CLAUS GERHARDT

Global regularity of the solutions to the capillarity problem


<http://www.numdam.org/item?id=ASNSP_1976_4_3_1_157_0>
Global Regularity of the Solutions
to the Capillarity Problem (*).

CLAUS GERHARDT (**) 

0. – Introduction.

Let \( \Omega \) be a bounded domain of \( \mathbb{R}^n \), \( n \geq 2 \), with smooth boundary \( \partial \Omega \), and let \( A \) be the minimal surface operator

\[
A = - D^i (a_i (p)) , \quad a_i = p^i \cdot (1 + |p|^2)^{-\frac{1}{2}} .
\]

Then, a (regular) solution of the capillarity problem can be looked at as a solution \( u \in C^\alpha (\overline{\Omega}) \) of the following equation

\[
Au + H(x, u) = 0 \quad \text{in} \quad \Omega
\]

subject to the boundary conditions

\[
a_i \cdot \gamma = \beta \quad \text{on} \quad \partial \Omega ,
\]

where \( H \) and \( \beta \) are given functions, and \( \gamma = (\gamma_1, \ldots, \gamma_n) \) is the exterior normal vector to \( \partial \Omega \).

Recently, SPRUCK [12] and URAL'CEVA [15] solved this question partially: In the case \( n = 2 \) Spruck could show the existence of a solution \( u \in C^{2, \alpha} (\overline{\Omega}) \) provided that \( \partial \Omega \) is of class \( C^4 \), \( \beta \) belongs to \( C^{1,\gamma} (\partial \Omega) \) such that \( 0 < \varepsilon < 1 \) and \( |\beta| < 1 \), and provided that \( H \) has the form \( H(x, t) = \kappa \cdot t, \ \kappa > 0 \). Spruck's methods are completely two-dimensional.

(*) During the preparation of this article the author was at the Université de Paris VI as fellow of the Deutsche Forschungsgemeinschaft.

(**) Mathematisches Institut, der Universität Heidelberg.

Pervenuto alla Redazione il 30 Novembre 1974 e in forma definitiva il 18 Luglio 1975.

(1) Here and in the following we sum over repeated indices from 1 to \( n \).
A different approach has been made by Ural’ceva which will be valid for arbitrary dimension. She proved the existence of a solution \( u \in C^{2,\alpha}(\Omega) \) under the assumptions

\[
\partial \Omega \in C^{2,\lambda}, \quad H \in C^{1,\lambda}(\mathbb{R}^n \times \mathbb{R}), \quad |\beta| < 1
\]

where \( H \) satisfies

\[
\frac{\partial H}{\partial t} > \kappa > 0
\]

and where \( \Omega \) is supposed to be convex and \( \beta \) is constant.

The last assumptions are rather restrictive, and it is the aim of this paper to exclude these restrictions by a suitable modification of Ural’ceva’s proof.

In the second part of this article we shall apply this result to the capillarity problem with constant volume—which is an obstacle problem—and we shall show that this problem has a solution \( u \in H^{2,p}(\Omega) \) for any \( p > n \).

Since the paper of Ural’ceva is written in Russian we shall repeat many proofs of that paper almost literally for the convenience of the reader.

1. - A priori estimates for \(|Du|\).

In this section we shall assume that \( u \in C^2(\Omega) \) is a solution to the differential equation (0.2), (0.3). Furthermore, let us suppose that \( \partial \Omega \) is of class \( C^2 \), and that the functions

\[
H \in C^{0,1}(\mathbb{R}^n \times \mathbb{R}) \quad \text{and} \quad \beta \in C^{0,1}(\partial \Omega)
\]

satisfy the conditions

\[
\frac{\partial H}{\partial t} > 0
\]

and

\[
|\beta| < 1 - a, \quad a > 0.
\]

Then, the following theorem is valid.

**Theorem 1.1.** Under the assumptions stated above the modulus of the gradient of \( u \) can be estimated by a constant depending on \(|u|_\Omega\), \(|H(x, u(x))|_\Omega\), \(|(\partial/\partial x)H(x, u(x))|_\Omega\), \(\partial \Omega\), \(n\), \(a\), and on the Lipschitz constant of \( \beta \).
PROOF. First of all, let us extend $u$ and $v$ into the whole domain $\Omega$ such that $u$ belonging to $C^{0,1}(\overline{\Omega})$ still satisfies (1.3), and such that the vector field $v$ is uniformly Lipschitz continuous in $\Omega$ and absolutely bounded by 1. These extensions are possible in view of the smoothness of $\partial \Omega$.

Then, following Ural'ceva's ideas, we are going to prove that the function

\[(1.4) \quad v = (1 + |Du|^2)^{-1} - \beta \cdot D^i u \cdot \gamma^i\]

is uniformly bounded in $\Omega$ by some constant which only depends on the quantities we have just mentioned in Theorem 1.1. Precisely, we shall show that $v$ is bounded locally near $\partial \Omega$. The global estimate then follows from well-known interior gradient bounds.

In order to prove the main result we need some lemmata which will be derived in the following.

We denote by $S$ the graph of $u$

\[(1.5) \quad S = \{ X = (x, x^{n+1}) : x \in \overline{\Omega}, x^{n+1} = u(x) \}\]

and by $\delta = (\delta^1, ..., \delta^{n+1})$ the usual differential operators on $S$, i.e. for $g \in C^1(\overline{\Omega}^{n+1})$ we have

\[(1.6) \quad \delta^i g = D^i g - v_i \cdot \sum_{k=1}^{n+1} v_k \cdot D^k g, \quad i = 1, ..., n + 1,\]

where $v = (v_1, ..., v_{n+1})$ is the exterior normal vector to $S$

\[(1.7) \quad v = \left(1 + |Du|^2\right)^{-\frac{1}{2}} \cdot (-D^1 u, ..., -D^n u, 1).\]

Then the following Sobolev Imbedding Lemma is valid:

**LEMMA 1.1.** For any function $g \in C^1(\overline{\Omega})$ the inequality

\[(1.8) \quad \left( \int_S |g|^{n/(n-1)} d\mathcal{H}_n \right)^{(n-1)/n} < c \cdot \left\{ \int_S (|\delta g| + |g|) d\mathcal{H}_n + \int_{\overline{\Omega}} |g| \left(1 + |Du|^2\right)^{\frac{1}{2}} d\mathcal{H}_n \right\}\]

holds, where $\mathcal{H}_n$ is the $n$-dimensional Hausdorff measure, and where the constant depends on $n$ and $|H(x, u(x))|_\Omega$.

