an iterative application of 6) (in the case already proved) shows that $(5.1)_{s,m}$ holds for all s and all integers greater or equal to -1. Applying successively 5), we get $(5.1)_{s,m}$ for all s and all $m \in Z/k$ with $m \geqslant -1$. When $m \geqslant 0$, then $m \in S$ and the corresponding $m_1 = -m-2$ runs over all $m \in Z/k$ with $m \leqslant -2$, hence by (5.7) we get $(5.1)_{s,m}$ for all s and all $m \leqslant -2$. All m such that $-3 \leqslant m \leqslant -2$ are in S, hence 6) gives $(5.1)_{s,m}$ for $-2 \leqslant m \leqslant -1$ and all s. Then, $(5.1)_{s,m}$ holds for all real s and all $m \in Z/k$. When $m \in S$, then by part a), $(5.2)_{s,m}$ and $(5.3)_{s,m}$ also hold.

END OF PROOF OF PROPOSITION 4.4. Let $m \in Z/k$ be arbitrary and suppose that $\partial_t u$ and $t^k A u$ belong to ${}^k\mathcal{H}_{\mathrm{loc}}^{s-1,m-1}(J)$. Then $Xu \in {}^k\mathcal{H}_{\mathrm{loc}}^{s-1,m-1}(J)$, hence $\mathcal{T}u = X^+ X u \in {}^k\mathcal{H}_{\mathrm{loc}}^{s-2,m-2}(J)$ if $m-2 \geqslant 0$ or $m-1 \leqslant 0$, so by lemma 5.1

we have $u \in {}^k\mathcal{H}^{s,m}_{\rm loc}(J)$. Hence, $\partial_t u$ and $t^k A u$ in ${}^k\mathcal{H}^{s-1,m-1}_{\rm loc}(J)$ imply $u \in {}^k\mathcal{H}^{s,m}_{\rm loc}(J)$ if $m \geqslant 2$ or $m \leqslant 1$. But for $m \geqslant 1$ it was already proved in section 4, so this happens for all $m \in Z/k$.

But now that we know that Prop. 4.4 is true without restrictions on m, the same reasoning of the beginning of 6) gives 6) without the restriction $m \geqslant -p$ (but only the restriction $m \in S$). This completes the proof of 6).

We come now to the last stage of the proof of Th. 5.1:

7) *If* $(5.1)_{s,m_0}$ *holds for all* s *and some* $m_0 \in Z$, *it holds for all* $m \in S$, *and also for* $m \geqslant -1$ *or* $m \leqslant -p+1$.

When P is selfadjoint, the proof is similar to that of Lemma 5.1, and we leave it to the reader.

For the operator X used to define $\mathcal{S}$, (4.28) holds, i.e., $(5.2)_{0,0}$ holds. By part *a*) and by 6), $(5.2)_{s,m}$ then holds for all s and all $m \in Z_+$. An easy induction then shows that, for every $m \in Z_+$, $(5.2)_{s,m_0}$ holds for all s and all $m_0 \in Z_+$, for the operator $X^m \in {}^k\mathcal{N}^{m,m}$.

For a general operator $P \in {}^k\mathcal{N}^{d,p}$, we may as well assume that $(5.2)_{m,m}$ holds for some integer $m \geqslant 0$. Setting $M = P^*(X^+)^m X^m P$, and thanks to $(5.2)_{0,0}$ which holds for X^m, we may rewrite $(5.2)_{m,m}$ (for P) in the form:

$$(5.8) \qquad \|\theta\varphi\|^2_{m+d,m+p} \leqslant C\big(|(M\theta\varphi,\theta\varphi)| + \|\varphi\|^2_{s'}\big),$$

which by Cauchy-Schwartz implies at once that $(5.2)_{-m-d,-m-p}$ holds for M; the same is therefore true of $(5.1)_{s,-m-p}$ for all s. Since M is selfadjoint and $-m-p \in Z$, 7) is true for M, i.e., $(5.1)_{s,n}$ is true for $n \in S_M$ or $n \geqslant -1$ or $n \leqslant -2p-2m+1$. Remark that $M \in {}^k\mathcal{N}^{2d+2m,2p+2m}$, hence $n \in S_M$ if and only if $n \geqslant 0$ or $n \leqslant -2p-2m$ or $n \in Z$.

Let now $u \in \mathcal{D}'(J; H^{-\infty})$ be such that $Pu \in {}^k\mathcal{H}^{s_1,m_1}_{\rm loc}(J)$. We derive from this (remarking that $P^*(X^+)^m X^m \in {}^k\mathcal{N}^{d+2m,p+2m}$) that $Mu \in {}^k\mathcal{H}^{s_1-d-2m,m_1-p-2m}_{\rm loc}(J)$, if $m_1-p-2m \geqslant 0$ or $m_1 \leqslant 0$ or $m_1 \in Z$.

Then, applying $(5.1)_{\dots,m_1-p-2m}$ when true for M, we get $u \in {}^k\mathcal{H}^{s_1+d,m_1+p}_{\rm loc}(J)$, if besides the earlier restrictions for m_1 we also have: either $m_1 \geqslant p+2m-1$ or $m_1 \leqslant -p+1$ or $m_1 \in Z$.

Hence, we have proved $(5.1)_{s_1,m_1}$ for P if either $m_1 \geqslant p+2m$ or $m_1 \leqslant -p+1$ or $m_1 \in Z$. Application of 5) now gives $(5.1)_{s_1,m_1}$ for all $m_1 \geqslant -1$.

The proof of Theorem 5.1 is complete.

Remark 5.2. Suppose that both P and P^* satisfy anyone of the conditions $(5.j)_{s,m}$ $(j=1,2,3)$ with $m \in Z$. Then, whatever $s \in R$, $m \in S$, the following stronger version of $(5.3)_{s,m}$ is valid:

$(5.9)_{s,m}$ Given any open set J' whose closure is compact and contained in J, and any $g \in {}^k\mathcal{H}_{\mathrm{loc}}^{-s-d,-m-\nu}(J)$, there is $f \in {}^k\mathcal{H}_{\mathrm{loc}}^{-s,-m}(J)$ such that $P^*f - g$ is C^∞ in J'.

This follows by exploitation of a priori estimates such as (5.2′) (or (5.4)) and standard Fréchet space techniques.

REMARK 5.3. If P is an operator of the form (2.1), satisfying (2.4) and (2.5) and also $\mathrm{Re}\, l_P \leqslant -k/2$, then (by Remark 4.1 and Th. 5.1) there is some open neighbourhood J' of 0 such that $(5.j)_{s,m}$ holds on J' for all $(s, m) \in R \times S$ and all $j \in \{1, 2, 3\}$. Moreover, $(5.1)_{s,m}$ holds on J' for all $(s, m) \in R \times Z/k$.
In the case of the operators considered in Remark 3.1, the same happens for every bounded open set J.

