D. M. Goldfeld
A. Schinzel

On Siegel’s zero

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4° série, tome 2, n° 4 (1975), p. 571-583

<http://www.numdam.org/item?id=ASNSP_1975_4_2_4_571_0>
On Siegel’s Zero.

D. M. GOLDFELD (*) - A. SCHINZEL (**) 

1. - Let \( d \) be fundamental discriminant, and let

\[
\chi(n) = \left( \frac{d}{n} \right) \quad \text{(Kronecker’s symbol)}.
\]

It is well known (see [1]) that \( L(s, \chi) \) has at most one zero \( \beta \) in the interval \((1 - c_1/\log |d|, 1)\) where \( c_1 \) is an absolute positive constant. The main aim of this paper is to prove:

**Theorem 1.** Let \( d, \chi \) and \( \beta \) have the meaning defined above. Then the following asymptotic relation holds

\[
1 - \beta = \frac{6}{\pi^2} \sum' \frac{L(1, \chi)}{1/a} \left[ 1 + O\left( \frac{(\log \log |d|)^2}{\log |d|} \right) + O((1 - \beta) \log |d|) \right]
\]

where \( \sum' \) is taken over all quadratic forms \((a, b, c)\) of discriminant \( d \) such that

\[
-\frac{a}{4} < b < a < \frac{1}{4} \sqrt{|d|},
\]

and the constants in the \( O \)-symbols are effectively computable.

In order to apply the above theorem we need some information about the size of the sum \( \sum' 1/a \). This is supplied by the following.

**Theorem 2.** If \((a, b, c)\) runs through a class \( C \) of properly equivalent primitive forms of discriminant \( d \), supposed fundamental, then

\[
\sum_{1/4 \sqrt{|d|} > |a| > |b| > -a} \frac{1}{|a|} \left\{ \begin{array}{ll}
\frac{1/m_0}{\log \varepsilon_0/ \log \left( \frac{1}{2} \sqrt{d} - 1 \right)} + \frac{4}{\sqrt{d}} & \text{if } d < 0, \\
\end{array} \right.
\]

\[
\sum_{(a, b, c) \in C} \frac{1}{|a|} \left\{ \begin{array}{ll}
\frac{1/m_0}{\log \varepsilon_0/ \log \left( \frac{1}{2} \sqrt{d} - 1 \right)} + \frac{4}{\sqrt{d}} & \text{if } d > 676,
\end{array} \right.
\]

(*) Scuola Normale Superiore, Pisa.

(**) Institut of Mathematics, Warsaw.

Pervenuto alla Redazione il 30 Maggio 1975.
where \( m_\alpha \) is the least positive integer represented by \( C \) and \( \varepsilon_\alpha \) is the least totally positive unit of the field \( \mathbb{Q}(\sqrt{d}) \).

Theorems 1 and 2 together imply

**Corollary.** For any \( \eta > 0 \) and \( |d| > c(\eta) \) (\( d \) fundamental) we have

\[
1 - \beta \geq \begin{cases} 
\left( \frac{6}{\pi} - \eta \right) \frac{1}{\sqrt{|d|}} & \text{if } d < 0, \\
\left( \frac{6}{\pi^2} - \eta \right) \frac{\log d}{\sqrt{d}} & \text{if } d > 0,
\end{cases}
\]

where \( c(\eta) \) is an effectively computable constant.

**Remark.** In the case \( d < 0 \), the constant \( 6/\pi \) could be improved by using the knowledge of all fields with class number \( \leq 2 \).

Similar inequalities with \( 6/\pi \) and \( 6/\pi^2 \) replaced by unspecified positive constants have been claimed by Hanecke [3], however, as pointed out by Pintz [8], Hanecke’s proof is defective and when corrected gives inequalities weaker by a factor \( \log \log |d| \). Pintz himself has proved the first inequality of the corollary with the constant \( 6/\pi \) replaced by \( 12/\pi \) (see [8]).

For \( d < 0 \), the first named author [2] has obtained (1) with a better error term by an entirely different method. M. Huxley has also found a proof in the case \( d < 0 \) by a more elementary method different, however, from the method of the present paper.

The authors wish to thank Scuola Normale Superiore which gave them the opportunity for this joint work.

