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1. – Introduction.

Let S be a portion of a smooth real $(2n-1)$-dimensional hypersurface in an n-dimensional complex analytic manifold M. If the Levi form of S does not vanish at a point z_0, then by H. Lewy's extension theorem smooth solutions of the tangential Cauchy-Riemann equations on a sufficiently small neighborhood of z_0 in S can be extended to at least one side of the hypersurface. In [2] and [3] Andreotti and Hill generalize this theorem in the following way: Let U be a neighborhood of S in M such that $S=\{z\in U: \varrho(z)=0\}$ for a smooth function ϱ with $d\varrho|S\neq 0$ and let $W\pm=\{z\in U: \varrho(z)\gtreqless 0\}$. Then cohomology groups $H^{pq}(W\pm)$ are defined by means of the Dolbeault sequence for (p, q)-forms on $W\pm$ which are smooth up to S, and certain boundary cohomology groups $H^{pq}(S)$ on S are also introduced. The generalization of Lewy's phenomenon consists in showing that given a boundary cohomology class $\xi_0\in H^{pq}(S)$, there exists a cohomology class $\xi\in H^{pq}(W^-)$ (or $H^{pq}(W^+)$) such that ξ_0 is the restriction of ξ to S. The following problem is also considered: Given $\xi_0\in H^{pq}(S)$, try to find $\xi^+\in H^{pq}(W^+)$ and $\xi^-\in H^{pq}(W^-)$ such that ξ_0 is the jump between ξ^+ and ξ^- across S.

In [2] Andreotti-Hill first show the existence of a nontrivial Mayer-Vietoris sequence, and by means of this the problems above are reduced to showing certain vanishing theorems for $H^{pq}(W^{\pm})$. Since these cohomology groups are not the usual ones (because they involve the behavior at part of the boundary), the proofs of the vanishing theorems are not standard; they are to be found in [3].

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In this note it is shown that if one is ready to accept arbitrary boundary values (and not just smooth), the arguments of Andreotti-Hill are simplified a lot: the Mayer-Vietoris sequence here is just the long exact sequence for local cohomology, and the vanishing theorems that we need are nothing but the classical theorems of Andreotti and Grauert. We also use some results due to Andreotti and Norguet about the infinite dimensionality of certain cohomology groups. In order to make the « arbitrary boundary values » more concrete, we show in § 3 how they at least locally can be interpreted as hyperfunctions satisfying the tangential Cauchy-Riemann equations for S.

There is also a connection between this work and (a special case of) a theorem due to Martineau in [11] (see also [9]), which says that a hyperplane in C^n is purely 1-codimensional relative to the sheaf $\mathcal{O}$ of germs of holomorphic functions in C^n. In fact the methods of Andreotti-Hill make it possible to investigate this question for arbitrary smooth hypersurfaces, and it turns out for instance that Martineau's theorem is no longer valid if the Levi form of S is non-degenerate at some point.

Finally we remark that the questions treated here are but very special cases of a deep theorem due to Kashiwara and Kawai in [7] (see in particular example 3) and [8]. However, the methods used in this note are quite elementary—which unfortunately is more than one can say about the work of Kashiwara and Kawai—and hence it might nevertheless be justified.

2. – The Mayer-Vietoris sequence.

Consider a connected complex analytic manifold M of complex dimension n and an open connected subset U of M (M and U are assumed to be paracompact). Let S be a smooth real hypersurface of U defined by $S = \{z \in U: \varrho(z) = 0\}$, where $\varrho \in C^\infty(U)$ and $d\varrho|S \neq 0$. Define $U^- = \{z \in U: \varrho(z) < 0\}$ and $U^+ = \{z \in U: \varrho(z) > 0\}$. For a locally free analytic sheaf $\mathcal{F}$ on U, we then have the usual exact sequence for local cohomology:

$$0 \to H^0_S(U, \mathcal{F}) \to H^0(U, \mathcal{F}) \to H^0(U - S, \mathcal{F}) \to H^1_S(U, \mathcal{F}) \to$$
$$\to H^1(U, \mathcal{F}) \to H^1(U - S, \mathcal{F}) \to \dots .$$

Now $H^0_S(U, \mathcal{F}) = 0$ by the unique continuation property for holomorphic functions, and

$$H^r(U - S, \mathcal{F}) \cong H^r(U^-, \mathcal{F}) \oplus H(U^+, \mathcal{F})$$

for all r since $U - S = U^+ \cup U^-$ and $U^+ \cap U^- = \emptyset$. Hence we arrive at the following exact « Mayer-Vietoris sequence »:

$$0 \to H^0(U, \mathcal{F}) \to H^0(U^-, \mathcal{F}) \oplus H^0(U^+, \mathcal{F}) \to H^1_S(U, \mathcal{F}) \to$$
$$\to H^1(U, \mathcal{F}) \to H^1(U^-, \mathcal{F}) \oplus H^1(U^+, \mathcal{F}) \to H^2_S(U, \mathcal{F}) \to \dots .$$

