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A Class of Pseudo Differential Operators on the Product of Two Manifolds and Applications.

LUIGI RODINO (*) (**)

Introduction.

Some particular linear singular integral operators on the product of two compact manifolds have been recently studied in [7], [8], [9] under the name of « bisingular operators ». These operators appear in connection with a boundary value problem for functions of two complex variables in bicylinders, which leads to a « bisingular equation » on the distinguished boundary (see [7], [10]). We can easily express a bisingular operator A in the form of pseudo differential operator, that is, in local coordinates and with the usual notations:

$$Au = (2\pi)^{-n}\int \exp\,[i\langle x, \xi\rangle] a(x, \xi)\hat{u}(\xi)\, d\xi \tag{0.1}$$

where, in the present case, $x = (x_1, x_2)$, $x_1 \in \Omega_1 \subset \boldsymbol{R}^{n_1}$, $x_2 \in \Omega_2 \subset \boldsymbol{R}^{n_2}$ and $\xi = (\xi_1, \xi_2) \in \boldsymbol{R}^n$, $n = n_1 + n_2$.

However, A need not be a classical pseudo differential operator; particularly, the symbol $a(x, \xi)$ is not in general in any of the classes of Hörmander [5], $S^m_{\varrho,\delta}(\Omega_1 \times \Omega_2 \times \boldsymbol{R}^{n_1+n_2})$, $\varrho > \delta$.

Our aim is to construct an algebra of pseudo differential operators containing the bisingular operators in [7], [8], [9]. We shall introduce in $\Omega_1 \times \Omega_2$ the classes of symbols $S^{m_1,m_2}(\Omega_1 \times \Omega_2)$ defined by the inequalities:

$$|D^{\alpha_1}_{x_1} D^{\alpha_2}_{x_2} D^{\beta_1}_{\xi_1} D^{\beta_2}_{\xi_2} a(x_1, x_2, \xi_1, \xi_2)| \leqslant c_{\alpha_1,\alpha_2,\beta_1,\beta_2,K}(1 + |\xi_1|)^{m_1-|\beta_1|}(1 + |\xi_2|)^{m_2-|\beta_2|} \tag{0.2}$$

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for x_1, x_2 in a fixed compact subset $K \subset \Omega_1 \times \Omega_2$. In section 1 we develop the symbolic calculus and we study the boundedness of pseudo differential operators with symbol in the preceding classes; in section 2 the symbolic calculus is specified for operators with homogeneous principal symbol (see def. 2.2).

All the statement and the proofs are modelled on the theory of classical pseudo differential operators and in particular on Hörmander [5], chapter II; however, it will be not convenient to identify the symbol with the function $a(x_1, x_2, \xi_1, \xi_2)$ in (0.2) but rather with the two maps:

$$\begin{cases} \sigma_1\colon\ x_1, \xi_1 \to a(x_1, x_2, \xi_1, D_2) \\ \sigma_2\colon\ x_2, \xi_2 \to a(x_1, x_2, D_1, \xi_2) \end{cases} \tag{0.3}$$

from $\Omega_1 \times \boldsymbol{R}^{n_1}$ to $L^{m_2}_{1,0}(\Omega_2)$ and from $\Omega_2 \times \boldsymbol{R}^{n_2}$ to $L^{m_1}_{1,0}(\Omega_1)$ (see the related definition of symbol of a bisingular operator in [7] and [8]). With this understanding we will find the usual property, that the symbol of the product of two operators is the product of the symbols (def. 2.3, th. 2.5).

In section 3 we consider operators on the product of two compact manifolds $X_1 \times X_2$; the principal symbol of an operator A will be a couple of homogeneous maps:

$$\begin{cases} \sigma_1\colon\ T^*(X_1) \to L^{m_2}_{1,0}(X_2) \\ \sigma_2\colon\ T^*(X_2) \to L^{m_1}_{1,0}(X_1)\,. \end{cases} \tag{0.3'}$$

It follows from the symbolic calculus that A is Fredholm if each operator of the two families σ_1 and σ_2 is exactly invertible in $L^{m_2}_{1,0}(X_2)$ and $L^{m_1}_{1,0}(X_1)$ respectively (th. 3.2): when we assume $m_1 = m_2 = 0$ in (0.2), this gives the results in [7], [9] about the bisingular operators.

In section 4 we present two applications. First we study the tensor product of complexes as in Atiyah-Singer [1]: we shall check that the tensor product of two elliptic complexes of pseudo differential operators of order zero is actually an elliptic complex in our algebra (note that the method of approximation in [1], proposition (5.4), fails for operators of order zero). In the second application we extend to systems the results about the boundary value problem in [7], [10].

Finally, I thank Professor G. Geymonat, who suggested the argument of the research, and Professor L. Hörmander, who guided the work.

1. – We write $x_i = (x_i^1, \dots, x_i^{n_i})$ for the coordinate in $\boldsymbol{R}^{n_i}$, $i = 1, 2$, and $\xi_i = (\xi_i^1, \dots, \xi_i^{n_i})$ for the dual coordinate; $\alpha_i = (\alpha_i^1, \dots, \alpha_i^{n_i})$ is an n_i-tuple of nonnegative integers; likewise we use in $\boldsymbol{R}^{n_i}$ the other standard notation of the theory of pseudo differential operators, as in [4], [5]. Particularly, $S^m_{\varrho,\delta}$ and $L^m_{\varrho,\delta}$ are the classes of symbols and operators in [5], chapters I and II.

DEFINITION 1.1. *Let Ω_i be an open subset of $\boldsymbol{R}^{n_i}$, $i=1,2$; we denote by $S^{m_1,m_2}(\Omega_1\times\Omega_2)$ the set of all $a\in C^\infty(\Omega_1\times\Omega_2\times\boldsymbol{R}^{n_1+n_2})$ such that for every compact set $K\subset\Omega_1\times\Omega_2$ and all multiorders α_1, α_2, β_1, β_2, the estimate:*

$$|D^{\alpha_1}_{x_1}D^{\alpha_2}_{x_2}D^{\beta_1}_{\xi_1}D^{\beta_2}_{\xi_2}a(x_1,x_2,\xi_1,\xi_2)|\leqslant c_{\alpha_1,\alpha_2,\beta_1,\beta_2,K}(1+|\xi_1|)^{m_1-|\beta_1|}(1+|\xi_2|)^{m_2-|\beta_2|} \tag{1.1}$$

is valid, for some constant $c_{\alpha_1,\alpha_2,\beta_1,\beta_2,K}$, $x_1,x_2\in K$, $\xi_1\in\boldsymbol{R}^{n_1}$, $\xi_2\in\boldsymbol{R}^{n_2}$.

We associate to every $a\in S^{m_1,m_2}(\Omega_1\times\Omega_2)$ the two maps:

$$x_1,\xi_1\to a(x_1,x_2,\xi_1,D_2)\,,\qquad \text{from } \Omega_1\times\boldsymbol{R}^{n_1} \text{ into } L^{m_2}_{1.0}(\Omega_2)$$

and

$$x_2,\xi_2\to a(x_1,x_2,D_1,\xi_2)\,,\qquad \text{from } \Omega_2\times\boldsymbol{R}^{n_2} \text{ into } L^{m_1}_{1.0}(\Omega_1)\,,$$

where:

$$\left\{\begin{aligned}&a(x_1,x_2,\xi_1,D_2)\varphi=(2\pi)^{-n_2}\!\int\exp[i\langle x_2,\xi_2\rangle]a(x_1,x_2,\xi_1,\xi_2)\hat\varphi(\xi_2)d\xi_2\\&a(x_1,x_2,D_1,\xi_2)\psi=(2\pi)^{-n_1}\!\int\exp[i\langle x_1,\xi_1\rangle]a(x_1,x_2,\xi_1,\xi_2)\hat\psi(\xi_1)d\xi_1\\&\varphi\in C^\infty_0(\Omega_2)\,,\qquad \psi\in C^\infty_0(\Omega_1)\,.\end{aligned}\right. \tag{1.1'}$$

Reciprocally, the symbol a is uniquely determined by one of these maps.