**PROOF OF LEMMA 1.1.** This Sobolev inequality for functions equal to zero on all of $\partial \Omega$ was established in [9] for solutions of (0.2). Here, we shall not assume that $g$ is equal to zero on $\partial \Omega$. 
Denote by \(d, d(x) = \text{dist}(x, \partial \Omega)\), the distance function to \(\partial \Omega\), and let

\[
\eta_k = \min(1, kd)
\]

for \(k \in \mathbb{N}\).

Let \(g \in C^1(\overline{\Omega})\) be given. Then

\[
g_k = g \cdot \eta_k
\]

has boundary values equal to zero, so that in view of the result in [9] inequality (1.8) is valid for \(g_k\).

If \(k\) goes to infinity the integrals

\[
\left( \int_{\mathcal{S}} |g_k|^n/(n-1) d\mathcal{H}^n \right)^{(n-1)/n} \quad \text{and} \quad \int_{\mathcal{S}} |g_k| d\mathcal{H}^n
\]

tend to the respective integrals with \(g_k\) replaced by \(g\), while

\[
\int_{\mathcal{S}} |\delta g_k| d\mathcal{H}^n
\]

is estimated by

\[
\int_{\mathcal{S}} |\delta g| \cdot \eta_k d\mathcal{H}^n + \int_{\Omega} |g| \cdot |D\eta_k| \cdot (1 + |Du|^2)^{1/2} dx.
\]

The last integral converges to

\[
\int_{\partial \Omega} |g| \cdot (1 + |Du|^2)^{1/2} d\mathcal{H}^n_{n-1}
\]

(cf. [5; Appendix III]), hence the result.

Next we need to technical lemmata. Let us denote by \(a_{ij}\)

\[
a_{ij} = \frac{\partial a_i}{\partial p^j}
\]

then we have

**Lemma 1.2.** On the boundary of \(\Omega\) we have the following estimate

\[
|\gamma \cdot a_{ij}(D^iv + D^i(\beta \cdot \gamma_k) \cdot D^k u)| < c_2
\]

where the constant \(c_2\) depends on \(\partial \Omega\) and the Lipschitz constant of \(\beta\).
Proof of Lemma 1.2. Let $x_0$ be an arbitrary boundary point and let us introduce new coordinates $y = y(x)$ which are related to $x$ by an orthogonal transformation such that the $y^n$-axis is directed along the exterior normal vector at $x_0$. Assume, furthermore, that in a neighbourhood of $x_0$ the surface $\partial \Omega$ is specified by

\[(1.17) \quad y^n = \omega(y^1, \ldots, y^{n-1}).\]

If we now differentiate the equation (0.3) with respect to the operator

\[(1.18) \quad (a_k - \beta \cdot \gamma_k) \sum_{s=1}^{n-1} \frac{\partial y^s}{\partial x^k} \frac{\partial}{\partial y^s},\]

then we obtain at $x_0$

\[(1.19) \quad \left| (a_k - \beta \cdot \gamma_k) \sum_{s=1}^{n-1} \frac{\partial y^s}{\partial x^k} (\gamma_i \cdot a_{is} D^i u + a_i \cdot D^i \gamma_{is}) \right| \leq \text{const}.

Moreover, since $\partial \omega/\partial y_i = 0$ et $x_0$ for $s = 1, \ldots, n-1$ we deduce

\[(1.20) \quad \gamma_k = \frac{\partial y^n}{\partial x_k}, \quad k = 1, \ldots, n.\]

Thus, we have in view of (0.3)

\[(1.21) \quad (a_k - \beta \cdot \gamma_k) \frac{\partial y^n}{\partial x_k} = 0,

and hence the relation (1.19) is also true for $s = n$.

On the other hand, combining (1.19) and

\[(1.22) \quad D^i v = (a_k - \beta \cdot \gamma_k) D^i D^k u - D^i (\beta \cdot \gamma_k) \cdot D^k u\]

we derive that the left side of (1.16) is bounded at $x_0$ by

\[(1.23) \quad \text{const} + |(a_k - \beta \cdot \gamma_k) a_i \cdot D^i \gamma_{is}|\]

by which the assertion is proved.

Lemma 1.3. In the whole domain $\Omega$ the following pointwise estimate is valid

\[(1.24) \quad |a_{is} D^k D^i u [a_{is} D^i D^k u - D^i (\beta \cdot \gamma_k)]| \leq c_0 \cdot \left[ |\delta v| \cdot (1 + |D u|^2)^{-1} + 1 \right],\]

where the constant depends on $|D \gamma|_\alpha$, $|D \beta|_\alpha$, and on $a$. 

11 - Annali della Scuola Norm. Sup. di Pisa
Proof of Lemma 1.3. During the proof we have to work in the \((n + 1)\)-dimensional Euclidean space rather than in the \(n\)-dimensional one; therefore we regard a function \(g = g(x^1, \ldots, x^n)\) as being defined in \(\mathbb{R}^{n+1}\) via the mapping \(x \to (x, 0)\). Let us introduce the notation \(\Phi\) for the Euclidean norm in \(\mathbb{R}^{n+1}\)

\[
\Phi(q) = |q|, \quad q \in \mathbb{R}^{n+1}.
\]

We shall consider the Hessian matrix of \(\Phi\), \((\Phi_{ij})_{i,j=1,\ldots,n+1}\), evaluated at \(q_0 = (-Du(x_0), 1)\) where \(x_0\) is an arbitrary but fixed point in \(\Omega\). Let \(z^1, \ldots, z^{n+1}\) be the eigenvectors of that matrix and \(\lambda_1, \ldots, \lambda_{n+1}\) be the corresponding eigenvalues. Evidently, \(q_0/|q_0|\) is itself an eigenvector, which is just the exterior normal vector of the surface \(S\) at the point \((x_0, u(x_0))\).

