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Homology theory for real analytic and semianalytic sets


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Homology Theory for Real Analytic and Semianalytic Sets. (*)

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the subgroup of real analytic boundaries
\[ \mathcal{B}_j(A, B) = \{ R + \partial S : R \in \mathcal{Z}_j(B, B) \text{ and } S \in \mathcal{Z}_j(A, A) \} , \]
and the real analytic homology groups
\[ H_j(A, B) = \mathcal{Z}_j(A, B) / \mathcal{B}_j(A, B) , \quad H_j(A) = H_j(A, \emptyset) , \]
we prove in § 4 our main results.

**Theorem.** If \( A \supset B \) are semianalytic sets, then there exists an arbitrarily small open neighborhood \( W \) of \( B \) such that \( H_j(A \cap W, B) \simeq 0 \) for all \( j \).

**Corollary.** There exist arbitrarily small open neighborhoods \( U \) of \( A \) in \( M \) and \( V \) of \( B \) in \( U \) such that the inclusion map of \( \mathcal{Z}_j(A, B) \) into \( \mathcal{Z}_j(U, V) \) induces an isomorphism, \( H_j(A, B) \simeq H_j(U, V) \), for all \( j \).

The corollary has two consequences. First in § 5 we define, by approximation, the homomorphism
\[ H_j(f) : H_j(C, D) \to H_j(A, B) \]
for any continuous map \( f : (C, D) \to (A, B) \) where \( C \supset D \) are semianalytic subsets of an analytic manifold; the axioms of Eilenberg-Steenrod follow as in [3, 4.4.1]. Second, in § 6, a homology intersection product
\[ \cap : H_i(A, B) \times H_j(A, B) \to H_{i+j-m}(A, B) , \]
where \( i \) is any nonnegative integer with \( i + j > m \), results by use of the intersection theory for real analytic chains of [6, § 5].

In [1] A. Borel and A. Haefliger, employing the Borel-Moore homology for locally-compact spaces, proved the orientability modulo 2 of real analytic sets and established a formula equating the modulo 2 cycle of the real part of the intersection of two holomorphic varieties with the intersection of the modulo 2 cycles of the real parts of the varieties. These facts are reproven in § 6 and § 8, using analytic chains and Federer’s theory of slicing ([3, 4.3], [6, § 4]). We observe in Example 7.2 that analytic sets are not necessarily locally orientable over \( \mathbb{Z} \) even though those of dimension or codimension one are (7.1). We also note in 5.7 that the homology of a relatively compact pair of semianalytic sets is finitely generated.
The proofs of our main results in § 4 involve, for bounded semianalytic subsets of $\mathbb{R}^n$, a certain stratification (2.8) and system of neighborhoods (2.9) built up from finitely many local stratifications; the required local stratification (2.6) is established by Lojasiewicz in [11, §11-§15] or [13, §13] using the Weierstrass Preparation Theorem and classical elimination theory. The main complication in § 4 is that the projection of a bounded semianalytic set may fail to be semianalytic ([13, p.133]). Readers interested in other aspects of semianalytic sets and their projections are referred to [4], [6, §2], [8], [11], [12], [13], [14] and [18].

Replacing, for any integer $v \geq 2$, «analytic chain and spt» by «analytic chain modulo $v$ and spt» [[7]], we obtain the real analytic homology group $H_\ast(A, B; \mathbb{Z}_v)$ with coefficients in $\mathbb{Z}_v = \mathbb{Z}/v\mathbb{Z}$. All of the proofs and results of §2 through §6 carry over to the modulo $v$ case. We also note that, by replacing everywhere «(real) analytic set, semianalytic set, and analytic mapping» by «(real) algebraic set, semialgebraic set, and algebraic (polynomial) mapping» we may define real algebraic chains and transfer the methods and results of this paper to the real algebraic case. In fact here the situation is simpler because, by [16, Theorem 1], the projection of a bounded semialgebraic set in $\mathbb{R}^n$ is semialgebraic. Thus section 4.4 would be unnecessary.

Real analytic chains are suitable for studying the homology of real analytic objects because of their geometric content, their applicability to arbitrary semianalytic sets, and their economy as the smallest group of singular chains containing the orienting cycles of orientable semianalytic sets. However, the fact that they are singular chains, i.e., that semianalytic sets are triangulable ([5], [12]), will not be used here.

Our notation, except for the symbols, $\mathcal{I}_\ast(A, B)$, $\mathcal{B}_\ast(A, B)$, $H_\ast(A, B)$, $H'_\ast(A, B; \mathbb{Z}_v)$, defined above, is consistent with [3] and [6] (See the glossaries on [6, pp. 669-671]). In addition we define, for any subset $G$ of a topological space, the frontier of $G$, denoted $\text{Fr } G$, as $(\text{Clos } G) \sim G$. The author wishes to thank Herbert Federer for suggesting many of the problems treated here, showing him Example 7.2, and offering needed encouragement and criticism.

2. – Semianalytic Sets.

Observing that the product, the sum of squares, or the cartesian product of two analytic functions is analytic, we readily verify that the union, intersection, difference, or cartesian product of two semianalytic sets is semianalytic. Moreover, a connected component of or the inverse image
under an analytic map of a semianalytic set is semianalytic. However, the direct image under an analytic map of even a compact analytic set may fail to be semianalytic ([13, p. 133]).

2.1. Real analytic dimension. The real analytic dimension of a subset of $M$, which is defined in [6, 2.2], may be described as follows:

If $A$ is a semianalytic subset of $M$, then

$$\dim A = \sup \{-1, k: A \text{ contains a } k \text{ dimensional analytic submanifold of } M\}$$

(hence, $\dim \emptyset = -1$).

If $E$ is an arbitrary subset of $M$, then

$$\dim E = \inf \{\dim A: A \supset E \text{ and } A \text{ is semianalytic}\}.$$

2.2. Semianalytic subsets of $\mathbb{R}^n$. Let $n$ be a fixed positive integer. We will use the following notations. With $\mathbb{R}^0 = \{0\}$, let $p_k: \mathbb{R}^n \rightarrow \mathbb{R}^k$ for $k \in \{0, \ldots, n\}$ and $q_l: \mathbb{R}^{n-1} \rightarrow \mathbb{R}^l$ for $l \in \{0, \ldots, n-1\}$ be given by $p_k(x_1, \ldots, x_n) = 0$, $g_0(x_1, \ldots, x_n) = 0$, $p_k(x_1, \ldots, x_n) = (x_1, \ldots, x_k)$, and $g_1(x_1, \ldots, x_{n-1}) = (x_1, \ldots, x_1)$ for $k \in \{1, \ldots, n\}$, $l \in \{1, \ldots, n-1\}$, and $(x_1, \ldots, x_n) \in \mathbb{R}^n$. We also abbreviate $p = p_{n-1}$, and let $q: \mathbb{R}^n \rightarrow \mathbb{R}$ be the complementary projection, $g(x_1, \ldots, x_n) = x_n$ for $(x_1, \ldots, x_n) \in \mathbb{R}^n$.

An affine line $L$ in $\mathbb{R}^n$ is nonsingular for a semianalytic subset $A$ of $\mathbb{R}^n$ if $A$ may be described locally using $U, \mathcal{F}$ as in §1 so that $f|(U \cap L) \neq 0$ whenever $f \in \mathcal{F}$ and $f \neq 0$. If $L$ is nonsingular for two semianalytic sets, then it is clearly nonsingular for their union, intersection, or difference. Moreover, if $\mathcal{A}_0 \subset A$ are semianalytic and $L$ is nonsingular for $A$, then $L$ is nonsingular for $\mathcal{A}_0$. In fact, if $C \subset A$ is a component of $f^{-1}\{0\} \sim g^{-1}\{0\}$ for some $f, g \in \mathcal{F}$, $f_0$ and $g_0$ are analytic in an open subset $U_0$ of $\mathbb{R}^n$, and $C_0 \subset \mathcal{A}_0$ is a component of $f_0^{-1}\{0\} \sim g_0^{-1}\{0\}$, then $C_0 \cap C$ is a union of components of

$$U_0 \cap U \cap [(f_0^2 + f^2)^{-1}\{0\} \sim (g_0^2 + f^2)^{-1}\{0\}],$$

and

$$(f_0^2 + f^2)|(U_0 \cap U \cap L) \neq 0 \quad \text{and} \quad (g_0^2 + f^2)|(U_0 \cap U \cap L) \neq 0.$$

**Lemma 2.3.** If $f$ is analytic in a connected open subset $U$ of $\mathbb{R}^n$ and $f \neq 0$, then, for $\mathcal{H}^{n-1}$ almost all $\xi \in S^{n-1}$,

$$f|(U \cap \{x + t\xi: t \in \mathbb{R}\}) \neq 0 \quad \text{for all } x \in \mathbb{R}^n.$$

**Proof.** The proof of [12, Lemma 3] (or even of [10, Theorem 51]) shows that

$$Z = S^{n-1} \cap \{\xi: f|(U \cap \{x + t\xi: t \in \mathbb{R}\}) \equiv 0 \text{ for some } x \in \mathbb{R}^n\}$$
is contained in the countable union of sets $\varphi(A^*)$ where $A^*$ is some connected analytic manifold, $\varphi$ is an analytic map, and $\dim D\varphi(y)[\Tan(A^*, y)] < n - 2$ for $y \in A^*$. By partitioning, as in [6, 2.9], $A^*$ into submanifolds on which $\varphi$ has constant rank and using [3, 3.1.18], we may obtain a countable cover $C$ of $A^*$, consisting of submanifolds $C_i$ of various dimensions, such that $\mathcal{K}^{n-1}[\varphi(C)]$ is finite; hence

$$\mathcal{K}^{n-1}[\varphi(A^*)] = \mathcal{K}^{n-1}[\varphi(\cup C)] = 0.$$ 

Thus $\mathcal{K}^{n-1}(Z) = 0$.

**Corollary 2.4.** If $B$ is a countable family of semianalytic sets in $\mathbb{R}^n$, then, for $\mathcal{K}^{n-1}$ almost all $\xi \in S^{n-1}$, $p^{-1}\{y\}$ is nonsingular for $b(B)$ whenever $y \in \mathbb{R}^{n-1}$, $b \in O(n)$, $B \in B$, and $b(\xi) = (0, \ldots, 0, 1)$.

2.5. For any open subset $Y$ of $\mathbb{R}^{n-1}$ a function $H$ on $Y \times \mathbb{R}$ is called a **monic pseudo-polynomial** if there exists a positive integer $d$ and real functions $c_1, \ldots, c_d$ analytic in a neighborhood of $\text{Clos} Y$ such that

$$H(y, z) = z^d + c_1(y)z^{d-1} + \ldots + c_d(y) \quad \text{for} \ (y, z) \in Y \times \mathbb{R}.$$ 

If for every $y \in Y$, $D(y)$ is the discriminant of $H(y, \cdot)$ and $\varrho_1(y) < \varrho_2(y) < \ldots < \varrho_d(y)$ is a complete list, counting multiplicities of the real parts of the complex roots of $H(y, \cdot)$, then $D$ is analytic on $Y$ ([19, 5.7]) and $\varrho_1, \ldots, \varrho_d$ are continuous on $Y$ ([3, p. 450]) and analytic on $Y \sim D^{-1}\{0\}$.

We will say that a family $J$ of sets is compatible with a set $A$ if for every $I \in J$, either $I \cap A = \emptyset$ or $I \subset A$. In addition we will call a semianalytic set that is a connected analytic submanifold a **semianalytic stratum**.

**Theorem 2.6.** (Local stratification) If $B$ is a finite family of semianalytic subsets of $\mathbb{R}^n$ and $p^{-1}\{0\}$ is nonsingular for every member of $B$, then there exists an $h \in O(n)$ with $h(0, \ldots, 0, 1) = (0, \ldots, 0, 1)$, $Q_0 = \mathbb{R}^n$, $H_0 = 1$, and, for $i \in \{1, \ldots, n\}$, positive $\delta_i$, $Q_i = \mathbb{R}^i \cap \{(x_1, \ldots, x_i): |x_1| < \delta_1, \ldots, |x_i| < \delta_i\}$, and monic pseudo-polynomials $H_i$ on $Q_{i-1} \times \mathbb{R}$ with discriminants $D_i$ on $Q_{i-1}$ such that:

1. $H_i(y, 0) = 0$ for $y \in Q_{i-1}$,

2. $D_i^{-1}\{0\} \subset Q_{i-1} \cap \{(x_1, \ldots, x_{i-1}): H_{i-1}[\langle x_1, \ldots, x_{i-2}, x_{i-1} \rangle] = 0\}$.

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(3) the partition $Q_n = \bigcup \mathcal{J}$, where $\mathcal{J}$ is the family of connected components of

$$Q_n \cap \{(x_1, \ldots, x_n) : H_n[(x_1, \ldots, x_{n-1}), x_n] = \ldots = H_{i+1}[(x_1, \ldots, x_i), x_{i+1}] = 0,$$

$$H_i[(x_1, \ldots, x_{i-1}), x_i] \neq 0\}$$

for $i \in \{0, \ldots, n\}$, is finite and compatible with $h(B)$ for every $B \in \mathcal{B}$,

(4) each $\Delta \in \mathcal{J}$ is a semianalytic stratum, $0 \in \text{Clos} \Delta$, $p_{\text{dim} \Delta}|\Delta$ is an analytic isomorphism, and

$$Q_n \cap \text{Fr} \Delta = \bigcup \mathcal{J} \cap \{\Gamma : \text{dim} \Gamma < \text{dim} \Delta \text{ and } \Gamma \cap \text{Fr} \Delta \neq \emptyset\}.$$ 

**Proof.** Either [11, § 11–§ 15] or [13, § 13] using the family $\mathcal{B} \cup \{q^{-1}(0)\}$.

2.7. From 2.6 we infer that if $\varrho : Q_{n-1} \rightarrow \mathbb{R}$ is continuous and $H_n[y, \varrho(y)] = 0$ for $y \in Q_{n-1}$, then set

$$E = p^{-1}(Q_{n-1}) \cap \{x : \varrho(x) = \varrho[p(x)]\}$$

is semianalytic because it equals

$$\bigcup \mathcal{J} \cap \{\Gamma : \Gamma \subset \text{Clos} \Delta \text{ for some } \Delta \in \mathcal{J} \text{ with } \Delta \subset E \text{ and } \text{dim} \Delta = n-1\}.$$ 

We also deduce from 2.4 and 2.6 that each of the semianalytic sets $B \in \mathcal{B}$ have the following properties.

