

ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

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**Integral representation of solutions of first-order linear
partial differential equations, I**

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4^e série, tome 3, n° 1
(1976), p. 1-35

http://www.numdam.org/item?id=ASNSP_1976_4_3_1_1_0

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Integral Representation of Solutions of First-Order Linear Partial Differential Equations, I (*).

FRANÇOIS TREVES (**)

dedicated to Hans Lewy

Introduction.

Until now the proofs of the local solvability of the linear PDEs with simple real characteristics satisfying Condition (P) are strictly « existential » and based on a priori estimates (see [1], [5]). In the present paper we give an integral representation of solutions (in a small open set) of the equation

$$(1) \quad Lu = f,$$

in the (very) particular case where L is a first-order linear partial differential operator with analytic coefficients, nondegenerate, satisfying Condition (P) (which, in these circumstances, is equivalent to being locally solvable). The right-hand side f is assumed to be C^∞ and the found solution u is also C^∞ (all this is local, in a fixed open neighborhood of the central point). Thus we answer, in this particular case, the question of the solvability within C^∞ -still open in the more general cases.

The techniques introduced in the present paper compel us to look at the local solvability of first-order linear PDEs from a new viewpoint, and to analyze in finer detail its geometric implications (see Ch. I). I believe they should lead to interesting results in more general set-ups. An outline of the main ideas of the paper can be found in [7].

(*) Work partly supported by NSF Grant No. MPS 75 - 07064.

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Pervenuto alla Redazione il 28 Aprile 1975.

CHAPTER I

GENERAL CONSIDERATIONS

1. – A new formulation of the solvability condition (P) for first-order linear PDEs.

Let Ω be an open subset of \mathbb{R}^N ($N \geq 2$); the coordinates in \mathbb{R}^N are denoted (at least for the moment) by y^1, \dots, y^N . We consider a first-order linear partial differential operator

$$(1.1) \quad L = \sum_{j=1}^N \gamma^j(y) \frac{\partial}{\partial y^j} + \gamma^0(y),$$

where the γ^j ($0 \leq j \leq N$) are complex-valued C^∞ functions in Ω . We make throughout the hypothesis that L is nowhere degenerate, that is,

$$(1.2) \quad \forall y \in \Omega, \quad \sum_{j=1}^N |\gamma^j(y)| \neq 0.$$

We denote by L_0 the principal part of L , that is,

$$(1.2') \quad L_0 = \sum_{j=1}^N \gamma^j(y) \frac{\partial}{\partial y^j}.$$

Let then y_0 be an arbitrary point of Ω . There is an open neighborhood $U \subset \Omega$ of y_0 and local coordinates in U , x^1, \dots, x^n, t (we set $n = N - 1$ and write $x = (x^1, \dots, x^n)$) such that, in U ,

$$(1.3) \quad L = \zeta(x, t) \left\{ \frac{\partial}{\partial t} + i \sum_{j=1}^n b^j(x, t) \frac{\partial}{\partial x^j} + c(x, t) \right\},$$

where ζ , b^j ($1 \leq j \leq n$), c are C^∞ functions in U ; furthermore, ζ does not vanish at any point of U and the b^j are real-valued. We may in fact suppose that U is a product-set

$$(1.4) \quad U = \{(x, t) \in \mathbb{R}^{n+1}; x \in U_0, |t| < T\},$$

where U_0 is an open neighborhood of the origin and T a number > 0 (we are assuming, for simplicity, that the coordinates x^1, \dots, x^n, t all vanish at y_0). We set: $\mathbf{b}(x, t) = (b^1(x, t), \dots, b^n(x, t)) \in \mathbb{R}^n$.

DEFINITION 1.1. We say that L satisfies Condition (P) at the point y_0 if there is an open neighborhood $V \subset U$ of y_0 such that the following is true:

(1.5) In V , the vector field $\mathbf{b}(x, t)/|\mathbf{b}(x, t)|$ is independent of t .

Although we have formulated (P) at y_0 in terms of particular coordinates, it can be proved (see [4]) that it does not depend on our choice of the coordinates (provided that the expression of L be of the kind (1.3)). We say that L satisfies Condition (P) in Ω if it so does at every point of Ω .

Let us assume that (P) holds at y_0 and take the neighborhood U equal to V in Def. 1.1. Let us introduce the singular set

$$(1.6) \quad \mathcal{N}_0 = \{x \in U_0; \forall t, |t| < T, \mathbf{b}(x, t) = 0\}.$$

In $U_0 \setminus \mathcal{N}_0$, $\mathbf{b}/|\mathbf{b}|$ is a C^∞ vector field, nowhere zero; we shall regard it as a first-order linear partial differential operator, without zero-order term, which we denote by

$$(1.7) \quad X = \sum_{j=1}^n v^j(x) \frac{\partial}{\partial x^j}.$$

We extend $\mathbf{v} = (v^1, \dots, v^n)$ as an arbitrary unit-vector in \mathcal{N}_0 (in general, \mathbf{v} will not be smooth throughout U_0). In U we may then write

$$(1.8) \quad L = \zeta(x, t) \left\{ \frac{\partial}{\partial t} + i|\mathbf{b}(x, t)|X + c(x, t) \right\}.$$

We find ourselves in the following situation: U is the union of the « vertical » lines $x = x_0 \in \mathcal{N}_0$, along which $\zeta^{-1}L_0 = \partial/\partial t$, and of the cylinders $\Sigma = \Gamma \times]-T, T[$, where Γ is any integral curve of X in $U_0 \setminus \mathcal{N}_0$, cylinders on which $\zeta^{-1}L_0 = \partial/\partial t + i|\mathbf{b}(x, t)|X$.

Assuming now that (P) holds at every point of Ω , this leads to a foliation of Ω , which is best seen when the coefficients of L_0 are analytic. Let us write $L_0 = A + \sqrt{-1}B$, where A and B are real vector fields. Let us denote by $\mathfrak{g}(A, B)$ the real Lie algebra generated by A and B (for the commutation bracket $[A, B] = AB - BA$). When A and B are analytic vector fields, it is known (see [2]) that each point y_0 of Ω belongs to one, and only one, subset \mathcal{M} of Ω having the following properties:

(1.9) \mathcal{M} is a connected analytic submanifold of Ω ;

(1.10) the tangent space to \mathcal{M} at anyone of its points, y , is exactly equal to the « freezing » of $\mathfrak{g}(A, B)$ at y ;

(1.11) \mathcal{M} is maximal for Properties (1.9) and (1.10).

We shall refer to the submanifolds \mathcal{M} as the *leaves defined by L* . When the coefficients of L_0 are analytic, Condition (P) is equivalent with the local solvability of L (here, at each point of Ω). It can be subdivided into two parts:

(P1) *the dimension of each leaf \mathcal{M} , defined by L , is either one or two.*

Observe that (P1) is always true when the dimension of the surrounding space, i.e., of Ω , is ≤ 2 —even when L is not locally solvable.

On an arbitrary two-dimensional leaf \mathcal{M} the complex vector field L_0 defines a skew-symmetric real two-tensor, which it is natural to denote by

$$(1.12) \quad A \wedge B = -\frac{1}{2i} L_0 \wedge \bar{L}_0.$$

(If we set $\alpha^j = \operatorname{Re} \gamma^j$, $\beta^j = \operatorname{Im} \gamma^j$, $j = 1, \dots, N$, the coordinates of $A \wedge B$ on the canonical basis $\partial/\partial y^j \wedge \partial/\partial y^k$, $1 \leq j < k \leq N$, are $\alpha^j \beta^k - \alpha^k \beta^j$.) Locally on \mathcal{M} , we may always find a generator of the line bundle $\wedge^2 T\mathcal{M}$ (in the coordinates x, t used earlier we may take $\partial/\partial t \wedge X$). Let θ be such a generator, say over an open subset \mathcal{O} of \mathcal{M} . Then $A \wedge B = \varrho \theta$, where ϱ is a real-valued (analytic) function in \mathcal{O} . Observe that the property that ϱ does not change sign in \mathcal{O} is independent of our choice of θ ; observe also that the zero-set of ϱ is a proper analytic subset of \mathcal{O} . This gives a meaning to the second « part » of Property (P):

(P2) *Restricted to any two-dimensional leaf \mathcal{M} (defined by L) in Ω , $-(1/2i)L_0 \wedge \bar{L}_0$ does not change sign.*

Indeed, in the form (1.7), we have

$$(1.13) \quad -\frac{1}{2i} L_0 \wedge \bar{L}_0 = |\mathbf{b}(x, t)| \left(\frac{\partial}{\partial t} \wedge X \right).$$

There is no need to point out that (P), stated as the conjunction of (P1) and (P2), is invariant under coordinates changes and also under multiplication of L by nowhere vanishing complex-valued (analytic) functions.

Along any one-dimensional leaf (parametrized by t) the principal part L_0 is essentially $\partial/\partial t$. We shall see, in the course of the proof of our main result, that, by virtue of (P2), on any two-dimensional leaf, L_0 can be transformed (up to a scalar factor) into the « Cauchy-Riemann operator » $\partial/\partial t + iX$, via a homeomorphism naturally associated with L_0 .

In the case of C^∞ coefficients, the reformulation of Property (P) is more delicate. First of all it is not true, in general, that through each point y_0 of Ω passes a unique C^∞ submanifold \mathcal{M} with property (1.10). There is another

consequential difference: let us return to the local representation (1.8) and consider the restriction of L to a cylinder $\Sigma = \Gamma \times]-T, T[$, where Γ is an integral curve of X in $U_0 \setminus \mathcal{N}_0$. When the coefficients of L are analytic, $|\mathbf{b}(x, t)|$ is an analytic function on Σ ; its zero-set intersects any vertical line $x = x_0 \in \Gamma$ only on a *locally finite* set, otherwise $|\mathbf{b}|$ would vanish identically on such a line and we should have $x_0 \in \mathcal{N}_0$, contrary to the definition of Γ . In particular the zero-set of $|\mathbf{b}|$ is a proper analytic subset of Σ . In the case of C^∞ coefficients the zero-set of $|\mathbf{b}|$ might have a nonempty interior. It is submitted to the sole condition of not containing any vertical segment $\{x_0\} \times]-T, T[$. Requirement which is of course dependent on the length T of the « time interval ».

Nevertheless we may formulate (P) as the conjunction of (P1) and (P2), provided we incorporate to (P1) a statement as to the existence of the foliation $\{\mathcal{M}\}$. Leaves cannot anymore be required to have property (1.10) (*example*: $L = \partial/\partial t + ib(x, t)(\partial/\partial x)$, where $b \in C^\infty(\mathbb{R}^2)$ is real-valued and does not vanish anywhere, except at the origin, where all derivatives of b vanish; there can be but one leaf, the whole plane, although $\mathfrak{g}(A, B)$ is one-dimensional at the origin).

DEFINITION 1.2. *Let k be any integer such that $0 \leq k \leq N$. A k -dimensional leaf, defined by L in Ω , is a subset \mathcal{M} of Ω having the following properties:*

- (1.14) \mathcal{M} is a connected C^∞ submanifold (*) of Ω , of dimension k ;
- (1.15) the tangent space to \mathcal{M} at anyone of its points, y , contains the « freezing » of $\mathfrak{g}(A, B)$ at y ;
- (1.16) the boundary (**) of \mathcal{M} is a union of leaves defined by L in Ω whose dimension is $< k$;
- (1.17) \mathcal{M} does not contain any leaf defined by L in Ω whose dimension is $< k$.