Assume, that the eigenvectors are numbered in such a way that \(z^{n+1} = q_0/|q_0|\). Then, we have \(\lambda_{n+1} = 0\). Furthermore, the eigenvectors \(z_i, i < n + 1\), are orthogonal to \(z^{n+1}\), and we easily derive

\[
\lambda_i = (1 + |Du|^2)^{-\frac{1}{4}}, \quad i = 1, \ldots, n.
\]

Finally, if we denote by \(\cos(z^i, x^k)\) the scalar product of the corresponding vectors, where the indices run from 1 to \(n + 1\), then we obtain

\[
\cos(z^{n+1}, x^k) = \begin{cases} 
-D^k u (1 + |Du|^2)^{-\frac{1}{4}}, & k = 1, \ldots, n, \\
(1 + |Du|^2)^{-\frac{1}{4}}, & k = n + 1.
\end{cases}
\]

To estimate

\[
A = a_{ij} \cdot D^k D^i u \cdot a_{kl} \cdot D^l D^i u = \Phi_{ij} \cdot D^k D^i u \cdot \Phi_{kl} \cdot D^l D^i u
\]

at \(x_0\), let us observe that we may sum from 1 to \(n + 1\) in this expression since \(u\) does not depend on \(x^{n+1}\). Moreover, as \(A\) is the trace of a product of matrices it is invariant under orthogonal transformations of the coordinate system. Thus, we derive,

\[
A = \lambda_i \lambda_j |D^i_s D^j_s u|^2
\]

having in mind that \(\lambda_{n+1} = 0\).

From (1.28) we conclude in view of (1.26)

\[
A = (1 + |Du|^2)^{-1} \cdot |\delta^2 u|^2,
\]

where

\[
|\delta^2 u| = \left(\sum_{i,j=1}^{n} |D^i_s D^j_s u|^2\right)^{\frac{1}{4}}.
\]
On the other hand, we have

\begin{equation}
(1.30) \quad a_{ij} \cdot D^k D^l u \cdot D^i(\beta \gamma_k) = \Phi_{ij} \cdot D^k D^l u \cdot D^i(\beta \gamma_k) = \sum_{r,s,t=1}^{n+1} \Phi_{ij} \cdot \cos(z^r, x^r) \cdot \cos(z^t, x^t) \cdot \cos(z^k, x^k) \cdot D_z^r D_z^l u \cdot D_z^i(\beta \gamma_k) = \\
= \sum_{r,s=1}^{n} \lambda_s \cdot \cos(z^r, x^k) \cdot D_z^r D_z^l u \cdot D_z^i(\beta \gamma_k) + \\
+ \sum_{s=1}^{n} \lambda_s \cdot \cos(z^{n+1}, x^k) \cdot D_z^{n+1} D_z^l u \cdot D_z^i(\beta \gamma_k),
\end{equation}

where now we also sum over the repeated indices \(i, j, \) and \(k\) from 1 to \(n+1\).

Furthermore, to estimate \(D_z^{n+1} D_z^l u\) for \(s \neq n+1\) we observe that for any \(C^1\)-function \(g\) which does not depend on \(x^{n+1}\) there holds

\begin{equation}
(1.31) \quad D_z^{n+1} g = -D_z^k g \cdot D_z^l u \cdot (1 + |D_u|^2)^{-1}
\end{equation}

and

\begin{equation}
(1.32) \quad |\delta g|^2 = \sum_{s=1}^{n} |D_z^s g|^2 = |D_z g|^2 - |D_z^{n+1} g|^2 > |D_z g|^2 \cdot (1 + |D_u|^2)^{-1},
\end{equation}

hence

\begin{equation}
(1.33) \quad \sum_{s=1}^{n} |D_z^{n+1} D_z^s u|^2 \lesssim (1 + |D_u|^2) \cdot |\delta u|^2.
\end{equation}

Finally, if differentiate \(v\) with respect to \(z^s\) for \(s \neq n+1\) we obtain

\begin{equation}
(1.34) \quad D_z^s v = (a_k - \beta \cdot \gamma_k) \cdot \sum_{l=1}^{n+1} D_z^k D_z^l u \cdot \cos(z^r, x^k) - D_z^k u \cdot D_z^i(\beta \cdot \gamma_k) = \\
\equiv \sum_{l=1}^{n+1} \alpha_l \cdot D_z^k D_z^l u - D_z^k u \cdot D_z^i(\beta \cdot \gamma_k),
\end{equation}

where the \(\alpha_l\)'s satisfy

\begin{equation}
(1.35) \quad |\alpha_l| < 2
\end{equation}

and—in view of (1.27)—

\begin{equation}
(1.36) \quad \alpha_{n+1} = -(a_k - \beta \cdot \gamma_k) \cdot D_z^k u \cdot (1 + |D_u|^2)^{-\frac{1}{2}}.
\end{equation}

Taking the estimate

\begin{equation}
(1.37) \quad (a_k - \beta \cdot \gamma_k) \cdot D_z^k u \approx a \cdot (1 + |D_u|^2)^{\frac{1}{4}} - c_4
\end{equation}
with some suitable constant \( c_4 \) into account, we thus deduce from (1.33) and (1.36)

\[
(1.38) \quad |x_{n+1} \cdot D_x^{n+1} D_z^s u| > a \cdot |D_z^{n+1} D_z^s u| - c_4 \cdot |\delta^2 u|.
\]

Combining the relations (1.34), (1.35), and (1.38) we then conclude

\[
(1.39) \quad |D_z^s D_z^{n+1} u| < a^{-1} \cdot \left[ |D_z^s u| + (2 \cdot n + c_4) \cdot |\delta^2 u| + |Du| \cdot |D(\beta y)| \right].
\]

Hence, there exists a constant \( c_3 \) depending on \( a \), \( |D(\beta y)| \), and known quantities such that in view of (1.29)

\[
(1.40) \quad |a_{ij} D^k D^j u \cdot D^i(\beta y)| < A + c_3 \cdot \left[ |\delta^2 u| \cdot (1 + |Du|^2)^{-\frac{1}{2}} + 1 \right]
\]

from which the assertion (1.24) immediately follows.

