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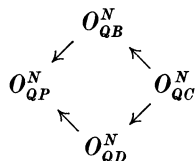
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Quasiharmonic L^p -Functions on Riemannian Manifolds (*).

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Let Q be the class of quasiharmonic functions q , defined by $\Delta q = 1$, with $\Delta = d\delta + \delta d$ the Laplace-Beltrami operator. Denote by P, B, D, C the classes of functions which are positive, bounded, Dirichlet finite, or bounded Dirichlet finite, respectively. For $X = P, B, D, C$, let O_{qX}^N be the classes of Riemannian N -manifolds, $N \geq 2$, for which $QX = Q \cap X = \emptyset$. These classes are known to be related by the strict inclusion relations



for each N , whereas there is no inclusion between O_{qB}^N and O_{qD}^N [3, 5].

In the present paper we introduce the class QL^p of quasiharmonic functions in L^p , with $1 < p < \infty$; the value $p = \infty$ will be excluded since $O_{qL^\infty}^N$ is nothing but O_{qB}^N . If \tilde{O}^N signifies the complement of O^N with respect to the totality of Riemannian N -manifolds, then we shall show that $O_{qL^p}^N \cap O_{qX}^N \neq \emptyset$, $\tilde{O}_{qL^p}^N \cap \tilde{O}_{qX}^N \neq \emptyset$, $\tilde{O}_{qL^p}^N \cap O_{qX}^N \neq \emptyset$ for $p \geq 1$, $X = P, B, D, C$; $O_{qL^p}^N \cap \tilde{O}_{qX}^N \neq \emptyset$ for $p \geq 1$, $X = P, B$; and $O_{qL^p}^N \cap \tilde{O}_{qD}^N \neq \emptyset$ for $p > 1$. In striking contrast with these noninclusions, we shall establish the strict inclusions $O_{qL^1}^N < O_{qD}^N$, and $O_{qL^p}^N < O_{qC}^N$ for $p \geq 1$.

1. – We first prove the existence of N -manifolds which carry neither QL^p nor QX functions:

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THEOREM 1. $O_{QL^p}^N \cap O_{QX}^N \neq \emptyset$ for $N \geq 2$, $1 < p < \infty$, $X = P, B, D, C$.

PROOF. An example is simply the Euclidean N -space E^N . We first show that $QL^1 = \emptyset$. Every $q \in Q$ can be written as $q = q_0 + h$, where $q_0 = -r^2/2N \in Q$ and h belongs to the class H of harmonic functions. Set $h = h(0) + k$, with $h(0)$ the value of h at the origin. Since $\int_{E^N} k dV = ck(0) = 0$, we have

$$\|q\|_1 = \int_{E^N} |q_0 + h(0) + k| dV \geq \left| \int_{E^N} (q_0 + h(0)) dV \right| = \infty;$$

in fact, $q_0 + h(0) \sim cr^2$ and $dV = r^{N-1} \lambda(\theta) dr d\theta_1 \dots d\theta_{N-1}$, with λ a trigonometric function of $\theta = (\theta_1, \dots, \theta_{N-1})$.

To see that $QL^p = \emptyset$ for $p > 1$, take a function $\varphi \in C^\infty(E^N)$ with $\varphi = r^\alpha$ on $\{r > 1\}$, α a real constant to be specified later. If p' is determined by $1/p + 1/p' = 1$, then

$$\|\varphi\|_{p'}^{p'} = a + b \int_1^\infty r^{\alpha p' + N - 1} dr < \infty,$$

that is, $\varphi \in L^{p'}$, for $\alpha < -N/p'$. Suppose there exists a function $q \in QL^p$. Then $|(q, \varphi)| < \infty$. For some $h \in H$,

$$|(q, \varphi)| = |(q_0 + h(0) + k, \varphi)| = |(q_0 + h(0), \varphi)| = \infty$$

if $\alpha \geq -N - 2$. Thus any $\alpha \in [-N - 2, -N/p']$ gives a contradiction, and we have $QL^p = \emptyset$ for every $p \geq 1$.