Note that if $a\in S^{m_1,m_2}(\Omega_1\times\Omega_2)$, then $D^{\alpha_1}_{x_1}D^{\alpha_2}_{x_2}D^{\beta_1}_{\xi_1}D^{\beta_2}_{\xi_2}a$ is in $S^{m_1-|\beta_1|,m_2-|\beta_2|}(\Omega_1\times\Omega_2)$; if b is in $S^{p_1,p_2}(\Omega_1\times\Omega_2)$ then ba is in $S^{m_1+p_1,m_2+p_2}(\Omega_1\times\Omega_2)$.

Note also that:

$$S^{m_1,m_2}(\Omega_1\times\Omega_2)\subset S^m_{0.0}(\Omega_1\times\Omega_2\times\boldsymbol{R}^{n_1+n_2})\,,\qquad m=\max(m_1,m_2,m_1+m_2)\,,$$

and this is in general the best possible inclusion in the classes of Hörmander [5] on $\Omega_1\times\Omega_2\times\boldsymbol{R}^{n_1+n_2}$.

EXAMPLE 1.2. If a_i is in the class $S^{m_i}_{1.0}(\Omega_i\times\boldsymbol{R}^{n_i})$, $i=1,2$, the product $a=a_1a_2$ is in $S^{m_1,m_2}(\Omega_1\times\Omega_2)$.

Let $u(x_1,x_2)\in C^\infty_0(\Omega_1\times\Omega_2)$; we define the operator:

$$\begin{aligned}&a(x_1,x_2,D_1,D_2)u=\\&\quad=(2\pi)^{-n_1-n_2}\!\int\exp\big[i(\langle x_1,\xi_1\rangle+\langle x_2,\xi_2\rangle)\big]\,a(x_1,x_2,\xi_1,\xi_2)\hat u(\xi_1,\xi_2)\,d\xi_1d\xi_2\end{aligned} \tag{1.2}$$

where $a(x_1, x_2, \xi_1, \xi_2)$ is in $S^{m_1,m_2}(\Omega_1 \times \Omega_2)$ and $\hat{u}(\xi_1, \xi_2)$ is the Fourier transform of u in $\boldsymbol{R}^{n_1+n_2}$; $a(x_1, x_2, D_1, D_2)$ is a continuous linear map of $C_0^\infty(\Omega_1 \times \Omega_2)$ into $C^\infty(\Omega_1 \times \Omega_2)$.

DEFINITION 1.3. *We write $L^{m_1,m_2}(\Omega_1 \times \Omega_2)$ for the class of operators of the form* (1.2); *$a(x_1, x_2, \xi_1, \xi_2)$ is called the symbol of $a(x_1, x_2, D_1, D_2)$.*

Now we shall study the composition of two operators in the classes $L^{m_1,m_2}(\Omega_1 \times \Omega_2)$ and the effect of a change of variables. For simplicity we assume that symbols have compact support in the x_1, x_2 variables; then, $a(x_1, x_2, D_1, D_2)$ is a continuous linear map of $C_0^\infty(\Omega_1 \times \Omega_2)$ into $C_0^\infty(\Omega_1 \times \Omega_2)$ and the composition is well defined. At first we introduce some operations on symbols.

DEFINITION 1.4. *Let $a \in S^{m_1,m_2}(\Omega_1 \times \Omega_2)$ and $b \in S^{p_1,p_2}(\Omega_1 \times \Omega_2)$ with compact support in the x_1, x_2 variables; the two symbols $(b \circ_{\xi_1} a)(x_1, x_2, \xi_1, \xi_2)$ and $(b \circ_{\xi_2} a)(x_1, x_2, \xi_1, \xi_2)$ are defined in $S^{m_1+p_1,m_2+p_2}(\Omega_1 \times \Omega_2)$ by the relations:*

$$
(1.3) \qquad \begin{cases} (b \circ_{\xi_1} a)(x_1, x_2, D_1, \xi_2)\psi = b(x_1, x_2, D_1, \xi_2)a(x_1, x_2, D_1, \xi_2)\psi \\ (b \circ_{\xi_2} a)(x_1, x_2, \xi_1, D_2)\varphi = b(x_1, x_2, \xi_1, D_2)a(x_1, x_2, \xi_1, D_2)\varphi \\ \psi \in C_0^\infty(\Omega_1), \qquad \varphi \in C_0^\infty(\Omega_2) \end{cases}
$$

where in the right products of operators in $L_{1,0}^{m_i}(\Omega_i)$ and $L_{1,0}^{p_i}(\Omega_i)$ are considered.

DEFINITION 1.5. *Let $a \in S^{m_1,m_2}(\Omega_1 \times \Omega_2)$ with compact support in the x_1, x_2 variables; let Ω_1' be an open subset of $\boldsymbol{R}^{n_1}$ and let $k_1 \colon \Omega_1 \to \Omega_1'$ be a diffeomorphism of Ω_1 into Ω_1'; the symbol $a_{k_1}(x_1' x_2, \xi_1', \xi_2)$ is defined in $S^{m_1,m_2}(\Omega_1' \times \Omega_2)$ by the relation:*

$$
(1.4) \qquad a_{k_1}(x_1', x_2, D_1, \xi_2)\psi = [a(x_1, x_2, D_1, \xi_2)(\psi \circ k_1)] \circ k_1^{-1} \qquad \psi \in C_0^\infty(\Omega_1')
$$

where $\psi(x_1') \circ k_1 = \psi(k_1(x_1))$ and k_1^{-1} is the inverse of k_1. If Ω_2' is an open subset of $\boldsymbol{R}^{n_2}$ and $k_2 \colon \Omega_2 \to \Omega_2'$ is a diffeomorphism, we define in the same way $a_{k_2}(x_1, x_2', \xi_1, \xi_2')$ in $S^{m_1,m_2}(\Omega_1 \times \Omega_2')$.

REMARKS. It is easy to verify that (1.3) and (1.4) define actually symbols in the sense of definition 1.1.

Note also that, in view of formula (2.1.9) in [5], the symbol

$$
b \circ_{\xi_1} a - \sum_{|\alpha_1| < N_1} \frac{1}{\alpha_1!} \partial_{\xi_1}^{\alpha_1} b D_{x_1}^{\alpha_1} a
$$

is in $S^{m_1+p_1-N_1,m_2+p_2}(\Omega_1\times\Omega_2)$. In view of formula (2.1.14) in [5], the symbol:

$$a_{k_1}(x_1',x_2,\xi_1',\xi_2)-a(k_1^{-1}(x_1'),x_2,{}^tk_1'\xi_1',\xi_2)$$

is in $S^{m_1-1,m_2}(\Omega_1\times\Omega_2)$.