REMARK 5.4. The anomalies which appeared in sections 4 and 5 are all due to the fact that $\partial_t({}^k\mathcal{H}_{\mathrm{loc}}^{s,m})$ is not contained in ${}^k\mathcal{H}_{\mathrm{loc}}^{s-1,m-1}$ if $0 < m < 1$. If instead of Def. 4.3 we would define the spaces ${}^k\mathcal{H}_{\mathrm{loc}}^{s,-m}(J)$ inductively as the set of distributions u on J that can be written as $\partial_t v + t^k A w$, with v and w in ${}^k\mathcal{H}_{\mathrm{loc}}^{s+1,-m+1}(J)$, then probably all these anomalies would disappear. This, however, would require more work, and the results of sections 4 and 5 suffice to our purposes.

6. – Study of the hypoellipticity of a particular class of operators.

In this section, we present a complete study (except for the proof) of the hypoellipticity of operators of the form (2.1), satisfying (2.4) and (2.5) and also a very restrictive condition. If we write $a(t, A) = \sum_{i=0}^{\infty} a_i(t) A^{-i}$, $b(t, A) = \sum_{i=0}^{\infty} b_i(t) A^{-i}$, $c(t, A) = \sum_{i=0}^{\infty} c_i(t) A^{-i}$, then we will require:

(6.1) Whatever the integer $i \geqslant 0$, there are complex numbers a_i, b_i, c_i such that $a_i(t) = a_i t^k$, $b_i(t) = b_i t^k$, $c_i(t) = c_i t^{k-1}$.

With this notation, a_0, b_0, c_0 coincide respectively with a, b, c in the notation (2.4). We call now:

$$a(A) = \sum_{i=0}^{\infty} a_i A^{-i}, \qquad b(A) = \sum_{i=0}^{\infty} b_i A^{-i}, \qquad c(A) = \sum_{i=0}^{\infty} c_i A^{-i}.$$

Since $a(t, A)$, $b(t, A)$ and $c(t, A)$ belong to $\mathcal{Q}_A(J)$, the same happens to $a(A)$, $b(A)$ and $c(A)$. Remark that we have:

$$a(t, A) = t^k a(A)\,, \qquad b(t, A) = t^k b(A)\,, \qquad c(t, A) = t^{k-1} c(A)\,.$$

We keep the notation of section 2, particularly $X = \partial_t - a(t, A)A$, $Y = \partial_t - b(t, A)A$, $\delta(t, A) = a(t, A) - b(t, A)$. Then, δ (or δ_0) is $a_0 - b_0$ and $l_P = c_0/\delta_0$. In the case (6.1) holds, we also call $\delta(A) = a(A) - b(A)$, hence $\delta(t, A) = t^k \delta(A)$.

The main result of this section is then the following:

THEOREM 6.1. *Let P be an operator of the form* (2.1), *satisfying* (2.4), (2.5) *and* (6.1). *Then, the following two properties are equivalent*:

(6.2) *P is hypoelliptic at $t = 0$;*

(6.3) *whatever the integer $m \geqslant 0$, none of the power series $c(A) - m(k+1)\delta(A)$ and $c(A) - \big(m(k+1) + 1\big)\delta(A)$ are identically zero.*

COROLLARY 6.1. *Let P satisfy the hypothesis of* Th. 6.1 *and assume also that $a_i = b_i = c_i = 0$ for all $i \geqslant 1$. If for some integer $m \geqslant 0$ we have $l_P = m(k+1)$ or $l_P = m(k+1) + 1$, then P is not hypoelliptic at $t = 0$.*

It should be remarked that, when (6.1) holds, then (6.2) and (6.3) are also equivalent to the property of P being locally solvable at $t = 0$. Although the proof of this fact is very easy (at least in what we will soon call the convergent case), we wish to concentrate the attention on hypoellipticity.

As a matter of fact, only Corollary 6.1 will be used in the proof of Th. 2.1 and in the sequel. There are two reasons why we give the more general result which is embodied in Th. 6.1. The first one is to show that when l_P assumes one of the critical values $m(k+1)$ or $m(k+1)+1$, where m is a non-negative integer, then P may be either hypoelliptic or not at $t = 0$, depending on the lower order terms. The second is to provide a better comparison of the technique of application of the next Lemma 6.1 (which was already used, in a similar context, in [3]) in the proof of Th. 6.1, and the method of proof of Th. 7.1 (which is a stronger version of Th. 2.1).

Since $\operatorname{Re}\delta_0 > 0$, we may consider $d(A) = c(A)/\delta(A)$ as a formal power series. We will prove Th. 6.1 only when $d(A)$ is a convergent power series. This case is enough to get Corollary 6.1 and to show the existence of hypoelliptic operators whose l_P assume some critical value. The complete proof of Th. 6.1 could be done with the same technique of using approximating operators P_N, for P, where $P_N = (\partial_t - t^k a_N(A)A)(\partial_t - t^k b_N(A)A) + t^{k-1}c_N(A)A$, where $a_N(A) = \sum_{i=0}^{N} a_i A^{-i}$, and similarly for $b_N(A)$ and $c_N(A)$, as it was done in [3], section (II.7).

Let E and F be abelian groups, F being a subgroup of E. Let P, Q, U, V be four endomorphisms of E, each one of which maps F into itself. We will make the hypothesis that:

$$UP = QV. \tag{6.4}$$

LEMMA 6.1. *Suppose that* (6.4) *holds and also that*

(6.5) *F contains the intersection of $V^{-1}(F)$ and $P^{-1}(F)$.*

Then, if F contains $Q^{-1}(F)$, F must also contain $P^{-1}(F)$.

Note that if F contains $P^{-1}(F)$, then $F = P^{-1}(F)$.

PROOF. Let $u \in E$ be such that $Pu = f \in F$. Then $QVu = UPu = Uf \in F$, and therefore, if F contains $Q^{-1}(F)$, we must have $Vu \in F$, i.e., u belongs to the intersection of $V^{-1}(F)$ and $P^{-1}(F)$, which is F.

REMARK 6.1. Usually the easiest way to show that (6.5) is verified is to show (if possible) that there are two other endomorphisms of E, T and S, which map F into itself, such that the endomorphism $W = TV + SP$ satisfies the condition: « F contains $W^{-1}(F)$ ».

We will always apply Lemma 6.1 taking some bounded open set J containing the origin, and $E = \mathfrak{D}'(J; H^{-\infty})$, F being the set of functions in $C^\infty(J; H^\infty)$ which can be extended as a C^∞ function in a neighborhood of $\bar{J}$. Then, if P is of the form (2.1), defined on J_1, and J is relatively compact in J_1, P must map F into itself.