We proceed to show that $E^N \in O_{QX}^N$ for all X . It suffices to establish $QP = \emptyset$. Write again an arbitrary $q \in Q$ as $q = q_0 + h$. Take an increasing sequence $\{r_n\}_1^\infty$ with $r_n \rightarrow \infty$. For every n there exists an $\theta^n = (\theta_1^n, \dots, \theta_{N-1}^n)$ such that $h(r_n, \theta^n) = h(0)$. Therefore $q(r_n, \theta^n) = q_0(r_n) + h(0) \rightarrow -\infty$, and $q \notin QP$.

2. - The relation $\tilde{O}_{QL^p}^N \cap \tilde{O}_{QX}^N \neq \emptyset$ is trivial in view of the Euclidean N -ball. We proceed to give a Riemannian N -manifold which carries QL^p functions but no QX functions.

THEOREM 2. $\tilde{O}_{QL^p}^N \cap \tilde{O}_{QX}^N \neq \emptyset$ for $N \geq 2$, $1 < p < \infty$, $X = P, B, D, C$.

PROOF. Consider the manifold

$$T: \{(x, y_1, \dots, y_{N-1}) \mid |x| < \infty, |y_i| \leq 1, i = 1, \dots, N - 1\},$$

with the opposite faces $y_i = 1$ and $y_i = -1$ identified for each i by a parallel translation perpendicular to the x -axis. Endow T with the metric

$$ds^2 = e^{-x^2} dx^2 + e^{-x^2/(N-1)} \sum_{i=1}^{N-1} dy_i^2.$$

For a quasiharmonic function $q_0(x)$ we have

$$\Delta q_0(x) = -e^{x^2} (e^{-x^2} e^{x^2} q_0'(x))' = 1,$$

which is satisfied by

$$q_0(x) = - \int_0^x \int_0^t e^{-t^2} dt dx.$$

To estimate

$$\|q_0\|_p^p = e \int_{-\infty}^{\infty} \left| \int_0^x \int_0^t e^{-t^2} dt dx \right|^p e^{-x^2} dx$$

we set $a = \int_0^{\infty} e^{-t^2} dt$ and obtain

$$\|q_0\|_p^p < a^p e \int_{-\infty}^{\infty} |x|^p e^{-x^2} dx < \infty.$$

Therefore $T \in \tilde{O}_{QL^p}^N$ for all p .

A harmonic function $h_0(x)$ satisfies $\Delta h_0(x) = -e^{x^2} (e^{-x^2} e^{x^2} h_0'(x))' = 0$, which gives $h_0(x) = ax + b$. The harmonic measure ω of the boundary component at $x = \infty$ on $\{x \geq 0\}$ is $\omega = \lim_n \omega_n$, with $\omega_n = x/n$ harmonic on $\{0 \leq x \leq n\}$. Thus $\omega \equiv 0$, the same is true of the harmonic measure of the boundary component at $x = -\infty$, and therefore T belongs to the class O_G^N of parabolic N -manifolds. In view of $O_G^N \subset O_{QP}^N$ (loc. cit.), we obtain $T \in O_{QX}^N$ for all X .

3. - Our next problem is to find an N -manifold which admits QX functions for $X = P, B$, but no QL^p functions.

THEOREM 3. $O_{QL^p}^N \cap \tilde{O}_{QX}^N \neq \emptyset$ for $N \geq 2, 1 < p < \infty, X = P, B$.

PROOF. Let M be the N -space equipped with the metric

$$ds^2 = dr^2 + \psi(r)^{2/(N-1)} \sum_1^{N-1} \lambda_i(\theta) d\theta_i^2,$$

where $\psi \in C^\infty[0, \infty)$,

$$\psi(r) = \begin{cases} r^{N-1} & \text{for } r \leq \frac{1}{2}, \\ e^{e^r} e^r & \text{for } r \geq 1, \end{cases}$$

and the λ_i are trigonometric functions of θ such that the metric is Euclidean on $\{r \leq \frac{1}{2}\}$. The function

$$q_0(r) = - \int_0^r \psi^{-1}(t) \int_0^t \psi(s) ds dt$$

satisfies the quasiharmonic equation $\Delta q_0 = -\psi^{-1}(\psi q_0)' = 1$. For $r > 1$,

$$\begin{aligned} q_0(r) &= q_0(1) - \int_1^r e^{-e^t} e^{-t} \left(a + \int_1^t e^{e^s} e^s ds \right) dt \\ &= q_0(1) - \int_1^r (e^{-t} + a_1 e^{-e^t} e^{-t}) dt = O(1) \end{aligned}$$

as $r \rightarrow \infty$. Therefore $q_0 \in QB$, and $M \in \tilde{O}_{QB}^N < \tilde{O}_{QP}^N$.