THEOREM 1.6. *Let $a\in S^{m_1,m_2}(\Omega_1\times\Omega_2)$ and $b\in S^{p_1,p_2}(\Omega_1\times\Omega_2)$ with compact support in the x_1, x_2 variables; the composition $b(x_1,x_2,D_1,D_2)a(x_1,x_2,D_1,D_2)$ defined on $C_0^\infty(\Omega_1\times\Omega_2)$ is an operator in $L^{m_1+p_1,m_2+p_2}(\Omega_1\times\Omega_2)$. Moreover, its symbol $c(x_1,x_2,\xi_1,\xi_2)$ has the asymptotic expansion:*

$$c\sim\sum_{j=0}^{\infty}c_{m_1+p_1-j,m_2+p_2-j} \tag{1.5}$$

where:

$$c_{m_1+p_1-j,m_2+p_2-j}=d'_{m_1+p_1-j,m_2+p_2-j}+d''_{m_1+p_1-j-1,m_2+p_2-j}+d'''_{m_1+p_1-j,m_2+p_2-j-1} \tag{1.6}$$

$$d'_{m_1+p_1-j,m_2+p_2-j}=\sum_{|\alpha_1|=|\alpha_2|=j}\frac{1}{\alpha_1!\,\alpha_2!}\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}b\,D_{x_1}^{\alpha_1}D_{x_2}^{\alpha_2}a$$

$$d''_{m_1+p_1-j-1,m_2+p_2-j}=\sum_{|\alpha_2|=j}\frac{1}{\alpha_2!}\left\{\partial_{\xi_2}^{\alpha_2}b\circ_{\xi_1}D_{x_2}^{\alpha_2}a-\sum_{|\alpha_1|\leqslant j}\frac{1}{\alpha_1!}\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}b\,D_{x_1}^{\alpha_1}D_{x_2}^{\alpha_2}a\right\}$$

$$d'''_{m_1+p_1-j,m_2+p_2-j-1}=\sum_{|\alpha_1|=j}\frac{1}{\alpha_1!}\left\{\partial_{\xi_1}^{\alpha_1}b\circ_{\xi_2}D_{x_1}^{\alpha_1}a-\sum_{|\alpha_2|\leqslant j}\frac{1}{\alpha_2!}\partial_{\xi_1}^{\alpha_1}\partial_{\xi_2}^{\alpha_2}b\,D_{x_1}^{\alpha_1}D_{x_2}^{\alpha_2}a\right\}$$

$c_{p,q}$ and $d'_{p,q}$, $d''_{p,q}$, $d'''_{p,q}$ are in $S^{p,q}(\Omega_1\times\Omega_2)$ and the (1.5) means that, for any

$$N>0,\ r_N=c-\sum_{j<N}c_{m_1+p_1-j,m_2+p_2-j}\ \textit{is in}\ S^{m_1+p_1-N,m_2+p_2-N}(\Omega_1\times\Omega_2).$$

PROOF. Direct computation shows that:

$$b(x_1,x_2,D_1,D_2)a(x_1,x_2,D_1,D_2)u=$$
$$=(2\pi)^{-n_1-n_2}\int\exp\left[i(\langle x_1,\xi_1\rangle+\langle x_2,\xi_2\rangle)\right]c(x_1,x_2,\xi_1,\xi_2)\hat{u}(\xi_1,\xi_2)d\xi_1d\xi_2$$

where

$$c(x_1,x_2,\xi_1,\xi_2)= \tag{1.7}$$
$$=(2\pi)^{-n_1-n_2}\int\exp\left[-i(\mu_1+\mu_2)\right]b(x_1,x_2,\eta_1,\eta_2)a(y_1,y_2,\xi_1,\xi_2)dy_1dy_2d\eta_1d\eta_2$$

with

$$\mu_1=\langle y_1-x_1,\eta_1-\xi_1\rangle$$
$$\mu_2=\langle y_2-x_2,\eta_2-\xi_2\rangle.$$

To find the asymptotic expansion (1.5), we put in (1.7) the following development of $b(x_1, x_2, \eta_1, \eta_2)$:

$$b(x_1, x_2, \eta_1, \eta_2) = b_N^1 + b_N^2 + b_N^3 + R_N \tag{1.8}$$

where:

$$\left\{\begin{aligned} b_N^1 &= \sum_{|\alpha_1|<N} \frac{1}{\alpha_1!} (\eta_1 - \xi_1)^{\alpha_1} \partial_{\xi_1}^{\alpha_1} b(x_1, x_2, \xi_1, \eta_2) \\ b_N^2 &= \sum_{|\alpha_2|<N} \frac{1}{\alpha_2!} (\eta_2 - \xi_2)^{\alpha_2} \partial_{\xi_2}^{\alpha_2} b(x_1, x_2, \eta_1, \xi_2) \\ b_N^3 &= \sum_{\substack{|\alpha_1|<N \\ |\alpha_2|<N}} \frac{1}{\alpha_1!\,\alpha_2!} (\eta_1 - \xi_1)^{\alpha_1} (\eta_2 - \xi_2)^{\alpha_2} \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} b(x_1, x_2, \xi_1, \xi_2) \\ R_N &= \sum_{|\alpha_1|=|\alpha_2|=N} \frac{N^2}{\alpha_1!\,\alpha_2!} (\eta_1 - \xi_1)^{\alpha_1} (\eta_2 - \xi_2)^{\alpha_2} \cdot \\ &\cdot \int_0^1\!\!\int_0^1 (1-t_1)^{N-1}(1-t_2)^{N-1} \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} b\big(x_1, x_2, \xi_1 + t_1(\eta_1 - \\ &\qquad - \xi_1), \xi_2 + t_2(\eta_2 - \xi_2)\big)\, dt_1\, dt_2\,. \end{aligned}\right. \tag{1.9}$$

We have:

$$c = c_N^1 + c_N^2 + c_N^3 + r_N$$

where:

$$c_N^i = (2\pi)^{-n_1-n_2} \int \exp[-i(\mu_1 + \mu_2)]\, b_N^i a\, dy_1\, dy_2\, d\eta_1\, d\eta_2 \qquad i = 1, 2, 3 \tag{1.10}$$

$$r_N = (2\pi)^{-n_1-n_2} \int \exp[-i(\mu_1 + \mu_2)]\, R_N a\, dy_1\, dy_2\, d\eta_1\, d\eta_2\,. \tag{1.11}$$

To compute c_N^1, in (1.10) we use the formula

$$(\eta_1 - \xi_1)^{\alpha_1} \exp[-i(\mu_1 + \mu_2)] = (-1)^{|\alpha_1|} D_{y_1}^{\alpha_1} \exp[-i(\mu_1 + \mu_2)]\,.$$

Then we integrate by parts: the result is

$$c_N^1 = \sum_{|\alpha_1|<N} \frac{(2\pi)^{-n_2}}{\alpha_1!} \int \exp[-i\mu_2]\, \partial_{\xi_1}^{\alpha_1} b(x_1, x_2, \xi_1, \eta_2)\, D_{x_1}^{\alpha_1} a(x_1, y_2, \xi_1, \xi_2)\, dy_2\, d\eta_2\,.$$

Hence:

$$c_N^1 = \sum_{|\alpha_1|<N} \frac{1}{\alpha_1!} \partial_{\xi_1}^{\alpha_1} b \circ_{\xi_2} D_{x_1}^{\alpha_1} a\,.$$