(1) $B$ is locally finite.

(2) $\text{Clos} B$, $\text{Fr} B$, and $\text{Bdry} B$ are semianalytic sets with $\text{dim} \text{Clos} B = \text{dim} B$, $\text{dim} \text{Fr} B < \text{dim} B$, and $\text{dim} \text{Bdry} B < n$.

(3) If $\delta < \inf \{\delta_1, \ldots, \delta_{n-1}\}$, then $p[B \cap p^{-1}U(0, \delta) \cap q^{-1}U(0, \delta_n)]$ is an at most $\text{dim} B$ dimensional semianalytic set in $\mathbb{R}^{n-1}$.

**Theorem 2.8.** (Global stratification) If $\mathcal{A}$ is a finite family of bounded semianalytic subsets of $\mathbb{R}^n$ and $p^{-1}(y)$ is nonsingular for $A$ whenever $y \in \mathbb{R}^{n-1}$ and $A \in \mathcal{A}$, then there exist $g \in \mathcal{O}(n)$ with $g(0, \ldots, 0, 1) = (0, \ldots, 0, 1)$ and finite partitions $\mathcal{S}$, $\mathcal{S}'$ of $\mathbb{R}^n$ into semianalytic strata with the following six properties:

(1) $\mathcal{G} = \mathcal{S} \cap \{\Gamma : \text{dim} \Gamma < n\} \subset \overline{\mathcal{G}} = \mathcal{S} \cap \{\Gamma : \text{dim} \Gamma < n\}$.

(2) For each $\Gamma \in \mathcal{G}$, $p_{\text{dim} \Gamma}|\Gamma'$ is an analytic isomorphism, and there are an open neighborhood $Y_{\Gamma}$ of $\text{Clos} p(\Gamma')$, a monic pseudopolynomial $H_{\Gamma}$ on
\(Y_r \times \mathbb{R}\) having zero as one root, and a continuous function \(\varrho: Y_r \to \mathbb{R}\) such that \(H_r[y, \varrho(y)] = 0\) for \(y \in Y_r\) and

\[I' \subset E_r = p^{-1}(Y_r) \cap \{x: q(x) = \varrho(p(x))\}\]

(hence \(p^{-1}(y)\) is nonsingular for \(I'\) whenever \(y \in \mathbb{R}^{n-1}\)).

(3) For each \(I' \in \mathcal{G}\), there is an open semianalytic neighborhood \(Z_{r'}\) of \(\text{Clos } p(I')\) in \(Y_r\) such that

\[p^{-1}(Z_{r'}) \cap \bar{\mathcal{G}} = p^{-1}(Z_{r'}) \cap \{x: H_r[p(x), q(x)] = 0\}.

(4) For each \(I' \in \mathcal{G}\) and \(\Lambda \in \mathcal{S} \cup \overline{\mathcal{S}}\), \(I' \subset \text{Clos } \Lambda\) whenever \(I' \cap \text{Clos } \Lambda \neq \emptyset\) and \(p(I') \subset p(\Lambda)\) whenever \(p(I') \cap p(\Lambda) \neq \emptyset\).

(5) If \(\mathcal{S}^\#\) is the partition of \(\mathbb{R}^{n-1}\) consisting of \(\{p(I'): I' \in \mathcal{G}\}\) along with the family of connected components of \(\mathbb{R}^{n-1} \sim p(\cup \mathcal{G})\), then statements (1) through (4) hold with \(n, \mathcal{S}, \overline{\mathcal{S}}\) replaced by \(n-1, \overline{\mathcal{S}}^\#, \) and \(\mathcal{S}^\#\) for some partition \(\overline{\mathcal{S}}^\#\) of \(\mathbb{R}^{n-1}\).

(6) \(\mathcal{S}\), and hence \(\overline{\mathcal{S}}\), is compatible with \(g(A)\) for every \(A \in \mathcal{A}\).

**Proof.** We use induction on \(n\). Since the case \(n = 1\) readily follows from 2.6, we assume \(n > 2\). For each positive integer \(k\) and \(u \in \mathbb{R}^k\), let \(\tau_u: \mathbb{R}^k \to \mathbb{R}^k\), \(\tau_u(v) = u + v\) for \(v \in \mathbb{R}^k\).

For each fixed \(a \in \mathbb{R}^n\) we use the family \(\mathcal{B}_a = \{\tau_u(A): A \in \mathcal{A}\}\) to select \(h, \delta_i, Q_i, H_i,\) and \(J\) as in 2.6 and \(h^\delta \in \mathcal{O}(n-1)\) so that \(p \circ h = h^\delta \circ p\). Fixing positive numbers \(\delta < \delta < \inf\{\delta_1, \ldots, \delta_{n-1}\}\), we define

\[Q_a = \tau_u[h^{-1}(Q_a)], \quad Y_a = p(Q_a), \quad Z_a = U[p(a), \delta],
\]

\[H_a(y, z) = z \cdot H_a[h^\delta[y - p(a)], z - q(a)] \quad \text{for} \quad (y, z) \in Y_a \times \mathbb{R},
\]

\[J_a = \{(\tau_u \circ h^{-1})[J' \cap p^{-1}U(0, \delta)]: J' \in J\} \cup \{(\tau_u \circ h^{-1})[J' \cap p^{-1}F_{\mathbb{R}}U(0, \delta)]: J' \in J\},
\]

\[\bar{J}_a = \{(\tau_u \circ h^{-1})[J' \cap p^{-1}U(0, \delta)]: J' \in J\} \cup \{(\tau_u \circ h^{-1})[J' \cap p^{-1}F_{\mathbb{R}}U(0, \delta)]: J' \in J\}.
\]

There is a finite subset \(F\) of \(\mathbb{R}^n\) with \(\text{Clos } \mathcal{A} \subset \bigcup_{a \in F} Q_a \cap p^{-1}U[p(a), \delta]\). Then, by 2.2 and 2.7(3),

\[\mathcal{B} = \{p(G_{\lambda_{(1)}} \cap \cdots \cap G_{\lambda_{(m)}}): G_a \in J_a \cup \bar{J}_a, m \in \{1, \ldots, \text{card } F\},
\]

and \(\lambda: \{1, \ldots, m\} \to F\).
is a finite family of bounded semianalytic subsets of $\mathbf{R}^{n-1}$. Choosing, by 2.4, $b \in O(n-1)$ so that $g_{n-2}^{-1}[w]$ is nonsingular for $b(B)$ whenever $w \in \mathbf{R}^{n-2}$ and $B \in \mathcal{B}$, we find, by induction, an orthogonal transformation $f \in O(n-1)$ and partitions $\mathcal{S}^f, \mathcal{S}^f$ of $\mathbf{R}^{n-1}$ which satisfy the theorem with $n, \alpha, g, \mathcal{S}, \tilde{\mathcal{S}}$ replaced by $n-1$, $\{b(B): B \in \mathcal{B}\}$, $f$, $\mathcal{S}^f, \mathcal{S}^f$. Letting $g^f = f \circ b$, $g \in O(n)$ satisfy $p \circ g = g^f \circ p$ and $g(0, \ldots, 0, 1) = (0, \ldots, 0, 1)$, and $\mathcal{S}$ [respectively, $\tilde{\mathcal{S}}$] be the partition of $\mathbf{R}^n$ consisting of

$$\mathcal{C}$$

[respectively, $\tilde{\mathcal{C}}$] = $\{g(\mathcal{G}) \cap p^{-1}(I^f): G \in \cup \mathcal{A} \}$ [respectively, $\tilde{\mathcal{F}}$],

$$\dim G < n, \quad \text{and} \quad I^f \in \mathcal{S}^f$$

along with the family of connected components of $\mathbf{R}^n \sim \cup \mathcal{S}$ [resp., $\tilde{\mathcal{C}}$], and use 2.6 to verify (1), (4), (5), and (6).

From 2.6 we also infer that, for each $I' \in \tilde{\mathcal{C}}$, $p|I'$ is an analytic isomorphism and obtain (2) by letting

$$F_r = F \cap \{a: p(I') \subset p[g(\cup \tilde{J}_a)]\},$$

$$Y_r = g^f \left( \bigcap_{a \in F_r} Y_a \right), \quad H_r = \prod_{a \in F_r} (H_a \circ ([g^f]^{-1} \mathbf{1}_r))(Y_r \times \mathbf{R}).$$

Finally for $I' \in \mathcal{C}$, (3) follows with $Z_r = g^f \left( \bigcap_{a \in F_r} Z_a \right)$.

**THEOREM 2.9 (System of Neighborhoods).** If $\mathcal{S}, \tilde{\mathcal{S}}, \mathcal{G}, \tilde{\mathcal{G}}, Z_r, \mathcal{S}^f, \tilde{\mathcal{S}}^f$ are as in 2.8, and, for each $I' \in \mathcal{S} \cup \tilde{\mathcal{S}}$, $V_r$ is a neighborhood of $I'$, then there exists a family $\{U_r: I' \in \mathcal{S} \cup \tilde{\mathcal{S}}\}$ of open sets with the following four properties:

1. For each $I' \in \mathcal{S} \cup \tilde{\mathcal{S}}$, $I' \subset U_r \subset V_r$.
2. For each $I' \in \mathcal{C}$, $p(U_r) \subset Z_r$.
3. For each $I'$ and $\Delta$ both belonging to either $\mathcal{S}$ or $\tilde{\mathcal{S}}$, $U_r \cap U_\Delta = \emptyset$ whenever $I' \cap \text{Clos} \Delta = \emptyset = I' \cap \text{Clos} I'$, and $U_r \cap U_\Delta = p^{-1}(p(\text{dim } U_r) \cap \text{dim } U_\Delta$ whenever $I' \cap \text{Clos} \Delta \neq \emptyset$.
4. There exists a family $\{U_{I'^f}: I'^f \in \mathcal{S}^f \cup \tilde{\mathcal{S}}^f\}$ satisfying (1) through (3) with $n, \mathcal{S}$, $\tilde{\mathcal{S}}$ replaced by $n-1$, $\mathcal{S}$, $\tilde{\mathcal{S}}^f$ such that $U_{p(I')} = p(U_r)$ whenever $I' \in \tilde{\mathcal{C}}$.

**Proof.** Letting $U_\Delta = \Delta$ for every $n$ dimensional member of $\mathcal{S} \cup \tilde{\mathcal{S}}$, we will define $U_r$ for $I' \in \tilde{\mathcal{C}}$ and establish 2.9 by induction on $n$. The case $n = 1$ is easily treated.

To handle the inductive step, choose, by 2.8(2), for each $k \in \{0, \ldots, n-1\}$ and each $k$ dimensional $I' \in \tilde{\mathcal{C}}$ a continuous function $\alpha_r$ on $\mathbf{R}^k$ such that

$$I' = p^{-1}[p(I')] \cap \{x: q(x) = \alpha_r[p_a(x)]\},$$
let $\delta_r(u)$, for $u \in p_k(I')$, be the infemum of the four numbers,
\[
1, \text{dist}[u, Fr p_k(I')], \frac{1}{2} \text{dist}[I' \cap p_k^{-1}(u), Fr V_I], \frac{1}{2} \text{dist}[I' \cap p_k^{-1}(u), Clos \mathcal{S} \cap \{C: I' \cap Clos C = \emptyset\}],
\]
(here $\text{dist}(u, 0) = +\infty$), and let $\epsilon_r$ be the extension to $R^k$ of $\delta_r$ and $0[[R^k \sim p_k(I')]$. Then $\epsilon_r$ is continuous, and
\[
X_r = R^{n-1} \cap \{y: \text{dist}(p^{-1}(y) \cap q^{-1}[\mathcal{A}(q,y)], Fr V_I) < \epsilon_r[q_s(y)]\}
\]
is an open neighborhood of $p(I')$. Moreover for each $l \in \{k + 1, \ldots, n - 1\}$
and $l$ dimensional with $l$
\[
is also an open neighborhood of $p(I')$ because $\epsilon_r[q_s(y)] > 0$, $\epsilon_l[q_l(y)] = 0$, and $\mathcal{A}(q,y) = \mathcal{A}_l[q_l(y)]$ whenever $y \in p(I')$. With
\[
X^I_\epsilon = R^{n-1} \quad \text{for } I' \in \overline{\mathcal{C}} \text{ with } I' \subset Fr \Lambda,
\]
\[
V^{I, \#}_\epsilon = \{Z_{I'}, X_{I'}, X^{I, \#}_\epsilon: I' \in \overline{\mathcal{C}}, \text{dist}(p(I') = I', \Lambda \in \overline{\mathcal{S}}, I' \subset Fr \Lambda\} \quad \text{for } \epsilon_{I, \#} \in S^\#,
\]
we inductively choose a family $\{U^{I, \#}_\epsilon: I' \in S^\# \cup \overline{S^\#}\}$ as in (4) such that $I' \subset U^{I, \#}_\epsilon \subset V^{I, \#}_\epsilon$ for $I' \in S^\# \cup \overline{S^\#}$, define
\[
U_r = p^{-1}[U^{I, \#}_\epsilon] \cap \{x: |q(x) - \mathcal{A}(p_k(x))| < \epsilon_r[p_k(x)]\},
\]
and verify the theorem by using 2.8(4)-(5).

3. – Real Analytic Chains.