The idea now is to use $H^r_S(U, \mathcal{F})$ as boundary cohomology groups. In particular, if Ω^p denotes the sheaf of germs of holomorphic p-forms, $H^{q+1}_S(U, \Omega^p)$ corresponds to $H^{pq}(S)$ in [2].

3. – Interpretation of the cohomology classes in $H^1_S(U, \mathcal{O})$ as hyperfunctions satisfying the tangential Cauchy-Riemann equations.

To simplify the exposition we consider only the sheaf $\mathcal{O}$ of germs of holomorphic functions in this section. The sheaf $\mathcal{H}^r_S$ is determined by the presheaf $\{H^r_{V\cap S}(V, \mathcal{O})\}$, where V are open sets in U. $\mathcal{H}^0_S = 0$ by the unique continuation property, and hence $\Gamma(V \cap S, \mathcal{H}^1_S) = H^1_{V\cap S}(V, \mathcal{O})$ for each open set V in U. By means of a spectral sequence argument (cf. theorems 1.7 and 1.8 in [10]) one can moreover show that $H^r_{V\cap S}(V, \mathcal{O}) \cong H^{r-1}(V, \mathcal{H}^1_S)$ for $r \leqslant p$ if $\mathcal{H}^r_S = 0$ for $2 \leqslant r \leqslant p$. So in order to understand the cohomology groups $H^r_S(U, \mathcal{O})$, we take a closer look at the sheaf $\mathcal{H}^1_S$.

Let $z_0 \in S$. Near z_0 we can find local coordinates $\{z_i\}_{i=1}^n$ such that

$$\varrho(z) = \operatorname{Re} z_1 + \sum_{j,k=1}^{n} A_{jk} z_j \cdot \bar{z}_k + \mathcal{O}(|z|^3)$$

near z_0. With this choice of coordinates the tangent vectors $\{\partial/\partial z_2, \dots, \partial/\partial z_n\}$ span the holomorphic tangent space $H_{z_0}(S)$ at z_0. By shrinking U, if necessary, we may assume that $\{z_1, \dots, z_n\}$ are coordinates in all of U and that to each $z \in S$ one can find $n-1$ holomorphic tangent vectors which span $H_z(S)$, such that these together with $\partial/\partial z_1$ span the holomorphic tangent space of U at z. U can be considered as an open set in $\boldsymbol{C} \times \boldsymbol{R}^{2n-2}$ by the identification $(z_1, z_2, \dots, z_n) \mapsto (z_1, x_2, y_2, \dots, x_n, y_n)$, where $z_k = x_k + iy_k$. Then U is imbedded in $W = U \times i\boldsymbol{R}^{2n-2} \subset \boldsymbol{C} \times \boldsymbol{R}^{2n-2} \times i\boldsymbol{R}^{2n-2} \cong \boldsymbol{C} \times \boldsymbol{C}^{2n-2} = \boldsymbol{C}^{2n-1}$ by

$$(z_1, x_2, y_2, \dots, x_n, y_n) \mapsto (z_1, x_2 + iu_2, y_2 + iv_2, \dots, x_n + iu_n, y_n + iv_n),$$

where $iu_2, iv_2, \dots, iu_n, iv_n$ are the coordinates in $i\boldsymbol{R}^{2n-2}$.

If now S is considered as a submanifold of the complex manifold W, we see by the choice of coordinates that the tangent space at each point in S inherits no complex structure from W. Thus S is a totally real submanifold of W in the sense of [5] and [6]. Following Harvey (see [5] and [6]) we can therefore introduce the sheaf of hyperfunctions ${}^{2n-1}\mathcal{B} = \mathcal{H}_S^{2n-1}({}^{2n-1}\mathcal{O})$ determined by the presheaf $\{H^{2n-1}_{V\cap S}(V, {}^{2n-1}\mathcal{O})\}$, where V is open in W and ${}^{2n-1}\mathcal{O}$ is the sheaf of germs of holomorphic functions on W. Then ${}^{2n-1}\mathcal{B}$ is a flabby sheaf supported on S and so we identify ${}^{2n-1}\mathcal{B}$ with its restriction to S. For each compact set $K \subset S$ there is an isomorphism ${}^{2n-1}\mathcal{O}(K)' = \Gamma_K(S, {}^{2n-1}\mathcal{B})$ and there is also an injection

$$\Gamma_*(S, \mathcal{D}') \hookrightarrow \Gamma_*(S, {}^{2n-1}\mathcal{B}) \cong \mathcal{O}(S)' ,$$

that is, the compactly supported distributions are injected into the compactly supported hyperfunctions which in their turn are identified with the analytic functionals on S (we refer to [5] and [6] for the details).