2. - The proofs of Theorems 1 and 2 are based on several lemmata.

**Lemma 1.** Let \( f(d) = (\log |d|/\log \log |d|)^2 \). Then

\[
\sum_{Na \leq \frac{1}{4} \sqrt{|d|} f(d)} \frac{1}{N_a} = \frac{\pi^2}{6} \sum_a \frac{1}{a} \left( 1 + O\left( \frac{(\log \log |d|)^2}{\log |d|} \right) \right),
\]

where the left hand sum goes over all ideals \( a \in \mathbb{Q}(\sqrt{d}) \) with norm \( \leq \frac{1}{4} \sqrt{|d|} f(d) \) and the constant in the \( O \)-symbol is effectively computable.

**Proof.** Every ideal \( a \) of \( \mathbb{Q}(\sqrt{d}) \) can be represented in the form

\[
a = u \left[ a, \frac{b + \sqrt{d}}{2} \right]
\]
where $u$, $a$ are positive integers and $b^2 \equiv d \pmod{4a}$ (see [5], Theorem 59). If we impose the condition that

$$-a < b < a$$

then the representation becomes unique. Since $Na = u^2a$, it follows that

$$\sum_{Na \leq \frac{1}{4}|d|f(d)} \frac{1}{Na} = \sum_{1 \leq u^2 \leq \frac{1}{4}|d|f(d)} \frac{1}{u^2} + O\left(\sum_{1 \leq u < \frac{1}{4}|d|f(d)} \frac{1}{u}\right) =$$

$$= \sum_{1 \leq u^2 \leq \frac{1}{4}|d|f(d)} \frac{\pi^2}{6} + O\left((f(d))^{-1}\right) + O(\delta).$$

To estimate the sum $S$, we divide it into two sums $S_1$ and $S_2$. In the sum $S_1$, we gather all the terms $1/a$ such that $a$ has at least one prime power factor

$$p^x > \ell(d) = d^{1/2\log \log |d|},$$

$$p^x|a,$$

and in $S_2$ all the other terms.

Let $r(a)$ be the number of representations of $a$ as $Na$ where $a$ has no rational integer divisor $> 1$. Then $r(a)$ is a multiplicative function satisfying

$$r(p^x) = \begin{cases} 
1 + \left(\frac{d}{p^x}\right) & \text{if } p \nmid d \text{ or } x = 1, \\
0 & \text{otherwise}.
\end{cases}$$

Clearly

$$S_1 < \sum_{p^x \leq \ell(d)} \sum_{p^x \leq \sqrt{|d|/4a}} r(p^x) p^{-x}$$

$$< \sum_{p^x \leq \ell(d)} \sum_{p^x \leq \sqrt{|d|/4a}} 2p^{-x}$$

where $\sum''$ goes over all prime powers $p^x$ with

$$\max\left(\ell(d), \sqrt{|d|/4a}\right) < p^x < \sqrt{|d|f(d)/4a}.$$

Now, by a well known result of Mertens

$$\sum_{p^x < x} p^{-x} = \log \log x + c + O((\log x)^{-1})$$

where $c$ is a constant.
Hence
\[ \sum_{x < p^a < y} p^{-a} = \log \left( \frac{\log y}{\log x} \right) + O((\log x)^{-1}) \leq \frac{\log y}{\log x} - 1 + O((\log x)^{-1}) = \frac{\log y}{x} + O(1). \]

This gives
\[ \sum \frac{\log f(d) + O(1)}{\log l(d)} \leq \frac{(\log \log |d|)^2}{\log d} \]

and we get
\[ S_1 = O \left( \frac{(\log \log |d|)^2}{\log |d|} \right) \sum \frac{1}{a}. \] (4)

To estimate \( S_2 \), we notice that each \( a \) occurring in it must have at least \( k_0 = \frac{\log (\frac{1}{2} \sqrt{|d|})}{\log l(d)} \geq 10 \log \log |d| \) distinct prime factors. Therefore
\[ S_2 \leq \sum_{k \geq k_0} \frac{1}{k!} \left( \sum_{p^a < l(d)} v(p^a) p^{-a} \right)^k \leq \frac{1}{k_0!} \sigma^{k_0} e^\sigma \]

where
\[ \sigma = \sum_{p^a < l(d)} v(p^a) p^{-a} < 2 \log \log l(d) + O(1) = 2 \log \log |d| + O(1). \]