As we are treating the case of a non-convex domain and a variable \( \beta \) we need the following estimate

**Lemma 1.4.** For any positive function \( \eta \in H^{1,\infty}(\Omega) \) we have the estimate

\[
(1.41) \quad \int_{\Omega} \int_0^{\eta} d\mathcal{K}_{n-1} \leq c_5 \cdot \left[ |\delta^2 u| + \eta \right] d\mathcal{K}_n,
\]

where the constant depends on \( |\delta^2 u| \) and \( |H(x, u(x))| \).

**Proof of Lemma 1.4.** Let \( W = (1 + |Du|^2)^{\frac{1}{2}} \) and \( \varphi \in H^{1,\infty}(\Omega) \). Then, for \( i = 1, \ldots, n \), we have

\[
(1.42) \quad \int_{\Omega} \delta^i \varphi \, d\mathcal{K}_n = \int_{\Omega} \delta^i \varphi \cdot W \, dx = \int_{\Omega} \{D^i \varphi - a_i \cdot a_k \cdot D^k \varphi \} W \, dx = \\
= \int_{\Omega} \{D^i \varphi \cdot W + a_k \cdot D^k D^i u \cdot \varphi \} \, dx - \int_{\Omega} a_k \cdot D^k (\varphi \cdot D^i u) \, dx = \\
= \int_{\Omega} D^i (\varphi \cdot W) \, dx - \int_{\Omega} a_k \cdot D^k (\varphi \cdot D^i u) \, dx = \\
= \int_{\Omega} \gamma_i \cdot \varphi \cdot W \, d\mathcal{K}_{n-1} - \int_{\Omega} \int a_k \cdot \gamma_k \cdot D^i u \cdot \varphi \, d\mathcal{K}_{n-1} + \int_{\Omega} D^k a_k \cdot \varphi \cdot D^i u \, dx.
\]

Thus, we deduce the identity

\[
(1.43) \quad \int_{\Omega} \delta^i \varphi \, d\mathcal{K}_n = \int_{\Omega} \{\gamma_i \cdot W - a_k \cdot D^i u \cdot \gamma_k \} \cdot \varphi \, d\mathcal{K}_{n-1} - \int_{\Omega} \varphi \cdot H \cdot \gamma_i \, d\mathcal{K}_n.
\]
Inserting $\varphi = \eta \cdot \gamma_i$ in this equality and summing over $i$ from 1 to $n$ yields

$$\int_{\partial \Omega} \delta^i (\eta \cdot \gamma_i) \, d\mathcal{C}_n = \int_{\partial \Omega} \nu \cdot \eta \, d\mathcal{C}_{n-1} - \int_{\partial \Omega} H \cdot \nu \cdot \gamma_i \cdot \eta \, d\mathcal{C}_n,$$

hence the result.

Up to now we have only proved auxiliary propositions which we shall need for estimating certain expressions that will appear in the following calculations. As we mentioned at the beginning we are going to show that $v$, or better,

$$w = \log v$$

is uniformly bounded in $\Omega$. To accomplish this, let us look at the integral identity

$$\int_{\Omega} D^k a_i \cdot \varphi \, dx = -\int_{\partial \Omega} D^k D^i a_i \cdot \varphi \, dx + \int_{\partial \Omega} \gamma_i \cdot D^k a_i \varphi \, d\mathcal{C}_{n-1}.$$

If we choose $\varphi = (a_k - \beta \cdot \gamma_k) \eta$, $0 < \eta \in H^{1,\infty} (\Omega)$ and $\text{supp} \eta \subset \{ w > h \}$, where $h$ is large, then we obtain in view of (0.2) and (1.22)

$$\int_{\partial \Omega} \{ a_{ij} [D^j v + D^j (\beta \gamma_k) \cdot D^k u] D^i \eta + a_{ij} D^k D^j a_k \cdot D^i a_j \cdot D^i u - D^i (\beta \gamma_k) \} \eta \, dx$$

$$+ D^k H \cdot (a_k - \beta \gamma_k) \eta \, dx = \int_{\partial \Omega} \gamma_i \cdot a_{ij} [D^j v + D^j (\beta \gamma_k) \cdot D^k u] \cdot \eta \, d\mathcal{C}_{n-1}.$$

Moreover, observing that

$$D^k H = \frac{\partial H}{\partial x^k} + \frac{\partial H}{\partial t} \cdot D^k u$$

we deduce from (1.47) in view of the Lemmata 1.2 and 1.3, and in view of the assumption (1.3)

$$\int_{\Omega} a_{ij} [D^j v + D^j (\beta \gamma_k) \cdot D^k u] \cdot D^i \eta \, dx \leq c_2 \int_{\partial \Omega} \eta \, d\mathcal{C}_{n-1} + c_6 \int_{\Omega} \left[ \frac{\vert \partial \varphi \vert}{\mathcal{W} + 1} \right] \eta \, dx,$$

where $c_6$ is a suitable constant and $\eta$ any positive $C^1$-function.