H. Federer has proven in [3, 3.4.8 (13)] that, for any nonnegative integer $j$, the restriction of $j$ dimensional Hausdorff measure, $\mathcal{H}^j$, to any $j$ dimensional semianalytic set in $M$ is locally finite.

By [3, 4.2.28] a current $T$ is a $j$ dimensional analytic chain in $M$ if and only if it satisfies one of the two equivalent conditions:

(1) There exist a locally finite disjointed family $\mathcal{B}$ of $j$ dimensional orientable semianalytic strata, orienting $j$ vectorfields $\xi_B$ and integers $m_B$ for $B \in \mathcal{B}$, such that $T = \sum_{B \in \mathcal{B}} m_B(\mathcal{H}^j \cap B) \wedge \xi_B$; that is,
\[
T(\mathcal{H}) = \sum_{B \in \mathcal{B}} m_B \int_B \langle \varphi(x), \xi_B(x) \rangle d\mathcal{H}, \quad \text{for } \varphi \in \mathcal{D}(M).
\]
(2) \( T \in \mathcal{F}^\text{loc}_j(M) \), \( \dim(\text{spt } T) < j \), and \( \dim(\text{spt } \partial T) < j - 1 \) ([3, 4.1.24]).

From (2) it follows for positive \( j \) that the current \( \partial T \) where \( (\partial T)(\psi) = T(d\psi) \) for \( \psi \in \mathcal{D}^{j-1}(M) \) is a \( j - 1 \) dimensional analytic chain in \( M \).

From (1), 2.4, 2.7 (2), and 2.1 we infer that if \( T \neq 0 \), then

\[ \text{spt } T = \cup \{ \text{Clos } B : B \in \mathcal{B} \text{ and } m_B \neq 0 \} \]

is a \( j \) dimensional semianalytic subset of \( M \).

From (1) we also see that if \( A \) is semianalytic subset of \( M \), then the current \( T \setminus A \) is also an analytic chain in \( M \). In fact, for each \( B \in \mathcal{B} \), we may, by 2.4 and 2.6, choose a locally finite disjointed family \( C_B \) of \( j \) dimensional semianalytic strata \( C \subset A \setminus B \) such that \( \dim[(A \setminus B) \sim \cup C_B] < j \); hence

\[ T \setminus A = \sum_{B \in \mathcal{B}} m_B(\mathcal{K}^j \setminus A \cap B) \wedge \xi_B = \sum_{B \in \mathcal{B}} \sum_{C \in C_B} m_B(\mathcal{K}^j \setminus C) \wedge \xi_B. \]

From either (1) or (2) we infer that if \( N \) is an analytic submanifold of \( M \) with \( \text{spt } T \subset N \), then the above equation defining \( T(\psi) \) gives us, for \( \psi \in \mathcal{D}(N) \), an analytic chain \( T|N \) in \( N \), called the restriction of \( T \) to \( N \).

**Lemma 3.1.** Suppose \( f : M \rightarrow N \) is an analytic map of analytic manifolds and \( C \subset M \) and \( D \subset N \) are semianalytic. If \( \dim(C \cap f^{-1}(y)) < 0 \) for all \( y \in D \), then \( \dim[C \cap f^{-1}(D)] < \dim D \).

**Proof.** If \( x \) is a regular point of \( E = C \cap f^{-1}(D) \) such that \( \dim Df(x) \cdot [\text{Tan}(E, x)] \) is maximal, then by [6, 2.2 (4)] and [3, 3.1.18, 3.4.11],

\[ \dim E = \dim \text{Tan}(E, x) = \dim \text{Tan}(E \cap f^{-1}(f(x)), x) + \dim Df(x)[\text{Tan}(E, x)] \leq 0 + \dim \text{Tan}(D, x) < \dim D. \]

**Corollary 3.2.** If \( M \) and \( N \) are orientable, \( f \) maps \( C \) homeomorphically onto an open subset of \( N \), \( \dim[\text{im } Df(x)] = \dim N \) for \( x \in M \), and \( j \) is a non-negative integer, then there exists a unique homomorphism

\[ Y_j : \mathcal{Z}_j(f(C), j(C)) \rightarrow \mathcal{Z}_j(C, C) \]

such that \( f_\# \circ Y_j = 1_{\mathcal{Z}_j(f(C), j(C))}. \)

**Proof.** For any semianalytic subset \( A \) of \( C \) we infer from the proof of [6, 2.9] and [3, 3.1.18] that \( \dim[A \sim G(A)] < \dim A \) where

\[ G(A) = A \setminus \{ x : x \text{ is a regular point of } A \} \text{ with } \dim Df(x)[\text{Tan}(A, x)] = \dim A. \]
Inasmuch as \( \text{spt} f_\# T \cap f[G(\text{spt} T)] \) for \( T \in J_i(C, C) \) by [3, 4.1.30], the homomorphism \( f_\# J_i(C, C) \) is injective; thus \( Y_i \) is unique.

To prove existence, let \( k = \dim N, \omega \) and \( \eta \) be dual ([3, 1.7.5]) orienting \( k \) form and \( k \) vectorfield for \( N \), and \( N' = \mathcal{E} \wedge \eta \) the corresponding orienting cycle for \( N \). The submanifold \( G(C) \) is then oriented by the vectorfield \( Y \) which is dual to the \( k \) form \( [f|G(C)]^\# \omega / ||f|G(C)|^\# \omega |. \) By [3, 4.1.28] and the estimates

\[
\dim G(C) < k, \quad \dim \text{Fr} G(C) < \dim([C \sim \text{Fr} G(C)] \cup \text{Fr} C) < k - 1,
\]

\( J = [\mathcal{E} \setminus G(C)] \setminus Y \) is an analytic chain in \( M \); moreover, \( f_\# J = N' \setminus f(C) \) because \( \text{spt}[f_\# J - N' \setminus f(C)] \) is contained in the \( \mathcal{E} \) null subset \( f([C \sim G(C)] \cup \text{Fr} C) \) of \( N \). It follows that \( f^{-1}[f(C)] \cap \text{spt} \partial J = \emptyset \) because

\[
f(C) \cap \text{spt} f_\# \partial J = f(C) \cap \text{spt} \partial[N' \setminus f(C)] = \emptyset
\]

and \( f_\# J_{k-1}(C, C) \) is injective. For \( Q \in J_i[f(C), f(C)] \) we infer from [3, 3.1.18], 2.1, and 3.1 that

\[
\dim f^{-1}(\text{spt} Q) < j + \dim M - k, \quad \dim f^{-1}(\text{spt} \partial Q) < j + \dim M - k - 1, \\
\dim [f^{-1}(\text{spt} Q) \cap \text{spt} J] < j, \quad \dim [f^{-1}(\text{spt} \partial Q) \cap J] < j - 1, \\
f^{-1}(\text{spt} Q) \cap \text{spt} \partial J \subset f^{-1}[f(C)] \cap \text{spt} \partial J = \emptyset,
\]

and use [6, 5.8(11)] to define \( Y_i(Q) = (f^\# Q) \cap J \in J_i(C, C) \) and verify that

\[
f_\# Y_i(Q) = Q \cap f_\# J = Q \cap [N' \setminus f(C)] = Q \cap N = Q.
\]

4. - Homology Neighborhood Theorem.

4.1. If \( T \) is a \( j \) dimensional analytic chain in \( R^n \) satisfying condition (\#) \( \dim p(\text{spt} T) < j \) and \( \dim p(\text{spt} \partial T) < j - 1 \), then \( p_\# T \) is, by § 3 (2), a \( j \) dimensional analytic chain in \( R^{n-1} \). In particular, if \( E \) is an at most \( n - 1 \) dimensional semianalytic set in \( R^n \) and \( p^{-1}(y) \) is nonsingular for \( E \) whenever \( y \in R^{n-1} \), then, by 2.2 and 2.7 (3), any analytic chain with support in \( E \) satisfies condition (\#). For \( R^n \supset A \supset B \), let

\[
J_i^f(A, B) = J_i(A, B) \cap \{T: T \text{ satisfies condition (\#)}\}, \\
\mathcal{B}_i^f(A, B) = \{R + \partial S: R \in J_i^f(B, B) \text{ and } S \in J_{i+1}^f(A, A)\}, \\
H_i^f(A, B) = \mathcal{B}_i^f(A, B) / \mathcal{B}_{i+1}^f(A, B).
\]

We will prove by induction that the following two propositions are true for every positive integer \( n \).
PROPOSITION \(B_n\) [respectively, \(B^n\)]. Suppose \(S\) is a partition of \(R^n\) and \(\{U_T : T' \in S\}\) is a system of neighborhoods as in 2.8 and 2.9. If \(C \subset S\), \(D \subset S\), and \(V\) is an open subset of \(R^k\) where \(k = \inf \{\dim T' : T' \in C\}\), then
\[
H_\left[\bigcup_{T \in C} \bigcup_{D \in D} U_T \cap p_k^{-1}(V) \cap (T' \cup A), \bigcup_{T \in C} p_k^{-1}(V) \cap T \right] \simeq 0
\]
[respectively, \(H^n_i[, , ] \simeq 0\)] for all \(j\).

Using the homotopy formula for currents (3, 4.1.9)] we readily verify Propositions \(B_1\) and \(B^n\). Assuming now that \(n > 2\) and

PROPOSITION \(A_n\) [respectively \(A^n\)] is Proposition \(B_n\) [respectively \(B^n\)] in case \(C\) has only one member \(T'\),

we establish the induction in the following four sections:

4.2. Proposition \(A_{n-1}\) implies proposition \(A^n\).

PROOF. We assume \(T' \in D\) and \(k = \dim T' < n\) and abbreviate \(W = U_T \cap p_k^{-1}(V), T^n = p(T), W^n = p(W) = p(U_T) \cap q_k^{-1}(V)\).

First to treat the case \(\dim \cup D < n\), we will prove the stronger assertion,

PROPOSITION \(A^n\) is true if \(D\) is replaced by any subfamily \(\mathcal{D}\) of \(S\) with \(\dim \cup \mathcal{D} < n\),

by induction on \(\dim \cup \mathcal{D}\). If \(\dim \cup \mathcal{D} < k\), then \(W \cup \mathcal{D} \subset T'\), and the assertion is trivial. We now assume

\[
dim \cup \mathcal{D} \in \{k + 1, \ldots, n - 1\}, \quad T \in \mathcal{Z}_f(W \cup \mathcal{D}, W \cap T'),
\]

and

\[
\mathcal{F} = \mathcal{D} \cap \{A : \dim A = \dim \cup \mathcal{D}\}, \quad \mathcal{J} = \mathcal{D} \sim \mathcal{F}, \quad \text{and for each } A \in \mathcal{F}
\]

\[
\mathcal{D}_A = \mathcal{D} \cap \{D : D \subset \text{Clos } A\}, \quad \mathcal{J}_A = \mathcal{D}_A \cap \mathcal{J} = \mathcal{D}_A \sim \{A\},
\]

\[
\mathcal{D}^f = S^f \cap \{p(D) : D \in \mathcal{D}\}, \quad \mathcal{J}^f = \mathcal{D}^f \sim \{p(A)\}.
\]

For each \(A \in \mathcal{F}\), \(p|\text{Clos } A\) is a homeomorphism, by 2.8 (2), and the analytic chain \(T_A = T \cap A\) satisfies

\[
spt T_A \subset (spt T) \cap \text{Clos } A \subset W \cup \mathcal{D}_A,
\]

\[
spt \partial T_A \subset (spt T) \cap \text{Fr } A \subset W \cup \mathcal{J}_A,
\]

and condition (\#) by 2.8 (2) and 4.1. We apply Proposition \(A_{n-1}\) twice — with \(T'\), \(D\) replaced:

first, by \(T^n\), \(\mathcal{J}^f\), to choose an analytic chain \(P^n_A\) in \(R^{n-1}\) with

\[
spt P^n_A \subset W^n \cup \mathcal{J}^f_A, \quad spt (p^n_A T_A - P^n_A) \subset W^n \cap T^n,
\]

and
and second, by $I^g$, $D^g$, to choose an analytic chain $S_A^g$ in $R^{n-1}$ with

$$spt S_A^g \subset W^g \cap \cup D^g = p(W \cap \cup D_A),$$

$$spt(p_y T_A - \partial S_A^g) \subset (spt P_A^g) \cup spt(p_y T_A - P_A^g - \partial S_A^g) \subset p(W \cap \cup J_A).$$

By 2.8 (2), 4.1, and 3.2, $S_A^g$ lifts to an analytic chain $S_A$ in $R^n$ satisfying condition (2) and

$$spt S_A \subset W \cap \cup D_A, \quad spt(T_A - \partial S_A) \subset W \cap \cup J_A.$$

Inasmuch as $spt \left( \sum_{d \in \mathcal{F}} S_A^d \right) \subset W \cap \cup D$ and

$$spt(T - \partial \sum_{d \in \mathcal{F}} S_A^d) \subset spt(T - \sum_{d \in \mathcal{F}} T_A^d) \cup \cup spt(T_A - \partial S_A) \subset W \cap \cup J,$$

there is, by induction an analytic chain $S$ satisfying condition (2) and

$$spt S \subset W \cap \cup J, \quad spt(T - \partial (\sum_{d \in \mathcal{F}} S_A^d) - \partial S) \subset W \cap \Gamma;$$

thus $T \in B^g_1(W \cap \cup \overline{D}, W \cap \Gamma)$.