With this definition we may now state:

(P1) Ω is the union of leaves defined by L , whose dimension is either one or two.

Observe that, because of Hypothesis (1.2), the dimension of $\mathfrak{g}(A, B)$ at every point is at least one; consequently there are no zero-dimensional leaves

(*) It is perhaps worth recalling what a C^∞ submanifold \mathcal{M} of Ω is: every point of \mathcal{M} (and not necessarily every point of Ω !) has a neighborhood in which \mathcal{M} is defined by the vanishing of a number of smooth functions whose differentials are linearly independent.

(**) We use the word « boundary » as in « manifold with boundary », not with its point-set topology meaning.

and, in virtue of (1.16), the one-dimensional leaves defined by L must be without a boundary.

Let us prove that (P) is equivalent with the conjunction of (P1) and (P2). First, suppose (P) holds at every point of Ω . Let \mathcal{M} denote a connected one-dimensional C^∞ submanifold of Ω , having property (1.15), and also the following one:

(1.18) *Let (U, x^1, \dots, x^n, t) be a local chart in Ω , yielding the representation (1.8) of L . Any segment $\{x_0\} \times]-T, T[$, $x_0 \in U_0$, which intersects $\mathcal{M} \cap U$ is contained in \mathcal{M} .*

The manifold \mathcal{M} cannot have any boundary point y_0 in Ω , as one sees by taking a local chart (U, x^1, \dots, x^n, t) as in (1.18), containing y_0 . Thus, by Def. 1.2, \mathcal{M} is a leaf of L in Ω ; and every one-dimensional leaf of L must have property (1.18). Let F denote the union of all the one-dimensional leaves of L in Ω . Since, in any local chart like the one in (1.18), such leaves are unions of vertical segments $\{x_0\} \times]-T, T[$ with $x_0 \in \mathcal{N}_0$, every point $y_0 \in U$ which belongs to the closure of F must lie on such a segment, from which it follows at once that F must be closed.

Let now \mathcal{M} denote a connected two-dimensional C^∞ submanifold of $\Omega \setminus F$, satisfying (1.15) and (1.18), and maximal for these properties. Its boundary $\partial\mathcal{M}$ in Ω (see footnote (**)) to p. 5) must also have property (1.18). By the maximality of \mathcal{M} , at no point of $\partial\mathcal{M}$ can the Lie algebra $\mathfrak{g}(A, B)$ be two-dimensional. Thus $\partial\mathcal{M}$ must be a union of one-dimensional leaves of L , in fact of at most two of them, since \mathcal{M} is connected, and possibly of only one or of none. Thus \mathcal{M} is a two-dimensional leaf of L . A moment of thought shows that, conversely, every two-dimensional leaf of L in Ω is such a submanifold of $\Omega \setminus F$. Note that every C^∞ submanifold of $\Omega \setminus F$, of dimension ≤ 2 , satisfying (1.15), must be entirely contained in some such leaf; and that any two such leaves cannot intersect without being identical. In particular, every point of $\Omega \setminus F$ is contained in one, and only one, two-dimensional leaf of L .

That (P2) is a consequence of (P) has already been explained.

Suppose now that (P1) holds. Let y_0 be an arbitrary point of Ω . By (1.2) we may assume that L has the representation (1.3) in a local chart (U, x^1, \dots, x^n, t) centered at y_0 . This also implies that every leaf \mathcal{M} defined by L in Ω (according to Def. 1.2), is equal, in U , to the union of vertical lines $x = x_0$: As a consequence, every two-dimensional leaf \mathcal{M} intersects the piece of « hyperplane » $\{(x, t) \in U; t = 0\}$ along a smooth curve $\Gamma \subset U_0$; and \mathcal{M} is the cylinder $\Sigma = \Gamma \times]-T, T[$. Let \tilde{X} be a smooth vector field tangent to Γ and nowhere vanishing on Γ . From what we have just seen it follows

that, on Σ , $\zeta^{-1}L_0 = \partial/\partial t + i\lambda(x, t)\tilde{X}$, and (P2) means exactly that λ does not change sign in Σ (if \tilde{X} is a *unit* vector-field and if λ is nonnegative, in the notation of (1.8) we have $\lambda = |\mathbf{b}(x, t)|$ and $\tilde{X} = X$).

A final remark: in the C^∞ case, in contrast with what happens in the analytic case, the foliation defined by L in a proper open subset of Ω can be strictly finer than the one induced by the foliation of Ω .

2. – Reduction to flat right-hand sides.

We shall systematically use the following terminology:

DEFINITION 2.1. *Let M be a C^∞ manifold, ϱ a nonnegative continuous function in M . We say that a function $f \in C^\infty(M)$ is ϱ -flat in M if given any differential operator (with C^∞ coefficients) \mathbf{P} in M and any integer $k \geq 0$, $\varrho^{-k}\mathbf{P}f$ is a continuous function in M , and we write then $f \underset{\varrho}{\sim} 0$.*

It is convenient to introduce the notation:

$$(2.1) \quad \mathcal{E}_{\varrho\text{-flat}}(M) = \{f \in C^\infty(M); f \text{ is } \varrho\text{-flat in } M\},$$

$$(2.2) \quad \mathcal{D}_{\varrho\text{-flat}}(M) = \{f \in \mathcal{E}_{\varrho\text{-flat}}(M); \text{supp } f \text{ is a compact subset of } M\}.$$

We shall comply throughout with the notation of Sect. 1. In particular, we reason locally, in the neighborhood U given by (1.4). We assume that the « central point » is the origin in \mathbb{R}^{n+1} and that L is given by (1.3); we may even divide by ζ and take $L = L_0 + c(x, t)$, with

$$(2.3) \quad L_0 = \frac{\partial}{\partial t} + i \sum_{j=1}^n b^j(x, t) \frac{\partial}{\partial x^j}.$$

We suppose that $\mathbf{b} = (b^1, \dots, b^n)$ and c are C^∞ functions in an open neighborhood of \bar{U} (which is compact), valued in \mathbb{R}^n and \mathbb{C} respectively.

The function ϱ we use (in a neighborhood of \bar{U}) is the following:

$$(2.4) \quad \varrho(x, t) = \left| \int_{-T}^t |\mathbf{b}(x, s)| ds \right|.$$

Following a suggestion of L. Nirenberg we begin by proving

THEOREM 2.1. *If U is small enough, there is a continuous linear operator $E: C_c^\infty(U) \rightarrow C^\infty(U)$ such that $R = LE - I$ maps $C_c^\infty(U)$ into $\mathcal{E}_{\varrho\text{-flat}}(U)$.*

REMARK 2.1. In Th. 2.1 it is *not* assumed that L is locally solvable.

REMARK 2.2. By Def. 2.1 it is clear that $\mathcal{E}_{\varrho\text{-flat}}(M)$ carries a natural Fréchet space topology: the coarsest one which renders all the mappings $f \mapsto \varrho^{-k} P f$ continuous, from $\mathcal{E}_{\varrho\text{-flat}}(M)$ into $C^0(M)$, as k and P vary freely. The space $\mathcal{D}_{\varrho\text{-flat}}(M)$ carries the derived \mathcal{LF} -topology. It will be obvious that the operator R is continuous for these topologies.

PROOF. We apply Th. 1 of [6] (or else use almost-analytic extensions). If U is small enough, for each $j = 1, \dots, n$, there is a complex-valued C^∞ function $z^j = z^j(x, t, t')$ such that

$$(2.5) \quad L_0 z^j \sim_{\varrho} 0 \quad \text{in an open neighborhood of } \bar{U},$$

$$(2.6) \quad z^j|_{t=t'} = x^j$$

($t' \in [-T, T]$). If the coefficients of L_0 are analytic, the equivalence (2.5) can be replaced by an exact equality $L_0 z^j = 0$ (because of the Cauchy-Kovalevska theorem). Furthermore (according to Th. 1 of [6])

$$(2.7) \quad \left| z^j - x^j - i \int_{t'}^t b^j(x, s) ds \right| \leq C \varrho(x, t)$$

(where $C \rightarrow 0$ with T).

Application of Th. 1 of [6] yields (2.5) and (2.7) with $\varrho(x, t)$ replaced by $\left| \int_{t'}^t |\mathbf{b}(x, s)| ds \right|$; but if $-T \leq t' \leq t$, the latter quantity clearly does not exceed $\varrho(x, t)$. We write $z = (z^1, \dots, z^n)$ and set, for arbitrary $f \in C_c^\infty(U)$,

$$(2.8) \quad E_0 f(x, t) = (2\pi)^{-n} \int_{-T}^t \int_{\xi \in \mathbb{R}^n} \exp [iz(x, t, t') \cdot \xi] \cdot g(\xi \cdot \text{Im } z(x, t, t')) \hat{f}(\xi, t') dt' d\xi, (*)$$

where $\hat{f}(\xi, t)$ is the Fourier transform of f with respect to the x -variables, and $g \in C^\infty(\mathbb{R}^1)$, $g(\tau) = 1$ for $\tau > -1$, $g(\tau) = 0$ for $\tau < -2$. It is checked at once that E_0 maps continuously $C_c^\infty(U)$ into $C^\infty(U)$. Furthermore, if $R_0 = L_0 E_0 - I$, we have

$$(2.9) \quad R_0 f(x, t) = (2\pi)^{-n} \int_{-T}^t \int_{\mathbb{R}^n} \exp [iz \cdot \xi] R_0(x, t, t', \xi) \hat{f}(\xi, t') d\xi dt',$$

(*) This formula mimicks the construction of continuous almost-analytic extensions by Mather ([3]).

with

$$(2.10) \quad R_0(x, t, t', \xi) = \{ig(\xi \cdot \text{Im } z)L_0 z + g'(\xi \cdot \text{Im } z)L_0(\text{Im } z)\} \cdot \xi$$

($z = z(x, t, t')$). Whatever $k \in \mathbb{Z}_+$, for a suitable $C_k > 0$,

$$(2.11) \quad |g'(\xi \cdot \text{Im } z)| \leq C_k |\xi \cdot \text{Im } z|^k,$$

whence, combining (2.7) and (2.11),

$$(2.12) \quad |g'(\xi \cdot \text{Im } z)L_0(\text{Im } z) \cdot \xi| \leq C'_k |\xi|^{k+1} \varrho(x, t)^k.$$

On the other hand, by (2.5), we have

$$(2.13) \quad |L_0 z| \leq C''_k \varrho(x, t)^k.$$

Since on the support of $g(\xi \cdot \text{Im } x)$, we have $|\exp[iz \cdot \xi]| \leq e^2$, we see that

$$(2.14) \quad |R_0 f(x, t)| \leq C_k^m \varrho(x, t)^k \int_{-T}^t (1 + |\xi|)^{k+1} |\hat{f}(\xi, t')| d\xi dt'.$$

An analogous inequality can be obtained for every derivative of $R_0 f$ with respect to (x, t) , which shows that $R_0 f$ is ϱ -flat.