We shall use this relation with $\eta = v \cdot \max \{ w^2 - h, 0 \}$, where $h$ is a large positive number and $\zeta$, $0 < \zeta < 1$, a smooth function. Introducing
the notations \( z = \max \{ w \xi^2 - h, 0 \} \), \( A(h, \xi) = \{ X \in S : w(x) \xi^2(x) > h \} \), and \( |A(h, \xi)| = \mathcal{K}_n(A(h, \xi)) \) we then obtain in view of Lemma 1.4

\[
(1.50) \quad \int \left\{ a_{ij} D^j v \cdot D^i v \cdot z + a_{ij} v^2 \cdot D^j w \cdot D^i w \cdot \xi^2 \right\} W^{-1} d\mathcal{K}_n \leq - \int a_{ij} D^j w \cdot v^2 \cdot w \cdot \xi \cdot D^i \xi \cdot W^{-1} d\mathcal{K}_n + c_\xi \cdot c_\sigma \cdot \int \left[ |\delta w| \cdot \xi^2 + 2 w \xi \cdot |\delta \xi| + z \right] d\mathcal{K}_n - \\
- \int a_{ij} D^j (\beta \gamma)^i \cdot D^k u [D^i v \cdot z + v D^i w \xi^2 + 2 v w \xi \cdot D^i \xi] W^{-1} d\mathcal{K}_n + \\
+ c_\sigma \cdot \int \left[ \frac{|\delta v|}{W} + 1 \right] v \cdot z \cdot W^{-1} d\mathcal{K}_n .
\]

Thus, taking the relations

\[
(1.51) \quad a_{ij} D^i g D^j g = W^{-1} \cdot |\delta g|^2 \quad \forall g \in C^1(\Omega), \\
(1.52) \quad |a_{ij} D^i g \cdot D^j \phi| \leq W^{-1} \cdot |\delta g| \cdot |D\phi| \quad \forall \phi \in C^1(\Omega), \\
(1.53) \quad a \cdot W \leq v \leq 2 \cdot W, \\
(1.54) \quad a_{ij} p^i q^j \leq \frac{\varepsilon}{2} \cdot a_{ij} p^i p^j + \frac{1}{2\varepsilon} \cdot a_{ij} q^i q^j ,
\]

and

\[
(1.55) \quad |\delta (w \xi^2)|^2 \leq 8 \cdot \left\{ |\delta w|^2 \xi^2 + w^2 |\delta \xi|^2 \right\}
\]

into account, we derive

\[
(1.56) \quad \int \delta |z|^2 d\mathcal{K}_n = \int \delta (w \xi^2)|^2 d\mathcal{K}_n \leq \left[ c_\gamma + |\delta \xi|^2 \right] \left\{ A(h, \xi) \right\} + \int w^2 d\mathcal{K}_n .
\]

Moreover, from the Lemmata 1.1 and 1.4, and from (1.53) we conclude

\[
(1.57) \quad \left( \int S^n |z|^{n/(n-1)} d\mathcal{K}_{n} \right)^{(n-1)/n} \leq c_1 \cdot \left\{ \int \delta |z| + |z| d\mathcal{K}_n + \\
+ a^{-1} \cdot c_\sigma \cdot \int (|\delta z| + |z|) d\mathcal{K}_n \right\} .
\]

Thus, using the Hölder inequality

\[
(1.58) \quad \int S^n |z| d\mathcal{K}_n \leq |A(h, \xi)|^{1/n} \left( \int S^n |z|^{n/(n-1)} d\mathcal{K}_n \right)^{(n-1)/n}
\]
and choosing \( \text{supp } \zeta \) small enough we deduce

\[
\left( \int_S |z|^{n/(n-1)} \, d\mathcal{H}_n \right)^{(n-1)/n} < c_8 \cdot \int_S |\delta z| \, d\mathcal{H}_n
\]

from which we derive by a well-known argument

\[
\int_S |z|^2 \, d\mathcal{H}_n < c_9 \cdot |A(h, \zeta)|^{2/n} \cdot \int_S |\delta z|^2 \, d\mathcal{H}_n.
\]

Hence, we conclude

\[
\int_S |z| \, d\mathcal{H}_n < [c_{10} + |\delta \zeta|_2] \cdot \left\{ |A(h, \zeta)|^{1+1/n} + |A(h, \zeta)|^{1/2+1/n} \cdot \left( \int_{A(h, \zeta)} w^2 \, d\mathcal{H}_n \right)^{1/2} \right\}.
\]

Now, the boundedness of \( w \cdot \zeta^2 \) follows immediately provided that

\[
\int_{B(h_0)} w^2 \cdot W \, dx
\]

is bounded, where \( B(h_0) = \{ x \in \Omega : v(x) > h_0 \} \) and \( h_0 \) is sufficiently large (cf. [3; p. 195]).

As a first step we prove

**Lemma 1.5.** Suppose that the assumptions of Theorem 1.1 are satisfied. Then we have

\[
\int_{B(h_0)} [W^{-3} \cdot |Dv|^2 + v] \, dx < c_{11}.
\]

**Proof of Lemma 1.5.** We insert \( \eta = \max(v - h_0, 0) \) in the inequality (1.49) and conclude with the help of the relations (1.51)-(1.54)

\[
\int_{B(h_0)} W^{-3} \cdot |Dv|^2 \, dx < \int_{B(h_0)} W^{-1} |\delta v|^2 \, dx < c_{12} \int_{\Omega} W \, dx.
\]

Thus it remains to prove that \( \int_{\Omega} v \, dx \) or equivalently \( \int_{\Omega} W \, dx \) is bounded.

To accomplish this, we consider the identity

\[
\int_{\Omega} a_i \cdot D^i \eta \, dx + \int_{\Omega} H \cdot \eta \, dx - \int_{\partial \Omega} \beta \eta \, d\mathcal{H}_{n-1} = 0, \quad \forall \eta \in C^1(\Omega).
\]
Choosing \( \eta = u \) the result follows from the inequality

\[
\int_{\partial \Omega} |\beta \eta| d\mathcal{H}^{n-1} < (1 - a) \int_{\partial \Omega} |D\eta| \, dx + c_{12} \int_{\partial \Omega} |\eta| \, dx
\]

(cf. [6; Lemma 1]).