Likewise we find:

$$c_N^2 = \sum_{|\alpha_2|<N} \frac{1}{\alpha_2!} \partial_{\xi_2}^{\alpha_2} b \circ_{\xi_1} D_{x_2}^{\alpha_2} a \, .$$

To compute c_N^3, we use the formula

$$(\eta_1 - \xi_1)^{\alpha_1}(\eta_2 - \xi_2)^{\alpha_2} \exp[-i(\mu_1 + \mu_2)] = (-1)^{|\alpha_1|+|\alpha_2|} D_{y_1}^{\alpha_1} D_{y_2}^{\alpha_2} \exp[-i(\mu_1 + \mu_2)]$$

and we integrate by parts in (1.10). We have

$$c_N^3 = \sum_{\substack{|\alpha_1|<N \\ |\alpha_2|<N}} \frac{1}{\alpha_1!\,\alpha_2!} \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} b \, D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} a \, .$$

Now, if we re-arrange the terms in $c_N^1 + c_N^2 + c_N^3$, we obtain

$$\sum_{j<N} c_{m_1+p_1-j, m_2+p_2-j} \, .$$

To estimate r_N, write in (1.11):

$$\text{(1.12)} \qquad \exp[-i(\mu_1 + \mu_2)] =$$
$$= (1 + |\eta_1 - \xi_1|^2)^{-M} (1 + |\eta_2 - \xi_2|^2)^{-M} (1 - \Delta_{y_2})^M (1 - \Delta_{y_1})^M \exp[-i(\mu_1 + \mu_2)]$$

and integrate by parts. Then note that:

$$|R_N| \leqslant C(1 + |\xi_1|)^{m_1-N} (1 + |\xi_2|)^{m_2-N} (1 + |\eta_1 - \xi_1|^2)^N (1 + |\eta_2 - \xi_2|^2)^N \, .$$

If we choose $M > N + \max(n_1, n_2) + 1$, we have from (1.11):

$$|r_N(x_1, x_2, \xi_1, \xi_2)| \leqslant C(1 + |\xi_1|)^{m_1+p_1-N} (1 + |\xi_2|)^{m_2+p_2-N} \, .$$

Similar estimates provide bounds for $D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} D_{\xi_1}^{\beta_1} D_{\xi_2}^{\beta_2} r_N$: hence r_N is in $S^{m_1+p_1-N, m_2+p_2-N}(\Omega_1 \times \Omega_2)$ and the theorem 1.6 is proved.

Later on we shall only deal with the first term of the expansion (1.5):

$$\text{(1.13)} \qquad c(x_1, x_2, \xi_1, \xi_2) = d'_{m_1+p_1, m_2+p_2} + d''_{m_1+p_1-1, m_2+p_2} + d'''_{m_1+p_1, m_2+p_2-1} + r_1$$
$$= ba + (b \circ_{\xi_1} a - ba) + (b \circ_{\xi_2} a - ba) + r_1 \, .$$

THEOREM 1.7. *Let $a \in S^{m_1, m_2}(\Omega_1 \times \Omega_2)$ with compact support in the x_1, x_2 variables; let Ω_i' be an open subset of $\mathbf{R}^{n_i}$, $i = 1, 2$, and let $k_i \colon \Omega_i \to \Omega_i'$ be* *a diffeomorphism of* Ω_i *into* Ω_i'; *denote* $k_1 \times k_2 \colon \Omega_1 \times \Omega_2 \to \Omega_1' \times \Omega_2'$ *the diffeomorphism product. The operator*:

$$(1.14) \qquad [a(x_1, x_2, D_1, D_2)(v \circ k_1 \times k_2)] \circ (k_1 \times k_2)^{-1} \qquad v \in C_0^\infty(\Omega_1' \times \Omega_2')$$

is in $L^{m_1, m_2}(\Omega_1' \times \Omega_2')$ *with symbol in* $S^{m_1, m_2}(\Omega_1' \times \Omega_2')$:

$$(1.15) \qquad a'(x_1', x_2', \xi_1', \xi_2') = e_{m_1, m_2} + e_{m_1-1, m_2} + e_{m_1, m_2-1} + s_{m_1-1, m_2-1} \text{ where:}$$

$$(1.16) \qquad \begin{cases} e_{m_1, m_2} = a\big(k_1^{-1}(x_1'), k_2^{-1}(x_2'), {}^t k_1' \xi_1', {}^t k_2' \xi_2'\big) \\ e_{m_1-1, m_2} = a_{k_1}\big(x_1', k_2^{-1}(x_2'), \xi_1', {}^t k_2' \xi_2'\big) - e_{m_1, m_2} \\ e_{m_1, m_2-1} = a_{k_2}\big(k_1^{-1}(x_1'), x_2', {}^t k_1' \xi_1', \xi_2'\big) - e_{m_1, m_2} \end{cases}$$

and s_{m_1-1, m_2-1} *is in* $S^{m_1-1, m_2-1}(\Omega_1' \times \Omega_2')$.

Proof. By an usual argument due to M. Kuranishi the operator (1.14) can be expressed by:

$$(1.17) \qquad (2\pi)^{-n_1-n_2} \int \exp[i\nu] q v(y_1', y_2') \, dy_1' \, dy_2' \, d\xi_1' \, d\xi_2'$$

with:

$$\nu = \langle \xi_1', x_1' - y_1' \rangle + \langle \xi_2', x_2' - y_2' \rangle$$

and with:

$$q = q(x_1', x_2', y_1', y_2', \xi_1', \xi_2') = \\ = a\big(k_1^{-1}(x_1'), k_2^{-1}(x_2'), \psi_1(x_1', y_1')\xi_1', \psi_2(x_2', y_2')\xi_2'\big) \frac{|\det \psi_1 \det \psi_2|}{|\det k_1' \det k_2'|}$$

where ψ_i, $i = 1, 2$, is a convenient matrix and $\psi_i(x_i', x_i') = {}^t k_i'$: We put in (1.17) the following development of q:

$$q(x_1', x_2', y_1', y_2', \xi_1', \xi_2') = -q_0 + q_1 + q_2 + S$$

where:

$$(1.18) \qquad \begin{cases} q_0 = q(x_1', x_2', x_1', x_2', \xi_1', \xi_2') = a\big(k_1^{-1}(x_1'), k_2^{-1}(x_2'), {}^t k_1' \xi_1', {}^t k_2' \xi_2'\big) \\ q_1 = q(x_1', x_2', x_1', y_2', \xi_1', \xi_2') = \\ \qquad = a\big(k_1^{-1}(x_1'), k_2^{-1}(x_2'), {}^t k_1' \xi_1', \psi_2(x_2', y_2')\xi_2'\big) \dfrac{|\det \psi_2|}{|\det k_2'|} \\ q_2 = q(x_1', x_2', y_1', x_2', \xi_1', \xi_2') = \\ \qquad = a\big(k_1^{-1}(x_1'), k_1^{-1}(x_2'), \psi_1(x_1', y_1')\xi_1', {}^t k_2' \xi_2'\big) \dfrac{|\det \psi_1|}{|\det k_1'|} \end{cases}$$

and

$$(1.19)\qquad S = S(x_1', x_2', y_1', y_2', \xi_1', \xi_2') = \sum_{\substack{|\alpha_2|=1\\|\alpha_1|=1}} (y_1' - x_2')^{\alpha_1} (y_2' - x_2')^{\alpha_2} \cdot$$

$$\cdot \int_0^1\!\!\int_0^1 \partial_{y_1'}^{\alpha_1} \partial_{y_2'}^{\alpha_2} q(x_1', x_2', x_1' + t_1(y_1' - x_1'), x_2' + t_2(y_2' - x_2'), \xi_1', \xi_2')\, dt_1\, dt_2 .$$