Having verified the assertion, we now assume that dim $\cup D = n$ and $T \in B_1^g(W \cap \cup D, W \cap \Gamma)$, and define $\mathcal{F}$, $J$, $D_A$, $J_A$, and $D_A^g$ as above with $\overline{D} = \overline{S} \cap \{A: A \subset \cup D\}$; hence, $\cup \overline{D} = \cup D$. Thus for each $A \in \mathcal{F}$ the analytic chain $T_A = T \cap A$ satisfies

$$spt T_A \subset W \cap \cup D_A, \quad spt \partial T_A \subset W \cap \cup J_A,$$

and condition (2) by 2.8 (2), 4.1, and the inclusion $p(spt T_A) \subset p(spt T)$. We first apply the assertion with $D$ replaced by $J_A$, to obtain an analytic chain $P_A$ satisfying condition (2) and

$$spt P_A \subset W \cap \cup J_A, \quad spt \partial P_A \subset W \cap \Gamma,$$

second use Proposition $A_{n-1}$, with $\Gamma$, $D$ replaced by $I^g$, $D_A^g$ to obtain an analytic chain $S_A^g$ with

$$spt S_A^g \subset W^g \cap \cup D_A^g = p(W \cap \cup D_A),$$

$$spt[p_y(T_A - P_A) - \partial S_A^g] \subset W^g \cap \Gamma^g = p(W \cap \Gamma),$$

and third recall again 2.8 (2), 4.1, and 3.2 to select an analytic chain $Q_A$.
in $\mathbb{R}^n$ satisfying condition (\#), $\text{spt} \, Q_A \subset W \cap \Gamma$, and $p_\# Q_A = p_\# (T_A - P_A) - \partial S_A^t$; hence $\delta (T_A - P_A - Q_A) = 0$.

There exists a semianalytic set $C_\delta$ in $\mathbb{R}^n$ such that $\text{spt} \, S_\delta^t \subset p(C_\delta \cap W \cap \cup \mathcal{D}_A)$ and $p$ maps $C_\delta$ homeomorphically onto an open subset of $\mathbb{R}^{n-1}$. In fact, with $Y_r$, $H_r$ as in 2.8 (2),

$$\delta = \inf \{1, \text{dist}[\Gamma \cap p^{-1}(\text{spt} \, S_\delta^t), \text{Fr} \, U_r]\}$$

and

$$\epsilon = \sup \{|z|: H_r(y, z) = 0 \text{ for some } y \in Y_r\}$$

are finite positive numbers. If $r_1, r_2, \ldots, r_d$ is a complete list, counting multiplicities of the complex roots of $H_r$ such that $q_1 = \mathcal{R}r_1 < q_2 = \mathcal{R}r_2 < \ldots < q_\alpha = \mathcal{R}r_\alpha$, $q_0 = -\infty$, and $q_{\alpha+1} = +\infty$, then

$$p^{-1}(Y_r) = p^{-1}(Y_r) \cap \{x: q(x) = q_i[p(x)]\},$$

$$p^{-1}(Y_r) \cap A = p^{-1}(Y_r) \cap \{x: q(x) \text{ is strictly between } q_\alpha(x) \text{ and } q_m(x)\}$$

for some $l \in \{1, \ldots, d\}$ and $m \in \{l-1, l+1\}$. With $\sigma_0 = q_1 - 1$, $\sigma_1 = q_1$, $\ldots$, $\sigma_\alpha = q_\alpha$, $\sigma_{\alpha+1} = q_{\alpha} + 1$, the set

$$C_\delta = p^{-1}(Y_r) \cap \{x: q(x) = [\sigma_1 + (2\epsilon)^{-1}\delta(\sigma_m - \sigma_i)][p(x)]\}$$

satisfies the above inclusion. The function $I_r$ on $Y_r \times \mathbb{R}$ whose value at $(y, z) \in Y_r \times \mathbb{R}$ equals

$$\prod_{\lambda=1}^{d} \prod_{\mu=1}^{d} [z - [r_\lambda + (2\epsilon)^{-1}\delta(r_\mu - r_\lambda)](y)] \cdot [z - [r_\lambda(y) + (2\epsilon)^{-1}\delta]] [z - [r_\lambda(y) - (2\epsilon)^{-1}\delta]]$$

is a monic pseudo-polynomial because its coefficients, being symmetric polynomial functions of $r_1, \ldots, r_d$ are polynomial function of the coefficients of $H_r$ ([19, 5.7]), hence analytic in $Y_r$. Thus $C_\delta$, being the graph of a continuous root of $I_r$, is semianalytic by 2.6.

In the following, our construction (and our reason for using the stratification $\mathcal{S}$ of $S$) is based on the observation (2.8 (3), 2.9 (2)(3)):

$$tx + (1-t)y \in W \cap \cup \mathcal{D}_A \text{ whenever } 0 < t < 1, p(x) = p(y), \text{ and } x, y \in W \cap \cup \mathcal{D}_A.$$

Choosing $Y_i$ as in 3.2 with $f$, $M$, $N$, $C$ replaced by $p$, $\mathbb{R}^n$, $\mathbb{R}^{n-1}$, $C_\delta$, we let $S_A = Y_i(S_\delta^t)$,

$$\tilde{C} = (\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n) \cap \{(t, x, y, z): p(x) = p(y), y \in C_\delta, \quad z = tx + (1 - t)y\},$$
Then $S_\delta$ satisfies condition (\#) by 2.2 and 2.7 (3), $\delta$ is semianalytic, $f$ maps $\delta$ homeomorphically onto an open subset of $R^n \times R^n$, $h[(f|\delta)^{-1}(0, x)] \in C_\delta$ and $h[(f|\delta)^{-1}(1, x)] = x$ whenever $x \in p^{-1}[p(C_\delta)]$ and $p[h(t, x, y, z)] = p(x)$ whenever $(t, x, y, z) \in \delta$. Applying 3.2 again, this time with $f$, $M$, $N$, $C$ replaced by $f$, $R \times R^n \times R^n \times R^n$, $R \times R^n$, $\delta$ to obtain a lifting $\delta_i$, we let

$$J_\delta = h_\delta \delta_i[([0, 1] \times (T_\delta - P_\delta - Q_\delta)]$$

and compute, using [3, 4.1.15],

$$T_\delta - P_\delta - Q_\delta - \partial J_\delta = h_\delta \delta_i[(s_\delta \times (T_\delta - P_\delta - Q_\delta)] =$$

$$= (\delta_i \circ \delta_i \circ \delta_i)[(s_\delta \times (T_\delta - P_\delta - Q_\delta)] =$$

$$= \delta_i[(s_\delta \circ T_\delta - P_\delta - Q_\delta)] = \delta_i(\partial S_\delta) = \partial \delta_i(S_\delta) = \partial S_\delta.$$

From the inequalities

$$\dim(spt J_\delta) < \dim p(spt J_\delta) + 1 < \dim p(spt(T_\delta - P_\delta - Q_\delta)] + 1 < j + 1$$

we see that $J_\delta$ is an analytic chain satisfying condition (\#). Moreover,

$$spt J_\delta = \{tx + (1 - t)y: 0 < t < 1, x \in spt(T_\delta - P_\delta - Q_\delta), y = (p|C_\delta)^{-1}[p(x)]\} \subset W \cap \cup D_\delta,$$

$$spt S_\delta = C_\delta \cap p^{-1}(spt S_\delta) \subset W \cap \cup D_\delta,$$

Finally since $spt \sum_{\delta \in F} (J_\delta + S_\delta) \subset W \cap \cup D$ and

$$spt \left[T - \partial \sum_{\delta \in F} (J_\delta + S_\delta)\right] \subset spt \left(T - \sum_{\delta \in F} T_\delta\right) \cup \cup spt(T_\delta - \partial J_\delta - \partial S_\delta) \subset W \cap \cup \delta,$$

another application of the assertion, with $D$ replaced by $\delta$, provides an analytic chain $S$ satisfying condition (\#) and

$$spt S \subset W \cap \cup \delta, \quad spt \left[T - \partial \sum_{\delta \in F} (J_\delta + S_\delta) - \partial S\right] \subset W \cap \delta;$$

whence $T \in S_\delta^\#(W \cap \cup D, W \cap \delta)$. 
4.3. Proposition $A^e_n$ implies proposition $B^e_n$.

PROOF. We use induction on $i = \dim U \in C$. For each $I \in C$ we abbreviate $W_R = U_R \cap p^{-1}_k(V)$, hence $W_R \cap I = p^{-1}_k(V) \cap I$, and assume $0 \neq T \in \mathbb{Z}^n \cap (I \cup D) \cup W_R \cap I$.

Recalling 2.9 (3), we let

$$B = C \cap \{B: \text{dim } B < i\}, \quad \varepsilon = C \cap \{I: \text{dim } I = i\},$$

$$U_B = \bigcup_{B \in B} U_B, \quad U_{\varepsilon} = \bigcup_{I \in \varepsilon} U_R, \quad V' = p_*[U_B \cap p^{-1}_k(V)],$$

and for each $I \in \varepsilon$,

$$D_I = D \cap \{A: I \cap \text{Clos } A\}, \quad W' = U_B \cap W_R = U_R \cap p^{-1}_k(V'),$$

and observe that $\{U_R: I \in \varepsilon\}$ is disjointed by 2.8 (3) and 2.9 (3). Thus if $\text{spt } T \subset U_{\varepsilon}$, for example, if $i = 0$, then we may apply Proposition $A^e_n$ to each of the analytic chains $T \subset U_R$ for $I \in \varepsilon$.

We now assume $U_B \cap \text{spt } T \neq \emptyset$. Since $p^{-1}[p(U_B)] \cap U_{\varepsilon} = U_B \cap U_{\varepsilon}$ by 2.9 (3),

$$\varepsilon = \text{dist} \left[ \text{Fr } p(U_B), p(\text{spt } T \sim U_{\varepsilon}) \right]$$

is positive. We may choose, first by the Stone-Weierstrass theorem ([9, p. 244]), a polynomial $\alpha$ on $\mathbb{R}^{n-1}$ such that

$$|\alpha(y) - \text{dist} \left[ y, p(\text{spt } T \sim U_{\varepsilon}) \right]| < \varepsilon/3 \quad \text{for } y \in p(\text{spt } T),$$

and then, by [6, 2.2 (7)], a number $r$ between $\varepsilon/3$ and $2\varepsilon/3$ such that

\[
\dim \left[ (\alpha \circ p)^{-1}(r) \cap \text{spt } T \right] < j - 1, \quad \dim \left[ (\alpha \circ p)^{-1}(r) \cap \partial T \right] < j - 2,
\]

\[
\dim \left[ x^{-1}(r) \cap p(\text{spt } T) \right] < j - 1, \quad \dim \left[ x^{-1}(r) \cap p(\partial T) \right] < j - 2.
\]

We infer by [3, 4.2.1, 4.3.4] and [6, 4.3] that

$$\langle T, \alpha \circ p, r \rangle = (\partial T) \cap \{x: (\alpha \circ p)(x) > r\} - (\partial T) \cap \{x: (\alpha \circ p)(x) > r\}$$

$$= (\partial T) \cap \{x: (\alpha \circ p)(x) < r\} - (\partial T) \cap \{x: (\alpha \circ p)(x) < r\}$$

are all analytic chains. We obtain the decompositions

$$\langle T, \alpha \circ p, r \rangle = \sum_{r \in \varepsilon} T_r, \quad T \cap \{x: (\alpha \circ p)(x) > r\} = \sum_{r \in \varepsilon} T_r.$$
where, for each \( f \in \mathcal{E} \), \( R_f \) and \( T_f \) are analytic chains which have supports contained in \( U_f \) and which therefore satisfy

\[
\text{spt } R_f \subset U_f \cap W_f \cap (I' \cup \cup_2 D) \subset W_f' \cap (I' \cup \cup_2 D), \quad \text{spt } \partial R_f \subset W_f' \cap I',
\]

\[
\text{spt } T_f \subset W_f \cap (I' \cup \cup_2 D), \quad \partial T_f + R_f = (\partial T) \cup \cup_2 U_f \cap \{x: (x \circ p)(x) > r\},
\]

and condition (\#) because

\[
p(\text{spt } R_f) \subset p([x \circ p]^{-1}[r] \cap \text{spt } T) \subset x^{-1}[r] \cap p(\text{spt } T),
\]

\[
p(\text{spt } \partial R_f) \subset x^{-1}[r] \cap p(\text{spt } \partial T),
\]

\[
p(\text{spt } T_f) \subset p(\text{spt } T), \quad p(\text{spt } \partial T_f) \subset p(\text{spt } R_f) \cup p(\text{spt } \partial T).
\]

First, for each \( f' \in \mathcal{E} \), we use Proposition \( A^\#_n \), with \( k, V \) replaced by \( i, V' \) to choose analytic chains \( P_f, Q_f \) satisfying condition (\#) and

\[
P_f + \partial Q_f = R_f, \quad \text{spt } P_f \subset W_f' \cap I', \quad \text{spt } Q_f \subset W_f' \cap (I' \cup \cup_2 D). \]

Second, since \( \text{spt } (T_f + Q_f) \subset W_f \cap (I' \cup \cup_2 D) \) and \( \text{spt } \partial (T_f + Q_f) = \text{spt } (\partial T_f + R_f - P_f) \subset W_f \cap I \), we may again apply Proposition \( A^\#_n \), this time to select an analytic chain \( S_f \) satisfying condition (\#) and

\[
\text{spt } S_f \subset W_f \cap (I' \cup \cup_2 D), \quad \text{spt } (T_f + Q_f - \partial S_f) \subset W_f \cap I'.
\]

Third we observe that

\[
\text{spt } \partial (T \setminus \{x: (x \circ p)(x) < r\} - \sum Q_f) = \text{spt } [(\partial T) \setminus \{x: (x \circ p)(x) < r\} + \sum P_f] \subset \text{spt } \cup_{f \in E} \cup_{b \in B} W_f' \cap (B \cup I'),
\]

and that \( T \setminus \{x: (x \circ p)(x) < r\} - \sum Q_f \) satisfies condition (\#). Since \( \dim \cup B < i \), there exists, by induction, an analytic chain \( Q \) satisfying condition (\#) and

\[
\text{spt } Q \subset \cup_{f \in E} \cup_{b \in B} W_f \cap (B \cup I),
\]

\[
\text{spt } \partial (T \setminus \{x: (x \circ p)(x) < r\} - \sum Q_f - Q) \subset \cup_{b \in B} W_f \cap B.
\]