After having established the estimate (1.63) we use the relation (1.65) once more, this time with \( \eta = u \cdot \max(w^2 - h, 0) \) and we obtain in view of the preceding inequality

\[
\int_{\{w^2 > h\}} \left\{ a_i \cdot D^i u \cdot (w^2 - h) + u \cdot a_i \cdot D^i v^1 \cdot 2w + H \cdot u \cdot (w^2 - h) \right\} \, dx < (1 - a) \int_{\{w^2 > h\}} \left\{ |Du| \cdot (w^2 - h) + |u| \cdot 2w \cdot |Dv| \cdot v^1 \right\} \, dx + c_{13} \int_{\{w^2 > h\}} |u| \cdot (w^2 - h) \, dx.
\]

Hence, we deduce

\[
\int_{\{w^2 > h\}} |Du| \cdot (w^2 - h) \, dx < c_{14} \int_{\{w^2 > h\}} [w^2 + W^{-3} \cdot |Dv|^2] \, dx,
\]

where we used the inequality

\[
a \cdot b \leq \frac{\varepsilon}{2} \cdot a^2 + \frac{1}{2\varepsilon} \cdot b^2.
\]

To complete the proof of the boundedness of the integral (1.62) we observe that

\[
w^2 \leq \alpha \cdot W
\]

for some suitable constant \( \alpha \).

Thus, we have proved that, given a suitable boundary neighbourhood \( U \), \( |Du| \) is bounded in \( U \). Together with well-known interior gradient estimates (cf. [1, 8, 13]) this completes the proof of Theorem 1.1.

2. – Existence of a solution \( u \).

In view of the \( a \) priori estimates which we have just established the existence of a solution will be proved by a continuity method.

**Theorem 2.1.** Suppose that the boundary of \( \Omega \) is of class \( C^{\alpha,1} \), and that \( H \) and \( \beta \) are \( C^{1,1} \)-functions in their arguments. Furthermore, assume that \( H \)
satisfies

\begin{equation}
\frac{\partial H}{\partial t} > \kappa > 0.
\end{equation}

Then the boundary value problem (0.2), (0.3) has a unique solution \( u \in C^{2,\alpha}(\overline{\Omega}) \), where the exponent \( \alpha, \ 0 < \alpha < 1 \), is determined by the above quantities.

**Proof.** Let \( \tau \) be a real number with \( 0 < \tau < 1 \), and consider the boundary value problems

\begin{align*}
(2.2.\tau) & \quad Au_\tau + \tau \cdot H(x, u_\tau) = 0, \\
(2.3.\tau) & \quad a_i \cdot \gamma_i = \tau \cdot \beta.
\end{align*}

Let \( T \) be the set

\begin{equation}
T = \{ \tau: \text{there exists a solution } u_\tau \in C^0(\overline{\Omega}) \}.
\end{equation}

\( T \) is obviously not empty for \( u_0 = 0 \) belongs to it, and we shall show that it is both open and closed.

In view of the assumption (2.1) we obtain an a priori bound of \( |u_\tau|_\Omega \) for any \( \tau \in T \) independent of \( \tau \) (cf. [2]). Furthermore, let us remark that any solution \( u_\tau \in C^0(\overline{\Omega}) \) is of class \( C^{2,\alpha}(\overline{\Omega}) \) with some fixed \( \alpha, \ 0 < \alpha < 1 \), such that the norm of \( u_\tau \) in \( C^{2,\alpha}(\overline{\Omega}) \) is bounded independently of \( \tau \).

To prove this, we first deduce from Theorem 1.1 that \( |Du_\tau|_\Omega \) is uniformly bounded

\begin{equation}
|Du_\tau|_\Omega \leq K_1.
\end{equation}

Then, we choose a smooth vector field \( \tilde{a}_i \) such that \( \partial \tilde{a}_i / \partial p^j \) is uniformly elliptic, and such that

\begin{equation}
\tilde{a}_i(p) = a_i(p) \quad \text{for } |p| \leq 3 \cdot K_1.
\end{equation}

From [7; Chapter 10, Theorem 2.2] we conclude that the problem

\begin{align*}
(2.7.\tau) & \quad \tilde{A} \tilde{u}_\tau + \tau \cdot H(x, \tilde{u}_\tau) = 0 \quad \text{in } \Omega, \\
(2.8.\tau) & \quad \tilde{a}_i \cdot \gamma_i = \tau \cdot \beta \quad \text{on } \partial \Omega,
\end{align*}

has a solution \( \tilde{u}_\tau \in C^{2,\alpha}(\overline{\Omega}) \) for any \( \tau \). Moreover in view of (2.5) and (2.6) we derive

\begin{equation}
\tilde{A} u_\tau = Au_\tau.
\end{equation}
Hence, we obtain from the uniqueness of the solution

\[ \tilde{u}_\tau = u_\tau. \]

Thus, we can finally conclude that $|u_\tau|_{2,\alpha,\Omega}$ is uniformly bounded

\[ |u_\tau|_{2,\alpha,\Omega} < K_2 \]

where the constant is determined by known quantities.

From the estimate (2.11) it follows immediately that $T$ is closed.

On the other hand, let $\tau_0 \in T$. Then, we consider the boundary value problems (2.7.1), (2.8.1) as before. Since $|D\tilde{u}_\tau|_\Omega$ depends continuously on $\tau$, it turns out that

\[ |D\tilde{u}_\tau|_\Omega < 2 \cdot K_1 \quad \text{for} \quad |\tau - \tau_0| < \varepsilon. \]

But this yields $\tilde{u}_\tau = u_\tau$ for those $\tau$'s. Thus, the proof of Theorem 2.1 is completed.

For our considerations in the next section it will be necessary to bound the norm of $u$ in the function space $H^{2,p}(\Omega), \ p > n$, by a constant which only depends on $|Du|_\Omega, \ |u|_\Omega, \ p, \ n$, the $C^2$-norm of $\partial \Omega$, $|D\beta|_\partial$, and on the $L^p$-norm of $H(x, u(x))$.

**Theorem 2.2.** Under the assumptions of Theorem 2.1 the norm of $u$ in $H^{2,p}(\Omega), \ n < p < \infty$, is bounded by a constant being only determined by the quantities mentioned above.