Let now $c_1(x, t) \in C_c^\infty(U)$ equal $c(x, t)$ in a subneighborhood U_1 of the origin, and set

$$(2.15) \quad Ef = \exp[-E_0 c_1] E_0(\exp[E_0 c_1] f), \quad f \in C_c^\infty(U_1).$$

We have, in U_1 ,

$$\begin{aligned} LEf &= (L_0 + c_1)\{\exp[-E_0 c_1] E_0(\exp[E_0 c_1] f)\} \\ &= \exp[-E_0 c_1](L_0 + c_1 - L_0 E_0 c_1) E_0(\exp[E_0 c_1] f) \\ &= f - (R_0 c_1) f + \exp[-E_0 c_1] R_0(\exp[E_0 c_1] f), \end{aligned}$$

which shows that the statement of Th. 2.1 is verified, if U_1 is substituted for U and if we define R by

$$(2.16) \quad Rf = \exp[-E_0 c_1] R_0(\exp[E_0 c_1] f) - (R_0 c_1) f. \quad \text{Q.E.D.}$$

CHAPTER II

CONSTRUCTION OF FUNDAMENTAL SOLUTIONS
WHEN THE COEFFICIENTS ARE ANALYTIC**I. – Statement of the theorem.**

We use the notation of Ch. I. In particular, $L = L_0 + c(x, t)$ and L_0 has the form (2.3), Ch. I. The coefficients of L_0 are now analytic in an open neighborhood of the origin in \mathbb{R}^{n+1} , which we take to be Ω . In this chapter we prove:

THEOREM 1.1. *There is an open neighborhood $U \subset \Omega$ of the origin in \mathbb{R}^{n+1} and a continuous linear operator $K: C_c^\infty(U) \rightarrow C^\infty(U)$ such that $LK = I$, the identity of $C_c^\infty(U)$.*

Let us note right-away that it suffices to prove the result when $c(x, t) \equiv 0$. For if $L_0 K_0 = I$ and $h = K_0 c_1$, with $c_1 \in C_c^\infty(U)$, $c_1 = c$ in an open neighborhood $U' \subset U$ of 0, we may take $K = e^{-h} K_0 e^h$, which yields $LK = I$ in U' . From now on we assume that $L = L_0$, i.e.

$$(1.1) \quad L = \frac{\partial}{\partial t} + i \sum_{j=1}^n b^j(x, t) \frac{\partial}{\partial x^j}$$

in Ω .

If we take Th. 2.1, Ch. I, into account, we see that Th. 1.1 will follow from the result below. We introduce and shall use throughout the following function:

$$(1.2) \quad r(x) = \left\{ \int_{-T}^T |\mathbf{b}(x, s)|^2 ds \right\}^{\frac{1}{2}}.$$

If we compare with the function $\varrho(x, t)$ defined in (2.4), Ch. I, we see that

$$(1.3) \quad \varrho(x, t) \leq \sqrt{(T+t)} r(x) \leq \sqrt{2T} r(x)$$

for $(x, t) \in \Omega$, $|t| \leq T$. Thus if we regard r as a function in Ω , we see that any ϱ -flat (Def. 2.1, Ch. I) C^∞ function is also r -flat. In particular (see (2.1), Ch. I), $\mathcal{E}_{\varrho\text{-flat}}(U) \subset \mathcal{E}_{r\text{-flat}}(U)$. We shall prove the following result:

THEOREM 1.2. *If the open neighborhood $U \subset \subset \Omega$ of the origin is small enough, there is a linear operator $K_1: \mathcal{D}_{r\text{-flat}}(U) \rightarrow \mathcal{E}_{r\text{-flat}}(U)$ (see (2.1), (2.2), Ch. 1) such that $LK_1 = I$, the identity of $\mathcal{D}_{r\text{-flat}}(U)$.*

Let then $\zeta(x, t) \in C_c^\infty(U)$, $\zeta(x, t) = 1$ in an open subneighborhood U' of 0. Let R be the operator in Th. 2.1, Ch. I, and set, for any $f \in C_1(U)$,

$$(1.4) \quad Kf = Ef - K_1[\zeta(x, t)Rf],$$

where E is the operator in Th. 2.1, Ch. I. Obviously K satisfies the requirements of Th. 1.1, if we substitute U' for U (inspection of the proofs shows easily that K_1 , as well as R , are continuous when $\mathcal{D}_{r\text{-flat}}(U)$ and $\mathcal{E}_{r\text{-flat}}(U)$ carry their natural topologies, and this implies the continuity of K).

2. - Determination of an across-the-board phase function.

From now on we suppose that

$$(2.1) \quad U_0 = \{x \in \mathbb{R}^n; |x| < R\}, \quad R > 0,$$

recalling that $U = U_0 \times]-T, T[$. We also recall that

$$(2.2) \quad \mathcal{N}_0 = \{x \in \Omega; \forall t, |t| < T, \mathbf{b}(x, t) = 0\}.$$

Let us set

$$(2.3) \quad \sigma(z, t) = \sum_{k=1}^n [b^k(z, t)]^2,$$

where $z = x + iy$ varies in an open neighborhood of $\bar{U} + i\{y \in \mathbb{R}^n; |y| < r_0\}$. Since $\sigma(x, t) \geq 0$ (for $x \in \mathbb{R}^n$ in a neighborhood of \bar{U}), there is a constant $M \geq 0$ such that

$$(2.4) \quad |\sigma_x(x, t)| \leq M \sqrt{\sigma(x, t)}, \quad \forall x \in \bar{U}, |t| < T.$$

Of course, $\sqrt{\sigma(x, t)} = |\mathbf{b}(x, t)|$. We may write

$$(2.5) \quad \sigma(x + iy, t) = \sigma(x, t) + iy \cdot \sigma_x(x, t) + \tau_2(x, y, t),$$

where $|\tau_2(x, y, t)| \leq M_1|y|^2$. If $\varkappa > 0$ is small enough and if

$$(2.6) \quad |y| < \varkappa |\mathbf{b}(x, t)|,$$

we obtain:

$$(2.7) \quad |\sigma(x + iy, t) - \sigma(x, t)| < \frac{1}{2}\sigma(x, t).$$

LEMMA 2.1. *If $\kappa > 0$ is small enough, $\beta(x, t) = |\mathbf{b}(x, t)|$ can be extended as a holomorphic function of $z = x + iy$ in the region:*

$$(2.8) \quad x \in U_0 \setminus \mathcal{N}_0, \quad |y| < \kappa \sup_{|s| < T} |\mathbf{b}(x, s)|,$$

for all t , $|t| < T$ (κ is independent of t).

PROOF. According to what precedes the statement of Lemma 2.1, in the region

$$(2.9) \quad x \in U_0 \setminus \mathcal{N}_0, \quad |y| < \kappa |\mathbf{b}(x, t)|,$$

we may (see (2.7)) set $\beta(z, t) = \sqrt{\sigma(z, t)}$ (the second inequality in (2.9) demands $|\mathbf{b}(x, t)| \neq 0$). If we set, in (2.9),

$$(2.10) \quad \mathbf{v}(z, t) = \beta(z, t)^{-1} \mathbf{b}(z, t),$$

we see, by analytic continuation from $z = x$ real, that $\partial_t \mathbf{v}(z, t) = 0$, and we may write $\mathbf{v}(z)$ instead of $\mathbf{v}(z, t)$:

$$(2.11) \quad \beta(z, t) \mathbf{v}(z) = \mathbf{b}(z, t).$$

Let $x_0 \in U_0 \setminus \mathcal{N}_0$ be arbitrary. Consider any number $\eta > 0$ such that

$$(2.12) \quad \eta < \sup_{|t| < T} |\mathbf{b}(x_0, t)|.$$

There is t_0 , $|t_0| < T$, such that $\beta(x_0, t_0) > \eta$. Actually, we may find $\eta_1 > 0$ such that

$$(2.13) \quad |x - x_0| < \eta_1 \Rightarrow \beta(x, t_0) > \eta.$$

We apply (2.7) or, rather, its consequence, namely that

$$(2.14) \quad |y| < \kappa \beta(x, t_0) \Rightarrow |\beta(x + iy, t_0)| > \frac{1}{\sqrt{2}} \beta(x, t_0),$$

hence:

$$(2.15) \quad |x - x_0| < \eta_1, \quad |y| < \kappa \beta(x, t_0)$$

implies

$$(2.16) \quad \beta(x + iy, t_0) > \eta/\sqrt{2}.$$

But then, in the region (2.15),

$$(2.17) \quad v(x + iy) = \mathbf{b}(x + iy, t_0)/\beta(x + iy, t_0)$$

is holomorphic and, since $|\mathbf{b}(z, t_0)| \geq |\beta(z, t_0)|$, nowhere vanishes. In particular, every point in the set (2.15) has an open neighborhood in which, for some $j = 1, \dots, n$, $v^j(x + iy)$ is nowhere zero. Since

$$\beta(x + iy, t) = b^j(x + iy, t)/v^j(x + iy),$$

we see that $\beta(z, t)$ can be extended as a holomorphic function to such a neighborhood. Letting

$$\eta \rightarrow \sup_{|s| < T} \beta(x_0, s)$$

proves the assertion in Lemma 2.1. **Q.E.D.**

REMARK 2.1. Actually $\beta(x + iy, t)$ is a C^∞ function of t , $|t| < T$, valued in the space of holomorphic functions of $x + iy$ in the set (2.8).

REMARK 2.2. We have also shown that $|v(z + iy)| \geq 1$ in the set (2.8) provided that $\kappa > 0$ is small enough.

Assume, as we may, that $T < \frac{1}{2}$. Then (cf. (1.2)):

$$(2.18) \quad r(x) = \left\{ \int_{-T}^T |\mathbf{b}(x, s)|^2 ds \right\}^{\frac{1}{2}} \leq \sup_{|s| \leq T} |\mathbf{b}(x, s)|.$$

LEMMA 2.2. *If $\kappa > 0$ is small enough, the holomorphic function $\int_{-T}^T \beta(x + iy, s)^2 ds$ does not vanish at any point $x + iy$ such that*

$$(2.19) \quad x \in U_0 \setminus \mathcal{N}_0, \quad |y| < \kappa r(x).$$

PROOF. We have $\beta(z, s)^2 = \sigma(z, s)$ (see (2.3)). We have (cf. (2.4) and (2.5)):

$$(2.20) \quad \left| \int_{-T}^T \sigma(x + iy, s) ds - \int_{-T}^T \sigma(x, s) ds \right| \leq M\kappa r(x) \int_{-T}^T \beta(x, s) ds + 2TM_1\kappa^2 r(x)^2.$$

On the other hand,

$$\int_{-T}^T \beta(x, s) ds \leq \sqrt{2T} r(x),$$

whence, by (2.20),

$$(2.21) \quad \left| \int_{-T}^T \sigma(x + iy, s) ds - r(x)^2 \right| \leq M_2 \varkappa r(x)^2.$$

Lemma 2.2 follows from the fact that $r(x) > 0$ if $x \in U_0 \setminus \mathcal{N}_0$. Q.E.D.

REMARK 2.3. Actually, as we see in (2.21) where we take $M_2 \varkappa < \frac{1}{2}$, that is,

$$(2.22) \quad \left| \int_{-T}^T \sigma(x + iy, s) ds - r(x)^2 \right| < \frac{1}{2} r(x)^2,$$

we may define the square-root,

$$(2.23) \quad r(x + iy) = \left\{ \int_{-T}^T \beta(x + iy, s)^2 ds \right\}^{\frac{1}{2}},$$

which is holomorphic in the open set (2.19).