Fourth we note that

\[
\text{spt } (T \setminus \{x: (x \circ p)(x) < r\} - \sum Q_f - Q) \cup_{B \in B} \cup_{A \in DU \cup E} W_f \cap (B \cup A),
\]
apply induction to choose an analytic chain $S$ satisfying condition (\#) and

$$spt(S \subset \bigcup_{b \in B} \bigcup_{a \in \partial \mathcal{D} \cup \xi} W_B \cap (B \cup \Delta),$$

$$spt(T \bigcap \{x: (z \circ p)(x) < r\} - \sum_{T \in b} Q_T - Q - \partial S) \subset \bigcup_{b \in B} W_B \cap B,$$

and conclude that

$$T = T \bigcap \{x: (z \circ p)(x) \geq r\} + T \bigcap \{x: (z \circ p)(x) < r\}$$

$$= \sum_{T \in b} (T_T + Q_T - \partial S_T) + (T \bigcap \{x: (z \circ p)(x) < r\} - \sum_{T \in b} Q_T - Q - \partial S +$$

$$Q + \partial \left( S + \sum_{T \in b} S_T \right)$$

belongs to $B_\delta^f \left[ \bigcup_{T \in C} \bigcup_{a \in \mathcal{D}} W_T \cap (\Gamma \cup \Delta), \bigcup_{T \in C} W_T \cap \Gamma \right].$

4.4. Proposition $B^f_\delta$ implies proposition $A_\delta$.

**Proof.** We assume $\Gamma \in \mathcal{D}$, $k = \dim \Gamma < n$, $W = U_\Gamma \cap p_\delta^{-1}(V)$, and $T \in 3_\delta(W \cap \cup \mathcal{D}, W \cap \Gamma)$. Since, for $\dim \cup \mathcal{D} < n$,

$$3_\delta(W \cap \cup \mathcal{D}, W \cap \Gamma) = 3_\delta^f(W \cap \cup \mathcal{D}, W \cap \Gamma) =$$

$$= B_\delta^f(W \cap \cup \mathcal{D}, W \cap \Gamma) \subset B_\delta(W \cap \cup \mathcal{D}, W \cap \Gamma),$$

by 2.8 (2), 4.1, and Proposition $B^f_\delta$, we also assume $\dim \cup \mathcal{D} = n$. From 4.2 we recall the following notations

$$\overline{\mathcal{D}}, \mathcal{F}, \mathcal{J}, \mathcal{D}_\Delta, T_\Delta, E_\Gamma,$$

and $C_\delta$ (which depends on $\delta$).

It will be sufficient to find, for each $\Delta \in \mathcal{F}$, an analytic chain $S_\Delta$ such that $spt S_\Delta \subset W \cap \cup \mathcal{D}$ and $T_\Delta - \partial S_\Delta$ satisfies condition (\#) because then

$$T - \partial \sum_{\Delta \in \mathcal{F}} S_\Delta = \left( T - \sum_{\Delta \in \mathcal{F}} T_\Delta \right) + \sum_{\Delta \in \mathcal{F}} (T_\Delta - \partial S_\Delta)$$

would, by 2.8 (2) and 4.1, satisfy condition (\#) and belong to $B_\delta^f(W \cap \cup \mathcal{D}, W \cap \Gamma) \subset B_\delta(W \cap \cup \mathcal{D}, W \cap \Gamma)$ by Proposition $B^f_\delta$.

Fixing $\Delta \in \mathcal{F}$, we note that $spt \partial T_\Delta \subset W \cap \cup J_\Delta$ and that $\partial T_\Delta$ satisfies, by 2.8 (2) and 4.1, condition (\#) (even though $T_\Delta$ may not). Recalling the assertion in 4.2 (or repeating the proof of the assertion) with $\mathcal{D}$ replaced by $J_\Delta$, we choose an analytic chain $P_\Delta$ with

$$spt P_\Delta \subset W \cap \cup J_\Delta$$

and

$$spt(\partial(T_\Delta - P_\Delta) \subset W \cap \Gamma).$$
By 2.9 (1)(3) and 4.1 (1)(2) we may select an open semianalytic set $X$ which has compact closure in $W$ and contains

$$\{tx + (1-t)y: 0 < t < 1, \ x \in \text{spt}(T_A - P_3), \ y = (p|E_r)^{-1}[p(x)]\}.$$

Since any semianalytic set is, by 2.6 (4), a countable union of compact sets, there is a countable family $\mathcal{N}$ of open neighborhoods of $\cup J$ such that any neighborhood of $\cup J$ contains some member of $\mathcal{N}$.

As a first approximation to $S_A$ we will select for every $N \in \mathcal{N}$ analytic chains $R^N_A$ and $S^N_A$ such that $R^N_A$ satisfies condition $(\#)$,

$$\text{spt } R^N_A \cup \text{spt } S^N_A \subset X \cap \cup D, \quad \text{spt } R^N_A \subset X \cap \cup J,$$

$$\text{spt } (T_A - R^N_A - \partial S^N_A) \subset X \cap N.$$

Fixing $N \in \mathcal{N}$ and choosing, by 2.8 (2), 2.9, and 2.9, an orthogonal transformation $g'(0, ..., 0, 1) = (0, ..., 0, 1)$, and

$$N' = (g')^{-1}(\cup \{U'_{r'}: \Gamma' \in S' \} \cap \Gamma' \subset B'),$$

we infer from Proposition $B^\#_n$—with $S$, $C$, $D$, $V$ replaced by $S'$, $S' \cap \{\Gamma': \Gamma' \subset C\}$, $S' \cap \{\Gamma': \Gamma' \subset A\}$, $R^k$—that

$$\mathcal{S}^\#_{j-1}[(X \cap \cup D) \cap N', \ X \cap \cup J] = (g')^{-1}\left(\mathcal{S}^\#_{j-1}[\Gamma' \cap g'(N'), B']\right)$$

$$= (g')^{-1}(\mathcal{S}^\#_{j-1}[\Gamma' \cap g'(N'), B']) = \mathcal{S}^\#_{j-1}[(X \cap \cup D) \cap N', \ X \cap \cup J].$$

Recalling from 4.2 the construction of $C_A$, we may replace $\delta$ by a smaller positive number in order that $C_A$ be close enough to $E_{r'}$ so that

$$\{tx + (1-t)y: 0 < t < 1, \ x \in \text{spt } \partial(T_A - P_3), \ y = (p|C_3)^{-1}[p(x)]\} \subset N',$$

$$\{tx + (1-t)y: 0 < t < 1, \ x \in \text{spt } \partial(T_A - P_3), \ y = (p|C_3)^{-1}[p(x)]\} \subset X \cap \cup D.$$

By 2.4, 2.7 (3), and 4.2 (1), we may choose an orthogonal transformation $\gamma \in \mathcal{O}(n)$ near $1_{R^k}$ so that $\gamma^{-1}(C_3)$ is nonsingular for $p^{-1}[y]$ whenever $y \in R^{n-1},$

$$\dim(p \circ \gamma)[\text{spt } (T_A - P_3)] < k, \quad \dim(p \circ \gamma)[\text{spt } \partial(T_A - P_3)] < k - 1,$$

$$\{tx + (1-t)y: 0 < t < 1, \ x \in \text{spt } \partial(T_A - P_3), \ y = [p \circ \gamma \gamma^{-1}(C_3)]^{-1}(p \circ \gamma)(x)\} \subset N',$$

$$\{tx + (1-t)y: 0 < t < 1, \ x \in \text{spt } (T_A - P_3), \ y = [p \circ \gamma \gamma^{-1}(C_3)]^{-1}(p \circ \gamma)(x)\} \subset (X \cap A) \cup N'.$$
With \( f \) and \( h \) as in 4.2 and

\[
\mathcal{C} = (R \times R^n \times R^n \times R^n) \cap \\
\cap \{(t, x, y, z) : (p \circ \gamma)(x) = (p \circ \gamma)(y), \ y \in \gamma^{-1}(C_d), \ z = tx + (1 - t)y\}
\]

we infer that \( \mathcal{C} \) is semianalytic, that \( f \) maps \( \mathcal{C} \) homeomorphically onto an open subset of \( R \times R^n \), that \( h[(f(\mathcal{C}))^{-1}(0, x)] \in \gamma^{-1}(C_d) \) and \( x = h[(f(\mathcal{C}))^{-1}(1, x)] \) whenever \( x \in (p \circ \gamma)^{-1}(p(C_d)) \), and that \( (p \circ \gamma)[h(t, x, y, z)] = (p \circ \gamma)(x) \) whenever \( (t, x, y, z) \in \mathcal{C} \). Applying 3.2, twice, with \( f, M, N, C \) replaced by \( p \circ \gamma, R^n, R^{n-1}, \gamma^{-1}(C_d) \) and \( f, R \times R^n \times R^n \times R^n, R \times R^n, \mathcal{C} \) to obtain liftings \( Y_f \) and \( \mathcal{Y}_f \) respectively, we let

\[
R = Y_f[(p \circ \gamma)_p(T_A - P_A)], \\
I = h_y \mathcal{Y}_f[[0, 1] \times \partial(T_A - P_A)], \\
J = h_y \mathcal{Y}_f[[0, 1] \times (T_A - P_A)],
\]

observe that

\[
spt R \subset \gamma^{-1}(C_d) \cap [(X \cap A) \cup N'], \quad spt H \subset (X \cap A) \cup N', \quad spt I \subset N',
\]

and compute, using [3, 4.1.15], that

\[
T_A - P_A + I - \partial J = h_y \mathcal{Y}_f[\delta_0 \times (T_A - P_A)] = \\
= (Y_f \circ (p \circ \gamma)_p \circ h_y \circ \mathcal{Y}_f)[\delta_0 \times (T_A - P_A)] = R.
\]

From the inequalities

\[
\dim(spt I) \leq \dim(p \circ \gamma)(spt I) + 1 \leq \dim(p \circ \gamma)[spt \partial(T_A - P_A)] + 1 < j,
\]

\[
\dim(spt J) \leq \dim(p \circ \gamma)[spt(T_A - P_A)] + 1 < j + 1,
\]

we see that \( I \) and \( J \) are analytic chains in \( R^n \). Choosing an open semianalytic set \( D \) with

\[
\text{Clos } D \subset (X \cap A) \quad \text{and} \quad (spt R \cup spt J) \sim D \subset N',
\]

we infer from 4.1, with \( E = \gamma^{-1}(C_d) \), that \( R \perp D \) satisfies condition (\#). Then since \( \partial(R \perp D) \) belongs to \( \mathcal{J}_j \mathcal{J}_j^{-1}[(X \cap \cup D) \cap N', X \cap \cup J] \), we may select an analytic chain \( Q \) satisfying condition (\#) and

\[
spt Q \subset (X \cap \cup D) \cap N', \quad spt \partial[(R \perp D) - Q] \subset X \cap \cup J.
\]
With $R^N_A = R \sqcup D - Q$ and $S^N_A = J \sqcup D$ we obtain the desired inclusions:

$$(\text{spt } R^N_A) \cup \text{spt } S^N_A \subset X \cap \cup D,$$

$\text{spt } (T_A - R^N_A - \partial S^N_A) = \text{spt } [Q + (R - R \sqcup D) + \partial(J - J \sqcup D) + P_A + I] \subset N \subset X \cap N.$

Next we fix a bounded semianalytic set $Y$ with $\text{Clos } X \subset Y \subset W$ and use 2.4, 2.7 (3), 2.8, and 2.9 to select an orthogonal transformation $g^* \in O(n)$, a partition $S^*$ of $R^n$, and a system of neighborhoods $\{U^*_r : \Gamma^* \subset C^*\}$ such that $S^*$ is compatible with $A^* = g^*(Y \cap \cup D)$ and $B^* = g^*(Y \cap \cup J)$, $g^*_*(T_A - R^N_A - \partial S^N_A)$ satisfies condition ($\#$) for all $N \in N'$, and

$$U^* = \cup \{U^*_r : \Gamma^* \subset S^* \text{ and } \Gamma^* \subset B^*\} \subset g^*(W).$$

Finally choosing $N \in N$ so that

$$N \subset (g^*)^{-1}(U^*) \cup (R^n - \text{Clos } X); \quad \text{hence } g^*(X \cap N) \subset U^*,$$

we observe that $g^*_*(T_A - R^N_A - \partial S^N_A) \subset g^*(A^* \cap U^*, B^*)$ and apply Proposition $B_\#_n$—with $S, C, D, V$ replaced by $S^*, S^* \cap \{\Gamma^* : \Gamma^* \subset B^*\}, S^* \cap \{\Gamma^* : \Gamma^* \subset A^*\}, R^s$—to obtain an analytic chain $S^*_A$ such that

$$\text{spt } S^*_A \subset A^* \cap U^* \subset g^*(W \cap \cup D),$$

$$\text{spt } [g^*_*(T_A - R^N_A - \partial S^N_A) - \partial S^*_A] \subset B^* \subset g^*(W \cap \cup J).$$

With $S_A = S^N_A + (g^*)^{-1} S^*_A$, we conclude that $\text{spt } S_A \subset W \cap \cup D$ and, by 2.8 (2) and 4.1, that $T_A - \partial S_A$ satisfies condition ($\#$) because $R^N_A$ does and

$$\text{spt } (T_A - R^N_A - \partial S_A) \subset (g^*)^{-1}[g^*(W \cap \cup J)] \subset \cup J,$$

which completes the proof.

4.5. Proposition $A_n$ implies proposition $B_n$.

Proof. Here we may repeat the argument of 4.3. Specifically we should, from that proof, drop all superscripts $\#$ and omit any statements concerning condition ($\#$) and any dimensional estimates involving the projection $p$.

Corollary 4.6. If $A \cap B$ are semianalytic subsets of a real analytic manifold $M$, then there exists an arbitrarily small open neighborhood $W$ of $B$ such that $H_j(A \cap W, B) = 0$ for all $j$. 