Next consider the following result proved by Martineau [11] and Komatsu [9]: Let U be an open set in $\boldsymbol{R}^p \times i\boldsymbol{R}^q \subset \boldsymbol{R}^p \times i\boldsymbol{R}^p = \boldsymbol{C}^p$ and let ${}^p\mathcal{O}$ be the sheaf of germs of holomorphic functions on $\boldsymbol{C}^p$. Then $H^r_U(\boldsymbol{C}^p, {}^p\mathcal{O}) = 0$ if $r \neq p - q$ and $H^{p-q}_U(\boldsymbol{C}^p, {}^p\mathcal{O})$ is isomorphic to the space of hyperfunctions defined on U and holomorphic in the q complex variables of U, so in particular U is purely $(p-q)$-codimensional with respect to ${}^p\mathcal{O}$(i.e. $\mathcal{H}^r_U(\boldsymbol{C}^p, {}^p\mathcal{O}) = 0$ for $r \neq p - q$). We write $\mathcal{H}^{p-q}_U({}^p\mathcal{O}) = {}^{p-q}\mathcal{B}^q\mathcal{O}$, and regard this as a sheaf on U.

We also need the following theorem on local cohomology (theorem 1.9 in [10]): Let W be a topological space, let $\mathcal{F}$ be a sheaf on W, let U be a locally closed set in W purely m-codimensional with respect to $\mathcal{F}$ and let S be a subset of U. Then

$$H^p_{S\cap V}(V, \mathcal{F}) \cong \begin{cases} 0 , & p < m , \\ H^{p-m}_S(V, \mathcal{H}^m_U(\mathcal{F})) , & p \geqslant m , \end{cases}$$

for any open set V in W, and

$$\mathcal{H}^p_S(\mathcal{F}) \cong \begin{cases} 0 , & p < m , \\ \mathcal{H}^{p-m}_S(\mathcal{H}^m_U(\mathcal{F})), & p \geqslant m . \end{cases}$$

So in our situation (where $m = 2n-2$)

$$\mathcal{H}^p_S({}^{2n-1}\mathcal{O}) \cong \begin{cases} 0 , & p < 2n-2 , \\ \mathcal{H}^{p-2n+2}_S(\mathcal{H}^{2n-2}_U({}^{2n-1}\mathcal{O})) , & p \geqslant 2n-2 , \end{cases}$$

and in particular

$${}^{2n-1}\mathcal{B} = \mathcal{H}^{2n-1}_S({}^{2n-1}\mathcal{O}) \cong \mathcal{H}^1_S({}^{2n-2}\mathcal{B}^1\mathcal{O}) .$$

On U we have two sheaves: ${}^n\mathcal{O}$ coming from the imbedding $U \subset \boldsymbol{C}^n \cong \cong C \times \boldsymbol{R}^{2n-2}$ and ${}^{2n-2}\mathcal{B}\,{}^1\mathcal{O}$ coming from $U \subset \boldsymbol{C} \times \boldsymbol{R}^{2n-2} \times i\boldsymbol{R}^{2n-2} \cong \boldsymbol{C}^{2n-1}$, and clearly ${}^n\mathcal{O} \hookrightarrow {}^{2n-2}\mathcal{B}\,{}^1\mathcal{O}$. If the sheaf $\mathcal{A}$ is defined by the exact sequence

$$0 \to {}^n\mathcal{O} \to {}^{2n-2}\mathcal{B}\,{}^1\mathcal{O} \to \mathcal{A} \to 0\,,$$

we get the exact sequence

$$0 \to \mathcal{H}^0_S({}^n\mathcal{O}) \to \mathcal{H}^0_S({}^{2n-2}\mathcal{B}\,{}^1\mathcal{O}) \to \mathcal{H}^0_S(\mathcal{A}) \to \mathcal{H}^1_S({}^n\mathcal{O}) \to \mathcal{H}^1_S({}^{2n-2}\mathcal{B}\,{}^1\mathcal{O}) \to \dots .$$