We next prove that $M \in O_{QL}^N$. Suppose there exists a $q \in QL^1$. Then $|(q, e^{-r})| < \infty$. We may again write $q = q_0 + c + k$, $k \in H$, $k(0) = 0$, and

$$(q, e^{-r}) = (q_0 + c, e^{-r}) = a + b \int_1^\infty (q_0 + c) e^{-r} e^{e^r} dr.$$

Set

$$c_0 = -q_0(\infty) = \int_0^\infty \psi^{-1} \int_0^t \psi ds dt.$$

If $c \neq c_0$, then $\lim_{r \rightarrow \infty} (q_0 + c) = d \neq 0$, and $|(q, e^{-r})| = \infty$. If $c = c_0$, then for $r > 1$

$$q_0 + c_0 = \int_r^\infty (e^{-t} + a_1 e^{-e^t} e^{-t}) dt$$

and

$$\begin{aligned} (q, e^{-r}) &= a + b \int_1^\infty (q_0 + c_0) e^{e^r} dr \\ &= a + b \int_1^\infty \left(e^{-r} + a_1 \int_r^\infty e^{-e^t} e^{-t} dt \right) e^{e^r} dr \\ &= a + b \int_1^\infty e^{-r} e^{e^r} dr + a_1 b \int_1^\infty \left(\int_1^\infty e^{-e^t} e^{-t} dt \right) e^{e^r} e^{-r} dr. \end{aligned}$$

Here

$$\lim_{r \rightarrow \infty} \left(\int_r^\infty e^{-e^t} e^{-t} dt \right) e^{e^r} e^r = \lim_{r \rightarrow \infty} \frac{-e^{-e^r} e^{-r}}{-e^{-e^r} (1 + e^{-r})} = 0,$$

so that for some $R > 0$,

$$(q, e^{-r}) = c + b \int_1^\infty e^{-r} e^{e^r} dr + a_1 b \int_R^\infty o(1) e^{-r} dr.$$

The last integral converges, the first diverges, and therefore $|(q, e^{-r})| = \infty$. This contradiction shows that $q \notin L^1$, that is, $M \in O_{QL^1}^N$.

To see that $M \in O_{QL^p}^N$ for $p > 1$, let p' be determined by $1/p + 1/p' = 1$. For a function $\varphi \in C^\infty(M)$ with

$$\varphi\{r \geq 1\} = (e^{-e^r} e^{-r} r^{-2})^{1/p'}$$

we have

$$\|\varphi\|_{p'}^{p'} = a + b \int_1^\infty e^{-e^r} e^{-r} r^{-2} e^{e^r} e^r dr < \infty,$$

hence $\varphi \in L^{p'}$. If there exists a $q \in QL^p$, then $|(q, \varphi)| < \infty$. But $(q, \varphi) = (q_0 + c, \varphi)$, and if $c \neq c_0$, the integrand in $(q_0 + c, \varphi)$ is asymptotically

$$c_1 (e^{-e^r} e^{-r} r^{-2})^{1/p'} e^{e^r} e^r = c_1 e^{e^r/p} e^{r/p} r^{-2/p'},$$

so that $|(q_0 + c, \varphi)| = \infty$, a contradiction.

In the case $c = c_0$ we observe that for $r > 1$

$$q_0 + c_0 = e^{-r} + a_1 \int_r^\infty e^{-e^t} e^{-t} dt,$$

with

$$\int_r^\infty e^{-e^t} e^{-t} dt < e^{-e^r} \int_r^\infty e^{-t} dt = e^{-e^r} e^{-r},$$

so that $q_0 + c_0 \sim e^{-r}$ as $r \rightarrow \infty$. It follows that the integrand in $(q_0 + c_0, \varphi)$ is asymptotically

$$e^{-r} (e^{-e^r} e^{-r} r^{-2})^{1/p'} e^{e^r} e^r = e^{e^r/p} e^{-r/p'} r^{-2/p'},$$

and therefore $|(q_0 + c_0, \varphi)| = \infty$. This contradiction shows that $M \in O_{QL^p}^N$, and the proof of Theorem 3 is complete.