First we have

$$(2\pi)^{-n_1-n_2}\int \exp[iv] q_0 v(y_1', y_2')\, dy_1' dy_2' d\xi_1' d\xi_2' = (2\pi)^{-n_1-n_2}\int \exp\left[i(\langle x_1', \xi_1'\rangle + \right.$$

$$\left. + \langle x_2', \xi_2'\rangle)\right] a(k_1^{-1}(x_1'), k_2^{-1}(x_2'), {}^t k_1' \xi_1', {}^t k_2' \xi_2') \hat{v}(\xi_1', \xi_2')\, d\xi_1' d\xi_2' .$$

Secondly, if we keep in mind the effect of a change of variables for the pseudo differential operators in $L_{1,0}^{m_1}(\Omega_1)$, $L_{1,0}^{m_2}(\Omega_2)$ (see [5]), in view of the definition 1.5 we obtain:

$$(2\pi)^{-n_1-n_2}\int \exp[iv] q_1 v(y_1', y_2')\, dy_1' dy_2' d\xi_1' d\xi_2' =$$

$$= (2\pi)^{-n_1-n_2}\int \exp\left[i(\langle x_1', \xi_1'\rangle + \langle x_2', \xi_2'\rangle)\right] a_{k_2}(k_1^{-1}(x_1'), x_2', {}^t k_1' \xi_1', \xi_2') \hat{v}(\xi_1', \xi_2')\, d\xi_1' d\xi_2'$$

and

$$(2\pi)^{-n_1-n_2}\int \exp[iv] q_2 v(y_1', y_2')\, dy_1' dy_2' d\xi_1' d\xi_2' =$$

$$= (2\pi)^{-n_1-n_2}\int \exp\left[i(\langle x_1', \xi_1'\rangle + \langle x_2', \xi_2'\rangle) a_{k_1}(x_1', k_2^{-1}(x_2'), \xi_1', {}^t k_2' \xi_2') \hat{v}(\xi_1', \xi_2')\, d\xi_1' d\xi_2' .$$

Finally, if we use an argument similar to the proof of the theorem 2.1.1 in [5], it follows from (1.19):

$$(2\pi)^{-n_1-n_2}\int \exp[iv] S v(y_1', y_2')\, dy_1' dy_2' d\xi_1' d\xi_2' =$$

$$(2\pi)^{-n_1-n_2}\int \exp\left[i(\langle x_1', \xi_1'\rangle + \langle x_2', \xi_2'\rangle)\right] s_{m_1-1, m_2-1}(x_1', x_2', \xi_1', \xi_2') \hat{v}(\xi_1', \xi_2')\, d\xi_1' d\xi_2'$$

with $s_{m_1-1, m_2-1} \in S^{m_1-1, m_2-1}(\Omega_1' \times \Omega_2')$.

The proof of theorem 1.7 is complete.

Now we shall study the boundedness of operators in $L^{m_1, m_2}(\Omega_1 \times \Omega_2)$. $H^{s_1, s_2}(\boldsymbol{R}^{n_1+n_2})$ will denote the space of all $u \in \mathfrak{D}'(\boldsymbol{R}^{n_1+n_2})$ such that:

$$(1.20)\qquad \|u\|_{s_1, s_2} = (2\pi)^{-n_1-n_2}\int (1 + |\xi_1|)^{s_1} (1 + |\xi_2|)^{s_2} |\hat{u}(\xi_1, \xi_2)|^2\, d\xi_1\, d\xi_2 < \infty .$$

THEOREM 1.8. *Every* $a(x_1, x_2, D_1, D_2)$ *in* $L^{m_1,m_2}(\Omega_1\times\Omega_2)$ *is linear continuous from* $H^{s_1,s_2}_{\text{comp}}(\Omega_1\times\Omega_2)$ *to* $H^{s_1-m_1,s_2-m_2}_{\text{loc}}(\Omega_1\times\Omega_2)$.

PROOF. We can easily restrict ourselves to proof that $a(x_1, x_2, D_1, D_2)$ in $L^{0,0}(\Omega_1\times\Omega_2)$ is continuous from $H^{0,0}_{\text{comp}}(\Omega_1\times\Omega_2)=L^2_{\text{comp}}(\Omega_1\times\Omega_2)$ to $H^{0,0}_{\text{loc}}(\Omega_1\times\Omega_2)=L^2_{\text{loc}}(\Omega_1\times\Omega_2)$. Then, in view of the inclusion in the note after the definition 1.1, $a(x_1, x_2, \xi_1, \xi_2)$ is in $S^0_{0,0}(\Omega_1\times\Omega_2\times\mathbf{R}^{n_1+n_2})$ and the usual arguments for boundedness of pseudo differential operators hold.

2. – Now we shall study in more detail operators in $L^{m_1,m_2}(\Omega_1\times\Omega_2)$ with homogeneous principal symbol (see the following definitions 2.1 and 2.2).

Later on we shall note by $HL^m(\Omega_i)$, $i=1,2$, the set of all the $a(x_i, D_i)\in L^m_{1,0}(\Omega_i)$ with homogeneous principal symbol $a^0(x_i, \xi_i)$ (this means that we assume the existence of a C^∞ homogeneous function $a^0(x_i, \xi_i)$ of degree m on $\Omega_i\times\mathbf{R}^{n_i}$ such that $a-a^0\in S^{m-1}_{1,0}(\Omega_i\times\mathbf{R}^{n_i})$).

DEFINITION 2.1. *We denote by* $\Sigma^{m_1,m_2}(\Omega_1\times\Omega_2)$ *the set of all couples* $\{\sigma_1, \sigma_2\}$ *such that:*

1) $\sigma_1\in C^\infty(\Omega_1\times\Omega_2\times\{\mathbf{R}^{n_1}\setminus 0\}\times\mathbf{R}^{n_2})$ *and, for* $t>0$:

$$\sigma_1(x_1, x_2, t\xi_1, \xi_2)=t^{m_1}\sigma_1(x_1, x_2, \xi_1, \xi_2)\,.$$

Moreover, for all $x_1, \xi_1, \xi_1\neq 0$, $\sigma_1(x_1, x_2, \xi_1, D_2)$, *defined as in* (1.1′), *is in* $HL^{m_2}(\Omega_2)$ *with homogeneous principal symbol*

$$\sigma_1^0(x_1, x_2, \xi_1, \xi_2)\in C^\infty(\Omega_1\times\Omega_2\times\{\mathbf{R}^{n_1}\setminus 0\}\times\{\mathbf{R}^{n_2}\setminus 0\})\,.$$

2) $\sigma_2\in C^\infty(\Omega_1\times\Omega_2\times\mathbf{R}^{n_1}\times\{\mathbf{R}^{n_2}\setminus 0\})$ *and, for* $t>0$:

$$\sigma_2(x_1, x_2, \xi_1, t\xi_2)=t^{m_2}\sigma_2(x_1, x_2, \xi_1, \xi_2)\,.$$

Moreover, for all $x_2, \xi_2, \xi_2\neq 0$, $\sigma_2(x_1, x_2, D_1, \xi_2)$ *is in* $HL^{m_1}(\Omega_1)$ *with homogeneous principal symbol*

$$\sigma_2^0(x_1, x_2, \xi_1, \xi_2)\in C^\infty(\Omega_1\times\Omega_2\times\{\mathbf{R}^{n_1}\setminus 0\}\times\{\mathbf{R}^{n_2}\setminus 0\})\,.$$

3) *We impose:* $\sigma_1^0=\sigma_2^0=\sigma^0$.