We are now going to study a Cauchy problem

$$(2.24) \quad \frac{\partial \varphi}{\partial t} + i \sum_{k=1}^n b^k(z, t) \frac{\partial \varphi}{\partial z^k} = g(z, t),$$

in a region

$$(2.25) \quad x \in U_0 \setminus \mathcal{N}_0, \quad |y| < \varkappa r(x), \quad |t| < T' \leq T,$$

with initial condition,

$$(2.26) \quad \varphi|_{t=0} = 0,$$

where g will be eventually chosen, but in any event, is a C^∞ function of (x, y, t) in the region

$$(2.27) \quad x \in U_0 \setminus \mathcal{N}_0, \quad |y| < \varkappa r(x), \quad |t| < T,$$

holomorphic with respect to $x + iy$.

Let $x_0 \in U_0 \setminus \mathcal{N}_0$ be arbitrary. We denote by E_s ($0 < s < 1$) the space of continuous functions in the ball $\{z \in \mathbf{C}^n; |z - x_0| \leq s\kappa_0 r(x_0)\}$, holomorphic in the interior. Here κ_0 is chosen so as to have, if $z = x + iy$,

$$(2.28) \quad |z - x_0| \leq \kappa_0 r(x_0) \Rightarrow |y| < \kappa r(x),$$

where κ is the number in Lemma 2.2. Such a choice is indeed possible, for $|z - x_0| \leq \kappa_0 r(x_0)$ implies $|x - x_0| \leq \kappa_0 r(x_0)$ and $|y| \leq \kappa_0 r(x_0)$ ($z = x + iy$), therefore $|r(x) - r(x_0)| \leq \left(\int_{-T}^T |\mathbf{b}(x, s) - \mathbf{b}(x_0, s)|^2 ds \right)^{\frac{1}{2}} \leq M|x - x_0| \leq M\kappa_0 r(x_0) \leq \frac{1}{2}r(x_0)$ if $M\kappa_0 \leq \frac{1}{2}$, and thus it implies $r(x_0) \leq 2r(x)$, whence $|y| < \kappa r(x)$ if $2\kappa_0 < \kappa$.

We shall denote by $h \mapsto \|h\|_s$ the (maximum) norm in E_s . We solve (2.24)-(2.26) by the standard iteration method: we set

$$(2.29) \quad \varphi(z, t) = \sum_{\nu=0}^{\infty} \varphi_{\nu}(z, t),$$

with

$$(2.30) \quad \varphi_0(z, t) = \int_0^t g(z, s) ds,$$

$$(2.31) \quad \varphi_{\nu}(z, t) = - \int_0^t \sum_{k=1}^n b^k(z, s) \frac{\partial \varphi_{\nu-1}}{\partial z^k}(z, s) ds, \quad \text{for } \nu > 0,$$

We shall prove that the series (2.29) converges in E_{s_0} provided that s_0 is small enough (and that the same is true of the t -derivatives of the series). We are going to use the notation

$$(2.32) \quad B(x, t) = \int_0^t \{|\mathbf{b}(x, s)| + Mr(x)\} ds,$$

where $M > 0$ is chosen so as to have

$$(2.33) \quad |\mathbf{b}(z, t)| \leq \beta(x, t) + Mr(x)$$

if $|z - x| \leq \kappa_0 r(x)$ (cf. (2.21)).

We shall prove by induction on $\nu = 0, 1, \dots$, that for suitable constants $C_0, C > 0$, and all $s, 0 < s < 1$,

$$(2.34) \quad \|\varphi_{\nu}(\cdot, t)\|_s \leq C_0 \left(\frac{CB(x_0, t)}{1-s} \right)^{\nu}, \quad |t| < T.$$

For $\nu = 0$, (2.34) holds trivially. By Cauchy's inequalities and (2.33), we have for $\nu > 0$,

$$\begin{aligned}
 (2.35) \quad & \|\varphi_\nu(\cdot, t)\|_s \leq \\
 & \leq \frac{1}{\varepsilon \kappa_0 r(x_0)} \left| \int_0^t [\beta(x_0, t) + Mr(x_0)] \|\varphi_{\nu-1}(\cdot, t')\|_{s+\varepsilon} dt' \right| \leq \\
 & \leq C_0 \frac{C^{\nu-1}}{\kappa_0 r(x_0)} \frac{1}{\varepsilon} \frac{1}{(1-s-\varepsilon)^{\nu-1}} \left| \int_0^t B(x_0, t')^{\nu-1} \frac{\partial B}{\partial t}(x_0, t') dt' \right| \leq \\
 & \leq C_0 \frac{C^{\nu-1}}{\kappa_0 r(x_0)} \frac{1}{\nu \varepsilon (1-s-\varepsilon)^{\nu-1}} B(x_0, t)^\nu.
 \end{aligned}$$

We take $\varepsilon = (1-s)/(\nu+1)$. Then

$$(2.36) \quad \|\varphi_\nu(\cdot, t)\|_s \leq C_0 \left(\frac{CB(x_0, t)}{1-s} \right)^\nu \frac{1}{C \kappa_0 r(x_0)} \left(\frac{\nu+1}{\nu} \right)^\nu,$$

and we take

$$(2.37) \quad C \geq \frac{e}{\kappa_0 r(x_0)}.$$

This proves (2.34).

We note now that

$$B(x, t) \leq \sqrt{|t|} \left(\int_0^t |\mathbf{b}(x, s)|^2 ds \right)^{\frac{1}{2}} + Mr(x)|t| \leq (1 + M\sqrt{T})r(x)\sqrt{|t|},$$

hence, by (2.34) and (2.37),

$$(2.38) \quad \|\varphi_\nu(\cdot, t)\|_s \leq C_0 \left[\frac{e(1 + M\sqrt{T})}{(1-s)\kappa_0} \right]^\nu |t|^{\nu/2}.$$

Finally we take $s \leq \frac{1}{2}$, and

$$(2.39) \quad |t| \leq T' = \inf \left(T, \left[\frac{\kappa_0}{2e(1 + M\sqrt{T})} \right]^2 \right),$$

and conclude that the series (2.29) converges in $C_0([-T', T']; E_s)$. From this and from the eq. (2.24) we derive easily that

$$\varphi(z, t) \in C^\infty([-T', T']; E_s).$$

We apply what precedes to the case where $g(z, t) = \beta(z, t)/ir(z)$. By virtue of Lemma 2.1 (if $T < \frac{1}{2}$) we know that $\beta(z, t)$ is holomorphic in the set (2.19), and depends smoothly on (z, t) ($|t| < T$). By Lemma 2.2 and Remark 2.3 we know that $r(z)$ is holomorphic and nowhere zero in (2.19). We may state:

LEMMA 2.3. *Suppose that $\varkappa > 0$ and $0 < T' < T$ are small enough. Then, in the region*

$$(2.40) \quad x \in U_0 \setminus \mathcal{N}_0, \quad |y| < \varkappa r(x), \quad |t| < T',$$

the Cauchy problem

$$(2.41) \quad \frac{\partial \varphi}{\partial t} + i \sum_{k=1}^n b^k(z, t) \frac{\partial \varphi}{\partial z^k} = \frac{1}{i} \frac{\beta(z, t)}{r(z)},$$

$$(2.42) \quad \varphi|_{t=0} = 0,$$

has a unique solution which is C^∞ with respect to (x, y, t) and holomorphic with respect to $z = x + iy$. It satisfies in (2.40), for $y = 0$,

$$(2.43) \quad r(x) \left| \sum_{k=1}^n v^k(x) \frac{\partial \varphi}{\partial x^k}(x, t) \right| \leq C_1 |t|^{\frac{1}{2}},$$

and if t' is any other point in the interval $] - T', T' [$,

$$(2.44) \quad \left| \varphi(x, t) - \varphi(x, t') - \frac{1}{ir(x)} \int_{t'}^t |\mathbf{b}(x, s)| ds \right| \leq C_1 \sup_{s \in [t, t']} (|s|^{\frac{1}{2}}) \left| \int_{t'}^t \frac{|\mathbf{b}(x, s)|}{r(x)} ds \right|.$$

We recall that $\beta(x, t) = |\mathbf{b}(x, t)|$ and $\mathbf{v}(x, t) = \mathbf{b}(x, t)/|\mathbf{b}(x, t)|$ (see (2.10)).

PROOF. The existence has already been proved; the uniqueness is standard. We shall prove (2.43) and (2.44).

If we apply (2.30) and the estimates (2.38) we see that

$$\left| \varphi(z, t) - \frac{1}{i} \int_0^t \frac{\beta(z, s)}{r(z)} ds \right| \leq C' |t|^{\frac{1}{2}}.$$

If $|z - x_0| < \varkappa_0 r(x_0)$ and if \varkappa_0 is small enough, we have

$$|\beta(z, s)| \leq \beta(x_0, s) \leq M \varkappa r(x_0),$$

$$|r(z)| \geq \frac{1}{2} r(x_0),$$

whence:

$$|\varphi(z, t)| \leq 2 \left| \int_0^t \frac{\beta(x_0, s)}{r(x_0)} ds \right| + 2M^{\frac{1}{2}}|t| + C'|t|^{\frac{1}{2}}.$$

But $\left| \int_0^t \beta(x_0, s) ds \right| \leq \sqrt{|t|}r(x_0)$, and therefore:

$$(2.45) \quad |\varphi(z, t)| \leq C''|t|^{\frac{1}{2}}.$$

By Cauchy's inequalities:

$$\left| \sum_{k=1}^n v^k(x) \frac{\partial \varphi}{\partial x^k}(x, t) \right| \leq \frac{1}{\kappa_0 r(x)} \sup_{|z-x| \leq \kappa_0 r(x)} |\varphi(z, t)|,$$

and if κ_0 is small enough, we derive (2.43) from (2.45).

Let us denote by $w(x, t)$ the left-hand side in (2.43). By integration of (2.41) with respect to t , between t' and t , we derive:

$$\left| \varphi(x, t) - \varphi(x, t') + i \int_{t'}^t \frac{\beta(x, s)}{r(x)} ds \right| \leq \int_{t'}^t \frac{\beta(x, s)}{r(x)} w(x, s) ds.$$

Since, by (2.43), we have $w(x, s) \leq C_1|s|^{\frac{1}{2}}$, we get indeed (2.44). Q.E.D.

APPENDIX TO SECTION 2

Estimates for later use.

LEMMA 2.4. *To every $\alpha \in \mathbb{Z}_+^n$ there is a constant $C_\alpha > 0$ such that*

$$(2.46) \quad |\partial_x^\alpha \mathbf{v}(x)| \leq C_\alpha r(x)^{-|\alpha|}, \quad \forall x \in U_0 \setminus \mathcal{N}_0.$$

PROOF. Let $x_0 \in U_0 \setminus \mathcal{N}_0$ be arbitrary. By the mean value theorem there is t_0 , $|t_0| < T$, such that $|\mathbf{b}(x_0, t_0)| = (1/\sqrt{2T})r(x_0) (\neq 0)$.

For x sufficiently near x_0 we may write $\mathbf{v}(x) = \mathbf{b}(x, t_0)/|\mathbf{b}(x, t_0)|$. The numerator, and the square of the denominator are C^∞ functions in an open neighborhood of \bar{U}_0 . Consequently, there is a constant $C'_\alpha > 0$, depending only on the derivatives of these functions (and not on x_0), such that, for x near x_0 ,

$$(2.47) \quad |\partial^\alpha \mathbf{v}(x)| \leq C'_\alpha |\mathbf{b}(x, t_0)|^{-|\alpha|}.$$

Making $x = x_0$ in (2.47) yields (2.46) with $C_\alpha = C'_\alpha (2T)^{|\alpha|/2}$. Q.E.D.