Now, Stirling’s formula gives \( k! > k_0^{k_0} \exp \left( - k_0 \right) \). Hence
\[ \log S_2 \leq - k_0 \log k_0 + k_0 ((\log \sigma) + 1) + \sigma \leq - k_0 \left[ \log 10 + \log \log \log |d| - \log 2 - \log \log \log |d| - 1 \right] + \sigma \leq - 3 \log \log |d| + O(1) \]

and
\[ S_2 = O((\log |d|)^{-3}). \] (5)
The lemma now follows from equations (3), (4) and (5). The next lemma gives the growth conditions for the Riemann zeta-function and Dirichlet L-functions on the imaginary axis.

**Lemma 2.** For all real $t$

\begin{align*}
(6) \quad |\zeta(it)| & \ll (|t|^\frac{1}{2} + 1) \log(|t| + 2) \\
(7) \quad |L(it, \chi)| & \ll \sqrt{|d|}(|t|^\frac{1}{2} + 1) \log \left(|d|(|t| + 2)\right).
\end{align*}

**Proof.** If $|t| > t_0$, the estimate

\[ |\zeta(it)| \ll |t|^\frac{1}{2} \log |t| \]

holds (see [10], p. 19). Since $\zeta(s)$ has no pole on the imaginary axis, we have

\[ |\zeta(it)| \ll 1 \quad \text{for} \quad |t| < t_0 \]

and the inequality (6) now follows.

To prove (7), we note that

\[ |L(1 - it, \chi)| \ll \log(|d|(|t| + 2)) \]

(see [1], p. 17, lemma 2 with $q = |d|$, $x = 2|d|(|t| + 2)$).

Now, by the fundamental equation for $L$-functions

\[ |L(it, \chi)| = |L(1 - it, \chi)||d|^\frac{1}{2} \Gamma(\frac{1}{2} it + A) \Gamma(\frac{1}{2} it + A) \Gamma^{-1}(\frac{1}{2} - \frac{1}{2} it + A) \]

where

\[ A = \frac{1}{8}(1 - \chi(-1)). \]

Using the formula

\[ |\Gamma(s)| = \sqrt{2\pi}|s|^{\sigma - \frac{1}{2}} \exp\left[-\frac{1}{2}\pi t(1 + O(|t|^{-1}))\right] \]

valid for $s = \sigma + it$, $0 < \sigma < \frac{1}{2}$, $|t| > 1$ (see [9], p. 395), equation (7) follows, upon noting that

\[ |\Gamma(\frac{1}{2} it + A) \Gamma^{-1}(\frac{1}{2} - \frac{1}{2} it + A)| \ll 1 \quad \text{for} \quad |t| < 1. \]
PROOF OF THEOREM 1. By the standard argument ([4], p. 31)

\[ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{y^s}{(s+2)(s+3)} \, ds = \begin{cases} \frac{1}{6} - \frac{y^2}{2} + \frac{y^3}{3} & \text{if } y > 1, \\ 0 & \text{if } 0 < y < 1. \end{cases} \]

Since for \( \Re(s) > 1 \)

\[ \zeta(s) L(s, \chi) = \sum (Na)^{-s}, \]

it follows that for any \( x > 0 \)

\[ I = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \zeta(s + \beta) L(s + \beta, \chi) \frac{x^s}{(s+2)(s+3)} \, ds \]

\[ = \sum_{Na \leq x} (Na)^{-\beta} \left( \frac{1}{6} \left( \frac{(Na)^2}{2x^2} + \frac{(Na)^3}{3x^3} \right) \right). \]

Choose \( x = \frac{1}{4} \sqrt{d} f(d) \) with \( f(d) = (\log|d|/\log \log|d|)^2 \).