Let $\{\sigma_1, \sigma_2\}\in\Sigma^{m_1,m_2}$; let $\psi_i(\xi_i)\in C^\infty(\mathbf{R}^{n_i})$, $i=1,2$, $\psi_i(\xi_i)=0$ if $|\xi_i|\leqslant 1$, $\psi_i(\xi_i)=1$ if $|\xi_i|\geqslant 2$. We can construct in $S^{m_1,m_2}(\Omega_1\times\Omega_2)$ the symbol:

$$\sigma=\sigma_1\psi_1+\sigma_2\psi_2-\sigma^0\psi_1\psi_2\,. \tag{2.1}$$

Note that the symbol σ does not depend from the choise of ψ_i, except for addition of a term in $S^{m_1-1,m_2-1}(\Omega_1\times\Omega_2)$; then we can introduce the following definition:

DEFINITION 2.2. *We denote by* $HS^{m_1,m_2}(\Omega_1\times\Omega_2)$ *the set of all* $a\in S^{m_1,m_2}(\Omega_1\times\Omega_2)$ *such that for some* $\{\sigma_1,\sigma_2\}\in\Sigma^{m_1,m_2}(\Omega_1\times\Omega_2)$ $a-\sigma$ *is in* $S^{m_1-1,m_2-1}(\Omega_1\times\Omega_2)$. *We write* $HL^{m_1,m_2}(\Omega_1\times\Omega_2)$ *for the corresponding subset of* $L^{m_1,m_2}(\Omega_1\times\Omega_2)$ *and we call* $\{\sigma_1,\sigma_2\}$ *the homogeneous principal symbol of* $a(x_1,x_2,D_1,D_2)$.

DEFINITION 2.3. *Let* $\{\sigma_1,\sigma_2\}\in\Sigma^{m_1,m_2}(\Omega_1\times\Omega_2)$, $\{\tau_1,\tau_2\}\in\Sigma^{p_1,p_2}(\Omega_1\times\Omega_2)$ *and let* σ_1, σ_2, τ_1, τ_2 *have compact support in the* x_1, x_2 *variables; we define in* $\Sigma^{m_1+p_1,m_2+p_2}(\Omega_1\times\Omega_2)$ *the composition*:

$$\{\tau_1,\tau_2\}\circ\{\sigma_1,\sigma_2\}=\{\tau_1\circ_{\xi_2}\sigma_1,\ \tau_2\circ_{\xi_1}\sigma_2\} \tag{2.2}$$

where $\tau_1\circ_{\xi_2}\sigma_1$ *and* $\tau_2\circ_{\xi_1}\sigma_2$ *are defined as in* (1.3), *for* $\xi_1\neq 0$ *and* $\xi_2\neq 0$ *respectively.*

DEFINITION 2.4. *Let* $\{\sigma_1,\sigma_2\}\in\Sigma^{m_1,m_2}(\Omega_1\times\Omega_2)$ *and let* σ_1, σ_2 *have compact support in the* x_1, x_2 *variables; then, with the notations of theorem* 1.7, *we define in* $\Sigma^{m_1,m_2}(\Omega_1'\times\Omega_2')$:

$$\{\sigma_1,\sigma_2\}_{k_1\times k_2}=\{\sigma_{1_{k_1}}(x_1',k_2^{-1}(x_2'),\xi_1',{}^tk_2'\xi_2'),\ \sigma_{2_{k_2}}(k_1^{-1}(x_1'),x_2',{}^tk_1'\xi_1',\xi_2')\} \tag{2.3}$$

where $\sigma_{1_{k_1}}$ *and* $\sigma_{2_{k_2}}$ *are defined as in* (1.4), *for* $\xi_1'\neq 0$ *and* $\xi_2'\neq 0$ *respectively.*

THEOREM 2.5. *Let* $a(x_1,x_2,D_1,D_2)\in HL^{m_1,m_2}(\Omega_1\times\Omega_2)$ *and* $b(x_1,x_2,D_1,D_2)\in$ $\in HL^{p_1,p_2}(\Omega_1\times\Omega_2)$ *with compact support in the* x_1, x_2 *variables; let* $\{\sigma_1,\sigma_2\}\in$ $\in\Sigma^{m_1,m_2}(\Omega_1\times\Omega_2)$ *and* $\{\tau_1,\tau_2\}\in\Sigma^{p_1,p_2}(\Omega_1\times\Omega_2)$ *their homogeneous principal symbol. The composition* $b(x_1,x_2,D_1,D_2)a(x_1,x_2,D_1,D_2)$ *is in* $HL^{m_1+p_1,m_2+p_2}(\Omega_1\times\Omega_2)$ *and its homogeneous principal symbol is*:

$$\{\lambda_1,\lambda_2\}=\{\tau_1,\tau_2\}\circ\{\sigma_1,\sigma_2\}\,.$$

THEOREM 2.6. *Let* $a(x_1,x_2,D_1,D_2)\in HL^{m_1,m_2}(\Omega_1\times\Omega_2)$ *with compact support in the* x_1, x_2 *variables and let* $\{\sigma_1,\sigma_2\}\in\Sigma^{m_1,m_2}(\Omega_1\times\Omega_2)$ *its homogeneous principal symbol. The operator defined in* (1.9), *theorem* 1.7, *is in* $HL^{m_1,m_2}(\Omega_1'\times\Omega_2')$ *with homogeneous principal symbol in* $\Sigma^{m_1,m_2}(\Omega_1'\times\Omega_2')$:

$$\{\sigma_1',\sigma_2'\}=\{\sigma_1,\sigma_2\}_{k_1\times k_2}\,.$$

The theorems 2.5 and 2.6 follow easily from theorems 1.6 and 1.7 and the details of the proofs are left for the reader.

Note finally that there is no difficulty in extending the preceding results to matrices of operators in $L^{m_1,m_2}(\Omega_1\times\Omega_2)$ and $HL^{m_1,m_2}(\Omega_1\times\Omega_2)$.

3. – Now we shall consider operators with homogeneous principal symbol, as in definition 2.2, in the product of two compact manifolds, as operators between vector bundles.
Let X_i, $i=1,2$, be a compact manifold; we write $T^*(X_i)$ for the cotangent bundle and $S^*(X_i)$ for the unit sphere of $T^*(X_i)$ for some metric. E, F, G will denote vector bundles on $X_1\times X_2$. If $P_1\in X_1$, we denote by E_{P_1} the restriction $E|_{P_1\times X_2}$; likewise, if $P_2\in X_2$, $E_{P_2}=E|_{X_1\times P_2}$. Let π_i be the canonical projection of $S^*(X_i)$ into X_i; we note: $E^*=(\pi_1\times\pi_2)^*(E)$; E^* is a vector bundle on $S^*(X_1)\times S^*(X_2)$.

Moreover we shall write $HL^m(X_i,E_i,F_i)$, $i=1,2$, for the class of pseudo differential operators with homogeneous principal symbol between two vector bundles E_i, F_i on X_i.