Now $\mathcal{H}^0_s({}^{2n-2}\mathcal{B}\,{}^1\mathcal{O}) \cong \mathcal{H}^0_s(\mathcal{A})$ by the unique continuation property for ${}^n\mathcal{O}$ (actually $\mathcal{H}^0_s({}^{2n-2}\mathcal{B}\,{}^1\mathcal{O}) = 0$ thanks to the choice of coordinates and the unique continuation property for ${}^{2n-2}\mathcal{B}\,{}^1\mathcal{O}$, as proved in [9], lemma 5.7, but we don't need this), and therefore there results an injection $\mathcal{H}^1_s({}^n\mathcal{O}) \hookrightarrow {}^{2n-1}\mathcal{B}$. Hence

$$H^1_{V\cap S}(V, {}^n\mathcal{O}) = \Gamma(V \cap S, \mathcal{H}^1_S) \hookrightarrow \Gamma(V \cap S, {}^{2n-1}\mathcal{B})$$

for each open subset V in U, so the cohomology classes in $H^1_S(U, {}^n\mathcal{O})$ can be interpreted as hyperfunctions on S.

Next we want to characterize those hyperfunctions on S that belong to $H^1_s(U, {}^n\mathcal{O})$. Being totally real in W, S is purely $(2n-1)$-codimensional with respect to ${}^{2n-1}\mathcal{O}$, and by above U is purely $(2n-2)$-codimensional with respect to ${}^{2n-1}\mathcal{O}$. Therefore

$$\Gamma(S, {}^{2n-1}\mathcal{B}) \cong H^{2n-1}_S(W, {}^{2n-1}\mathcal{O}) \cong H^1_S(W, \mathcal{H}^{2n-2}_U({}^{2n-1}\mathcal{O})) \cong H^1_S(U, {}^{2n-2}\mathcal{B}\,{}^1\mathcal{O})\,,$$

so we shall characterize the subset $H^1_s(U, {}^n\mathcal{O})$ of $H^1_S(U, {}^{2n-2}\mathcal{B}\,{}^1\mathcal{O})$. By our choice of coordinates in U, $n-1$ anti-holomorphic vector fields $\{X_k\}^n_{k=2}$, inducing $\{\partial/\partial\bar{z}_2, \dots, \partial/\partial\bar{z}_n\}$ at z_0, span the anti-holomorphic tangent space at each point of S, when S is regarded as a subset of U. Thus the induced Chauchy-Riemann equations on S are $X_k f = 0$ for $k = 2, 3, \dots, n$. The action of $\{X_k\}^n_{k=2}$ on $H^1_S(U, {}^{2n-2}\mathcal{B}\,{}^1\mathcal{O})$ is defined by first letting extensions $\tilde{X}_k$ of them to U act on the sheaf ${}^{2n-2}\mathcal{B}\,{}^1\mathcal{O}$ de ned on $U \subset \boldsymbol{C} \div \boldsymbol{R}^{n-2}$ and then taking cohomology. But each germ in ${}^{2n-2}\mathcal{B}\,{}^1\mathcal{O}$ is holomorphic in z_1, and since $\tilde{X}_k$ together with $\partial/\partial\bar{z}_1$ span the anti-holomorphic tangent space at each point of U, if U is small enough, $\{g \in {}^{2n-2}\mathcal{B}\,{}^1\mathcal{O} : \tilde{X}_k g = 0,\ k = 2, \dots, n\} = {}^n\mathcal{O}$. Hence

$$H^1_s(U, {}^n\mathcal{O}) \supseteq \{\xi \in H^1_S(U, {}^{2n-2}\mathcal{B}\,{}^1\mathcal{O}) : \xi \text{ satisfies the tangential Cauchy-Riemann equations for } S\}\,,$$

and the inverse inclusion is trivial.

Thus $H^1_S(U, \mathcal{O})$ can be identified with the space of hyperfunctions on S satisfying the tangential Cauchy-Riemann equations, and $\mathcal{H}^1_s(U, \mathcal{O})$ is the corresponding sheaf. In particular $H^1_S(U, \mathcal{O})$ contains the distributions on S which satisfy the induced Cauchy-Riemann equations.

4. – The theorems of Hartogs and Bochner.

Let M be a Stein manifold of complex dimension $n \geqslant 2$, and let S be any connected closed real hypersurface (not necessarily smooth) in M, such that $M - S$ is the union of two connected open sets M^- and M^+ of which say M^- is relatively compact and $\overline{M^-} = M^- \cup S$. Let $\mathcal{F}$ be a locally free analytic sheaf on M.