As a first application of Lemma 6.1, we give:

PROPOSITION 6.1. *Suppose* (2.4) *and* (2.5) *hold for $a(t, A)$ and $b(t, A)$, and assume that $\delta(t, A)$ is divisible by t (i.e., that $\delta_i(0) = a_i(0) - b_i(0) = 0$ for every integer $i \geq 0$). Then, the operator $XY + (\delta(t, A)/t)A$ is not hypoelliptic at $t = 0$. In particular, if* (6.1) *also holds, then $XY + t^{k-1}\delta(A)A$ is not hypoelliptic at $t = 0$.*

PROOF. We have $(tX + 2)Y = (XY + (\delta(t, A)/t)A)t$. Calling $U = tX + 2$, $P = Y$, $Q = XY + (\delta(t, A)/t)A$, $V = t$, relation (6.4) holds. Moreover, setting $T = Y$, $S = -t$, we have $TV + SP = Yt - tY = I$, which clearly satisfies the condition: « F contains $I^{-1}(F)$ ». Hence, by Remark 6.1, relation (6.5) holds. If we assume that, for some open neighbourhood J_2 of 0, F contains $Q^{-1}(F)$, then F should contain $P^{-1}(F)$ by Lemma 6.1. But this is impossible, since $P = Y$ is not hypoelliptic at $t = 0$. Since, whatever the neighbourhood J_2 of 0, F does not contain $Q^{-1}(F)$, Q cannot be hypoelliptic at $t = 0$. Q.E.D.

In the proof of Th. 6.1, we will call $P(c(A))$ the operator $XY + t^{k-1}c(A)A$, for X and Y fixed.

PROOF OF TH. 6.1. *a*) (6.2) implies (6.3). (This is the part needed for Corollary 6.1).

Recalling that XY and $XY + t^{k-1}\delta(A)A$ are not hypoelliptic at $t = 0$, it is enough to prove that, if $P(c(A))$ is hypoelliptic at $t = 0$, so must be $P(c(A) - (k+1)\delta(A))$. We assume that $d(A) = c(A)/\delta(A)$ is a convergent series and apply Lemma 6.1, as in the proof of Prop. 6.1. We start remarking that the following identity holds, as can be easily verified by direct computation:

$$(tX + d(A) + 1)P(c(A) - (k+1)\delta(A)) = P(c(A))(tX + d(A) - 1). \tag{6.6}$$

In order to verify (6.5), we use Remark 6.1. In this case, $V = tX + d(A) - 1$, $P = XY + (c(A) - (k+1)\delta(A))t^{k-1}A$. A simple computation shows that $YV - tP = d(A)X$ and $td(A)X - d(A)V = d(A)(I - d(A))$, hence:

$$(tY - d(A))V - t^2P = d(A)(I - d(A)). \tag{6.7}$$

Let us call $e(A)$ the second member of (6.7): $e(A)$ is identically zero if and only if either $d(A) \equiv 0$ or $d(A) \equiv I$ (i.e., $c(A) \equiv 0$ or $c(A) \equiv \delta(A)$). Therefore, if $c(A) \not\equiv 0$ and $c(A) \not\equiv \delta(A)$, we may write $e(A)$ in the form $e(A) \equiv \alpha A^{-n}(I + A^{-1}g(A))$, where α is a complex number different from zero and $g(A) \in \mathfrak{Q}_A(J)$ is independent of t. Then, (6.7) gives:

$$\alpha^{-1}A^n(tY - d(A))V - \alpha^{-1}A^n t^2 P = I + A^{-1}g(A).$$

If we now take a real number r sufficiently close to $-\infty$, the operator $R = P - r\delta t^{k-1}A(I + A^{-1}g(A))$ will be such that $l_R \leqslant -k/2$, hence by Remark 5.3, R is hypoelliptic at $t = 0$. Since

$$R = -r\delta\alpha^{-1}t^{k-1}A^{n+1}(tY - d(A))V + (I + r\delta\alpha^{-1}t^{k-1}A^{n+1}t^2)P,$$

Remark 6.1 gives (6.5).

This proves that, if $c(A) \not\equiv 0$, $c(A) \not\equiv \delta(A)$ and $P(c(A))$ is hypoelliptic at $t = 0$, so must be $P(c(A) - (k+1)\delta(A))$. Since $P(0)$ and $P(\delta(A))$ are not hypoelliptic at $t=0$, as it was already seen, we have in fact proved the following: « if $P(c(A))$ is hypoelliptic at $t=0$, then so must be $P(c(A)-(k+1)\delta(A))$ ».

b) (6.3) implies (6.2).

Let us consider the following assertion:

$$(6.8)_J \qquad (6.3) \text{ implies } (6.2) \text{ if } \operatorname{Re} l_P \leqslant (J-1)(k+1).$$

We will prove *b*) by proving $(6.8)_J$ for every integer $J \geqslant 0$. When $J=0$ we have $\operatorname{Re} l_P \leqslant -(k+1) \leqslant -k/2$, hence, by Remark 5.3, $(6.8)_0$ holds. In order to prove that $(6.8)_J$ implies $(6.8)_{J+1}$, it is enough to prove (remarking that $l_{P_1}=l_{P_0}-(k+1)$, with P_0 and P_1 as in (6.9)) that:

(6.9) If $c(A)\not\equiv 0$, $c(A)\not\equiv\delta(A)$ and $P_1=P(c(A)-(k+1)\delta(A))$ is hypoelliptic at $t=0$, then so must be $P_0=P(c(A))$.

We again apply Lemma 6.1, this time using the following identity:

$$(6.10) \quad (tY-d(A)+2)P(c(A))=P(c(A)-(k+1)\delta(A))(tY-d(A)).$$

In this case, $V=tY-d(A)$, $P=XY+t^{k-1}c(A)A$, and we have:

$$(6.11) \quad XV-tP=(I-d(A))Y;\ t\big((I-d(A))Y\big)-(I-d(A))V= \\ =d(A)(I-d(A))$$

hence

$$(6.12) \qquad (tX+d(A)-1)V-t^2P=d(A)(I-d(A)).$$

Continuing as in the proof of *a*), we see that (6.5) is verified, provided that $d(A)\not\equiv 0$ and $d(A)\not\equiv I$, i.e., that $c(A)\not\equiv 0$ and $c(A)\not\equiv\delta(A)$. Hence, by Lemma 6.1, (6.9) holds. Q.E.D.

7. – Proof of the main theorem.