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PROOF. Since there exists ([5, Theorem 3]) a proper analytic embedding of $M$ into some Euclidean space, we assume that $M$ equals $R^m$ for some positive integer $m$.

Suppose $X$ is an open neighborhood of $B$. With $B_i = B(0, i)$ for $i \in \{1, 2, \ldots\}$ we use 2.4, 2.8, and 2.9 to choose inductively orthogonal transformations $g_1, g_2, \ldots \in O(m)$ and partitions $S_1, S_2, \ldots$ of $R^m$ with corresponding families $\{U_{i}\Gamma' \in S_{i}\}$ of neighborhoods such that $S_1$ is compatible with $B_1$, $g_1(A \cap B_1)$, and $g_i(B \cap B_i)$ and, for $i = 1, 2, \ldots$, $S_{i+1}$ is compatible with

$$B_{i+1}, \quad g_{i+1}(A \cap B_{i+1}), \quad g_{i+1}(B \cap B_{i+1}), \quad \text{and} \quad g_{i+1}[g_i^{-1}(I')]$$

for $I' \in S_i$.

$U_{i}^{i+1} \subseteq g_{i+1}(X)$ whenever $I' \in S_{i+1}$ and $I' \subseteq g_{i+1}(B \cap B_{i+1})$, and $W_{i}^{i+1} = g_{i+1}^{-1}(U_{i}^{i+1}) \subseteq U(0, i + 2) \cap g_i^{-1}(U_i')$ whenever $I' \in S_{i+1}$, $I'' \in S_i$, and $I' \subseteq g_{i+1}[g_i^{-1}(I'')]$; letting, for $i \in \{1, 2, \ldots\}$,

$$C_i = S_i \cap \{I': I' \subseteq g_i(B \cap B_i)\}, \quad D_i = S_i \cap \{A: A \subseteq g_i(A \cap B_i)\},$$

$$W_i = \bigcup \{W_{i}': I' \in C_i\},$$

we conclude from Proposition $B_0$ that

$$H_i(A \cap W_i, B \cap W_i) \simeq H_i[g_i(A \cap W_i), g_i(R \cap W_i)] =$$

$$= H_i \left( \bigcup_{I \in C_i} \bigcup_{A \in D_i} \bigcup_{I \in I_i} \cap (I \cup A) \cup I \right) \simeq 0 \quad \text{for all } j.$$

The set $W = \bigcup_{i=1}^{\infty} W_i$ is an open neighborhood $B$ in $X$. Suppose $T \in \mathcal{B}_0(A \cap W, B)$. To see that $T$ belongs to $\mathcal{B}_0(A \cap W, B)$, we will first let $W_0 = \emptyset$ and choose inductively, for $i = 0, 1, \ldots$, an analytic chain $S_i$ such that

$$\text{spt } S_i \subseteq A \cap W_i \quad \text{and} \quad \text{spt } (T - \partial \sum_{i=0}^{i} S_i) \subseteq \cup_{i=0}^{\infty} W_i.$$

Suppose $S_0 = 0$ and $S_1, S_2, \ldots, S_{i-1}$ have been chosen. Selecting an analytic chain $R$ such that

$$\text{spt } R \subseteq A \cap W_{i-1}, \quad \text{spt } \left[(T - \partial \sum_{i=0}^{i} S_i) - R\right] \subseteq \bigcup_{i=0}^{\infty} W_i;$$

hence $\text{spt } \partial R \subseteq A \cap B_i \cap \bigcup_{i=0}^{\infty} W_i \subseteq A \cap W_i$, we choose an analytic chain $Q$ with

$$\text{spt } Q \subseteq A \cap W_i, \quad \text{spt } \partial (R - Q) \subseteq B \cap W_i,$$
and then an analytic chain $S_i$ such that

$$\text{spt } S_i \subset A \cap W_i, \quad \text{spt}(R - Q - \partial S_i) \subset B \cap W_i;$$

thus $\text{spt}\left( T - \partial \sum_{i=0}^{i} S_i \right) \subset \bigcup_{i=0}^{\infty} W_i$.

Finally we take $i$ large enough so that $\text{spt } T \subset B_i$, infer that

$$\text{spt}\left( T - \partial \sum_{i=0}^{i} S_i \right) \subset A \cap \left( \bigcup_{i=0}^{\infty} W_i \right) \cap \left( B_i \cup \bigcup_{i=0}^{i} W_i \right) \subset A \cap W_i,$$

and select an analytic chain $S$ with

$$\text{spt } S \subset A \cap W_i \quad \text{and} \quad \text{spt}\left( T - \partial \sum_{i=0}^{i} S_i - \partial S \right) \subset B \cap W_i.$$

**Corollary 4.7.** There exist arbitrarily small open neighborhoods $U$ of $A$ in $M$ and $V$ of $B$ in $U$ such that the inclusion map of $\mathfrak{Z}_j(A, B)$ into $\mathfrak{Z}_j(U, V)$ induces an isomorphism $\Phi_j$ mapping $H_j(A, B)$ onto $H_j(U, V)$ for all $j$.

**Proof.** Apply 4.6 twice—with $A$, $B$ replaced:

first, by $M$, $A$ to obtain an open neighborhood $U$ of $A$ in $M$ with $H_j(U, A) = 0$ and

second, by $M$, $B$ to obtain an open neighborhood $V$ of $B$ in $U$ with $H_j(V, B) = 0$.

5. — Real Analytic Homology Theory.

Suppose $M$ and $N$ are $m$ and $n$ dimensional real analytic manifolds, $M \supset A \supset B$ and $N \supset C \supset D$ are semianalytic sets, and $f$ maps the pair $(C, D)$ continuously into $(A, B)$. Using 4.7 we will, by approximation, define the group homomorphism

$$H_j(f) \colon H_j(C, D) \rightarrow H_j(A, B) \quad \text{for } j \in \{0, 1, \ldots \}.$$

**Lemma 5.1.** If $U \supset V$ are open subsets of $M$, $j \in \{0, 1, \ldots \}$, $Q \subset I_j(M)$,\n
$spt \, Q \subset U$, and either $j = 0$ or $spt \, \partial Q \subset V$, then there exist an analytic chain $R \in \mathfrak{Z}_j(U, V)$ and an integral current $S \in I_{j+1}(M)$ such that $spt \, S \subset U$ and $\text{spt}(Q - R - \partial S) \subset V$. 


PROOF. Let $\alpha: M \to \mathbb{R}^s$ be a proper, real analytic embedding ([5]), and $A' \supset B'$ be relatively open semianalytic subsets of $\alpha(M)$ with $\alpha(\text{spt} Q) \subset A' \subset \alpha(U)$ and $\alpha(\text{spt} \partial Q) \subset B' \subset \alpha(V)$. Also let $\varrho$ be a class $\infty$ retraction mapping an open subset $W$ of $\mathbb{R}^s$ onto $A'$ ([3, 3.1.20]). We select open neighborhoods $U'$ of $A'$ in $W$ and $V'$ of $B'$ in $U' \cap \varrho^{-1}(B')$ such that $H_j(A', B') \simeq H_j(U', V')$ as in 4.7.

By the polyhedral approximation of [3, 4.2.9 (1)(4)(6)], there is a real analytic chain $R' \in I_j(U', V')$ and an integral current $S' \in I_{j+1}(M)$ with $\text{spt} S' \subset U'$ and $\text{spt}(\alpha_\varrho Q - R' - \partial S') \subset V'$. With analytic chains $R'' \in I_j(A', B')$ and $S'' \in I_{j+1}(U', V')$ chosen so that $\text{spt}(R'' - R' - \partial S'') \subset V'$, the lemma is satisfied by the two currents $R$ and $S$ in $M$ which are characterized by the conditions,

$$\alpha_\varrho R = R'' \quad \text{and} \quad \alpha_\varrho S = \varrho_\varrho(S' + S'').$$

COROLLARY 5.2. The inclusion map of $I_j(U, V)$ into $I_j(M) \cap \{Q: \text{spt} Q \subset U, \text{spt} \partial Q \subset V\}$ induces an isomorphism $\mathcal{P}$ mapping $H_j(U, V)$ onto the integral current homology group ([3, 4.4.5])

$$I_j(M) \cap \{Q: \text{spt} Q \subset U, \text{spt} \partial Q \subset V\}/\{R + \partial S: R \in I_j(M), S \in I_{j+1}(M), \text{spt} R \subset V, \text{spt} S \subset U\}.$$

LEMMA 5.3. If $K$ is a compact subset of $C$ and $\varepsilon > 0$, then there exists a class $\infty$ function $g$ mapping $N$ into $M$ such that $\text{dist}[f(x), g(x)] < \varepsilon$ for all $x \in K$.

PROOF. We consider the commutative diagram

$$
\begin{array}{ccc}
N \subset K & \xrightarrow{f|K} & M \\
\downarrow \text{id}_K & & \downarrow \alpha \\
\mathbb{R}^b & \xrightarrow{F} & \mathbb{R}^s
\end{array}
$$

where $\alpha: M \to \mathbb{R}^s$ and $\beta: N \to \mathbb{R}^b$ are class $\infty$ proper imbeddings ([20, p. 113]), and $F$ is a continuous extension to $\mathbb{R}^s$ of the map $(x \circ f \circ \beta^{-1})|\beta(K)$. We also choose a class $\infty$ retraction $\varrho$ of an open neighborhood $W$ of $\alpha(M)$ onto $\alpha(M)$ and a compact subset $L$ of $W$ with $\alpha[f(K)] \subset \text{Int} L$. With

$$\lambda = \sup \{\text{Lip}(\varrho|L), \text{Lip}[x^{-1}|\varrho(L)]\},$$

we may, by regularization ([3, 4.1.2]), choose a class $\infty$ mapping $G: \mathbb{R}^b \to \mathbb{R}^s$ such that $G[\beta(K)] \subset \text{Int} L$ and $\text{dist}[F(y), G(y)] < \varepsilon/\lambda^2$ for $y \in \beta(K)$; the lemma follows with $g = x^{-1} \circ \varrho \circ G \circ \beta$. 


LEMMA 5.4. For any $T \in I_c(N)$ with $\text{spt } T \subset C$ and open neighborhoods $U$ of $f(\text{spt } T)$ and $V$ of $f(\text{spt } \partial T)$ in $M$, there is an $\varepsilon > 0$ such that if $g$ and $h$ are class $\infty$ mappings of $N$ into $M$ with $\text{dist}[f(x), g(x)] < \varepsilon$ and $\text{dist}[f(x), h(x)] < \varepsilon$ for $x \in \text{spt } T$, then

$$(\text{spt } g_\# T) \cup (\text{spt } h_\# T) \subset U,$$

and there exists an integral current $S \in I_{i+1}(M)$ with $\text{spt } S \subset U$ and

$$\text{spt}(g_\# T - h_\# T - \partial S) \subset V.$$  

PROOF. With $\alpha$, $R^\alpha$, $\varrho$, $W$ as in 5.3, we choose $\varepsilon < 0$ so that

$$(\{y : \text{dist}[y, (x \circ f)(\text{spt } T)] < \varepsilon\} \subset W),$$

$$\varrho(\{y : \text{dist}[y, (x \circ f)(\text{spt } T)] < \varepsilon\}) \subset \alpha(U),$$

$$\varrho(\{y : \text{dist}[y, (x \circ f)(\text{spt } \partial T)] < \varepsilon\}) \subset \alpha(V).$$

If $g$ and $h$ satisfy the hypothesis and

$$\sigma : R \times N \to R^b, \quad \sigma(t, x) = (1 - t)g(x) + th(x) \quad \text{for } (t, x) \in R \times N,$$

then $S = \alpha_{R^b}^{-1} g_\# \sigma_{[0, 1]}(0, 1) \times T) \in I_{i+1}(M)$ satisfies, by [3, 4.1.9],

$$\alpha(\text{spt } S) \subset \varrho(\sigma([t : 0 < t < 1] \times \text{spt } T)) \subset \alpha(U),$$

$$\alpha(\text{spt}(g_\# T - h_\# T - \partial S)) \subset \varrho(\sigma([t : 0 < t < 1] \times \text{spt } \partial T)) \subset \alpha(V).$$

5.5. Let $T \in \mathcal{I}(C, D)$ and $K = \text{spt } T$. With $U$, $V$ as in 4.7, $\varepsilon$ as in 5.4, and $g$ as in 5.3, let $\omega$ be the integral current homology class (5.2) of the integral current $g_\# T$; the function which associates $\omega$ with $T$ is, by 5.4, a well-defined group homomorphism with kernel containing $\mathcal{B}_\varepsilon(C, D)$. Letting $\Omega$ denote the induced homomorphism on $H_i(C, D)$, we recall 4.7 and 5.2 and define the homomorphism

$$H_i(f) : H_i(C, D) \to H_i(A, B), \quad H_i(f) = \Phi_i^{-1} \circ \psi^{-1} \circ \Omega.$$ 

The axioms of Eilenberg and Steenrod ([2, p. 10]), which for integral current homology in the local Lipschitz category readily follow by elementary properties of integral currents as in [3, 4.4.1, 4.4.5], are also easily verified, by approximation, for our real analytic homology theory on the category of semianalytic sets and continuous maps.
5.6. The homology groups $H_j(A, B)$ for $j \in \{0, 1, \ldots\}$ are isomorphic to the homology groups of the chain complex ([2, p. 124]) with chain groups $C_j = Z_j(A, A)/B_j(B, B)$ for $j \geq 0$, $C_j = \{0\}$ for $j < 0$, and with boundary homomorphisms $\partial_j : C_j \to C_{j-1}$ induced by $\partial$ for $j > 0$.

**Theorem 5.7.** If $A \cap B$ are relatively compact semianalytic subsets of $M$, then $H_j(A, B)$ is finitely generated for all $j$.