To prepare for the Bochner theorem, we first give Harvey's proof of the theorem of Hartogs (see [5]). Let L be the holomorphically convex hull of $\overline{M^-}$ in M; then there are obvious injections

$$H^0(M, \mathcal{F}) \hookrightarrow H^0(M^+, \mathcal{F}) \hookrightarrow H_0(M - L, \mathcal{F}) .$$

Now

$$H^0(M, \mathcal{F}) \to H^0(M - L, \mathcal{F}) \to H^1_L(M, \mathcal{F})$$

is exact and $H^r_L(M, \mathcal{F}) = 0$ for $r \neq n$. Hence

$$H^0(M^+, \mathcal{F}) \cong H^0(M, \mathcal{F}) ,$$

which is the theorem of Hartogs.

To prove the Bochner theorem we use the Mayer-Vietoris sequence in § 2:

$$0 \to H^0(M, \mathcal{F}) \to H^0(M^+, \mathcal{F}) \oplus H^0(M^-, \mathcal{F}) \to H^1_S(M, \mathcal{F}) \to 0 .$$

Since $H^0(M, \mathcal{F}) \cong H^0(M^+, \mathcal{F})$, we get the desired result:

$$H^1_S(M, \mathcal{F}) \cong H^0(M^-, \mathcal{F}) ,$$

that is, each boundary cohomology class in $H^1_S(M, \mathcal{F})$ has a unique extension as a section in $\mathcal{F}$ over M^-.

5. – H. Lewy's theorem.

We return to the situation in § 2. If U is a Stein manifold (which might always be assumed if we are only interested in local properties), the Mayer-Vietoris sequence becomes

$$0 \to H^0(U, \mathcal{F}) \to H^0(U^+, \mathcal{F}) \oplus H^0(U^-, \mathcal{F}) \to H^1_S(U, \mathcal{F}) \to 0$$

and

$$0 \to H^r(U^+, \mathcal{F}) \oplus H^r(U^-, \mathcal{F}) \to H_S^{r+1}(U, \mathcal{F}) \to 0$$

for $1 \leqslant r \leqslant n$.

The first sequence shows that

$$H_S^1(U, \mathcal{F}) \cong \frac{H^0(U^+, \mathcal{F}) \oplus H^0(U^-, \mathcal{F})}{H^0(U, \mathcal{F})},$$

which we interpret by saying that each element in $H_S^1(U, \mathcal{F})$ is the difference of boundary values of sections in $\mathcal{F}$ over U^+ and U^- (this is analogous to the case of hyperfunctions in one variable).

Now let $\mathcal{F} = \mathcal{O}$ and assume that the Levi form of S (i.e. the Levi form of ϱ restricted to the analytic tangent plane of S) has at least one positive eigenvalue at each point in S. If $\tilde{U}^+$ denotes the envelope of holomorphy of U^+, the E. E. Levi theorem shows that $\hat{U}^+ = \tilde{U}^+ \cap U$ is a two-sided neighborhood of S in U. Since

$$H^0(\hat{U}^+, \mathcal{O}) \cong H^0(U^+, \mathcal{O}),$$

it follows that

$$0 \to H^0(U, \mathcal{O}) \to H^0(\hat{U}^+, \mathcal{O}) \oplus H^0(U^-, \mathcal{O}) \to H_S^1(U, \mathcal{O}) \to 0$$

is exact. As $\hat{U}^+ \cup U^- = U$ and $\hat{U}^+ \cap U^-$ is a non-empty open subset of U, we also have the ordinary Mayer-Vietoris sequence as in [1]:

$$0 \to H^0(U, \mathcal{O}) \to H^0(\hat{U}^+, \mathcal{O}) \oplus H^0(U^-, \mathcal{O}) \to H^0(\hat{U}^+ \cap U^-, \mathcal{O}) \to 0.$$

Comparing these two sequences, we see that

$$H_S^1(U, \mathcal{O}) \cong H^0(\hat{U}^+ \cap U^-, \mathcal{O}),$$

i.e. each cohomology class in $H_S^1(U, \mathcal{O})$ can be uniquely extended to a holomorphic function in the one-sided neighborhood $\hat{U}^+ \cap U^-$ of S in U. The isomorphism $H_S^1(U, \mathcal{O}) \xrightarrow{\sim} H^0(\hat{U}^+ \cap U^-, \mathcal{O})$ is defined in the following way. A cohomology class $\xi \in H_S^1(U, \mathcal{O})$ is lifted to $(f, g) \in H^0(\hat{U}^+, \mathcal{O}) \oplus H^0(U^-, \mathcal{O})$, where (f, g) is unique modulo $H^0(U, \mathcal{O})$, and then the difference $f - g$ gives the desired function in $H^0(\hat{U}^+ \cap U^-, \mathcal{O})$.