In the proofs of Theorems 1-3 we have actually shown somewhat more:

$$\begin{aligned} \bigcap_p O_{QL^p}^N \cap O_{QX}^N &\neq \emptyset, \\ \bigcap_p \tilde{O}_{QL^p}^N \cap O_{QX}^N &\neq \emptyset \end{aligned}$$

for $N \geq 2$, $1 < p < \infty$, $X = P, B, D, C$; and

$$\bigcap_p O_{QL^p}^N \cap \tilde{O}_{QX}^N \neq \emptyset$$

for $N \geq 2$, $1 < p < \infty$, $X = P, B$.

4. - Next we are to find an N -manifold which carries QD functions but no QL^p functions. First we consider the case $p > 1$.

THEOREM 4. $O_{QL^p}^N \cap \tilde{O}_{QD}^N \neq \emptyset$ for $N \geq 2$, $1 < p < \infty$.

PROOF. Take the manifold

$$T: \{(x, y_1, \dots, y_{N-1}) | 0 < x < 1, |y_i| \leq \pi, i = 1, \dots, N-1\},$$

with the opposite faces $y_i = \pi$, $y_i = -\pi$ identified as in No. 2, and choose the metric

$$ds^2 = x^{-2\alpha} dx^2 + x^{2\beta/(N-1)} \sum_{i=1}^{N-1} dy_i^2,$$

where α, β are real constants to be specified later; they will depend on p , so that we shall not have a generalization of the theorem in the same manner as at the end of No. 3.

The function

$$q_0(x) = -(\beta - \alpha + 1)^{-1}(-2\alpha + 2)^{-1}x^{-2\alpha+2}$$

satisfies the quasiharmonic equation

$$\Delta q_0 = -x^{\alpha-\beta}(x^{\beta-\alpha}x^{2\alpha}q_0')' = 1$$

provided

$$\beta - \alpha + 1 \neq 0, \quad \alpha \neq 1.$$

The Dirichlet integral is

$$D(q_0) = c \int_0^1 q_0'^2 x^{2\alpha} x^{\beta-\alpha} dx = c_1 \int_0^1 x^{\beta-3\alpha+2} dx < \infty$$

if

$$\beta - 3\alpha + 3 > 0.$$

For the L^p norm we have

$$\|q_0\|_p^p = c \int_0^1 x^{(-2\alpha+2)p} x^{\beta-\alpha} dx = \infty$$

if

$$\beta - (2p + 1)\alpha + 2p + 1 \leq 0.$$

An inspection of the last two inequalities shows that $p = 1$ is ruled out. For

$$p > 1, \quad \alpha > \frac{3}{2}, \quad \beta \in (3(\alpha - 1), (2p + 1)(\alpha - 1)],$$

all four inequalities are satisfied. In particular, $T \in \tilde{O}_{QD}^N$.

The exponent $\beta - \alpha$ in the volume element is positive, and the constant function 1 belongs to L^p for our $p' > 1$. Suppose there exists a $q \in QL^p$. Then $|(q, 1)| < \infty$.

Every $h \in H$ can be written

$$h(x, y) = h_0(x) + \sum'_n f_n(x) G_n(y),$$

where $h_0 \in H$, $f_n G_n \in H$, $n = (n_1, \dots, n_{N-1})$, the n_i integers ≥ 0 , the G_n products of the form

$$G_n(y) = \frac{\cos n_1 y_1}{\sin n_1 y_1} \frac{\cos n_2 y_2}{\sin n_2 y_2} \dots \frac{\cos n_{N-1} y_{N-1}}{\sin n_{N-1} y_{N-1}},$$

and the prime in \sum' indicates that in each term at least one n_i does not vanish. The harmonic equation $\Delta h_0(x) = -x^{\alpha-\beta}(x^{\beta-\alpha} x^{2\alpha} h_0')' = 0$ is satisfied by

$$h_0(x) = ax^{-\alpha-\beta+1} + b.$$

Suppose first $a \neq 0$. Since $-2\alpha + 2 > -\alpha - \beta + 1$,

$$q_0(x) + h_0(x) \sim h_0(x) \quad \text{as } x \rightarrow 0.$$

It follows that the integrand in $(q, 1) = (q_0 + h_0, 1)$ is asymptotically $x^{-\alpha-\beta+1+\beta-\alpha} = x^{-2\alpha+1}$. A fortiori, $|(q, 1)| = \infty$, a contradiction.