LEMMA 2.5. *To every $\alpha \in \mathbb{Z}_+^n$ there is a constant $\tilde{C}_\alpha > 0$ such that*

$$(2.48) \quad |\partial_x^\alpha r(x)| \leq \tilde{C}_\alpha r(x)^{1-|\alpha|}, \quad \forall x \in U_0 \setminus \mathcal{N}_0.$$

In particular, r is uniformly Lipschitz continuous in U_0 .

Indeed, $r^2 \in C^\infty(\bar{U}_0)$.

COROLLARY 2.1. *Given any $\alpha \in \mathbb{Z}_+^n$ there is a constant $\tilde{\tilde{C}}_\alpha > 0$ such that*

$$(2.49) \quad |\partial_x^\alpha [r(x)v(x)]| \leq \tilde{\tilde{C}}_\alpha r(x)^{1-|\alpha|}, \quad \forall x \in U_0 \setminus \mathcal{N}_0.$$

In particular, the function equal to $r(x)v(x)$ in $U_0 \setminus \mathcal{N}_0$ and to zero in \mathcal{N}_0 is uniformly Lipschitz continuous in U_0 .

In the statements below φ stands for the solution of (2.41)-(2.42).

LEMMA 2.6. *Given any $\alpha \in \mathbb{Z}_+^n$, there is a constant $C'_\alpha > 0$ such that*

$$(2.50) \quad |\partial_x^\alpha \{X\varphi(x, t)\}| \leq C'_\alpha |t|^{\frac{1}{2}} r(x)^{-\alpha-1}, \quad \forall x \in U_0 \setminus \mathcal{N}_0, |t| < T.$$

PROOF. By Cauchy's inequalities and (2.45) we have (provided that \varkappa_0 is small enough):

$$(2.51) \quad \frac{1}{\alpha!} |\partial_x^\alpha \varphi(x, t)| \leq [\varkappa_0 r(x)]^{-|\alpha|} \sup_{|z-x| \leq \varkappa_0 r(x)} |\varphi(z, t)| \leq C'' \varkappa_0^{-|\alpha|} |t|^{\frac{1}{2}} r(x)^{-|\alpha|}.$$

Combining (2.46) and (2.51) yields (2.50).

COROLLARY 2.2. — *$\forall \alpha \in \mathbb{Z}_+^n$, there is a constant $\tilde{C}'_\alpha > 0$ such that*

$$(2.52) \quad |\partial_x^\alpha \{r(x)X\varphi(x, t)\}| \leq \tilde{C}'_\alpha |t|^{\frac{1}{2}} r(x)^{-|\alpha|}, \quad \forall x \in U_0 \setminus \mathcal{N}_0, |t| < T.$$

Indeed, combine Lemmas 2.5 and 2.6.

COROLLARY 2.3. *Whatever $f \in \mathcal{D}_{r\text{-flat}}(U)$, we also have $(rX\varphi)f \in \mathcal{D}_{r\text{-flat}}(U)$.*

3. — Solution on the individual leaves.

In $U_0 \setminus \mathcal{N}_0$ the vector field X defined in (1.7), Ch. I,

$$(3.1) \quad X = \sum_{j=1}^n v^j(x) \frac{\partial}{\partial x^j}$$

is analytic and nowhere zero (see proof of Lemma 2.1). For any $x_0 \in U_0 \setminus \mathcal{N}_0$ we denote by Γ_{x_0} the *connected* integral curve of X through x_0 , by Σ_{x_0} the

two-dimensional « leaf » $\Gamma_{x_0} \times]-T, T[$. Writing $\mathbf{v} = (v^1, \dots, v^n)$ as before, let us denote by $x(\chi, x_0)$ the solution of the problem

$$(3.2) \quad \frac{dx}{d\chi} = r(x)\mathbf{v}(x), \quad x|_{\chi=0} = x_0.$$

The function r is defined in (1.2). The mapping

$$(3.3) \quad \chi \mapsto x(\chi, x_0)$$

is a C^∞ mapping and, in fact, a local diffeomorphism of an open interval $J(x_0)$ of \mathbb{R}^1 onto Γ_{x_0} . Conversely we may regard χ as a smooth function in a sufficiently small open arc in Γ_{x_0} , centered at x_0 , specifically the solution of

$$(3.4) \quad r(x)X\chi = 1, \quad \chi|_{x=x_0} = 0.$$

Let us recall the standard relations, valid if $x, x_1 \in \Gamma_{x_0}$:

$$(3.5) \quad \chi(x_0, x_1) = -\chi(x_1, x_0),$$

$$(3.6) \quad \chi(x, x_1) = \chi(x, x_0) + \chi(x_0, x_1).$$

It is perhaps worth distinguishing the various possibilities:

(I) Γ_{x_0} is a compact subset of $U_0 \setminus \mathcal{N}_0$. This means that $x(\chi, x_0)$ is periodic with respect to χ ; then $J(x_0) = \mathbb{R}^1$ and (3.3) is a covering map;

(II) Γ_{x_0} is noncompact in $U_0 \setminus \mathcal{N}_0$; then (3.3) is a global diffeomorphism of $J(x_0)$ onto Γ_{x_0} .

In Case (II), $J(x_0)$ might have one finite boundary point in \mathbb{R}^1 , two of them or none. But:

LEMMA 3.1. *If $J(x_0)$ has a finite boundary point χ_0 , then as $\chi \rightarrow \chi_0$, $x(\chi, x_0)$ goes to the boundary of U_0 (i.e., exits from every compact subset of U_0).*

PROOF. By Cor. 2.1 we know that $|r(x)\mathbf{v}(x) - r(x')\mathbf{v}(x')| \leq K|x - x'|$ for some $K < +\infty$ and all $x, x' \in U_0$. By the Picard iteration method applied to (3.2) we obtain

$$(3.7) \quad |x_1 - x_0| \leq r(x_1) (\exp [K|\chi(x_0, x_1)|] - 1)$$

(we have tacitly exploited (3.5)). Suppose then that $\chi(x_0, x_1) \rightarrow \chi_0 \in \partial J(x_0)$, $|\chi_0| < +\infty$; clearly we cannot have $r(x_1) \rightarrow 0$. On the other hand, since $r(x)X$ does not vanish in $U_0 \setminus \mathcal{N}_0$, x_1 must go to the boundary of $U_0 \setminus \mathcal{N}_0$, whence the conclusion.

An equivalent interpretation of Lemma 3.1 is that taking χ as the parameter on an integral curve Γ_{x_0} of X amounts to changing the metric along Γ_{x_0} in such a way that the distance (for the new metric) from x_0 to x tends to $+\infty$ whenever x tends to \mathcal{N}_0 . Actually the reasons for using $r(x)$ are even subtler than this, for using $\int_{-T}^T |\mathbf{b}(x, s)| ds$ instead of $r(x) = \left(\int_{-T}^T |\mathbf{b}(x, s)|^2 ds \right)^{\frac{1}{2}}$ would also have had the preceding implication, but would not have had the consequence, repeatedly needed (cf., e.g., (2.43) and its proof), that

$$(3.8) \quad \frac{1}{r(x)} \int_{-T}^T |\mathbf{b}(x, s)| ds$$

converges to zero with T .

Let then φ be the phase-function of Lemma 2.3. We set

$$z(x, t) = \chi(x, x_0) + \varphi(x, t), \quad x \in \Gamma_{x_0}, \quad |t| < T.$$

(We should rather write $z(x, x_0, t)$ for indeed z depends on x_0). Let us also write

$$\omega(x, t) = |\mathbf{b}(x, t)|/r(x),$$

whence

$$(3.9) \quad L = \frac{\partial}{\partial t} + i\omega(x, t)r(x)X.$$

If we take (2.41) and (3.4) into account we see that

$$(3.10) \quad Lz = 0,$$

and by (2.42), that

$$(3.11) \quad z|_{t=0} = \chi(x, x_0).$$

We consider then the mapping

$$(3.12) \quad (\chi, t) \mapsto z = \chi + \varphi(x(\chi, x_0), t).$$

It is a C^∞ mapping of the « slab »

$$(3.13) \quad S(x_0) = J(x_0) \times]-T, T[$$

into \mathbb{R}^2 . The following result provides, in a sense, another interpretation of the local solvability condition (P):

LEMMA 3.2. *If $T > 0$ is small enough, the mapping (3.12) is a homeomorphism of $S(x_0)$ onto an open subset of \mathbb{R}^2 . Its Jacobian determinant is given by*

$$(3.14) \quad \frac{D(\operatorname{Re} z, \operatorname{Im} z)}{D(\chi, t)} = -\omega |1 + rX\varphi|^2.$$

PROOF. By (3.10) we have $z_t = -i|\mathbf{b}|Xz$, hence $\bar{z}_t = i|\mathbf{b}|X\bar{z}$, and therefore

$$(3.15) \quad (\operatorname{Re} z)_t = |\mathbf{b}|X(\operatorname{Im} z), \quad (\operatorname{Im} z)_t = -|\mathbf{b}|X(\operatorname{Re} z).$$

By (3.4) we have $r(x)X = \partial/\partial\chi$, hence

$$(3.16) \quad (\operatorname{Re} z)_t = \omega \frac{\partial}{\partial\chi} (\operatorname{Im} z), \quad (\operatorname{Im} z)_t = -\omega \frac{\partial}{\partial\chi} (\operatorname{Re} z),$$

and therefore

$$(3.17) \quad \frac{D(\operatorname{Re} z, \operatorname{Im} z)}{D(\chi, t)} = -\omega \left| \frac{\partial z}{\partial\chi} \right|^2.$$

By (3.8), $\partial z/\partial\chi = 1 + rX\varphi$, whence (3.14).

In order to prove that (3.12) is a homeomorphism, it suffices to prove that it is injective. Suppose that

$$(3.18) \quad \chi + \varphi(x(\chi, x_0), t) = \chi' + \varphi(x(\chi', x_0), t'),$$

that is:

$$(3.19) \quad (\chi - \chi') + [\varphi(x(\chi, x_0), t) - \varphi(x(\chi', x_0), t)] + \\ + [\varphi(x(\chi', x_0), t) - \varphi(x(\chi', x_0), t')] = 0.$$

By (3.4) and (2.43) we have:

$$(3.20) \quad |\varphi(x(\chi, x_0), t) - \varphi(x(\chi', x_0), t)| \leq \left| \int_{\chi'}^{\chi} (rX\varphi)(x(s, x_0), t) ds \right| \leq C_1 |t|^{\frac{1}{2}} |\chi - \chi'|.$$

Concerning the third term in the left-hand side of (3.19) we apply (2.44) with $x = x(\chi', x_0)$, obtaining

$$\left| \varphi(x, t) - \varphi(x, t') - \frac{1}{i} \int_{t'}^t \omega(x, s) ds \right| \leq C_1 \sqrt{T} \left| \int_{t'}^t \omega(x, s) ds \right|,$$

which, combined with (3.19) and (3.20), yields

$$(3.21) \quad \left| (\chi - \chi') + i \int_{t'}^t \omega(x, s) ds \right| \leq C_1 \sqrt{T} \left\{ |\chi - \chi'| + \left| \int_{t'}^t \omega(x, s) ds \right| \right\}.$$

Therefore, if $C_1 \sqrt{T} < \frac{1}{2}$, we must have:

$$(3.22) \quad \chi = \chi', \quad \int_{t'}^t \omega(x, s) ds = 0.$$

But (by definition of $U_0 \setminus \mathcal{N}_0$ and by the analyticity of $\mathbf{b}(x, t)$ with respect to t) $\omega(x, s) > 0$ in any compact subinterval of $] - T, T[$, except possibly at a finite number of points. Therefore the second equation in (3.22) implies $t = t'$. **Q.E.D.**