Same notation as in the preceding section. All the operators considered in this section satisfy (2.1): they all belong to ${}^k\mathcal{N}^{2,2}$, and therefore the set S introduced on section 5 is the same for all of them: a number $m\in Z/k$ belongs to S if and only if either $m\geqslant 0$ or $m\leqslant -2$ or $m=-1$.

If P is an operator of the form (2.1) defined on J and satisfying (2.4), it is natural to call the operator $\hat{P}=(\partial_t-a_0(t)A)(\partial_t-b_0(t)A)+ct^{k-1}A$ as the « k-principal part of P », since $P-\hat{P}\in {}^k\mathcal{N}^{1,1}$, as it is easy to see, and there is no term in $\hat{P}$ belonging to ${}^k\mathcal{N}^{1,1}$. To every such operator P, we will also connect (as in section 3) the operator $\tilde{P}=(\partial_t-at^kA)(\partial_t-bt^kA)+ct^{k-1}A$, which we will call the « strong k-principal part of P ». We recall (cf. section 3), that

$$P-\tilde{P}=t^{k+1}f_1(t,A)A\partial_t+t^{2k+1}g_1(t,A)A^2+f_0(t,A)\partial_t+t^kg_0(t,A)A+h(t,A),$$

where $f_i(t,A)$, $g_i(t,A)$, for $i=1,2$, and $h(t,A)$ all belong to $\mathcal{Q}_A(J)$. Let us call $R_1=t^{k+1}f_1(t,A)A\partial_t$, $R_2=t^{2k+1}g_1(t,A)A^2$, $R_3=f_0(t,A)\partial_t+t^kg_0(t,A)A+h(t,A)$ Then, $R_3\in{}^k\mathcal{N}^{1,1}$, but the same does not happen to R_1 or R_2. In fact, for no $\varepsilon>0$ is it true that R_1 or R_2 belong to ${}^k\mathcal{N}^{2-\varepsilon,2}$. Nevertheless, R_1 and R_2 both belong to ${}^k\mathcal{N}^{2,2+1/k}$, which is a proper subspace of ${}^k\mathcal{N}^{2,2}$. Remark that, unlike P, $\tilde{P}$ is defined on the whole real line.

Throughout this section, it will be always assumed that P satisfies (2.1), (2.4), and that $(\operatorname{Re}a)(\operatorname{Re}b)\neq 0$ (and k is a nonnegative integer), and J_1 will always denote the greatest open interval (containing 0 and contained in the domain of definition J_0 of P) such that $\operatorname{Re}\big(a_0(t)/t^k\big)$ and $\operatorname{Re}\big(b_0(t)/t^k\big)$ never vanish on J_1.

We consider the following properties, that may hold or not in some open set J contained in J_1 (they are all related to (2.10) and (5.j) for $j=1,2,3$):

$(7.1)_{s,J}$ Let $u\in\mathcal{D}'(J;H^{-\infty})$. If $Pu\in\mathcal{K}^s_{\text{loc}}(J)$, then $u\in\mathcal{K}^{s+2/(k+1)}_{\text{loc}}(J)$.

$(7.2)_{s,m,j,J}$ $(5.j)_{s,m}$ holds for P on J.

(7.3) There is an open neighbourhood J of 0, contained in J_1, and there is $s\in R$ such that $(7.1)_{s,J}$ holds for P.

(7.4) For every open set J contained in J_1 and every $s\in R$, $(7.1)_{s,J}$ holds for P.

(7.5) There is an open neighbourhood J of 0, contained in J_1, there is $(s,m)\in R\times Z$, and there is $j\in\{1,2,3\}$ such that $(7.2)_{s,m,j,J}$ holds for P.

(7.6) For every open set J relatively compact in J_1, for every $(s,m)\in R\times S$, and every $j\in\{1,2,3\}$, $(7.2)_{s,m,j,J}$ holds for P. Also, for every $(s,m)\in R\times Z/k$ and every open set J contained in J_1, $(7.2)_{s,m,1,J}$ holds for P.

Of course, (7.6) implies (7.5), and (7.4) implies (7.3). Next we present two easy propositions. The first one shows that (7.5) and (7.6) for P are equivalent to (7.5) and (7.6) for its strong k-principal part $\tilde{P}$. The second shows that (7.3) and (7.4) are equivalent to (7.5) and (7.6).

PROPOSITION 7.1. *Let P be an operator satisfying* (2.1) *and* (2.4) (*but not necessarily* (2.5)), *and let $\tilde{P}$ be its strong k-principal part. Then, the following conditions are equivalent*:

a) (7.5) *holds for* P;

b) (7.6) *holds for* P;

c) (7.5) *holds for* $\tilde{P}$;

d) (7.6) *holds for* $\tilde{P}$ (*remark that, for* $\tilde{P}$, $J_1 = R$).

PROOF. Of course *b*) implies *a*), and *d*) implies *c*). If *a*) holds, then by Th. 5.1 we may as well assume that $j = 1$. Let K be a compact neighbourhood of 0, contained in J. Then on the open set $J_1 - K$ the operator P is elliptic (with the obvious meaning), so $(7.2)_{s.m.1.J_1-K}$ holds for P. This together with $(7.2)_{s.m.1.J}$ implies $(7.2)_{s.m.1.J_1}$. Applying Th. 5.1 with Remark 5.1, we get *b*). Similarly, *c*) implies *d*). It is enough now to show that *b*) implies *c*), since the same argument also shows that *d*) implies *a*).

The idea is to apply Prop. 5.1. This could be done if we had the k-principal part $\hat{P}$ of P instead of $\tilde{P}$ in condition *c*). We have trivially: *b*) implies that (7.5) holds for $\hat{P}$ *for the same sets* J contained in J_1.

However, the terms R_1 and R_2 of the difference $P - \tilde{P}$ are not in ${}^k\mathcal{N}^{1,1}$ or in ${}^k\mathcal{N}^{2-\varepsilon,2}$ for $\varepsilon > 0$, so Prop. 5.1 cannot be applied. In fact, we should not expect to be able to apply it: if $\tilde{P}$ satisfies (7.6) (with $J_1 = R$) we should not expect P to be also hypoelliptic on the whole real line, even if it were everywhere defined, but any application of Prop. 5.1 would yield hypoellipticity on the *same* sets. What we may hope to achieve is a result similar to Prop. 5.1, *provided* that we allow J to be shrinked.

Let $R = tU$, where $U \in {}^k\mathcal{N}^{2,2}$. We will prove that:

(7.7) If $(5.2)_{0,0}$ holds for P and the open neighbourhood J of 0, then $(5.2)_{0,0}$ also holds for $P - R$, on a possibly smaller neighborhood J' of 0.