Now let $a = 0$, $h_0 = b$. Since

$$\varphi(x) = x^{(-2\alpha+2)/p'} \in L^{p'},$$

$|(q, \varphi)| < \infty$. On the other hand,

$$|(q, \varphi)| = |(q_0 + b, \varphi)| = a_1 + b_1 \int_0^1 x^{-2\alpha+2} x^{(-2\alpha+2)/p'} x^{\beta-\alpha} dx = \infty$$

if

$$-2(\alpha-1) \left(1 + \frac{1}{p'}\right) + \beta - (\alpha-1) \leq 0,$$

i.e.,

$$\beta \leq \left(3 + \frac{2}{p'}\right) (\alpha-1).$$

Since $2p + 1 > 3 + 2/p'$ for $p > 1$, the choice

$$\beta \in \left(3(\alpha-1), \left(3 + \frac{2}{p'}\right) (\alpha-1)\right)$$

gives the contradiction $|(q, \varphi)| = \infty$ while preserving the earlier inequalities. We conclude that $O_{QL^p}^N \cap \tilde{O}_{QD}^N \neq \emptyset$ for all $p > 1$.

5. - For $p = 1$, $O_{QL^p}^N \cap \tilde{O}_{QD}^N \neq \emptyset$ is no longer true. In fact, we even have a strict inclusion:

THEOREM 5. $O_{QL^1}^N < O_{QD}^N$ for $N \geq 2$.

PROOF. To prove the inclusion relation $O_{QL^1}^N \subset O_{QD}^N$, suppose $u \in QD$. For any regular subregion Ω , the Riesz decomposition yields (cf. e.g. Nakai-Sario [3])

$$u(x) = h_\Omega(x) + \int_\Omega g_\Omega(x, y) dy$$

on Ω , where $h_\Omega(x)$ is the harmonic function on Ω with $h_\Omega = u$ on $\partial\Omega$, and $g_\Omega(x, y)$ is the Green's function on Ω with pole y . By Stokes' formula,

$$\int_\Omega \int_\Omega g_\Omega(x, y) dy dx \leq D_\Omega(u).$$

On letting $\Omega \rightarrow R$ we obtain

$$\int_R \int_R g(x, y) dy dx \leq D(u) < \infty,$$

where g is the Green function on R . Since $\Delta_x \int_R g(x, y) dy = 1$, we have $\int_R g(x, y) dy \in QL^1$, and therefore $O_{QL^1}^N \subset O_{QD}^N$. By Theorem 2, $\tilde{O}_{QL^1}^N \cap O_{QD}^N \neq \emptyset$, hence $O_{QL^1}^N \subset O_{QD}^N$.

6. – It remains to consider the class QC . Here we have the most elegant case, as there is strict inclusion for all p :

THEOREM 6. $O_{QL^p}^N \subset O_{QC}^N$ for $N \geq 2, p \geq 1$.

PROOF. In view of Theorem 2, it suffices to show that $O_{QL^p}^N \subset O_{QC}^N$. Suppose $R \notin O_{QC}^N$, and take a $u \in QC$. The Riesz decomposition of u on Ω implies

$$\int_{\Omega} g_{\Omega}(x, y) dy \leq |u(x)| + |h_{\Omega}(x)| \leq 2 \sup_{\Omega} |u(x)|.$$

On letting $\Omega \rightarrow R$ we obtain $\int_R g(x, y) dy \in B$. From the proof of Theorem 5 we conclude that $\int_R g(x, y) dy \in C$. Let

$$R_1 = \{x \in R \mid \int_R g(x, y) dy > 1\}.$$

Then

$$V(R_1) = \int_{R_1} dx \leq \int_R \int_R g(x, y) dy dx < \infty.$$

For $p \geq 1$,

$$\begin{aligned} \int_R \left(\int_R g(x, y) dy \right)^p dx &= \int_{R_1} \left(\int_R g(x, y) dy \right)^p dx + \int_{R-R_1} \left(\int_R g(x, y) dy \right)^p dx \\ &\leq M V(R_1) + \int_R \int_R g(x, y) dy dx < \infty, \end{aligned}$$

and therefore $\int_R g(x, y) dy \in QL^p$.

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