If the Levi form of S has at least one negative eigenvalue everywhere, it follows in the same way that

$$H_S^1(U, \mathcal{O}) \cong H^0(U^+ \cap \hat{U}^-, \mathcal{O}),$$

where $\hat{U}^-$ is the intersection of U and the envelope of holomorphy of U^-, and $U^+ \cap \hat{U}^-$ is a one-sided neighborhood of S in U.

And if the Levi form of S has at least one positive and one negative eigenvalue at each point,

$$H^1_S(U, \mathcal{O}) \cong H^0(\hat{U}^+ \cap \hat{U}^-, \mathcal{O}),$$

where $\hat{U}^+ \cap \hat{U}^-$ is a two-sided neighborhood of S in U.

By making local extensions and then glueing together, we can now for instance obtain the following result by means of the arguments in § 3:

Let S be a global smooth hypersurface in M such that the Levi form of S everywhere has at least one positive and one negative eigenvalue. Let f be a distribution on S which satisfies the tangential Cauchy-Riemann equations for S. Then f can be uniquely extended to a holomorphic function in an open neighborhood of S in M.

For observe first that if $V \subset U$ are open sets in S such that

$$\Gamma(V, \mathcal{H}^1_S) = H^1_S(V, \mathcal{O}) \cong H^0(\hat{V}, \mathcal{O}),$$
$$\Gamma(U, \mathcal{H}^1_S) = H^1_S(U, \mathcal{O}) \cong H^0(\hat{U}, \mathcal{O}),$$

where $\hat{V} \subset \hat{U}$ are open in M, then we have a commutative diagram

$$\begin{array}{ccc} \Gamma(U, \mathcal{H}^1_S) & \xrightarrow{\cong} & H^0(\hat{U}, \mathcal{O}) \\ \downarrow & & \downarrow \\ \Gamma(V, \mathcal{H}^1_S) & \xrightarrow{\cong} & H^0(\hat{V}, \mathcal{O}), \end{array}$$

where the vertical maps are restriction mappings.

So if U_1, U_2 are open sets in S such that $U_{12} = U_1 \cap U_2 \neq \emptyset$ and $f_i = f|_{U_i}$ has an extension $F_i \in \mathcal{O}(\hat{U}_i)$ for $i = 1, 2$, then $f_1|_{U_{12}} - f_2|_{U_{12}} = 0$ has the unique extension $F_1|_{\hat{U}_{12}} - F_2|_{\hat{U}_{12}}$, and therefore $F_1|_{\hat{U}_{12}} = F_2|_{\hat{U}_{12}}$. Consequently F_1 and F_2 are holomorphic continuations of each other.

Next we consider the case with $1 \leqslant r \leqslant n$. As in [2], we interpret the isomorphism

$$H^{r+1}_S(U, \mathcal{F}) \cong H^r(U^+, \mathcal{F}) \oplus H^r(U^-, \mathcal{F})$$

by saying that each element in $H^{r+1}_S(U, \mathcal{F})$ is a jump between a cohomology class on U^+ and a cohomology class on U^- across S. From this isomorphism it follows for instance that the extension theorem

$$H^{r+1}_S(U, \mathcal{F}) \cong H^r(U^+, \mathcal{F})$$

is equivalent to the vanishing theorem

$$H^r(U^-, \mathcal{F}) = 0 .$$

If the Levi form of S has at least $n-p+1$ positive eigenvalues at each point, then U^- is p-complete (since U is Stein), and by the Andreotti-Grauert theorems ([1])

$$H^r(U^-, \mathcal{F}) = 0 \quad \text{for } r \geqslant p ,$$

i.e. $H_S^{r+1}(U, \mathcal{F}) \cong H^r(U^+, \mathcal{F})$ for $r \geqslant p$.

6. – Global results.

Following Andreotti-Hill, we now consider a global example:

Let X be a compact connected manifold of complex dimension n, and let $\mathcal{F}$ be a locally free analytic sheaf on X. Let $\varrho: X \to \mathbf{R}$ be a C^∞ function, and assume that $S = \{x \in X: \varrho(x) = 0\}$ is a smooth hypersurface ($d\varrho \neq 0$ on S) dividing X into the two regions

$$X^- = \{x \in X: \varrho(x) < 0\} \quad \text{and} \quad X^+ = \{x \in X: \varrho(x) > 0\} .$$

We assume that the Levi form of S is nowhere degenerate and has p positive and q negative eigenvalues ($p + q = n - 1$). (An explicit example of such a situation is to be found in [4], pp. 217-219.)