**Proof.** By the fourth axiom (exactness) of Eilenberg-Steenrod, we assume that $B = \emptyset$. We also note that if $E$ and $F$ are semianalytic sets with $F \cap \text{Clos } E \subset E$, then the inclusion $\bigcup_j (E, E \cap F) \subset \bigcup_j (E \cup F, F)$ and the map sending $T \in \bigcup_j (E \cup F, F)$ to $T \cup E \in \bigcup_j (E, E \cap F)$ induce inverse isomorphisms between $H_j(E, E \cap F)$ and $H_j(E \cup F, F)$ for all $j$; thus if $F \cap \text{Clos } E \subset E$ and $E \cap \text{Clos } F \subset F$, then there is, by [2, 1.4.1, 15.3], an exact Mayer-Vietoris sequence

$$0 \leftarrow H_0(E \cup F) \leftarrow H_0(E) \oplus H_0(F) \leftarrow H_0(E \cap F) \leftarrow H_1(E \cup F) \leftarrow \ldots$$

$$\vdots \leftarrow H_{j-1}(E \cap F) \leftarrow H_j(E \cup F) \leftarrow H_j(E) \oplus H_j(F) \leftarrow \ldots$$

From this we observe, by induction, that if $\mathcal{E}$ is a finite family of semianalytic sets such that $E \cap \text{Clos } F \subset F$, $F \cap \text{Clos } E \subset E$, and $H_j(E \cap \bigcup \mathcal{F})$ is finitely generated whenever $E \in \mathcal{E}$, $F \in \mathcal{E}$, and $\mathcal{F} \subset \mathcal{E}$, then $H_j(\bigcup \mathcal{E})$ is finitely generated for all $j$. In particular, by covering $\text{Clos } A$ by finitely many closed balls contained in coordinate neighborhoods, we may assume $M$ is an open subset of $\mathbb{R}^n$.

We now use induction on $n$. For any interval or singleton set $I$ in $\mathbb{R}^1$ and $a \in I$, there is a strong deformation retraction ([17, p. 30]) of $I$ onto $\{a\}$; thus $H_0(I) \simeq H_0(\{a\}) \simeq \mathbb{Z}$ and $H_j(I) \simeq H_j(\{a\}) \simeq 0$ for $j > 0$ by the first, fifth and seventh axioms of Eilenberg-Steenrod. The case $n = 1$ follows because any bounded semianalytic subset of $\mathbb{R}^1$ is a finite disjoint union of intervals and singleton sets.

To handle the inductive step, we assume, after an orthogonal transformation of $\mathbb{R}^n$, that $\mathcal{A}$, $g$, $S$, $\mathcal{C}$, $\overline{\mathcal{C}}$, $H$, $Z_F$, $Z_F$ are as in 2.8 with $\mathcal{A} = \{A\}$ and $g = 1_{\mathbb{R}^n}$, and let

$$\mathcal{C} = \{A \cap \text{Clos } I'; I' \in \overline{\mathcal{C}} \text{ and } I' \subset A\},$$

$$\mathcal{D} = \{A \cap \text{Clos}(A \cap p^{-1}[p(I')] : I' \in \mathcal{C}, A \in \overline{\mathcal{A}} \sim \overline{\mathcal{C}}, \text{ and } A \subset A\},$$

and $\mathcal{E} = \mathcal{C} \cup \mathcal{D}$. Then, being bounded, $A = \cup \mathcal{E}$. Moreover $F \cap \text{Clos } E \subset E$ and $E \cap \text{Clos } F \subset F$ whenever $E, F \in \mathcal{E}$.
If $E \subset C$ and $\mathcal{F} \subset \mathcal{C}$, then, by 2.8 (2)(4)(6), $p$ maps $E \cap \cup \mathcal{F}$ homeomorphically onto the semianalytic subset $p(E \cap \cup \mathcal{F})$ of $R^{n-1}$; hence, $H_j(E \cap \cup \mathcal{F}) \simeq H_j[p(E \cap \cup \mathcal{F})]$ is finitely generated for all $j$. It follows, in particular, that for any $B \subset C$, $H_j(\cup \mathcal{B})$ is finitely generated for all $j$.

Next if $E \not\subset \mathcal{F}$ and $\mathcal{F} \subset \mathcal{C}$, then there are two possibilities. If $E \not\in \mathcal{F}$, then $E \cap \cup \mathcal{F}$ is, by 2.8 (4), the union of a subfamily of $C$; hence $H_j(E \cap \cup \mathcal{F})$ is finitely generated for all $j$. If however $E \in \mathcal{F}$, then $E \cap \cup \mathcal{F} = E$. By 2.8 (4)(5)(6), $p(E)$ is a semianalytic subset of $R^{n-1}$. Suppose $E = A \cap \text{Clos}(A \cap p^{-1}[p(I)])$ where $I \subset C$, $A \in \overline{C}$ and $A \subset A$. There are, by 2.8 (3), continuous functions $\sigma$ and $\tau$ on $Z_p$ such that $H_p[y, \sigma(y)] = H_p[y, \tau(y)] = 0$ for $y \in Z_p$ and $\sigma(y) \mapsto \tau(y)$.

Arguing as in 4.2 we see that

$$C = p^{-1}(Z_p) \cap \{x: q(x) = \frac{1}{2}(\sigma + \tau)[p(x)]\}$$

is a semianalytic set for which $p^{-1}[y]$ is nonsingular whenever $y \in R^{n-1}$; thus, by 2.2 and 2.7 (3), $p$ maps any semianalytic set in $C$ homeomorphically onto a semianalytic subset of $R^{n-1}$. Since by 2.8 (3)(4)(6)

$$h: \{t: 0 < t < 1\} \times E \to p^{-1}[p(E)]$$

$$h(t, x) = (1-t)x + t(p|C)^{-1}[p(x)]$$

for $0 < t < 1$ and $x \in E$,

is a strong deformation retract of $E$ onto $C \cap p^{-1}[p(E)]$, $H_j(E) \simeq H_j(C \cap p^{-1}[p(E)]) \simeq H_j[p(E)]$ are finitely generated for all $j$.

It now follows from our previous observation that $H_j(A) = H_j(\cup \mathcal{B})$ is finitely generated for all $j$.

6. - Intersection Theory for $H^*(A, B)$.

Suppose $M$ is an $m$ dimensional orientable real analytic manifold, $M \supset A \supset B$ are semianalytic, and $i$ and $j$ are nonnegative integers with $i + j \geq m$. Using 4.7 and [6, § 5] we will define, in 6.4, for any two homology classes $\varrho \in H_i(A, B)$ and $\tau \in H_j(A, B)$ the intersection class $\varrho \cap \tau \in H_{i+j-m}(A, B)$. Recall that for any $i$ dimensional analytic chain $R$ in $M$ and $j$ dimensional analytic chain $T$ in $M$ which intersect suitably, that is,

$$\dim(\text{spt } R \cap \text{spt } T) \leq i + j - m,$$

$$\dim[(\text{spt } \partial R \cap \text{spt } T) \cup (\text{spt } R \cap \text{spt } \partial T)] \leq i + j - m - 1,$$
an $i + j - m$ dimensional analytic chain $R \cap T$ has been defined and that
real analytic intersection theory in $M$ «at the chain level» has been treated
in [6, § 5].

To define $R \cap \tau$ we first observe that if $E$ and $F$ are subsets of $\mathbb{R}^m$ with
$\dim E + \dim F > m - 1$, then

$$\dim[\tau_z(E) \cap F] < \dim E + \dim F - m$$

for $\mathbb{R}^n$ almost all $z \in \mathbb{R}^m$.

In fact by using the maps $f: \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m$, $g_z: \mathbb{R}^m \to \mathbb{R}^m \times \mathbb{R}^m$,

$$f(x, y) = x - y, \quad g_z(x) = (x + z, x)$$

for $x, y, z \in \mathbb{R}^m$,

we may infer that $g_z$ is an analytic isomorphism mapping $\tau_z(E) \cap F$ onto
$(E \times F) \setminus f^{-1}(z)$ and then apply [6, 2.2 (7)].

**Lemma 6.1.** If $E$ and $F$ are semianalytic subsets of $\mathbb{R}^m$ with $\dim E + \dim F > m - 1$, then for $\mathbb{R}^{m-1}$ almost all $\xi \in S^{m-1}$

$$\dim\{x + t\xi: x \in E\} \cap F \leq \dim E + \dim F - m$$

for $\mathbb{R}^1$ almost all $t \in \mathbb{R}$ and

$$\dim\{x + t\xi: x \in E, t \in \mathbb{R}\} \cap F \leq 1 + \dim E + \dim F - m.$$ 

**Proof.** We abbreviate $l = 1 + \dim E + \dim F - m$ and for $(x, \xi) \in \mathbb{R}^m \times S^{m-1}$, $L_{x, \xi} = \mathbb{R}^m \cap \{x + t\xi: t \in \mathbb{R}\}$. From the above observation, 2.4, and Fubini’s theorem, we infer that, for $\mathbb{R}^{m-1}$ almost all $\xi \in S^{m-1}$, the line

$L_{x, \xi}$ is nonsingular for $E$ and

$$\dim[\tau_{\xi}(E) \cap F] < \dim E + \dim F - m.$$ 

for all $x \in \mathbb{R}^m$ and $\mathbb{R}^1$ almost most all $t \in \mathbb{R}$.

Fix such a $\xi \in S^{m-1}$, let $h: \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m$, $h(t, x) = x + t\xi$ for $(t, x) \in \mathbb{R} \times \mathbb{R}^m$,

and choose $b \in O(m)$ with $b(\xi) = (0, \ldots, 0, 1)$. Then, by 2.7 (3), $(p \circ b)(E)$ and

$$h(\mathbb{R} \times E) \cap F = (p \circ b)^{-1}[(p \circ b)(E)] \cap F$$

are semianalytic sets. Assuming, for contradiction, that $\dim[h(\mathbb{R} \times E) \cap \mathbb{R}^m] > l$, we choose, by [6, 2.2 (4)] a bounded interval $I \subset \mathbb{R}$ such that

$$\infty > \mathcal{H}^{l+1}[h(I \times E) \cap F] > 0.$$ 

Moreover by 2.6 (4) there is a semianalytic stratum $I'$ in $E$ such that $\mathcal{H}^{l+1}[h(I \times E) \cap F]$ is positive and $p|b(I')$ and
hence, \( h(I \times I') \) are analytic isomorphisms. Letting

\[
\varphi: I \times I' \to I, \quad \varphi(t, x) = t \quad \text{for} \ (t, x) \in I \times I',
\]

\[
\psi = \varphi \circ [h(I \times I')]^{-1} h(I \times I') \cap F,
\]

we infer that the approximate Jacobian \( \text{ap} \ J_1 \psi(x) \) ([3, 3.2.20]) is positive for every regular point \( x \) of \( h(I \times I') \cap F \). From [6, 2.2(3)(6)] and the coarea formula [3, 3.2.22 (3)] we deduce the contradiction

\[
0 = \int \mathcal{H}[\tau_{E}(E) \cap F] \, d\mathcal{H}^{1} t = \int \mathcal{H}[\tau_{E}(E) \cap F] \, d\mathcal{H}^{1} t
\]

\[
= \int_{I} \int_{\varphi^{-1}(t)} i \mathcal{H}^{1} d\mathcal{H}^{1} t = \int_{h(I \times I') \cap F} \text{ap} \ J_1 \psi \, d\mathcal{H}^{1+1} > 0.
\]

**Lemma 6.2.** Suppose \( Q \) is an \( i \) dimensional analytic chain in \( M \), \( \mathcal{F} \) is a countable collection of semianalytic sets in \( M \), \( U \), \( V \), \( W \), \( V \), \( W \) are open subsets of \( M \), \( \text{spt} \, Q \subset U \), \( \text{Clos} \, V \subset W \), \( W \cap \text{spt} \, \partial Q = \emptyset \), \( \text{Clos} \, W \) is compact, and there exists an analytic isomorphism mapping \( W \) into \( \mathbb{R}^m \). If \( \mathcal{R} \) and \( \mathcal{S} \) are analytic chains in \( M \), \( (\text{spt} \, \mathcal{R}) \cup \text{spt} \, \mathcal{S} \subset U \), \( \text{Clos} \, \mathcal{W} \cap \text{spt} \mathcal{Q} - \mathcal{R} - \mathcal{S} \), and \( \mathcal{S} \) is analytic chain in \( M \) such that \( \text{dim}(\mathcal{F} \cap \text{spt} \, \mathcal{R}) \leq i + (\text{dim} \, \mathcal{F}) - m \), \( \text{dim}(\mathcal{F} \cap \text{spt} \, \partial \mathcal{R}) \leq i + (\text{dim} \, \mathcal{F}) - m - 1 \)

\[
\text{for all } F \in \mathcal{F}, \text{then there exist analytic chains } \mathcal{R} \text{ and } \mathcal{S} \text{ in } M \text{ such that } (\text{spt} \, \mathcal{R}) \cup \text{spt} \, \mathcal{S} \subset U,
\]

\[
(\mathcal{V} \cup \mathcal{W}) \cap \text{spt} \mathcal{Q} = \emptyset,
\]

\[
\mathcal{V} \cap [\text{spt}(\mathcal{R} - \mathcal{R}) \cup \text{spt} \mathcal{S} - \mathcal{S}] = \emptyset,
\]

\[
\text{dim}(\mathcal{F} \cap \text{spt} \, \mathcal{R}) \leq j + (\text{dim} \, \mathcal{F}) - m \text{, } \text{dim}(\mathcal{F} \cap \text{spt} \, \partial \mathcal{R}) \leq j + (\text{dim} \, \mathcal{F}) - m - 1 \text{ for all } F \in \mathcal{F}.
\]

**Proof.** Choosing, by [6, 2.2 (7)], an open semianalytic set \( D \) with \( \text{Clos} \, V \subset D \subset \text{Clos} \, D \subset W \) and

\[
\text{dim}[(\mathcal{F} \cap \text{spt} \, \mathcal{R}) \cap \text{Fr} \mathcal{D}] \leq i + (\text{dim} \, \mathcal{F}) - m - 1,
\]

\[
\text{dim}[(\mathcal{F} \cap \text{spt} \, \partial \mathcal{R}) \cap \text{Fr} \mathcal{D}] \leq i + (\text{dim} \, \mathcal{F}) - m - 2
\]

for all \( F \in \mathcal{F} \), it suffices to prove the lemma with \( Q \), \( \mathcal{F} \), \( U \), \( V \), \( W \), \( R \), \( S \) replaced by \( Q \cap D - (\partial S) \cup \partial D \),

\[
\{F \cap \text{Clos} \mathcal{D}, F \cap \text{Fr} \mathcal{D} : F \in \mathcal{F}\}, \ U \cap W \text{, } V \cap W \text{, } W \cap W \text{, } \text{R} \cap \text{D} \text{, } S \cap \text{D}
\]
to obtain suitable analytic chains $R_i$, $S_i$ and then let

$$R = R_0 \sqcup (M \sim D) + R_1 \sqcup D, \quad S = S_0 \sqcup (M \sim D) + S_1 \sqcup D.$$ 

Thus we may assume $M$ equals $\mathbb{R}^n$ and $K = \text{spt} Q \cup \text{spt} R_0 \cup \text{spt} S_0$ is compact.