If \( Na \leq x \), we have

\[ (Na)^{-\beta} = (Na)^{-1} \left( 1 + O((1 - \beta) \log|d|) \right). \]

Hence

\[ I = \frac{1}{6} \sum_{Na \leq x} (Na)^{-1} \left( 1 + O((1 - \beta) \log|d|) \right) \]

\[ + O\left( \sum_{Na \leq x / f(d)} (Na)^{-1} f(d)^{-\beta} \right) + O\left( \sum_{f(d) \leq Na \leq x} (Na)^{-1} \right), \]

and by lemma (1) (cf. formula (3))

\[ I = \frac{1}{6} \sum_{Na \leq x} \frac{1}{a} \left| 1 + O\left( \frac{\log \log|d|}{\log|d|} \right) + O(1 - \beta) \log|d| \right| \]

On the other hand, after shifting the line of integration to \( \Re(s) = -\beta \)

\[ I = \frac{L(1, \chi)x^{1-\beta}}{(1-\beta)(3-\beta)(4-\beta)} + \frac{1}{2\pi i} \int_{-\beta - \infty}^{-\beta + \infty} \zeta(s + \beta) L(s + \beta, \chi) \frac{x^s}{(s+2)(s+3)} \, ds. \]

By lemma (2), the integral on the right does not exceed

\[ O(x^{-\beta} \sqrt{|d|} \log|d|) \]
and since
\[ x^{1-\beta} = 1 + O((1 - \beta) \log |d|) \]
\[ (1 - \beta)(3 - \beta)(4 - \beta) = 6 + O(1 - \beta) \]
we get from (8) and (9)
\[ \frac{6}{\pi^2} L(1, \chi) \sum \frac{1}{\alpha} 1 + O \left( \frac{(\log \log |d|)^2}{\log |d|} \right) + O((1 - \beta) \log |d|) \].

3. – Proof of Theorem 2. For \( d < 0 \) it is enough to prove that every class contains at most one form satisfying

(10) \[-|a| < b < |a| < \frac{1}{2} \sqrt{|d|} . \]

Now, since
\[ |d| = 4ac - b^2 \]
we infer from (10) that
\[ a < \sqrt{|d|} < |d| / 4a \cdot c , \]
thus every form satisfying (10) is reduced, and it is well known that every class contains at most one such form.

For \( d > 0 \), let us choose in the class \( C \) a form \((\alpha, \beta, \gamma) \) reduced in the sense of Gauss, i.e. such that

(11) \[ \beta + \sqrt{d} > 2|\alpha| > -\beta + \sqrt{d} > 0 . \]

We can assume without loss of generality that \( \alpha > 0 \). Now, for any form \( f \in C \), there exists a properly unimodular transformation
\[ T = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \]

taking \((\alpha, \beta, \gamma) \) into \( f \). The first column of this transformation can be made to consist of positive rational integers by Theorem 79 of [5]. If \( f \) satisfies (10), we infer from

(12) \[ \alpha p^2 + \beta pq + \gamma q^2 = a \]

\((\ast) \) \( \beta \) is not to be confused with Siegel’s zero.
that
\[
\left| p + \frac{\beta - \sqrt{\bar{d}}}{2x} q \right| = a \left| \alpha p + \frac{\beta + \sqrt{\bar{d}}}{2} q \right|^{-1} \leq \frac{1}{4} \sqrt{\bar{d}} \cdot 2(\sqrt{\bar{d}} q)^{-1} = \frac{1}{2} q^{-1}
\]

and by lemma (16), p. 175 from [5], \(p/q\) is a convergent of the continued fraction expansion for
\[
\omega = -\frac{\beta + \sqrt{\bar{d}}}{2x}.
\]

From this point onwards, we shall use the notation of Perron’s monograph [7]. Since by (11)
\[
\omega^{-1} > 1 \quad \text{and} \quad O > (\omega')^{-1} > -1,
\]
\(\omega^{-1}\) is a reduced quadratic surd and it has a pure periodic expansion into a continued fraction. Hence
\[
\omega = [0, \overline{b_1, b_2 \ldots b_k}]
\]
where the bar denotes the primitive period. The corresponding complete quotients form again a periodic sequence
\[
\omega_v = \frac{P_v + \sqrt{\bar{d}}}{Q_v}, \quad \omega_0 = \omega
\]
where for all \(v > 1\), \(\omega_v\) is reduced,

(13) \(\omega_v = \omega_{v+k}\),

and \(k\) is the least number with the said property.