Then $\dim_{\mathbf{C}} H^r(X, \mathcal{F}) < \infty$ for all r since X is compact, $\dim_{\mathbf{C}} H^r(X^-, \mathcal{F}) < \infty$ for $r \neq q$ and $\dim_{\mathbf{C}} H^r(X^+, \mathcal{F}) < \infty$ for $r \neq p$ by the fundamental theorem of Andreotti and Grauert in [1]. Moreover, both $H^q(X^-, \mathcal{F})$ and $H^p(X^+, \mathcal{F})$ are infinite-dimensional vector spaces over $\mathbf{C}$ according to theorem 2 in the paper [4] by Andreotti and Norguet.

A straightforward application of the Mayer-Vietoris sequence now leads to the following conclusion:

If $p \neq q$ there are homomorphisms

$$H^q(X^-, \mathcal{F}) \to H_S^{q+1}(X, \mathcal{F}) \quad \text{and} \quad H^p(X^+, \mathcal{F}) \to H_S^{p+1}(X, \mathcal{F})$$

with finite dimensional kernels and cokernels, and all other cohomology groups in the Mayer-Vietoris sequence have finite dimension.

If $p = q$ (which is possible if n is odd), there is a homomorphism

$$H^p(X^-, \mathcal{F}) \oplus H^p(X^+, \mathcal{F}) \to H_S^{p+1}(X, \mathcal{F})$$

with finite dimensional kernel and cokernel, while the remaining cohomology groups are of finite dimension over $\boldsymbol{C}$.

7. – The sheaves $\mathcal{H}_S^r(\mathcal{F})$.

We use the same assumptions as in § 2. The sheaf $\mathcal{H}_S^r(\mathcal{F})$ is determined by the presheaf $\{H^r_{V\cap S}(V, \mathcal{F})\}$; we know that $\mathcal{H}_S^0(\mathcal{F}) = 0$, $\mathcal{H}_S^1(\mathcal{F}) \neq 0$ and want to investigate what happens for $r \geqslant 2$. Introducing the stalks

$$\mathcal{H}^r(S^\pm, x, \mathcal{F}) = \varinjlim_{V \ni x} H^r(V^\pm, \mathcal{F}),$$

where V runs through a fundamental system of open neighborhoods in U of $x \in S$, a direct application of the Mayer-Vietoris sequence shows that

$$\big(\mathcal{H}_S^{r+1}(\mathcal{F})\big)_x \cong \mathcal{H}^r(S^+, x, \mathcal{F}) \oplus \mathcal{H}^r(S^-, x, \mathcal{F})$$

for $x \in S$ and $r \geqslant 1$. Hence $\big(\mathcal{H}_S^{r+1}(\mathcal{F})\big)_x$ is zero if and only if both $\mathcal{H}^r(S^+, x, \mathcal{F})$ and $\mathcal{H}^r(S^-, x, \mathcal{F})$ are zero.

Consider first the case when the Levi form of S is zero for all $x \in S$. In a local coordinate system at x we can choose V as a domain of holomorphy, and it follows that V^+ and V^- are domains of holomorphy too, so that $\mathcal{H}^r(S\pm, x, \mathcal{F}) = 0$ for $r > 0$. Hence the sheaves $\mathcal{H}_S^r(\mathcal{F})$ are zero in this case except when $r = 1$.

Next assume that the Levi form of S has p positive and q negative eigenvalues at x $(p + q \leqslant n - 1)$. Then the following is proved in [1]:

There exists a fundamental sequence of neighborhoods V_k of x in U such that for any locally free analytic sheaf $\mathcal{F}$ on U we have

$$H^r(V_k^-, \mathcal{F}) = 0 \qquad \text{if} \quad \begin{cases} r > n - p - 1 \\ \text{or} \\ 0 < r < q, \end{cases}$$

and there exists a fundamental sequence of neighborhoods W_k of x in U such that

$$H^r(W_k^+, \mathcal{F}) = 0 \qquad \text{if} \quad \begin{cases} r > n - q - 1 \\ \text{or} \\ 0 < r < p. \end{cases}$$

Hence

$$\mathcal{H}^r(S^-, x, \mathcal{F}) = 0 \quad \text{if} \begin{cases} r > n - p - 1 \\ \text{or} \\ 0 < r < q , \end{cases}$$

and

$$\mathcal{H}^r(S^+, x, \mathcal{F}) = 0 \quad \text{if} \begin{cases} r > n - q - 1 \\ \text{or} \\ 0 < r < p . \end{cases}$$