Let $k_i^{\omega_i}\colon X_i^{\omega_i}\to\Omega_i^{\omega_i}$, $X_i^{\omega_i}\subset X_i$, $\Omega_i^{\omega_i}\subset \boldsymbol{R}^{n_i}$, be a complete family of C^∞ coordinate systems in X_i; then:

$$k_1^{\omega_1}\times k_2^{\omega_2}\colon\; X_1^{\omega_1}\times X_2^{\omega_2}\to\Omega_1^{\omega_1}\times\Omega_2^{\omega_2} \tag{3.1}$$

is a complete family in $X_1\times X_2$.

We denote by $H^{s_1,s_2}(X_1\times X_2)$ the Hilbert space of all $u\in\mathcal{D}'(X_1\times X_2)$ such that $u\circ(k_1^{\omega_1}\times k_2^{\omega_2})^{-1}\in H^{s_1,s_2}_{\text{loc}}(\Omega_1^{\omega_1}\times\Omega_2^{\omega_2})$ for all $k_1^{\omega_1}\times k_2^{\omega_2}$ in (3.1); in the same way we define $H^{s_1,s_2}(X_1\times X_2,E)$ on the vector bundle E. (The properties of the spaces $H^{s_1,s_2}(X_1\times X_2)$ can be easily deduced from the results in [6], chapter II). Note that the inclusion mapping of $H^{s_1,s_2}(X_1\times X_2)$ into $H^{s_1-1,s_2-1}(X_1\times X_2)$ is completely continuous (see theorem 2.2.3 in [6]).

We denote by $HL^{m_1,m_2}(X_1\times X_2)$ the set of all linear maps of $C^\infty(X_1\times X_2)$ to $C^\infty(X_1\times X_2)$ such that, for every coordinate system $k_1^{\omega_1}\times k_2^{\omega_2}$ of the form (3.1), the associated operator from $C_0^\infty(\Omega_1^{\omega_1}\times\Omega_2^{\omega_2})$ to $C^\infty(\Omega_1^{\omega_1}\times\Omega_2^{\omega_2})$ is in $HL^{m_1,m_2}(\Omega_1^{\omega_1}\times\Omega_2^{\omega_2})$; in the same way, we can define $HL^{m_1,m_2}(X_1\times X_2,E,F)$ in the set of the linear maps from $C^\infty(X_1\times X_2,E)$ to $C^\infty(X_1\times X_2,F)$. In view of theorem 1.8, an operator in $HL^{m_1,m_2}(X_1\times X_2,\,E,\,F)$ extends to a continuous map of $H^{s_1,s_2}(X_1\times X_2,E)$ to $H^{s_1-m_1,s_2-m_2}(X_1\times X_2,F)$.

We denote by $\Sigma^{m_1,m_2}(X_1\times X_2,E,F)$ the set of all couples $\{\sigma_1,\sigma_2\}$ such that (we use the terminology of Atiyah-Singer [2]):

1) σ_1 is a C^∞ family of pseudo differential operators on $S^*(X_1)$ such that: if $v_1\in S^*(X_1)$, $P_1=\pi_1(v_1)$, $\sigma_1(v_1)$ is in $HL^{m_2}(X_2,E_{P_1}\colon F_{P_1})$. We can identify the symbol σ_1^0 of the family σ_1 with an element in $\mathrm{HOM}(E^*,F^*)$.

2) σ_2 is a C^∞ family of pseudo differential operators on $S^*(X_2)$ such that: if $v_2 \in S^*(X_2)$, $P_2 = \pi_2(v_2)$, $\sigma_2(v_2)$ is in $HL^{m_1}(X_1, E_{P_2}, F_{P_2})$. We can identify the symbol σ_2^0 of the family σ_2 with an element in $\mathrm{HOM}(E^*, F^*)$.

3) We impose: $\sigma_1^0 = \sigma_2^0 = \sigma^0$ in $\mathrm{HOM}(E^*, F^*)$.

Let $\{\sigma_1, \sigma_2\}$ be in $\Sigma^{m_1,m_2}(X_1 \times X_2, E, F)$: for a coordinate system $k_1^{\omega_1} \times k_2^{\omega_2}$ of the form 3.1, its local expression is in $\Sigma^{m_1,m_2}(\Omega_1^{\omega_1} \times \Omega_2^{\omega_2})$ (we understand the generalization of definition 2.1 to matrices). If we change the coordinate system, the local expression of $\{\sigma_1, \sigma_2\}$ changes as in definition 2.4; hence, in view of theorem 2.6, the principal symbol of every operator in $HL^{m_1,m_2}(X_1 \times X_2, E, F)$ is well defined in $\Sigma^{m_1,m_2}(X_1 \times X_2, E, F)$.

DEFINITION 3.1. *The symbol* $\{\sigma_1, \sigma_2\}$ *in* $\Sigma^{m_1,m_2}(X_1 \times X_2, E, F)$ *is elliptic if, for each* $v_1 \in S^*(X_1)$, $\sigma_1(v_1)$ *as an operator in* $HL^{m_2}(X_2, E_{P_1}, F_{P_1})$ *is exactly invertible and, for each* $v_2 \in S^*(X_2)$, $\sigma_2(v_2)$ *as an operator in* $HL^{m_1}(X_1, E_{P_2}, F_{P_2})$ *is exactly invertible.*

If $\{\sigma_1, \sigma_2\}$ is elliptic, we can construct its «inverse» in $\Sigma^{\;m_1,-m_2}(X_1 \times X_2, F, E)$. Theorem 2.5 immediately gives:

THEOREM 3.2. *Let A in $HL^{m_1,m_2}(X_1 \times X_2, E, F)$ have elliptic principal symbol. There exists B in $HL^{-m_1,-m_2}(X_1 \times X_2, F, E)$ such that*

$$\left\{\begin{array}{l} AB = I_F + K_F \\ BA = I_E + K_E \end{array}\right. \tag{3.2}$$

where I_F is the identity on $C^\infty(X_1 \times X_2, F)$, I_E is the identity on $C^\infty(X_1 \times X_2, E)$, K_F is compact on $H^{s_1,s_2}(X_1 \times X_2, F)$ and K_E is compact on $H^{s_1,s_2}(X_1 \times X_2, E)$. Then A, as a map from $H^{s_1,s_2}(X_1 \times X_2, E)$ to $H^{s_1-m_1,s_2-m_2}(X_1 \times X_2, F)$, is a Fredholm operator (it has closed range of finite codimension and a finite dimensional null space).

A, as a map of $C^\infty(X_1 \times X_2, E)$ to $C^\infty(X_1 \times X_2, F)$ or as a map of $\mathcal{D}'(X_1 \times X_2, E)$ to $\mathcal{D}'(X_1 \times X_2, F)$, has also range $R(A)$ of finite codimension and finite dimensional null space $N(A)$; the index:

$$i(A) = \dim N(A) - \operatorname{codim} R(A) \tag{3.3}$$

depends only on the homotopy class of the principal symbol of A in the space of elliptic symbols in $\Sigma^{m_1,m_2}(X_1 \times X_2, E, F)$.

Note that, when we assume $m_1 = m_2 = 0$, theorem 3.2 gives the results in [7], [9] about the bisingular operators.

4. – Now we present two applications of the theorem 3.2.

a) First we shall study the tensor product of pseudo differential operators as in Atiyah-Singer [1], pp. 512-515. Particularly, we consider the tensor product of two operators of order zero.