**Lemma 3.** Let \([0, \overline{b_1, b_2, \ldots, b_k}]\) be the continued fraction for \(\omega\) defined above. Then
\[
\sum_{(a,b,c) \in \mathcal{C}} \frac{1}{|a|} \leq \frac{2}{\sqrt{\bar{d}}} \left( \sum_{v=2}^{[b_k]} \min\left( \frac{\sqrt{\bar{d}}}{2}, b_v + 1 \right) \right)
\]
where the sum on the left is taken over all \((a, b, c)\) in the class \(\mathcal{C}\) satisfying (10).
PROOF. If \( A_j/B_j \) is the \( j \)-th convergent of \( \omega \), we have by formula (18), § 20 of [7]

\[
(A_{v-1}Q_v - B_{v-1}P_v)^2 - d(B_{v-1})^2 = (-1)^v Q_v Q_v
\]

which gives on simplification

\[
(14) \quad \alpha A_{v-1}^2 + \beta A_{v-1} B_{v-1} + \gamma B_{v-1}^2 = (-1)^v Q_v/2. \tag{14}
\]

Similarly, eliminating \( Q_v \) from formulae (16) and (17) in § 20 of [7], we get

\[
(15) \quad 2\alpha A_{v-1} A_{v-2} + \beta (A_{v-1} B_{v-2} + B_{v-2} A_{v-1}) + 2\gamma B_{v-1} B_{v-2} = (-1)^{v-1} P_v. \tag{15}
\]

Let \( p = A_{v-1}, \ q = B_{v-1} \ (v \geq 1) \). By (12)

\[ a = (-1)^v Q_v/2. \]

Hence, by formula (1) of § 6 of [7]

\[
\begin{vmatrix}
A_{v-1} & A_{v-2} \\
B_{v-1} & B_{v-2}
\end{vmatrix} = (-1)^v.
\]

and since

\[
\begin{vmatrix}
A_{v-1} & r \\
B_{v-1} & s
\end{vmatrix} = 1
\]

it follows that

\[
T = \begin{pmatrix}
A_{v-1} & A_{v-2} \\
B_{v-1} & B_{v-2}
\end{pmatrix} \begin{pmatrix}
1 & t \\
0 & (-1)^v
\end{pmatrix}, \quad t \in \mathbb{Z}.
\]

Thus we find using (14) and (15)

\[
f = (\alpha, \beta, \gamma) \begin{pmatrix}
A_{v-1} & A_{v-2} \\
B_{v-1} & B_{v-2}
\end{pmatrix} \begin{pmatrix}
1 & t \\
0 & (-1)^v
\end{pmatrix} =
\]

\[
= \begin{pmatrix}
(-1)^v Q_v/2, \ (-1)^{v-1} P_v, \ (-1)^v Q_{v-1}/2
\end{pmatrix} \begin{pmatrix}
1 & t \\
0 & (-1)^v
\end{pmatrix}.
\]

In order to make \( f \) satisfy (10) we must choose

\[
t = (-1)^v \left[ \frac{P_v}{Q_v} + \frac{1}{2} \right].
\]
Thus $f$ is uniquely determined by $\omega_v$ and in view of (13), we have

\begin{equation}
\sum_{(a,b,c)\in C} \frac{1}{|a|} < \sum_{\substack{v=1

\text{Since } \omega_v \text{ is reduced, we have further for } v \text{ in question}
\sqrt{d} \frac{2\sqrt{d}}{Q_v} > \frac{P_v + \sqrt{d}}{Q_v} > \frac{\sqrt{d}}{Q_v} \geq 2 .

Hence for

\begin{equation}
b_v = [\omega_v],
\end{equation}

we get the inequalities

\begin{equation}
\sqrt{d} > b_v > 2, \quad b_v + 1 > \sqrt{d}/Q_v,
\end{equation}

and by (16), lemma (3) follows.