If we moreover assume that the Levi form is nondegenerate at x, so that $p + q = n - 1$, then

$$\mathcal{H}^r(S^-, x, \mathcal{F}) = 0 \quad \text{for } r \neq 0, q ,$$

and

$$\mathcal{H}^r(S^+, x, \mathcal{F}) = 0 \quad \text{for } r \neq 0, p .$$

By proposition 6 in [4], both $\mathcal{H}^q(S^-, x, \mathcal{F})$ and $\mathcal{H}^p(S^+, x, \mathcal{F})$ are infinite dimensional in this case. So if $p \neq q$,

$$(\mathcal{H}_S^{q+1}(\mathcal{F}))_x \cong \mathcal{H}^q(S^+, x, \mathcal{F}) , \qquad (\mathcal{H}_S^{p+1}(\mathcal{F}))_x \cong \mathcal{H}^p(S^-, x, \mathcal{F})$$

and $(\mathcal{H}_S^r(\mathcal{F}))_x = 0$ for $r \neq 1,\ p+1,\ q+1$.

If $p = q = (n-1)/2$ (which is possible if n is odd),

$$(\mathcal{H}_S^{p+1}(\mathcal{F}))_x \cong \mathcal{H}^p(S^-, x, \mathcal{F}) \oplus \mathcal{H}^p(S^+, x, \mathcal{F})$$

and

$$(\mathcal{H}_S^r(\mathcal{F}))_x = 0 \quad \text{for } r \neq 1,\ p+1 .$$

8. – Exactness and non-exactness of certain complexes.

In this section we assume that $M = \boldsymbol{C}^n$. If Ω^j denotes the sheaf of germs of holomorphic differential forms of the type $(j, 0)$ on $\boldsymbol{C}^n$, and $\mathcal{B}^{j,k}$ denotes the sheaf of germs of differential forms of the type (j, k) with hyperfunction coefficients, we have the following flabby resolution of Ω^j (theorem 142 in [12]):

$$0 \to \Omega^j \to \mathcal{B}^{j,0} \xrightarrow{\bar{\partial}_0} \mathcal{B}^{j,1} \xrightarrow{\bar{\partial}_1} \dots\ \mathcal{B}^{jn} \to 0 .$$

For a real hypersurface S and an open set V in $\boldsymbol{C}^n$ the following sequence is induced:

$$\text{(I)} \qquad 0 \to \Gamma_{V\cap S}(V, \mathcal{B}^{j,0}) \xrightarrow{\bar\partial_0} \Gamma_{V\cap S}(V, \mathcal{B}^{j,1}) \xrightarrow{\bar\partial_1} \dots \xrightarrow{\bar\partial_{n-1}} \Gamma_{V\cap S}(V, \mathcal{B}^{j,n}) \to 0\,.$$

The cohomology groups of this complex are consequently $H^r_{V\cap S}(V, \Omega^j)$. So if V is Stein and $V - S = V^+ \cup V^-$, the inexactness of (I) is measured by

$$\ker \bar\partial_r / im \bar\partial_{r-1} \cong H^r_{V\cap S}(V, \Omega^j)\,,$$

which for $r \geqslant 2$ equals

$$H^{r-1}(V^-, \Omega^j) \oplus H^{r-1}(V^+, \Omega^j)\,.$$

Taking inductive limit in (I), we also have

$$\text{(II)} \qquad 0 \to \mathcal{H}^0_S(\mathcal{B}^{j,0}) \xrightarrow{\bar\partial_0} \mathcal{H}^0_S(\mathcal{B}^{j,1}) \xrightarrow{\bar\partial_1} \dots \xrightarrow{\bar\partial_{n-1}} \mathcal{H}^0_S(\mathcal{B}^{j,n}) \to 0\,,$$

where $\mathcal{H}^0_S(\mathcal{B}^{j,k})$ is the sheaf associated to the presheaf $\{\Gamma_{V\cap S}(V, \mathcal{B}^{j,k})\}$. Since the inductive limit is an exact functor, the cohomology groups of (II) are the sheaves $\mathcal{H}^r_S(\Omega^j)$.

If the Levi form of S is non-degenerate and has p positive and q negative eigenvalues everywhere ($p + q = n - 1$), then the results in § 7 show that the Poincaré lemma for the complex (II) is not valid in dimensions 1, $p+1$ and $q+1$, but holds in any other dimension.

Conversely one might of course deduce properties of $H^r_{V\cap S}(V, \Omega^j)$ and $\mathcal{H}^r_S(\Omega^j)$ from knowledge about the complexes (I) and (II), i.e. knowledge about systems of partial differential equations.

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