**Proof.** Since in the interior this result follows from the well-known Calderon-Zygmund–Inequalities, we have only to prove it near the boundary.

Let $\Gamma$ be a part of the boundary and suppose that an open subset $\Omega^*$ of $\Omega$ adjacent to $\Gamma$ is transformed into some open subset $G$ of the half-space $\{ y \in \mathbb{R}^n : y^n > 0 \}$ via a $C^2$-diffeomorphism $y = y(x)$ such that $y(\Gamma) \subset \{ y \in \mathbb{R}^n : y^n = 0 \}$.

In $G$ the equation assumes the form

\[ -D_y^k(a_i) \cdot D^i y^k + H = 0 \]

and the boundary condition (0.3) is transformed into

\[ a_i \cdot D^i y^n = \beta \cdot |D_y y^n| = \beta \quad \text{on} \ y(\Gamma), \]

\[ |D_y y^n| = \left( \sum_{i=1}^n \left| \frac{\partial y^n}{\partial x^i} \right|^2 \right)^{\frac{1}{2}}. \]
We are going to prove that the norm of $\tilde{u}(y) = u(y(x))$ in $H^{2,p}(G)$ can be estimated by the quantities mentioned in the theorem, where we assume that $u$ — and hence $\tilde{u}$ — is of class $C^2$.

It will be sufficient to bound the $L^p$-norm of $D^k_x D^r_y \tilde{u}$ where $k$ ranges from 1 to $n$ and $r$ from 1 to $n - 1$. The estimate for $D^k_x D^r_y \tilde{u}$ then follows from the equation.

We already know from the results of [7; p. 468] that the norm of $\tilde{u}$ in $H^{2,2}(G) \cap C^1(\overline{G})$ with some suitable $\alpha$ can be estimated appropriately.

Let $\xi$ and $\eta$ be arbitrary functions in $C^1(\overline{G})$ vanishing on $\partial G - y(I)$. Then we obtain from (2.13) and (2.14)

$$\int_{\partial} [a_i \cdot D^i_x y^k \cdot D^k_x \eta + a_i \cdot D^i_x y^k \cdot \eta + H \cdot \eta] dy = \int_{\{v^n = 0\}} \tilde{\beta} \cdot \eta dy^n. \quad (2.15)$$

Inserting $\eta = D^r_y \xi, \ r \neq n$, in this identity and integrating by parts yields

$$\int_{\partial} [a_i \cdot D^i_x y^k \cdot D^j_x y^l \cdot D^l_x D^r_y \tilde{u} \cdot D^k_x \xi - H \cdot D^r_y \xi] dy = \int_{\{v^n = 0\}} D^r_y \tilde{\beta} \cdot \xi dy^n. \quad (2.16)$$

Thus, we deduce

$$\int_{\partial} b_{kl} \cdot D^l_x D^r_y \tilde{u} \cdot D^k_x \xi dy < K_3 \cdot \|\xi\|_{1,\alpha, G} \quad (2.17)$$

where we have set

$$b_{kl} = a_{ij} \cdot D^i_x y^k \cdot D^j_x y^l. \quad (2.18)$$

and where $q$ is the conjugate exponent to $p$, and $\xi$ is any function belonging to $H^{1,\alpha}(G)$ vanishing on $\partial G - y(I)$. The constant $K_3$ depends on the quantities mentioned in the theorem.

If $\xi \in H^{1,\alpha}(G)$ is arbitrary, choose $\phi \in C^1(\overline{G}), \ 0 < \phi < 1$, which vanishes on $\partial G - y(I')$. Then the inequality (2.17) is satisfied for $\xi \cdot \phi$, so that we obtain

$$\int_{\partial} b_{kl} \cdot D^l_x (D^r_y \tilde{u} \cdot \phi) \cdot D^k_x \xi dy < K_4 \cdot \|\xi\|_{1,\alpha, G} \quad (2.19)$$

with some constant $K_4$ depending on $K_3, |D\tilde{u}|_{G'}, |D\phi|_{G'}$, and on $|b_{kl}|_G$.

Thus, we finally deduce

$$\int_{\partial} [b_{kl} \cdot D^l_x (D^r_y \tilde{u} \cdot \phi) \cdot D^k_x \xi + D^r_y \tilde{u} \cdot \phi \cdot \xi] dy < K_5 \cdot \|\xi\|_{1,\alpha, G} \quad (2.20).$$
Since the bilinear form

\[(2.21) \quad (u, v) = \int_{\tilde{Q}} [b_{kl} \cdot D^1_y u \cdot D^k_y v + u \cdot v] dy\]

is coercive and non-degenerate, and since the coefficients \(b_{kl}\) are continuous we conclude from [10; Theorem 5.2] that \(D^1_y \tilde{u} \cdot \phi\) belongs to \(H^{1,p}(G)\) and that its norm can be estimated by a constant depending on \(K_\phi\), the ellipticity constant of the \(b_{kl}\)'s, and on the modulus of continuity of the coefficients.

3. - The capillarity problem with constant volume.

In a former paper [4] we considered the variational problem

\[(3.1) \quad J(v) = \int_{\Omega} (1 + |Dv|^2) dx + \int_0^v \int_{\Omega} H(x, t) \, dt \, dx - \int_{\partial \Omega} \beta v d\mathcal{H}_{n-1} \to \min\]

in \(BV(\Omega) \cap \{v > \psi\} \cap \left\{ \int_{\Omega} (v - \psi) \, dx = V \right\} .