Now, let $\varepsilon_0$ be the least totally positive unit $\varepsilon_0 > 1$ of the ring $\mathbb{Z}(\sigma)$ where

\begin{equation}
\sigma = \begin{cases} 
\frac{1}{2} \sqrt{d} & \text{if } d \equiv 0 \pmod{4}, \\
\frac{1 + \sqrt{d}}{2} & \text{if } d \equiv 1 \pmod{4}.
\end{cases}
\end{equation}

By Theorem (7) of Chapter IV of [6]

\begin{equation}
\varepsilon_0 = \frac{u + v \sqrt{d}}{2},
\end{equation}

where for $l = [k, 2],
\begin{equation}
v = (q_{l-1}, p_{l-1} - q_{l-2}, p_{l-1}) , \quad u = p_{l-1} + q_{l-2}
\end{equation}

and $p_j, q_j$ are the numerator and denominator, respectively, of the $j$-th convergent for $\omega^{-1}$. Moreover, since $\omega^{-1}$ satisfies the equation

\begin{equation}
-\gamma \omega^2 - \beta \omega^{-1} - \alpha = 0, \quad (-\gamma > 0)
\end{equation}

we find from formula (1) of § 2 of Chapter IV of [6] that

\begin{equation}
q_{l-2} - p_{l-1} = -\beta v, \quad -p_{l-2} = -\alpha v.
\end{equation}

Hence

\begin{equation}
\varepsilon_0 = \frac{p_{l-1} + q_{l-2}}{2} + \frac{p_{l-2} \sqrt{d}}{2} = q_{l-2} + \frac{\beta + \sqrt{d}}{2} \cdot p_{l-2}.
\end{equation}
Since \( p_i = B_{i+1}, \ q_i = A_{i+1} \), we get

\[
\varepsilon_0 = B_{i-1} \left( \frac{A_{i-1}}{B_{i-1}} + \frac{\beta + \sqrt{d}}{2\alpha} \right) > B_{i-1} \left( \omega + \frac{\beta + \sqrt{d}}{2\alpha} \right) = \frac{\sqrt{d}}{\alpha} B_{i-1}.
\]

Now,

\[
\omega_i = b_i + \omega_{i+1} = b_i + \omega_i^{-1} = b_i + \omega, \quad \omega_1 = b_i + \omega'
\]

and since \( \omega \) is reduced \( 0 > b_i + \omega' > -1 \)

\[
b_i = \left\lfloor -\omega' \right\rfloor = \left\lfloor \frac{\beta + \sqrt{d}}{2\alpha} \right\rfloor < \frac{\sqrt{d}}{\alpha}.
\]

Thus (17) gives

\[
\varepsilon_0 > b_i B_{i-1} \prod_{v=1}^{i} b_v,
\]

and by (16)

\[
\sum_{(a,b,c) \in \mathcal{E}} \frac{1}{|a|} \leq \frac{2}{\sqrt{d}} \max \sum (x_i + 1) = \frac{2}{\sqrt{d}} M
\]

where maximum is taken over all non-decreasing sequence of at most \( l \) numbers satisfying

\[
2 < x_i < \frac{1}{2} \sqrt{d} - 1 = D, \quad \prod x_i < \varepsilon_0.
\]

Let \((x_1, x_2, \ldots, x_m)\) be a point in which the maximum is taken with the least number \( m \). We assert that the sequence contains at most one term \( x \) with \( 2 < x < D \). Indeed, if we had \( 2 < x_i < x_{i+1} < D \), we could replace the numbers \( x_i, x_{i+1} \) by

\[
\frac{x_i}{\min (x_i/2, D/x_{i+1})}, \quad x_{i+1} \min \left( \frac{x_i}{2}, \frac{D}{x_{i+1}} \right)
\]

and the sum \( \sum (x_i + 1) \) would increase. Also, if we had \( x_1 = x_2 = x_3 = 2 \), we could replace them by \( x_1 = 8 \), and the sum \( \sum (x_i + 1) \) would remain the same while \( m \) would decrease.