We look now at the effect of the mapping (3.12) on the operator L . Since $L = (Lz) \partial_z + (L\bar{z}) \partial_{\bar{z}}$, hence, by (3.10),

$$(3.23) \quad L = (L\bar{z}) \partial_{\bar{z}}.$$

But (cf. (3.8)-(3.16)), $L\bar{z} = -2i\omega X\bar{z}$, hence

$$(3.24) \quad L = -2i\omega(1 + rX\bar{\varphi}) \partial_{\bar{z}}.$$

Consider now an arbitrary function $f \in \mathcal{D}_{r\text{-flat}}(U)$. If $x_0 \in U_0 \setminus \mathcal{N}_0$, by restriction f defines a smooth function $f^{\natural}(x, t; x_0)$ on the leaf Σ_x . Via the mapping (3.3), $\chi \mapsto x(\chi, x_0)$, $f^{\natural}(x, t; x_0)$ defines a function $f^{\natural\natural}(\chi, t; x_0)$ in the slab (3.13), $S(x_0) = J(x_0) \times] - T, T[$. We observe that $f^{\natural\natural} \in C^\infty(S(x_0))$ and we list a number of properties of $f^{\natural\natural}$:

(3.25) *the projection of the support of $f^{\natural\natural}$ on the t -axis is contained in a compact subset of $] - T, T[$;*

$$(3.26) \quad \sup_{(\chi, t) \in S(x_0)} |f^{\natural\natural}(\chi, t; x_0)| < +\infty;$$

(3.27) *if $J(x_0)$ has a finite boundary point χ_0 , $f^{\natural\natural} \equiv 0$ in a full neighborhood of $\{\chi_0\} \times] - T, T[$.*

Suppose now that x_1 is another point on the orbit Γ_{x_0} of x_0 : In view of (3.7) we have:

$$(3.28) \quad f^{\natural\natural}(\chi, t; x_1) = f^{\natural\natural}(\chi + \chi(x_0, x_1), t; x_0).$$

Let now $O(x_0)$ be the open subset of \mathbf{C} which is the image of the slab $S(x_0)$ (in (3.13)) under the mapping (3.12). By Lemma 3.1 we may define a *continuous bounded* function in $O(x_0)$,

$$(3.29) \quad \tilde{f}(z; x_0) = f^{\natural\natural}(\chi, t; x_0)$$

where $(\chi, t) \in S(x_0)$ corresponds to z by the inverse homeomorphism of (3.12).

We come now to the solution, in the leaf Σ_{x_0} , of the equation

$$(3.30) \quad Lu = f \in \mathcal{D}_{r\text{-flat}}(U).$$

Let us set $u(x, t) = v(x, t) + \int_{-T}^t f(x, s) ds$. The function v must be a solution of

$$(3.31) \quad Lv = -i \sum_{j=1}^n b^j(x, t) \int_{-T}^t \frac{\partial f}{\partial x^j}(x, s) ds.$$

We observe that the right-hand side in (3.31) belongs to $\mathcal{E}_{r\text{-flat}}(U)$ (it might fail to have a compact support with respect to t) but also that it vanishes wherever $|\mathbf{b}(x, t)|$ does. After multiplication by a cut-off function of t , we may assume that, in (3.31), the right-hand side is equal to

$$(3.32) \quad |\mathbf{b}|Xg, \quad g \in \mathcal{D}_{r\text{-flat}}(U).$$

We switch now to the coordinates $(\operatorname{Re} z, \operatorname{Im} z)$ on the leaf Σ_{x_0} . By virtue of (3.24) we must now solve

$$(3.33) \quad \partial_{\bar{z}} \tilde{v} = \frac{i}{2} [(1 + rX\bar{\varphi})^{-1}(rXg)]^{\sim} \quad \text{in } O(x_0).$$

It follows at once from Cor. 2.3 that the right-hand side in (3.33) is of the form $\tilde{F}(z; x_0)$ for an obvious $F \in \mathcal{D}_{r\text{-flat}}(U)$. We shall therefore write the integral expression of a solution \tilde{v} of (3.33), that is, of

$$(3.34) \quad \partial_{\bar{z}} \tilde{v} = \tilde{F} \quad \text{in } O(x_0).$$

In order to handle in a « uniform » manner all possible kinds of right-hand sides \tilde{F} , periodic, almost-periodic (at one end or at both), fastly decaying at infinity (also at one end or at both), etc., corresponding to the various kinds of orbits Γ_{x_0} (cf. Fig. 1), we shall use a special fundamental solution

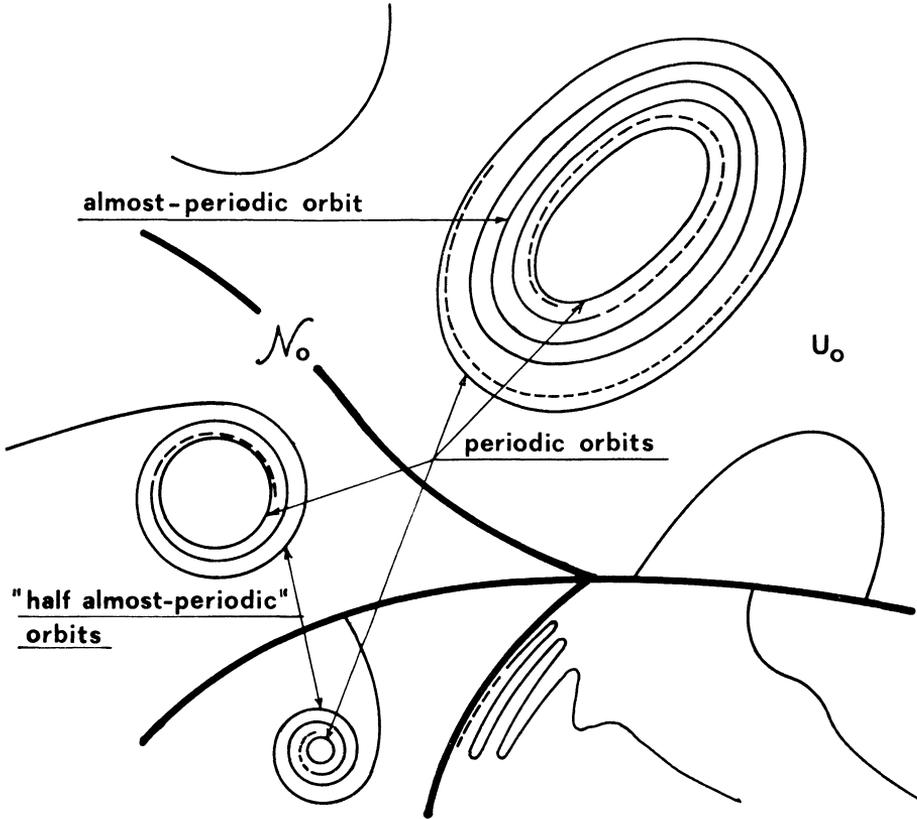


Fig. 1. – Examples of orbits of the vector field X in two space dimension.

of $\partial/\partial\bar{z}$. This will be possible thanks to Property (3.25) and to the fact that the open set $O(x_0)$ is contained in a slab $|\text{Im } z| \leq \text{const. } T$. We observe that, if $h(z)$ is any entire function such that $h(0) = 1$, we have

$$(3.35) \quad \frac{\partial}{\partial\bar{z}} \left(\frac{h(z)}{\pi z} \right) = \delta \quad (\text{the Dirac measure in } \mathbf{R}^2),$$

and we choose

$$h(z) = \exp[-z^2],$$

solving then (3.34) by

$$(3.36) \quad \tilde{v}(z; x_0) = \frac{1}{\pi} \iint \frac{\exp[-(z-z')^2]}{z-z'} \tilde{F}(z'; x_0) d(\operatorname{Re} z') d(\operatorname{Im} z').$$

1) *Convergence of the integral in (3.36).*

We take a closer look at the open set $O(x_0)$, image of $S(x_0) = J(x_0) \times]-\infty, T[$ under the map $(\chi, t) \mapsto z = \chi + \varphi(x(\chi, x_0), t)$. From (2.44) (where we take $t' = 0$) we derive

$$(3.37) \quad \left| \varphi(x, t) + i \int_0^t \omega(x, s) ds \right| \leq C_1 \sqrt{|t|} \left| \int_0^t \omega(x, s) ds \right|.$$

Note also that, whatever $x \in U_0 \setminus \mathcal{N}_0$ and $t, 0 < |t| < T$,

$$(3.38) \quad 0 < \left| \int_0^t \omega(x, s) ds \right| \leq \sqrt{|t|} \left(\int_{-T}^T \omega(x, s)^2 ds \right)^{\frac{1}{2}} = \sqrt{|t|}.$$

We reach easily the following conclusions:

$$(3.39) \quad O(x_0) \text{ is contained in the slab } \{z; |\operatorname{Im} z| \leq \sqrt{T} + C_1 T\};$$

$$(3.40) \quad \text{If } J(x_0) \subset]\chi_0, +\infty[, O(x_0) \text{ is contained in a half-space } \operatorname{Re} z > \tilde{\chi}_0 > -\infty; \text{ if } J(x_0) \subset]-\infty, \chi_0[, O(x_0) \subset \{z; \operatorname{Re} z < \tilde{\chi}_0 < +\infty\}.$$

$$(3.41) \quad \text{If } J(x_0) = \mathbb{R}^1, O(x_0) \text{ is an open neighborhood of the real axis in the } z\text{-plane.}$$

Suppose that z tends to the boundary of $O(x_0)$; this means (by Lemma 3.2) that (χ, t) tends to the boundary of $S(x_0)$. If then $F \in \mathcal{D}_{r\text{-flat}}(U)$ and $|t| \rightarrow T$, $F(x(\chi, t), t)$ will eventually be zero. If $|t|$ remains $\leq T' < T$ then χ must tend either to $\pm\infty$ or to a finite boundary point χ_0 of $J(x_0)$. By (3.27) we see that, also in the latter case, $F(x(\chi, t), t)$ will eventually be zero. This is not necessarily so when $\chi_0 \rightarrow \pm\infty$ for then $x(\chi, x_0)$ might very well remain in a compact subset of $U_0 \setminus \mathcal{N}_0$. But at any rate, if we denote by $\partial O(x_0)$ the boundary of $O(x_0)$ in the complex plane, that is, excluding the points at infinity, we see that

$$(3.42) \quad \text{whatever } F \in \mathcal{D}_{r\text{-flat}}(U), \tilde{F}(z, t) \text{ vanishes in a neighborhood of } \partial O(x_0).$$

We see that \tilde{F} can then be extended as a C^∞ function in \mathbb{R}^2 , which is bounded. Since $\exp[-z^2]/\pi z$ is integrable in every slab $|\operatorname{Im} z| < \text{const.}$ the convolu-

tion at the right in (3.36) defines a C^∞ function in \mathbb{R}^2 , which is bounded in every horizontal slab.