We could prove that this problem has a solution \(u \in C^{0,1}(\Omega) \cap L^\infty(\Omega)\), and \(u\) also minimizes the functional

\[(3.2) \quad J_\lambda(v) = J(v) + \lambda \cdot \int_{\Omega} v \, dx\]

in the convex set \(BV(\Omega) \cap \{v > \psi\}\), where \(\lambda\) is a suitable Lagrange multiplier. We had only to assume

\[(3.3) \quad \psi \in C^{0,1}(\bar{\Omega}) , \quad H \in C^{0,1}(\mathbb{R}^n \times \mathbb{R}) ,\]

and

\[(3.4) \quad \frac{\partial H}{\partial t} \geq 0 \quad \text{and} \quad |\beta| < 1 - a \, , \, a > 0 .\]

Here, we shall give sufficient conditions which imply that the variational problem \((3.1)\) has a (unique) solution \(u \in H^{2,p}(\Omega)\) for any finite \(p\). It will be important to remark that \(H\) need not be strictly monotone in \(t\). However, we deduce from [4] that in the following we may consider the variational problem

\[(3.5) \quad J(v) \to \min \text{ in } BV(\Omega) \cap \{v > \psi\} ,\]
where $H$ is supposed to satisfy the inequality (2.1). Moreover, we shall obviously assume that the conditions of Theorem 1.1 are fulfilled, and that $\psi$ belongs to $H^{2,\infty}(\Omega)$. But we still have to impose a further conditions on $\psi$ which ensures, that a solution $u \in H^{2,\infty}(\Omega)$ of (3.5) satisfies the boundary condition (0.3) which is absolutely necessary in order to obtain a priori estimates for the gradient. Therefore, we suppose that the relation

$$(3.6) \quad a_i(D\psi) \cdot \gamma_i < \beta$$

is valid on $\partial \Omega$.

Then, we have the following result

**Theorem 3.1.** Under the above assumptions the variational problem (3.5) has a solution $u \in H^{2,\infty}(\Omega)$ for any finite $p$, which is uniquely determined in that function class.

**Proof.** Let $\theta$ be the maximal monotone graph

$$(3.7) \quad \theta(t) = \begin{cases} -1, & t < 0, \\ [-1, 0], & t = 0, \\ 0, & t > 0, \end{cases}$$

and let $\theta_\varepsilon$ be a sequence of smooth monotone graphs tending to $\theta$ in such a way that

$$(3.8) \quad \theta_\varepsilon(t) = \begin{cases} -1, & t \leq -\varepsilon, \\ 0, & t > 0. \end{cases}$$

Furthermore, let $\mu$ be a positive constant such that

$$(3.9) \quad A\psi + H(x, \psi) < \mu \quad \text{in} \ \Omega.$$ 

Then, we consider the approximating boundary value problems

$$(3.10) \quad Au_\varepsilon + H(x, u_\varepsilon) + \mu \cdot \theta_\varepsilon(u_\varepsilon - \psi) = 0 \quad \text{in} \ \Omega, \quad a_i \gamma_i = \beta \quad \text{on} \ \partial \Omega.$$ 

We shall show that the a priori estimates of Section 1 are still valid in this case, where the estimate depends on $|D\psi|_2$, $\mu$, $|u_\varepsilon|_2$, and on known quantities.

1) The Lemmata 1.1-1.5 are still valid, but the constants might depend on $\mu$. 
2) The only difficulty arises in the estimate (1.49). But, since we apply this estimate with \( \eta = v \cdot \max \{ w \zeta^2 - h, 0 \} \), we deduce that for \( h > h_0 \) the critical term

\[
(3.12) \quad \mu \cdot \int_{\Omega} \theta'_e \cdot D^e(u_e - \psi) \cdot [a_e - \beta \cdot \gamma_h] \cdot \eta \, dx
\]

in (1.47) is positive and can therefore be neglected, where \( h_0 \) depends on \( |D\psi|_2 \) and \( a \).

Hence, we obtain

\[
(3.13) \quad |Du_e|_2 \leq K_2,
\]

where the constant is independent of \( \epsilon \), provided that \( |u_e|_2 \) is uniformly bounded. But this follows from the strict monotonicity of \( H \).

Thus, we deduce from Theorem 2.3 that for any \( p \|u_e\|_{2,p} \) is uniformly bounded, where we may assume without loss of generality that \( \partial \Omega, H \), and \( \beta \) satisfy the further smoothness conditions of Theorem 2.2.

To complete the proof of Theorem 2.1, we shall show that the relation

\[
(3.14) \quad \psi - \epsilon \leq u_e
\]

is valid in \( \Omega \).

But this estimate is an immediate consequence of the assumptions (3.6), (3.8), and (3.9). Indeed, set \( \psi_e = \psi - \epsilon \) and \( \eta = \min (u_e - \psi, 0) \). Then, we deduce from

\[
(3.15) \quad A \psi_e + H(x, \psi_e) + \mu \cdot \theta_e(\psi_e - \psi) = A \psi_e + H(x, \psi_e) - \mu \leq 0,
\]

\[
(3.16) \quad \int_{\Omega} \{ a_i (Du_e) - a_i (D\psi_e) \} D^i \eta + \\
+ \{ H(x, u_e) + \mu \cdot \theta_e (u_e - \psi) - H(x, \psi_e) - \mu \cdot \theta_e (\psi_e - \psi) \} \eta \, dx + \\
+ \int_{\partial \Omega} \{ a_i (D\psi_e) \cdot \gamma_i - \beta \} \eta \, d\mathcal{H}^{n-1} \leq 0.
\]

Hence, we obtain \( \eta = 0 \) in view of the strict monotonicity of \( H \).

In the limit case a subsequence of the \( u_e \)'s converges uniformly to some function \( u \in H^{2,p}(\Omega) \) satisfying

\[
(3.17) \quad u \geq \psi
\]

\[
(3.18) \quad Au + H(x, u) + \mu \cdot \theta(u - \psi) \geq 0 \quad \text{in} \quad \Omega
\]
and

\[(3.19) \quad a_i \cdot \gamma_i = \beta \quad \text{on } \partial \Omega.\]

But this is an equivalent formulation of the variational problem (3.5), if we restrict the variations to the convex set \(H^{2,p}(\Omega) \cap \{v > \varphi\}\), as one easily checks.

After having finished the present article the author became acquainted with a paper of Simon and Spruck [11] who proved similar results.

**REFERENCES**