Of course, if (7.7) is true, this finishes the proof that *b*) implies *c*). To prove (7.7) it is clearly enough to prove:

(7.8) Given any $\varepsilon > 0$, there is an $\varepsilon' > 0$ such that, for all $\varphi \in C_c^\infty((-\varepsilon', \varepsilon'); H^\infty)$, we have:

$$\|R\varphi\|_{0,0} \leqslant \varepsilon \|\varphi\|_{2,2}\,. \tag{7.8'}$$

Now, we trivially have:

$$(7.9)\qquad \|t\varphi\|_{0,0} \leqslant \varepsilon \|\varphi\|_{0,0} \quad \text{if } \varphi \in C_c^\infty((-\varepsilon, \varepsilon); H^\infty).$$

Since the operator $U: {}^k\mathcal{H}^{2,2}_{\mathrm{loc}}(J) \to {}^k\mathcal{H}^{0,0}_{\mathrm{loc}}(J)$ is continuous, taking a fixed open set J'' relatively compact in J, such that $0 \in J''$, there is a constant $C > 0$ such that $\|U\varphi\|_{0,0} \leqslant C\|\varphi\|_{2,2}$, for every $\varphi \in C_c^\infty(J''; H^\infty)$.

Hence, denoting by J''' the intersection of J'' and $(-\varepsilon/C, \varepsilon/C)$, we have for every $\varphi \in C_c^\infty(J'''; H^\infty)$:

$$\|R\varphi\|_{0,0} = \|tU\varphi\|_{0,0} \leqslant (\varepsilon/C)\|U\varphi\|_{0,0} \leqslant \varepsilon\|\varphi\|_{2,2},$$

as desired. Q.E.D.

PROPOSITION 7.2. *Let P be an operator of the form* (2.1), *satisfying* (2.4) *and* (2.5). *Then, conditions* (7.3), (7.4), (7.5) *and* (7.6) *are all equivalent. Moreover, we also have*:

(7.10) *If $u \in \mathcal{D}'(J; H^{-\infty})$ is such that $Pu \in {}^k\mathcal{H}^{s,m}_{\mathrm{loc}}(J)$ but $u \notin {}^k\mathcal{H}^{s+2,m+2}_{\mathrm{loc}}(J)$, then $u \notin {}^k\mathcal{H}^{s+2-j,m+2-j(k+1)/k}_{\mathrm{loc}}(J)$, whatever the integer $j \geqslant 0$.*

PROOF. By Prop. 7.1, we know that (7.5) is equivalent to (7.6), and it is evident that (7.4) implies (7.3). A trivial consequence of the inclusion ${}^k\mathcal{H}^{s+2,2}_{\mathrm{loc}}(J) \to \mathcal{H}^{s+2/(k+1)}_{\mathrm{loc}}(J)$ is that (7.6) implies (7.4). It remains to prove (7.10) and that (7.3) implies (7.5).

From (7.10) we easily derive that (7.3) implies (7.5): by hypothesis $(7.1)_{s,J}$ holds for some s and some J, with $0 \in J$; in order to prove (7.5) it suffices to prove $(7.2)_{s,0,1,J}$ with the same s and same J. Suppose that $(7.1)_{s,J}$ holds but not $(7.2)_{s,0,1,J}$: then, there would be some $u \in \mathcal{D}'(J; H^{-\infty})$, such that $Pu \in \mathcal{H}^s_{\mathrm{loc}}(J)$, $u \in \mathcal{H}^{s+2/(k+1)}_{\mathrm{loc}}(J)$ but $u \notin {}^k\mathcal{H}^{s+2,2}_{\mathrm{loc}}(J)$. By (7.10) we would derive (with $j = 2$ and $m = 0$) that $u \notin {}^k\mathcal{H}^{s,-2/k}_{\mathrm{loc}}(J)$. But $\mathcal{H}^{s+2/(k+1)}_{\mathrm{loc}}(J) \to {}^k\mathcal{H}^{s,-2/k}_{\mathrm{loc}}(J)$ (cf. Corollary 4.1) so we should have $u \notin \mathcal{H}^{s+2/(k+1)}_{\mathrm{loc}}(J)$, which gives a contradiction.

This shows that it suffices to prove (7.10). We begin by claiming:

(7.11) If $u \in \mathcal{D}'(J; H^{-\infty})$ is such that $Pu \in {}^k\mathcal{H}^{s,m}_{\mathrm{loc}}(J)$ and $t^{k-1}Au \in {}^k\mathcal{H}^{s,m}_{\mathrm{loc}}(J)$, then $u \in {}^k\mathcal{H}^{s+2,m+2}_{\mathrm{loc}}(J)$.

In fact, if α is a real number sufficiently close to $+\infty$, the operator $P_1 = P - \alpha\delta t^{k-1}A$ is such that $\operatorname{Re} l_{P_1} \leqslant -k/2$, hence by Remark 5.3, P_1 satisfies $(5.1)_{s,m}$. Since the hypothesis of (7.11) implies that $P_1 u \in {}^k\mathcal{H}^{s,m}_{\mathrm{loc}}(J)$, it also implies that $u \in {}^k\mathcal{H}^{s+2,m+2}_{\mathrm{loc}}(J)$, and (7.11) is proved.

If $u \in {}^k\mathcal{H}^{s+1,m+2-(k+1)/k}_{\text{loc}}(J)$, then $t^{k-1} Au \in {}^k\mathcal{H}^{s,m}_{\text{loc}}(J)$, so (7.11) may be rewritten as:

(7.12) If $u \in \mathcal{D}'(J; H^{-\infty})$ is such that $Pu \in {}^k\mathcal{H}^{s,m}_{\text{loc}}(J)$ but $u \notin {}^k\mathcal{H}^{s+2,m+2}_{\text{loc}}(J)$, then $u \notin {}^k\mathcal{H}^{s+1,m+2-(k+1)/k}_{\text{loc}}(J)$.

Remark that (7.12) is (7.10) for $j=1$, and that (7.10) is trivial for $j=0$. From (7.12) we get inductively (7.10): suppose (7.10) proved for j and let us prove it for $j+1$. Assume that $Pu \in {}^k\mathcal{H}^{s,m}_{\text{loc}}(J)$ but $u \notin {}^k\mathcal{H}^{s+2,m+2}_{\text{loc}}(J)$: then, by the inductive hypothesis, $u \notin {}^k\mathcal{H}^{s+2-j,m+2-j(k+1)/k}_{\text{loc}}(J)$. Since $Pu \in {}^k\mathcal{H}^{s,m}_{\text{loc}}(J)$, which is contained in ${}^k\mathcal{H}^{s-j,m-j(k+1)/k}_{\text{loc}}(J)$, if we apply (7.12) with $s-j$ substituted for s and $m-j(k+1)/k$ substituted for m, we get (7.10) for $j+1$.