2) *Formula (3.36) defines a function on the leaf Σ_{x_0} .*

In the double integral at the right, in (3.36), we switch from the coordinates $(\operatorname{Re} z, \operatorname{Im} z)$ to (χ, t) . We set $\chi = \chi(x, x_0)$, $\chi' = \chi(x', x_0)$. The Jacobian determinant is given by (3.14) and we have $F = -(1/2i) \cdot (1 + rX\bar{\varphi})^{-1}(rXg)$; the integral under consideration can be rewritten as

$$(3.43) \quad \frac{-1}{2\pi i} \iint \frac{\exp[-(z-z')^2]}{z-z'} (rXg)^{\natural\natural}(\chi', t'; x_0) \cdot [1 + (rXg)(x(\chi', x_0), t')] d\chi' dt',$$

where

$$(3.44) \quad z = \chi + \varphi(x(\chi, x_0), t), \quad z' = \chi' + \varphi(x(\chi', x_0), t').$$

The integration in (3.43) is performed over $S(x_0)$ or, which amounts to the same, over \mathbb{R}^2 (since $(rXg)^{\natural\natural}$ vanishes identically in a neighborhood of $\partial S(x_0)$). Let us denote by $v^{\natural\natural}(\chi, t; x_0)$ the transfer of $\tilde{v}(z; x_0)$ under the homeomorphism (3.12); it is equal to (3.43). We must check that $v^{\natural\natural}(\chi, t; x_0)$ is indeed the image, under the map (3.3), of a function $v^{\natural}(x, t; x_0)$ on the leaf Σ_{x_0} .

For simplicity let us set

$$(3.45) \quad E(z) = \frac{1}{2\pi i} \frac{\exp[-z^2]}{z}, \quad x = x(\chi, x_0), \quad x' = x(\chi', x_0).$$

We have:

$$(3.46) \quad v^{\natural\natural}(\chi, t; x_0) = -\iint E[\chi - \chi' + \varphi(x, t) - \varphi(x', t')](rXg)(x', t')[1 + r(x')X\varphi(x', t')] d\chi' dt'.$$

We apply (3.5)-(3.6):

$$(3.47) \quad \chi - \chi' = \chi(x, x').$$

We may also write

$$(3.48) \quad d\chi' = -d[\chi(x, x')],$$

regarding x as fixed (or as a parameter). Since $\Sigma_{x_0} = \Sigma_x$ in what precedes, we may define the following function of (x, t) , $x \in U_0 \setminus \mathcal{N}_0$, $|t| < T$:

$$(3.49) \quad v(x, t) = \int_{\Sigma_x} \int E[\chi(x, x') + \varphi(x, t) - \varphi(x', t')](rXg)(x', t') \cdot [1 + r(x')X\varphi(x', t')] d\chi(x, x') dt'.$$

Then $v^{\natural}(\chi, t; x_0)$ is the image under (3.3) of the restriction $v^{\natural}(x, t; x_0)$ of $v(x, t)$ to Σ_{x_0} .

In the next section we prove that, after setting $v(x, t) = 0$ when $x \in \mathcal{N}_0$, we have $v \in \mathcal{E}_{r\text{-flat}}(U)$. By taking then, in (3.49),

$$g(x, t) = -i\zeta(t) \int_{-T}^t f(x, s) ds$$

[$\zeta \in C^\infty(\mathbb{R}^1)$, $\zeta(t) = 1$ if $t \leq T' < T$, $\zeta(t) = 0$ if $t > \frac{1}{2}(T + T')$], we obtain the solution $u = v + g$ of (3.30) in $U' = U_0 \times]-T', T'[$, with the required properties ($f \mapsto u$ defines a linear operator $\mathcal{D}_{r\text{-flat}}(U') \rightarrow \mathcal{E}_{r\text{-flat}}(U')$, which can be given an explicit integral representation; the estimates in Sect. 4 easily imply the continuity of this operator).

4. - Smoothness and flatness of the solution.

We begin by simplifying the expression (3.49). We take $\chi = \chi(x, x')$ as one of the two integration variables at the right (the other remains t'), and regard x' as a function of (χ, x) , defined as the solution of

$$(4.1) \quad \frac{dx'}{d\chi} = -r(x')\mathbf{v}(x'), \quad x'|_{\chi=0} = x.$$

It is convenient to let (χ, t') vary in the whole plane, although x' is not defined for all χ but only for those belonging to $-J(x)$. But we have seen (Lemma 3.1) that if $J(x) \neq \mathbb{R}^1$ and if χ_0 is a boundary point of $J(x)$, then as $\chi \rightarrow -\chi_0$ (while x remains fixed), x' will exit from any compact subset of U_0 , in particular from the support of $g(\cdot, t)$. And by the standard results on ODES, as long as x' remains away from the boundary of U_0 , it is a C^∞ function of (χ, x) (x remains in $U_0 \setminus \mathcal{N}_0$). Let us set

$$(4.2) \quad \mathcal{W} = \mathcal{W}(x; \chi, t) = [1 + r(x')X\varphi(x', t)]r(x')Xg(x', t);$$

also

$$(4.3) \quad Z = Z(x, t; \chi, t') = \chi + \varphi(x, t) - \varphi(x', t').$$

With this notation (3.49) reads

$$(4.4) \quad v(x, t) = \iint E[Z(x, t; \chi, t')] \mathcal{W}(x; \chi, t') d\chi dt'.$$

The integration in (4.4) is performed over \mathbb{R}^2 ; (x, t) ranges over $(U_0 \setminus \mathcal{N}_0) \times]-T, T[$.

We avail ourselves of the following facts:

- (4.5) \mathcal{W} is a C^∞ function of (x, χ, t) in $(U_0 \setminus \mathcal{N}_0) \times \mathbb{R}^1 \times]-T, T[$;
- (4.6) \mathcal{W} is bounded;
- (4.7) the projection on the t -axis of $\text{supp } \mathcal{W}$ is contained in a compact sub-interval of $]-T, T[$.

On the other hand, we derive from (2.43)

$$(4.8) \quad |\varphi(x, t') - \varphi(x', t')| \leq C_0' |t|^{\frac{1}{2}} \chi(x, x'),$$

where $x \in U_0 \setminus \mathcal{N}_0$, $x' \in \Gamma_x$, $|t'| < T$; we may rewrite (2.44) as

$$(4.9) \quad \left| \varphi(x, t) - \varphi(x, t') + i \int_{t'}^t \omega(x, s) ds \right| \leq C_1 \sqrt{|T|} \left| \int_{t'}^t \omega(x, s) ds \right|$$

(here $|t| < T$), and by combining (4.8) and (4.9) we see that

$$(4.10) \quad \left| Z - \left\{ \chi - i \int_{t'}^t \omega(x, s) ds \right\} \right| \leq C \sqrt{|T|} \left| \chi - i \int_{t'}^t \omega(x, s) ds \right|.$$

From all this it follows easily that v , given by (4.4), is a *continuous* function of (x, t) in $(U_0 \setminus \mathcal{N}_0) \times]-T, T[$ (provided that $C \sqrt{|T|}$, in (4.10), be < 1 , which we may assume). We are now going to prove that it is a C^∞ function of (x, t) in that set and, at the end, that it is r -flat.

We shall need the following:

LEMMA 4.1. *To every $\alpha \in \mathbb{Z}_+^n$, $k \in \mathbb{Z}_+$, there are constants $C_{\alpha,k}$, $M_{\alpha,k} > 0$ and an integer $d_{\alpha,k} \in \mathbb{Z}$ such that*

$$(4.11) \quad |\partial_x^\alpha \partial_{\chi'}^k x'| \leq C_{\alpha,k} r(x')^{-d_{\alpha,k}} \exp(M_{\alpha,k} |\chi|)$$

for all $x \in U_0 \setminus \mathcal{N}_0$, $\chi \in -J(x)$.

PROOF. The case $\alpha = 0$, $k = 0$ follows at once from (3.7), since

$$(4.12) \quad |x' - x| \leq r(x') (\exp[K|\chi|] - 1).$$

Let us differentiate (4.1) with respect to χ and to x . If we set $\theta_{\alpha,k} = \partial_x^\alpha \partial_\chi^k x'$, we obtain

$$(4.13) \quad \partial_\chi \theta_{\alpha,k} = [\partial_x(r\mathbf{v})](x')\theta_{\alpha,k} + \sum_{|\beta| \leq |\alpha|+k} [\partial_x^\beta(r\mathbf{v})](x') \mathfrak{F}_{\alpha,\beta,k}(\theta),$$

where $\mathfrak{F}_{\alpha,\beta,k}(\theta)$ denotes a *polynomial* with respect to the variables $\theta_{\gamma,l}$ with $\gamma_j \leq \alpha_j$ ($j = 1, \dots, n$), $l \leq k$, and $|\gamma| + l < |\alpha| + k$. We apply Cor. 2.1, obtaining

$$(4.14) \quad |\partial_\chi \theta_{\alpha,k}| \leq C|\theta_{\alpha,k}| + \sum_{|\beta| \leq |\alpha|+k} \tilde{C} r(x')^{1-|\beta|} |\mathfrak{F}_{\alpha,\beta,k}(\theta)|.$$

We shall reason by induction on $|\alpha|$ and on k . First assume $k = 0$. Then to (4.14) we may adjoin the initial conditions

$$(4.15) \quad \theta_{\alpha,0}|_{\chi=0} = \partial_x^\alpha x.$$

Thus, if $|\alpha| = 1$ (in which case $\mathfrak{F}_{\alpha,\beta,0}(\theta) \equiv 0$), we obtain at once

$$(4.16) \quad |\theta_{\alpha,0}| \leq \exp[C|\chi|].$$

For $|\alpha| > 1$, the right-hand side in (4.15) is zero. By induction on $|\alpha|$ we derive from (4.14)

$$(4.17) \quad |\partial_\chi \theta_{\alpha,0}| \leq C|\theta_{\alpha,0}| + C'_\alpha r(x')^{-d_\alpha} \exp[M_\alpha|\chi|].$$

We apply then Gronwall's inequality and obtain

$$(4.18) \quad |\theta_{\alpha,0}| \leq C_{\alpha,0} r(x')^{-d_{\alpha,0}} \exp[M_{\alpha,0}|\chi|].$$

It suffices now to observe that the left-hand side in (4.14) can be rewritten as $|\theta_{\alpha,k+1}|$ and to reason by induction on k . Q.E.D.