Let A_i in $HL^0(X_i)$, X_i, $i=1,2$, compact manifolds. For simplicity we assume that A_i is a scalar operator; denote a_i^0 the principal symbol of A_i in $C^\infty(S^*(X_i))$. We define:

$$A_1 \# A_2 = \begin{pmatrix} A_1 \otimes I & -I \otimes A_2^* \\ I \otimes A_2 & A_1^* \otimes I \end{pmatrix} \tag{4.1}$$

$A_1 \# A_2$ is actually in $HL^{0,0}(X_1 \times X_2, E^2, E^2)$, where E^2 is the trivial 2-dimensional vector bundle over $X_1 \times X_2$. Its principal symbol $\{\sigma_1, \sigma_2\}$ in $\Sigma^{0,0}(X_1 \times X_2, E^2, E^2)$ is:

$$\begin{cases} v_1 \in S^*(X_1); \quad \sigma_1(v_1) = \begin{pmatrix} a_1^0(v_1) & -A_2^* \\ A_2 & \overline{a_1^0(v_1)} \end{pmatrix} \in HL^0(X_2, E^2, E^2) \\ v_2 \in S^*(X_2); \quad \sigma_2(v_2) = \begin{pmatrix} A_1 & -\overline{a_2^0(v_2)} \\ a_2^0(v_2) & A_1^* \end{pmatrix} \in HL^0(X_1, E^2, E^2) . \end{cases} \tag{4.2}$$

Direct computation shows that $\sigma_1(v_1)$ and $\sigma_2(v_2)$ are invertible in $HL^0(X_2, E^2, E^2)$ and $HL^0(X_1, E^2, E^2)$ for each $v_1 \in S^*(X_1)$ and $v_2 \in S^*(X_2)$. In view of theorem 3.2 we have proved:

THEOREM 4.1. $A_1 \# A_2$ *in* (4.1), *as a map of* $H^{s_1,s_2}(X_1 \times X_2, E^2)$ *to* $H^{s_1,s_2}(X_1 \times X_2, E^2)$, *is a Fredholm operator.*

This is the expected result, according to [1]; in [1] is also proved that

$$i(A_1 \# A_2) = i(A_1)\, i(A_2) \tag{4.3}$$

b) In the second application we extend to systems the results in [7], [10] about a boundary value problem for functions of two complex variables.

In the complex plane C_{z_i} we note

$$D_i^1 = \{z_i,\ |z_i| < 1\}, \qquad D_i^2 = \{z_i,\ |z_i| > 1\}, \qquad X_i = \{z_i,\ |z_i| = 1\} .$$

In $C^2 = C_{z_1} \times C_{z_2}$ we write $D^{h,k} = D_1^h \times D_2^k$, $h, k = 1, 2$, for the four complementary bicylinders with common distinguished boundary $X_1 \times X_2$. Consider the following boundary value problem:

PROBLEM 4.2. *Let $\mathcal{A}_{h,k}(z_1, z_2)$, $h, k = 1, 2$, be four $m \times m$ matrices of functions in $C^\infty(X_1 \times X_2)$. Find $f^{h,k}(z_1, z_2)$, $h, k = 1, 2$, m-tuples of functions in $C^\infty(\overline{D^{h,k}})$, such that:*

(I) *for* $h, k = 1, 2$: $\dfrac{\partial f^{h,k}}{\partial \bar{z}_1} = 0$, $\dfrac{\partial f^{h,k}}{\partial \bar{z}_2} = 0$ *in* $D^{h,k}$

and $f^{h,k}$ has a zero at infinity.

(II) $\sum_{h,k=1,2} \mathcal{A}_{h,k} f^{h,k}|_{X_1 \times X_2} = g$

where g is a given m-tuple of functions in $C^\infty(X_1 \times X_2)$.

The problem 4.2 can be reduced to the study of the following operator on $X_1 \times X_2$ (see [7], [10]):

$$P = \sum_{h,k=1,2} \mathcal{A}_{h,k} P^h_{z_1} \otimes P^k_{z_2} \tag{4.4}$$

where

$$P^r_{z_i} \varphi_i(z_i) = \frac{1}{2}\varphi_i(z_i) + \frac{(-1)^{r-1}}{+\sqrt{-1}\,2\pi} \int_{X_i} \frac{\varphi_i(\zeta_i)\,d\zeta_i}{\zeta_i - z_i} \tag{4.5}$$
$$\varphi_i \in C^\infty(X_i), \quad r = 1, 2$$

are the Plemelj's projections.

P is a map of $C^\infty(X_1 \times X_2, E^m)$ to $C^\infty(X_1 \times X_2, E^m)$, where E^m is the trivial m-dimensional vector bundle on $X_1 \times X_2$. Actually $P \in HL^{0,0}(X_1 \times X_2, E^m, E^m)$ with principal symbol $\{\sigma_1, \sigma_2\}$ in $\Sigma^{0,0}(X_1 \times X_2, E^m, E^m)$:

$$\begin{cases} \sigma_1(z_1^+) = \sum_{k=1,2} \mathcal{A}_{1,k} P^k_{z_2} \in HL^0(X_2, E^m, E^m) \\ \sigma_1(z_1^-) = \sum_{k=1,2} \mathcal{A}_{2,k} P^k_{z_2} \in HL^0(X_2, E^m, E^m) \\ \sigma_2(z_2^+) = \sum_{h=1,2} \mathcal{A}_{h,1} P^h_{z_1} \in HL^0(X_1, E^m, E^m) \\ \sigma_2(z_2^-) = \sum_{h=1,2} \mathcal{A}_{h,2} P^h_{z_1} \in HL^0(X_1, E^m, E^m) \end{cases} \tag{4.6}$$

where we identify $S^*(X_i)$ with two copies of X_i, X_i^+ and X_i^-, and $v_i \in S^*(X_i)$ with $z_i^+ \in X_i^+$ or $z_i^- \in X_i^-$. Now use the terminology in [3] and note $\delta_t(\mathcal{A})$, $t = 1, 2, \ldots, m$, the partial indices of a matrix $\mathcal{A}$ of functions on the unit circle in $\boldsymbol{C}$.

THEOREM 4.3. *Let* $\det \mathcal{A}_{h,k} \neq 0$ *for all* $h, k = 1, 2$ *and all* $z_1, z_2 \in X_1 \times X_2$. *Suppose that, if we consider partial indices with respect to* z_2:

$$\delta_t(\mathcal{A}_{12}^{-1}\mathcal{A}_{11}) = 0\,, \qquad \delta_t(\mathcal{A}_{22}^{-1}\mathcal{A}_{21}) = 0\,, \qquad t = 1, 2, \ldots, m \tag{4.7}$$

for each $z_1 \in X_1$; *moreover, if we consider partial indices with respect to* z_1:

$$\delta_t(\mathcal{A}_{21}^{-1}\mathcal{A}_{11}) = 0\,, \qquad \delta_t(\mathcal{A}_{22}^{-1}\mathcal{A}_{12}) = 0\,, \qquad t = 1, 2, \ldots, m \tag{4.8}$$

for all $z_2 \in X_2$. *Then* P *in* (4.4), *as a map of* $H^{s_1,s_2}(X_1 \times X_2, E^m)$ *to* $H^{s_1,s_2}(X_1 \times X_2, E^m)$, *is a Fredholm operator and the problem* 4.2 *has a finite index.*

In fact, in view of the results in [3], the conditions (4.7) and (4.8) imply that the symbol in (4.6) is elliptic; hence, if we apply theorem 3.2, we obtain theorem 4.3.

The index of P, in the case $m = 1$, is computed in [8].

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