REMARK 7.1. We only used (2.5) to prove (7.11) and thereby (7.12). Of course such requirement can be weakened: we only need that, given $a(t, A)$ and $b(t, A)$, there exists at least one $c(t, A)$ such that $P(c(t, A))$ satisfies (5.1). Then, an assertion similar to (7.11) can be proved, hence (7.12). For instance, if $\operatorname{Re} a > 0$, $\operatorname{Re} b > 0$ and k is an odd integer, then XY trivially satisfies (5.1). Hence, Prop. 7.2 is also true when $\operatorname{Re} a > 0$, $\operatorname{Re} b > 0$ and k is an odd integer.

COROLLARY 7.1. Same hypothesis of Prop. 7.2. If we denote by *a*), *b*), *c*), and *d*) respectively the conditions (7.3), (7.4), (7.5) and (7.6) when $\tilde{P}$ is substituted for P, then the following eight conditions are all equivalent: (7.3), (7.4), (7.5), (7.6), *a*), *b*), *c*) and *d*).

We will prove a more precise version of Th. 2.1, namely:

THEOREM 7.1. *Let P be an operator defined on J, satisfying* (2.1), (2.4) *and* (2.5). *Then, the following conditions are equivalent*:

a_1) (7.3) *for* P; a_2) (7.4) *for* P; a_3) (7.5) *for* P; a_4) (7.6) *for* P;

b_1) (7.3) *for* P^*; b_2) (7.4) *for* P^*; b_3) (7.5) *for* P^*; b_4) (7.6) *for* P^*;

c) *whatever the integer* $m \geq 0$, $l_P \neq m(k+1)$ *and* $l_P \neq m(k+1)+1$;

d) *whatever* $(s, m) \in R \times S$, *whatever the open set* J' *relatively compact in* J_1, *the operator* P *defines an isomorphism from* ${}^k\mathcal{H}^{s+2,m+2}_{\text{loc}}(J')/C^\infty(J'; H^\infty)$ *onto* ${}^k\mathcal{H}^{s,m}_{\text{loc}}(J')/C^\infty(J'; H^\infty)$;

e) *same as d*), *but for* P^*.

PROOF. Remark first that it is enough to prove that a_2) implies c) and c) implies a_4). In fact, since $l_{P^*} = \overline{l_P}$, if a_2) implies c), then b_2) implies c), and if c) implies a_4), then c) implies b_4). By Prop. 7.2, conditions a_1), a_2), a_3) and a_4) are equivalent; conditions b_1), b_2), b_3) and b_4) are equivalent; and it is trivial that d) implies a_3), and e) implies b_3).

We would then have a_i), c) and b_i) equivalent for $i = 1, 2, 3, 4$.

We also have: « a_4) implies e) ». In fact, it is already known now that a_4) implies b_4), and if both a_4) and b_4) hold, then by Remark 5.2, we get $(5.9)_{s,m}$ for all $(s, m) \in R \times S$ and all J' relatively compact in J_1. The open mapping theorem now gives e). Similarly, we get d).

Let us then prove that a_2) implies c) and that c) implies a_4).

a_2) implies c)

From a_2) and Corollary 7.1, we have (7.4) holding for $\tilde{P}$, which implies that $\tilde{P}$ is hypoelliptic (on the whole real line). By Corollary 6.1, and the remark that $l_{\tilde{P}} = l_P$, we get c).

c) implies a_4)

Since c) for P is the same as c) for $\tilde{P}$ and a_4) for P is equivalent to a_4) for $\tilde{P}$ (by Corollary 7.1), it is enough to prove that c) implies a_4) *for* $\tilde{P}$.

Let us fix the constants a and b, and consider the operator $P(c) = (\partial_t - at^k A)(\partial_t - bt^k A) + ct^{k-1}A$, (we call $X = \partial_t - at^k A$, $Y = \partial_t - bt^k A$, as usually). In order to prove that c) implies a_4) for $P(c)$, it is enough to prove that, for every integer $j \geq 0$, we have:

$$(7.13)_j \qquad c) \text{ implies } a_4) \quad \text{if } \operatorname{Re} l_{P(c)} \leq (j-1)(k+1)\,.$$

When $j = 0$, we have $\operatorname{Re} l_{P(c)} \leq -(k+1) \leq -k/2$, hence, by Remark 5.3 (and Corollary 7.1), $(7.13)_0$ holds. Remarking that $l_{P(c-(k+1)\delta)} = l_{P(c)} - (k+1)$, in order to prove that $(7.13)_j$ implies $(7.13)_{j+1}$ it is enough to show that, if $c \neq 0$, $c \neq \delta$ and $P\big(c - (k+1)\delta\big)$ satisfies a_4), then $P(c)$ also satisfies a_4), or, since a_3) is equivalent to a_4), that

(7.14) If $c \neq 0$, $c \neq \delta$ and $P\big(c - (k+1)\delta\big)$ satisfies a_4), then $P(c)$ satisfies a_3).

We use the following particular case of (6.10):

$$(7.15) \qquad P\big(c - (k+1)\delta\big)(tY - c/\delta) = (tY - c/\delta + 2)P(c)\,.$$

We will show that $(5.1)_{s,m}$ holds for $P(c)$, if $m \geqslant 5$.
Let $u \in \mathcal{D}'(J; H^{-\infty})$ be such that

$$(7.16) \qquad P(c)u \in {}^k\mathcal{H}^{s,m}_{\mathrm{loc}}(J)\,.$$

Since $(tY - c/\delta + 2) \in {}^k\mathcal{N}^{1,1+1/k}$, we have

$$(tY - c/\delta + 2)P(c)u \in {}^k\mathcal{H}^{s-1,m-(k+1)/k}_{\mathrm{loc}}(J)\,,$$

(because $m \geqslant 5$). Since by hypothesis $P\big(c-(k+1)\delta\big)$ satisfies $(5.1)_{s,m}$ for all $(s, m) \in R \times Z/k$, we get from (7.15):

$$(7.17) \qquad (tY - c/\delta)u \in {}^k\mathcal{H}^{s+1,m+1-1/k}_{\mathrm{loc}}(J)\,.$$

In our case, (6.12) gives:

$$(7.18) \qquad (tX + c/\delta - 1)(tY - c/\delta) - t^2 P(c) = (c/\delta)(1 - c/\delta)\,.$$