Let

\[
\frac{\varepsilon_0}{4} = D^* \theta, \quad \text{where } e = \log \left( \frac{\varepsilon_0/4}{\log D} \right).
\]

Using \( d > 676 \), we get

\[
M = \begin{cases} 
\frac{1}{2} e \sqrt{d} + \max (4\theta + 1, 2\theta + 4) & \text{if } 4\theta < D, \\
\frac{1}{2} e \sqrt{d} + 2\theta + 4 & \text{if } 2\theta < D < 4\theta, \\
\frac{1}{2} e \sqrt{d} + \theta + 7 & \text{if } D < 2\theta.
\end{cases}
\]
Now, \[
    e = \frac{\log \epsilon_0}{\log D} - \frac{\log 4\theta}{\log D}.
\]

Since for \(1 < y, \ y(\log x/\log y) > x - 1\), and for \(d > 676, \ D/\log D > 12/\log 12 > 4.8\), we obtain if \(4\theta < D\).

\[
    M - \frac{1}{2} \sqrt{d} \frac{\log \epsilon_0}{\log D} = \max (4\theta + 1, 2\theta + 4) - D \frac{\log 4\theta}{\log D} = \frac{\log 4\theta}{\log D} < \\
    < \max (4\theta + 1, 2\theta + 4) - \max (4\theta - 1, 6) < 2,
\]

if \(2\theta < D < 4\theta\)

\[
    M - \frac{1}{2} \sqrt{d} \frac{\log \epsilon_0}{\log D} = 2\theta + 4 - D \frac{\log 2\theta}{\log D} - D \frac{\log 2}{\log D} - \frac{\log 4\theta}{\log D} < \\
    < 2\theta + 4 - 2\theta + 1 - 3 - 1 = 1,
\]

if \(D < 2\theta\)

\[
    M - \frac{1}{2} \sqrt{d} \frac{\log \epsilon_0}{\log D} = \theta + 7 - D \frac{\log \theta}{\log D} - D \frac{\log 4}{\log D} - \frac{\log 4\theta}{\log D} < \\
    < \theta + 7 - \theta + 1 - 6 - 1 = 1.
\]

This together with (18) gives the theorem.

4. - Proof of Corollary. We can assume \(1 - \beta < (\log |d|)^{-2}\). It then by Theorem (1) that for every \(\eta > 0\), there exists \(c(\eta)\) such that if \(d > c(\eta)\)

\[
    1 - \beta > \frac{6}{\pi^2} \sum^{\prime} 1/a \left(1 - \eta \frac{1}{2}\right).
\]

Let \(h_0\) be the number of classes of forms in question. For \(d < -4\), we have

\[
    L(1, \chi) = \frac{\pi h_0}{\sqrt{|d|}},
\]

and by Theorem (2)

\[
    \sum^{\prime} \frac{1}{a} < h_0.
\]

Hence by (19)

\[
    1 - \beta > \frac{6}{\pi^2} \frac{h_0}{h_0 \sqrt{|d|}} \left(1 - \eta \frac{1}{2}\right) > \left(\frac{6}{\pi} - \eta\right) \frac{1}{\sqrt{|d|}}.
\]
For $d > 0$, we have

$$L(1, \chi) = \frac{h_0 \log \varepsilon_0}{\sqrt{d}}.$$  

Now, for any class $C$ of forms

$$\sum_{(a,b,c) \in C} \frac{1}{|a|} = \sum_{(a,b,c) \in C} \frac{1}{a} + \sum_{-a,b,c \in C} \frac{1}{\sqrt{|d|} + a}.$$  

If $(a, b, c)$ runs through $C$, $(-a, b, -c)$ runs through another class which we denote by $-C$ (It may happen that $-C = C$). If $C_1 \neq C_2$, then $-C_1 \neq -C_2$. Hence

$$\sum_{C} \sum_{(a,b,c) \in C} \frac{1}{|a|} = 2 \sum_{\sqrt{|d|} + a > -a} \frac{1}{a}$$

and by Theorem (2)

$$\sum_{a} \frac{1}{a} \leq \frac{h_0}{2} \left( \frac{\log \varepsilon_0}{\log (1/2 \sqrt{d} - 1)} + \frac{4}{\sqrt{d}} \right) \leq \frac{h_0 \log \varepsilon_0}{\log d} \left( 1 + O \left( \frac{1}{\sqrt{d}} \right) \right),$$

where the constant in the $O$-symbol is effective. (Note that $\varepsilon_0 > \frac{1}{2} \sqrt{d}$). This together with (19) gives the corollary.

REFERENCES