We shall now differentiate v (given by (4.4)) with respect to (x, t) . But since

$$(4.19) \quad \partial_t v - i|\mathbf{b}(x, t)|Xv = |\mathbf{b}(x, t)|Xg$$

(see (3.31), (3.32)), it suffices to consider the derivatives of v with respect to x . We have

$$(4.20) \quad \partial_x^\alpha v = \sum_{\alpha' \leq \alpha} \binom{\alpha}{\alpha'} \iint \partial_\alpha^\alpha [E(\mathbf{Z})] \partial_x^{\alpha-\alpha'} \mathfrak{W}(x; \mathbf{Z}, t) d\chi dt,$$

where $\alpha' \leq \alpha$ means $\alpha'_j \leq \alpha_j$ for all $j = 1, \dots, n$. First, if $|\alpha'| \geq 1$,

$$(4.21) \quad \partial_x^{\alpha'} E(Z) = \sum_{j=1}^{|\alpha'|} E^{(j)}(Z) \mathfrak{F}_j(\partial_x^\beta Z),$$

where $\mathfrak{F}_j(\partial_x^\beta Z)$ is a polynomial with respect to the derivatives of Z with respect to x , of order β , $0 < |\beta| \leq |\alpha'|$. Since $Z = \chi + \varphi(x, t) - \varphi(x', t')$, we have

$$(4.22) \quad \partial_x = [1 - \partial_x \varphi(x', t')] \partial_Z,$$

and, by (4.1),

$$\partial_x \varphi(x', t') = \varphi_x(x', t') \frac{\partial x'}{\partial \chi} = -r(x') \sum v^j(x') \frac{\partial \varphi}{\partial x^j}(x', t').$$

We apply once again (2.43); if $T > 0$ is small enough,

$$(4.23) \quad |\partial_x \varphi(x', t')| \leq C_1 \sqrt{T} < 1,$$

and $[1 - \partial_x \varphi(x', t')]$ is invertible. Thus, by iteration,

$$(4.24) \quad E^{(j)}(Z) = \left(\frac{1}{1 - \partial_x \varphi(x', t')} \frac{\partial}{\partial \chi} \right)^j E(Z),$$

and thus, after integration by parts [which is legitimate: see remark following (4.33)], we see that $\partial_x^\alpha v$ is a linear combination of terms of the form

$$(4.25) \quad \int \int E(Z) \left\{ \frac{\partial}{\partial \chi} \left[\frac{-1}{1 - \partial_x \varphi(x', t')} \right] \right\}^j [\mathfrak{F}_j(\partial_x^\beta Z) \partial_x^{\alpha-\alpha'} \mathcal{W}(x; \chi, t')] d\chi dt'.$$

We note that

$$(4.26) \quad \partial_x^\beta Z = \partial_x^\beta \varphi(x, t) - \sum_{\gamma \leq \beta} (\partial_x^\gamma \varphi)(x', t') \mathcal{Q}_\gamma(\partial_x^\lambda x'),$$

where \mathcal{Q}_γ is a polynomial with respect to the derivatives $\partial_x^\lambda x'$ of x' with respect to x of appropriate orders λ . If $k > 0$,

$$(4.27) \quad \partial_x^\beta \partial_x^k Z = - \sum_{|\gamma| \leq |\beta| + k} (\partial_x^\gamma \varphi)(x', t') \mathcal{Q}_{\gamma, k}(\partial_x^\lambda \partial_x^l x').$$

Applying Leibniz formula in (4.25) and taking (4.26)-(4.27) into account

shows that (4.25), hence also $\partial_x^\alpha v$, is a linear combination of terms of the form

$$(4.28) \quad \int \int E(Z) \mathfrak{R}_{\mu,j} [\partial_x^\beta \varphi(x, t), (\partial_x^\gamma \varphi)(x', t'), \partial_x^\lambda \partial_x^l x'] \cdot \\ \cdot \partial_x^\mu \mathcal{W}(x; \chi, t') [1 - \partial_x \varphi(x', t')]^{-2j} d\chi dt';$$

in (4.28), $\mathfrak{R}_{\mu,j}$ is a polynomial with respect to the indicated arguments. Let us rewrite (4.28) as

$$(4.29) \quad v_{\mu,j}(x, t) = \int \int E(Z) w_{\mu,j}(x, t; \chi, t') d\chi dt'.$$

Note that $w_{\mu,j}$ is a C^∞ function of (x, t, χ, t') in the region

$$(4.30) \quad x \in U_0 \setminus \mathcal{N}_0, \quad t, t' \in]-T, T[, \quad \chi \in \mathbb{R}^1$$

(cf. the remarks following (4.1)). The t' -projection of $\text{supp } w_{\mu,j}$ is contained in a compact subinterval of $]-T, T[$. Furthermore, as inspection of (4.28) shows, by virtue of (2.51) and (4.11), for suitable constants $C'_{\mu,j}$, $M'_{\mu,j} > 0$ and an integer $d'_{\mu,j} \geq 0$,

$$(4.31) \quad |w_{\mu,j}(x, t; \chi, t')| \leq \\ \leq C'_{\mu,j} \left[1 + \frac{1}{r(x)} + \frac{1}{r(x')} \right]^{d'_{\mu,j}} \exp [M'_{\mu,j} |\chi|] |\partial_x^\mu \mathcal{W}(x; \chi, t')|.$$

By (4.2) and the results of the appendix of Sect. 2, we see immediately that W is $r(x')$ -flat. If we combine this fact with Lemma 4.1 we deduce that, for a suitable constant $M_\mu > 0$ (depending only on μ), for any $\nu \in \mathbb{Z}_+$ and a suitable constant $C''_{\mu,\nu} > 0$,

$$(4.32) \quad |\partial_x^\mu \mathcal{W}(x; \chi, t')| \leq C''_{\mu,\nu} r(x')^\nu \exp [M_\mu |\chi|].$$

Combining this with (4.31) yields

$$(4.33) \quad |w_{\mu,j}(x, t; \chi, t')| \leq C''_{\mu,\nu,j} r(x')^\nu \left[1 + \frac{1}{r(x)} \right]^{d'_{\mu,j}} \exp [M''_{\mu,j} |\chi|].$$

In (4.33) $\nu \in \mathbb{Z}_+$ is arbitrary.

The estimate (4.33) allows us to change slightly the integral representation (4.29) of $v_{\mu,j}$. Indeed, by (4.22),

$$\frac{1}{Z} = \left(\frac{1}{1 - \partial_x \varphi(x', t')} \frac{\partial}{\partial \chi} \right)^2 [Z(\log^\pm Z - 1)],$$

where \log^\pm denotes suitable branches of the logarithmic function. Since $w_{\mu,j}$ grows at most exponentially with $|\chi|$, whereas $E(Z)$ behaves like $\exp[-\chi^2]$ at infinity, we may integrate by parts, obtaining

$$(4.34) \quad v_{\mu,j}(x, t) = \iint \exp[-Z^2][Z(\log^\pm Z - 1)]w_{\mu,j}^h(x, t; \chi, t') d\chi dt',$$

where $w_{\mu,j}^h$ is a function analogous to $w_{\mu,j}$ (cf. (4.25) and subsequent reasoning), in particular satisfies an inequality similar to (4.33):

$$(4.35) \quad |w_{\mu,j}^h(x, t; \chi, t')| \leq C_{\mu,v,j}^h r(x')^v \left[1 + \frac{1}{r(x)}\right]^{a_{\mu,j}^h} \exp[M_{\mu,j}^h |\chi|]$$

($v \in \mathbb{Z}_+$ arbitrary).

For any $M > 0$ we denote by $v_{\mu,j}^M(x, t)$ the value of the integral at the right in (4.34), when the integration with respect to χ is restricted to the region

$$(4.36) \quad |\chi| > 2\sqrt{M}(1 + |\log r(x)|)^{\frac{1}{2}}.$$

If we choose M large enough, we see, by (4.10), that

$$(4.37) \quad \operatorname{Re} Z^2 \geq \frac{3}{4}\chi^2 - \text{const.} \geq \frac{1}{2}\chi^2 + M|\log r(x)|$$

in the region (4.36), whatever $x \in U_0 \setminus \mathcal{N}_0$, $|t| < T$. Thus, there,

$$(4.38) \quad |\exp[-Z^2]| \leq r(x)^{\varepsilon M} \exp[-\chi^2/2],$$

where $\varepsilon = +1$ if $r(x) \leq 1$, $\varepsilon = -1$ if $r(x) > 1$.

Suppose first that x remains in a compact subset K of $U_0 \setminus \mathcal{N}_0$. We have, by (4.34), (4.35), (4.38),

$$(4.39) \quad |v_{\mu,j}^M(x, t)| \leq \text{const.} \int_{|\chi| > 2\sqrt{M}} \exp[-\frac{1}{2}\chi^2 + M_{\mu,j}^h |\chi|] \chi^2 d\chi.$$

Of course μ, j are fixed and so is $M_{\mu,j}^h$. We may choose M so large that the right-hand side in (4.39) will not exceed any number $\delta > 0$ given in advance. Thus, in order to study the continuity of $v_{\mu,j}(x, t)$, $x \in K$, $|t| < T$, it suffices to study that of $(v_{\mu,j} - v_{\mu,j}^M)(x, t)$. But the latter continuity is evident, as the integrand and the domain of integration, in the integral expression of $(v_{\mu,j} - v_{\mu,j}^M)(x, t)$, depend continuously on (x, t) , and the integrand is uniformly L^1 with respect to χ, t' .

This proves that v is a C^∞ function in $(U_0 \setminus \mathcal{N}_0) \times]-T, T[$. In order to complete the proof of Th. 1.2 we must prove that v is r -flat.

We suppose now that x is close to \mathcal{N}_0 , in particular that $r(x) < 1$. Using once more (4.35) and (4.38) we get

$$(4.40) \quad |v_{\mu,j}^M(x, t)| \leq \text{const. } r(x)^{M-d_{\mu,j}^h} \int_{|\chi| > 2\sqrt{M}} \exp[-\frac{1}{2}\chi^2 + M_{\mu,j}^h] \chi^2 d\chi,$$

thus:

$$(4.41) \quad |v_{\mu,j}^M(x, t)| \leq C(M, \mu, j) r(x)^{M'} \quad \text{if } M \geq M' + d_{\mu,j}^h.$$

We shall now estimate $v_{\mu,j} - v_{\mu,j}^M$. This difference is equal to the integral at the right in (4.34) but where the integration with respect to χ is restricted to the interval

$$(4.42) \quad |\chi| \leq 2\sqrt{M}(1 + |\log r(x)|)^{\frac{1}{2}}.$$

LEMMA 4.2. *There is a constant $C > 0$ such that, for all $M > 0$, if (4.42) holds, then*

$$(4.43) \quad 1 - \Theta \leq \left| \frac{\log r(x')}{\log r(x)} \right| \leq 1 + \Theta,$$

where $x' = x'(\chi, x)$ and $\Theta = 2\sqrt{M}(1 + |\log r(x)|)^{\frac{1}{2}}/|\log r(x)|$.

PROOF. We have (Lemma 2.5) $|\partial_\chi r| = |rXr| \leq Cr$, hence

$$|\log r(x) - \log r(x')| = \left| \int_0^{\chi} \frac{1}{r} \frac{\partial r}{\partial \chi} d\chi \right| \leq C|\chi|.$$

Combining this with (4.42) yields (4.43). **Q.E.D.**

Keeping M fixed, we may find a neighborhood \mathcal{U}_0 of \mathcal{N}_0 such that if $x \in \mathcal{U}_0 \setminus \mathcal{N}_0$, $r(x)$ is so small that $\Theta < \frac{1}{2}$. But then, from the first inequality (4.43), we draw

$$(4.44) \quad r(x') \leq r(x)^{\frac{1}{2}} \quad \text{if } x' = x'(\chi, x) \text{ and (4.42) holds.}$$

By (4.35) we see that, in the region (4.42), we have

$$(4.45) \quad |w_{\mu,j}^h(x, t; \chi, t')| \leq C_{\mu,j}^h r(x)^{\frac{1}{2} - d_{\mu,j}^h} \exp[M_{\mu,j}^h |\chi|].$$

Reporting this in the integral at the right of (4.34), where the integration in χ is restricted to (4.42), we obtain

$$\begin{aligned}
 (4.46) \quad & |v_{\mu,j}(x, t) - v_{\mu,j}^M(x, t)| \leq \\
 & \leq \text{const. } r(x)^{\frac{1}{2}v - d_{\mu,j}^h} \int_{-\infty}^{+\infty} \exp[-\chi^2 + M_{\mu,j}^h |\chi|] \chi^2 d\chi \\
 & \leq O^h(\mu, j) r(x)^{M'} \quad \text{if } v \geq 2(M' + d_{\mu,j}^h). \quad \text{Q.E.D.}
 \end{aligned}$$

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