Since $(tX + c/\delta - 1) \in {}^k\mathcal{N}^{1,1+1/k}$ and $t^2 \in {}^k\mathcal{N}^{0,2/k}$, (7.16) and (7.17) yield, thanks to (7.18): $(c/\delta)(1 - c/\delta)u \in {}^k\mathcal{H}^{s,m-2/k}_{\mathrm{loc}}(J)$, and since we are assuming $c \neq 0$ and $c \neq \delta$, we get:

$$(7.19) \qquad u \in {}^k\mathcal{H}^{s,m-2/k}_{\mathrm{loc}}(J)\,.$$

Now we use the trivial remark:

$$(7.20) \qquad P\big(c-(k+1)\delta\big) = P(c) - (k+1)\delta t^{k-1}A\,.$$

From (7.19) and (7.16) we conclude, thanks to (7.20), that

$$P\big(c-(k+1)\delta\big)u \in {}^k\mathcal{H}^{s-1,m-1-1/k}_{\mathrm{loc}}(J)\,,$$

and since $P\big(c-(k+1)\delta\big)$ satisfies (5.1), we get:

$$(7.21) \qquad u \in {}^k\mathcal{H}^{s+1,m+1-1/k}_{\mathrm{loc}}(J)\,.$$

Applying once more (7.20), with (7.16) and (7.21), we get $P\big(c-(k+1)\delta\big)u \in {}^k\mathcal{H}^{s,m}_{\mathrm{loc}}(J)$, hence by (5.1), $u \in {}^k\mathcal{H}^{s+2,m+2}_{\mathrm{loc}}(J)$. Q.E.D.

REMARK 7.2. If we apply the full strength of the proof of Prop. 7.1, we get in fact a much more general result than Th. 7.1: there is no reason to restrict ourselves to operators P satisfying (2.1): we may add to it any operator in ${}^k\mathcal{N}^{2-\varepsilon,2}$ with $\varepsilon > 0$ or any operator of the form tU, with $U \in {}^k\mathcal{N}^{2,2}$ (for instance: $td(t, A)\partial_t^2$, with $d(t, A) \in \mathfrak{Q}_A(J)$, or $e(t, A)A^r$, with $r < 2$), that properties (7.3) and (7.5) remain unchanged, if l_P is not any critical value.

REMARK 7.3. Remembering Remark 2.1, we get a further obvious generalization of Th. 7.1.

Appendix

We give here the proof of Proposition 1.1.

a) We prove first the following statement:

(A.1) If $\partial_t u$ and $|t|^a A^r u$ both belong to $L^2(R; H^s)$, then so does $|t|^{(a-1)/2} A^{r/2} u$. Moreover, there is a constant $C > 0$ such that (if $\| \, \|$ denotes the norm in $L^2(R; H^s)$) $\| |t|^{(a-1)/2} A^{r/2} u \|^2 \leqslant C(\| \partial_t u \|^2 + \| |t|^a A^r u \|^2)$.

In fact, for every $\varphi \in C_c^\infty(R; H^\infty)$ we have:

$$2\int_0^\infty \operatorname{Re}(\varphi_t, |t|^a A^r \varphi)_s \, dt = (\varphi, |t|^a A^r \varphi)_s \Big|_0^\infty - a\int_0^\infty (\varphi, |t|^{a-1} A^r \varphi)_s \, dt$$

and

$$2\int_{-\infty}^0 \operatorname{Re}(\varphi_t, |t|^a A^r \varphi)_s \, dt = (\varphi, |t|^a A^r \varphi)_s \Big|_{-\infty}^0 + a\int_{-\infty}^0 (\varphi, |t|^{a-1} A^r \varphi)_s \, dt,$$

so that

$$a\int_{-\infty}^\infty (\varphi, |t|^{a-1} A^r \varphi)_s \, dt = -2\int_{-\infty}^\infty (t/|t|) \operatorname{Re}(\varphi_t, |t|^a A^r \varphi)_s \, dt,$$

hence

$$a\int_R \| |t|^{(a-1)/2} A^{r/2} \varphi \|_s^2 \, dt \leqslant 2\int_R |(\varphi_t, |t|^a A^r \varphi)_s| \, dt \leqslant$$
$$\int_R \|\varphi_t\|_s \, \| |t|^a A^r \varphi \|_s \, dt \leqslant \int_R \|\varphi_t\|_s^2 \, dt + \int_R \| |t|^a A^r \varphi \|_s^2 \, dt,$$

which yields (A.1).

By iteration of (A.1) we get the following result:

(A.2) If $\partial_t u$ and $|t|^a A^r u$ both belong to $L^2(R; H^s)$, and m_0 is the first nonnegative integer such that $(a+1)/2^{m_0}-1 \leqslant 0$ (hence $m_0 \geqslant 1$), then for all integers m such that $0 \leqslant m \leqslant m_0$, $|t|^{(a+1)/2^m-1} A^{r/2^m} u$ belongs to $L^2(R; H^s)$. Moreover:

(A.3) There is $C>0$ (depending on a, m, r and s) such that

$$\left\| |t|^{(a+1)/2^m-1} A^{r/2^m} u \right\|^2 \leqslant C\left(\|\partial_t u\|^2 + \left\||t|^a A^r u\right\|^2 \right),$$

where $\| \ \|$ denotes again the norm of $L^2(R; H^s)$.

In order to prove a), it is enough to show that (A.3) is true for all *real* m satisfying $0 \leqslant m \leqslant m_0$. This can be done by an interpolation argument, using the fact that $\varphi(s) = \|u\|_s$ is a logarithmically convex function, for all u in $H^{-\infty}$. Let $0 < m < m_0$ and let us take λ real such that $\lambda + (1-\lambda)(1/2^{m_0}) = 1/2^m$. Then, $0 < \lambda < 1$ and:

$$\begin{aligned}
&\left(\int \left\||t|^a A^r u\right\|_s^2 dt\right)^{\lambda} \cdot \left(\int \left\||t|^{(a+1)/2^{m_0}-1} A^{r/2^{m_0}} u\right\|_s^2 dt\right)^{1-\lambda} \geqslant \\
&\int \left(\left\||t|^a A^r u\right\|_s^{2\lambda} \cdot \left\||t|^{(a+1)/2^{m_0}-1} A^{r/2^{m_0}} u\right\|_s^{2(1-\lambda)} \right) dt = \\
&\int \left(|t|^{2(a+1)/2^m-2} \cdot \|u\|_{s+r}^{2\lambda} \cdot \|u\|_{s+r/2^{m_0}}^{2(1-\lambda)} \right) dt \geqslant \\
&\int |t|^{2(a+1)/2^m-2} \|u\|_{s+r/2^m}^2 \, dt = \int \left\||t|^{(a+1)/2^m-1} A^{r/2^m} u\right\|_s^2 dt,
\end{aligned}$$

which ends the proof of a).

b) Remark that, since $t^k A u$ has compact support, so does Au, hence by (1.9) so does u. Since $\partial_t u \in \mathcal{K}_c^s(R)$, we must then have $u \in \mathcal{K}_c^s(R)$, as we easily realize, remarking that $u = \partial_t^{-1}(\partial_t u)$. Here, $\partial_t^{-1} v$ is defined for distributions v which vanish in some halfline $t < t_0$, as the integration from $-\infty$ to t, i.e., $\partial_t^{-1} v$ is the convolution of v with the Heaviside function.

Since u and $\partial_t u$ both belong to $\mathcal{K}_c^s(R)$, we must then have $\partial_t(t^j u) = t^j \partial_t u + j t^{j-1} u \in \mathcal{K}_c^s(R)$.

We prove b) first in the case $s \geqslant 0$. Then $\mathcal{K}^s$ is contained in $L^2(R; H^s)$, so that $\partial_t u$ and $t^k A u$ both belong to $L^2(R; H^s)$ and therefore, by a), $t^j A^{(j+1)/(k+1)} u \in L^2(R; H^s)$, hence $A^{s+(j+1)/(k+1)}(t^j u) \in \mathcal{K}_c^0(R)$, when $j = 0, 1, \ldots, k$. This, together with $\partial_t(t^j u) \in \mathcal{K}_c^s(R)$, immediately implies that

$$t^j u \in \mathcal{K}_c^{s+(j+1)/(k+1)}(R).$$

Let us now consider any real s, and let p be a positive integer such that $s+p \geqslant 0$. We apply (1.14): since $\partial_t u \in \mathcal{H}_c^s(R)$, there are v and w in $\mathcal{H}_c^{s+p}(R)$ such that $\partial_t u = \partial_t^p v + A^p w$. Hence, applying ∂_t^{-1} to both members, we get

$$u = \partial_t^{p-1} v + \partial_t^{-1} A^p w . \tag{A.4}$$

Remark that, since u and $\partial_t^{p-1} v$ have compact support, so does $\partial_t^{-1} A^p w$, and therefore also $\partial_t^{-1} w$. Let us call $h = \partial_t^{-1} w$: then $w = \partial_t h$, so $\partial_t h \in \mathcal{H}_c^{s+p}(R)$, which implies, as we saw in the beginning, that $h \in \mathcal{H}_c^{s+p}(R)$ and $\partial_t(t^k h) \in \mathcal{H}_c^{s+p}(R)$.

From (A.4) we get $t^k A u = t^k A \partial_t^{p-1} v + t^k \partial_t^{-1} A^{p+1} w$. Since by hypothesis $t^k A u \in \mathcal{H}_c^s(R)$, and so does $t^k A \partial_t^{p-1} v$, the above equality yields (remembering that $\partial_t^{-1} w = h$):

$$t^k A^{p+1} h \in \mathcal{H}_c^s(R) . \tag{A.5}$$

Since $\partial_t(t^k h) \in \mathcal{H}_c^{s+p}(R)$, (A.5) implies, as it is easy to see, that $t^k h \in \mathcal{H}_c^{s+p+1}(R)$, hence, $t^k A h \in \mathcal{H}_c^{s+p}(R)$. But $\partial_t h = w \in \mathcal{H}_c^{s+p}(R)$, and we can use the case $s \geqslant 0$ to get: $t^j h \in \mathcal{H}_c^{s+p+(j+1)/(k+1)}(R)$, for $j = 0, 1, \ldots, k$. Going back to (A.4), we have $u = \partial_t^{p-1} v + A^p h$, hence, for $j = 0, 1, \ldots, k$, we have: $t^j u = t^j \partial_t^{p-1} v + A^p(t^j h) \in \mathcal{H}_c^{s+(j+1)/(k+1)}(R)$.

c) Let $\partial_t u$ and $t^k A u$ both belong to $\mathcal{H}_{\mathrm{loc}}^s(J)$. We start claiming that $t^{k-j} A u \in \mathcal{H}_{\mathrm{loc}}^{s-j}(J)$ for $j = 0, 1, \ldots, k$. In fact, this is true for $j = 0$, and if it is true for some $j < k$, then $\partial_t(t^{k-j} A u) \in \mathcal{H}_{\mathrm{loc}}^{s-(j+1)}(J)$. Since

$$\partial_t(t^{k-j} A u) = t^{k-j} A \partial_t u + (k-j) t^{k-(j+1)} A u ,$$

and $t^{k-j} A \partial_t u \in \mathcal{H}_{\mathrm{loc}}^{s-(j+1)}(J)$ $\big($because $\partial_t u \in \mathcal{H}_{\mathrm{loc}}^s(J)\big)$, the statement is true also for $j+1$. In particular, we have then $A u \in \mathcal{H}_{\mathrm{loc}}^{s-k}(J)$, which together with $\partial_t u \in \mathcal{H}_{\mathrm{loc}}^s(J)$, implies that $u \in \mathcal{H}_{\mathrm{loc}}^{s-k+1}(J)$.

We claim now that $u \in \mathcal{H}_{\mathrm{loc}}^{s-k+1+j/(k+1)}(J)$, for $j = 0, 1, \ldots, k^2$. In fact, we already know this for $j = 0$. Suppose it is true for some $j < k^2$ $\big($which implies that $s - k + 1 + j/(k+1) \leqslant s\big)$; then, if $\varphi \in C_c^\infty(J)$, we have $\partial_t(\varphi u) = \varphi \partial_t u + (\partial_t \varphi) u \in \mathcal{H}_c^{s-k+1+j/(k+1)}(R)$, since $\partial_t u \in \mathcal{H}_{\mathrm{loc}}^s(J)$. But we also have $t^k A \varphi u \in \mathcal{H}_c^s(R)$, hence by *b*), $\varphi u \in \mathcal{H}_c^{s-k+1+(j+1)/(k+1)}(R)$. Since $\varphi \in C_c^\infty(J)$ is arbitrary, the statement is true for $j+1$. In particular, when $j = k^2$ we get $u \in \mathcal{H}_{\mathrm{loc}}^{s+1/(k+1)}(J)$.

Finally, we show that $t^j u \in \mathcal{H}_{\mathrm{loc}}^{s+(j+1)/(k+1)}(J)$, for $j = 0, 1, \ldots, k$. The case $j = 0$ was just proved. Let $\varphi \in C_c^\infty(J)$: then $\partial_t(\varphi u) = \varphi \partial_t u + (\partial_t \varphi) u \in \mathcal{H}_c^s(R)$ (using the case $j = 0$). But we also have $t^k A \varphi u \in \mathcal{H}_c^s(R)$, hence by *b*), $t^j \varphi u \in \mathcal{H}_c^{s+(j+1)/(k+1)}(R)$. Since $\varphi \in C_c^\infty(J)$ is arbitrary, that means that $t^j u \in \mathcal{H}_{\mathrm{loc}}^{s+(j+1)/(k+1)}(J)$. Q.E.D.

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