Let $\alpha$ be a polynomial on $\mathbb{R}^n$ with $\alpha(x) < 0$ for $x \in V_0 \cap K$ and $\alpha(x) > 1$ for $x \in K \sim W_0$ and choose, by [6, 2.2 (7)] a $r$ so that $0 < r < 1$ and

$$\dim(\alpha^{-1}(r) \cap \text{spt} R_0) < i - 1, \quad \dim(\alpha^{-1}(r) \cap \text{spt} \partial R_0) < i - 2,$$

$$\dim(\alpha^{-1}(r) \cap F \cap \text{spt} R_0) < i + (\dim F) - m - 1,$$

$$\dim(\alpha^{-1}(r) \cap F \cap \text{spt} \partial R_0) < i + (\dim F) - m - 2$$

for all $F \in \mathcal{F}$. Thus $(Q - R_0 - \partial S_0) \sqcup \{x: \alpha(x) < r\} = 0$. With

$$E = \{\text{spt} [(Q - \partial S_0) \sqcup \{x: \alpha(x) > r\}], \text{spt} <R_0, \alpha, r>, \text{spt} \partial [Q - \partial S_0) \sqcup \{x: \alpha(x) > r\}], \text{spt} \partial <R_0, \alpha, r>\},$$

we use 6.1 to select $\xi \in S^{n-1}$, $\varepsilon > 0$, and $h: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ so that

$$h(t, x) = x + t\xi \quad \text{for} \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n,$$

$$h(\{t: 0 < t < \varepsilon\} \times K) \subseteq U,$$

$$V \cap h(\{t: 0 < t < \varepsilon\} \times \text{spt} \partial Q) = \emptyset,$$

$$V_0 \cap h(\{t: 0 < t < \varepsilon\} \times K \cap \{x: \alpha(x) > r\}) = \emptyset,$$

$$\dim[\tau_{\varepsilon\xi}(E) \cap F] < \dim E + \dim F - m,$$

$$\dim[h(\mathbb{R} \times E) \cap F] < 1 + \dim E + \dim F - m$$

for all $E \in E$ and $F \in \{\mathbb{R}^m\} \cup \mathcal{F}$. From these last two estimates we infer that

$$R = R_0 \sqcup \{x: \alpha(x) < r\} + h_{\varepsilon}[0, \varepsilon] \times \langle R_0, \alpha, r\rangle + \delta_{\varepsilon\xi\varepsilon}[(Q - \partial S_0) \sqcup \{x: \alpha(x) > r\}],$$

$$S = S_0 + h_{\varepsilon}[0, \varepsilon] \times (Q - \partial S_0) \sqcup \{x: \alpha(x) > r\}$$

are analytic chains in $\mathbb{R}^n$ and

$$\dim(F \cap \text{spt} R) < i + (\dim F) - m,$$

$$\dim(F \cap \text{spt} \partial R) < i + (\dim F) - m - 1.$$
for all $\mathcal{F} \in \mathcal{F}$. Using [3, 4.1.9, 4.2.1, 4.3.4] we conclude

$$(\text{spt } R) \cup \text{spt } S \subset U, \quad V_0 \cap [\text{spt } (R - R_0) \cup \text{spt } (S - S_0)] = \emptyset,$$

$$(V_0 \cup V) \cap \text{spt } (Q - R - \partial S) = \emptyset.$$

**Theorem 6.3.** If $U \cup V$ are open subsets of $M$, $q \in H_i(U, V)$, and $\mathcal{C}$ is a countable collection of analytic chains in $M$, then there exists an analytic chain $R \in \mathfrak{R}$ such that $R$ and $T$ intersect suitably for all $T \in \mathcal{C}$.

**Proof.** Let $Q \in \mathcal{C}$ and choose first finite open covers $\{U_1, U_2, ..., U_L\}$ and $\{V_1, V_2, ..., V_L\}$ of spt $Q$ such that Clos($U_1 \cup ... \cup U_L$) is a compact subset of $U$, $V_L \subset V$, ($\text{spt } Q) \cap (U_1 \cup ... \cup U_{L-1}) = \emptyset$ and Clos $V_L \subset U_l$ for $l \in \{1, ..., L\}$, and then open sets $W_0, W_1, ..., W_L$ so that $W_0 = \emptyset$, for $l \in \{1, 2, ..., L\}$. We inductively apply 6.2, for each $l \in \{1, 2, ..., L-1\}$, with $\mathcal{F}, U, V_0, W_0, V, W, R_0, S_0$ replaced by $\{spt T, \text{spt } \partial T : T \in \mathcal{C}\}$, $V_1 \cup ... \cup V_L$, $W_{l+1} \cup W_{l-1} \cap U_l$, $V_l$, $U_l, R_{l+1}, S_{l+1}$ to obtain analytic chains $R_l$ and $S_l$ such that

$$(\text{spt } R_l) \cup \text{spt } S_l \subset V_1 \cup V_2 \cup ... \cup V_L,$$

$$\text{spt } (Q - R_l - \partial S_l) \subset M \sim (W_{l+1} \cup V_l) \subset V_{l+1} \cup ... \cup V_L,$$

and $R_l$ and $T$ intersect suitably for all $T \in \mathcal{C}$, and then take $R = R_{L-1}$.

**6.4. $q \cap \tau$.** Let $U$ and $V$ be as in 4.7. For any homology classes $q \in H_i(A, B)$ and $\tau \in H_i(A, B)$ we use 6.3 to choose analytic chains $R \in \Phi_i(q)$ and $T \in \Phi_i(\tau)$ which intersect suitably and define the intersection class $q \cap \tau \in H_{i+j-m}(A, B)$ as the $\Phi_{i+j-m}$ inverse image of the homology class in $H_{i+j-m}(U, V)$ of $R \cap T$.

The homology intersection class $q \cap \tau$ is then well-defined. In fact suppose $R' \in \Phi_i(q)$ and $T' \in \Phi_i(\tau)$ also intersect suitably. Then there are analytic chains $Q$ and $S$ such that (spt $Q) \cup \text{spt } S$ is a compact subset of $U$ and

$$\text{spt } (R - R' - \partial Q) \cup \text{spt } (T - T' - \partial S') \subset V.$$

Using 6.3 to change, if necessary, first $S$ and then $Q$, we may assume $\{S, R\}, \{S, R'\}, \{Q, T\}, \{Q, T'\}, \{Q, \partial S\}$, and hence $\{R - R' - \partial Q, T\}$ and $\{T - T' - \partial S, R\}$ intersect suitably. Thus, by [6, 5.8 (9)],

$$\text{spt } (R \cap T - R' \cap T' - \partial [(1 - 4)^{m - 4} R \cap S + T' \cap Q + Q \cap \partial S]) \subset V.$$
Moreover we also infer that $\partial \cap \tau$ is independent of the initial choice of $U$, $V$; if $U'$, $V'$ is a different pair of open sets satisfying 4.7, then we may by the previous paragraph, choose

$$R' \in \mathcal{Z}_i(U \cap U', V \cap V') = \mathcal{Z}_i(U, V) \cap \mathcal{Z}_i(U', V'),$$

$$T' \in \mathcal{Z}_j(U \cap U', V \cap V') = \mathcal{Z}_j(U, V) \cap \mathcal{Z}_j(U', V')$$

to compute $\partial \cap \tau$.


To prove associativity suppose $\varrho \in H_i(A, B)$, $\sigma \in H_j(A, B)$, and $\tau \in H_k(A, B)$ where $i + j > m$, $j + k > m$, and $i + j + k > 2m$. Choose analytic chains $S \in \Phi_i(\varrho)$ and $T \in \Phi_j(\sigma)$ which intersect suitably, apply 2.6 and 3.0 to construct a $j + k - m$ dimensional analytic chain $P$ and a $j + k - m - 1$ dimensional analytic chain $Q$ such that

$$(\text{spt } S) \cap \text{spt } T \subseteq \text{spt } P,$$

$$(\text{spt } \partial S \cap \text{spt } T) \cup (\text{spt } S \cap \text{spt } \partial T) \subseteq \text{spt } Q,$$

and then select an analytic chain $R \in \Phi_k(\tau)$ which intersects suitably with $S$, $P$, and $Q$. Thus $\{R, S\}$, $\{S, T\}$, and $\{R, S, T\}$ (See [6, 5.10]) intersect suitably; hence $(R \cap S) \cap T$ equals, by [6, 5.11 (6)], $R \cap (S \cap T)$.

7. - Real Analytic Sets.

In this section we assume that $E$ is a $k$ dimensional real analytic set in $M$ and let

$$\text{Reg } E = E \cap \{x: x \text{ has a neighborhood } U \text{ such that } U \cap E$$

is a $k$ dimensional analytic submanifold of $M\}.$

Thus $E$ is closed and $\dim (E \sim \text{Reg } E) \leq k$ by 2.6. We first study the extent to which $E$ is locally orientable.

**Theorem 7.1.** If $k$ equals either $m - 1$ or 1 and $y \in \text{Clos } \text{Reg } E$, then there exist an analytic chain $T$ in $M$ and an open ball $U$ about $x$ such that

$$U \cap \text{spt } T = U \cap \text{Clos } E,$$

$$U \cap \text{spt } \partial T = \emptyset$$

and for every $x \in U \cap E$, $\Theta^k(\|T\|, x) = 1$; hence ([3, 4.1.31 (2)]) the homology class of $T$ generates $H_k(E, E \sim \{x\})$. 

PROOF. We assume that \( k \geq 1 \), that \( \mathcal{M} \) is an open subset of \( \mathbb{R}^n \), that \( y = 0 \), and that the germ of \( E \) at 0, \( \gamma_0(E) \), is irreducible ([3, 3.4.5]).

Case 1, \( k = m - 1 \). Recalling 2.6 (or [3, 3.4.8 (13)(10)]), we choose a connected neighborhood \( V \) of 0 in \( \mathcal{M} \), a function \( f \) analytic in \( V \) and a subset \( F \) of \( V \) such that \( \dim F < k - 1 \), \( V \cap E \subset f^{-1}(0) \), \( 0 \in \text{Clos}(f^{-1}(0) \sim F) \), and \( Df(a) \neq 0 \) for all \( a \in f^{-1}(0) \sim F \); by [3, 3.4.5, 3.4.7] we may also assume that \( \gamma_0(f^{-1}(0)) \) is irreducible. Then the inclusions

\[
\gamma_0(E) \subset \gamma_0(f^{-1}(0)) \subset \gamma_0(\mathbb{R}^n)
\]

and [3, 3.4.8 (15)] imply that \( \gamma_0(E) = \gamma_0(f^{-1}(0)) \). Choosing an open ball \( U \) about 0 such that \( \text{Clos } U \subset V \) and \( U \cap E = U \cap f^{-1}(0) \), we define \( T \) to be the extension ([6, 3.3]) of \( \langle \mathbb{R}^n \setminus V, f, 0 \rangle \) to \( \mathcal{M} \); therefore \( U \cap \text{spt } \partial T \subset \text{spt } \mathbb{E}^n = \emptyset \). Noting that for all points \( a \in U \cap (f^{-1}(0) \sim F) \), \( \text{im } Df(a) = \mathbb{R} \), hence \( \Theta^1(\|T\|, a) = 1 \) by [3, 4.3.11], and that \( U \cap \text{Reg } E \subset \text{Clos}(f^{-1}(0) \sim F) \), we conclude first that

\[
U \cap \text{spt } T = U \cap \text{Clos}(f^{-1}(0) \sim F) = U \cap \text{Clos } \text{Reg } E
\]

and second, by [3, 4.1.31 (2)], that \( \Theta(\|T\|, x) = 1 \) for \( x \in \text{Reg } E \).

Case 2, \( k = 1 \). Here we use [3, 3.4.8 (10)] to choose \( r > 0 \), orthogonal projections \( \mu: \mathbb{R}^n \to \mathbb{R}^2 \), \( v: \mathbb{R}^2 \to \mathbb{R} \) with \( v(s, t) = s \) for \( (s, t) \in \mathbb{R}^2 \), and a finite family \( \mathcal{J} \) of one dimensional semianalytic strata in \( W = (v \circ \mu)^{-1}U(0, r) \) such that \( \gamma_0(\{0\} \cup \cup \mathcal{J}) = \gamma_0(E) \), \( E' = \mu(\{0\} \cup \cup \mathcal{J}) \) is an analytic subset of \( M' = v^{-1}U(0, r) \), \( \mu \) maps \( \cup \mathcal{J} \) isomorphically onto an analytic submanifold of \( M' \), and \( v \circ \mu \) maps each \( G \in \mathcal{J} \) isomorphically onto either \( \mathbb{R} \cap \{t: r < t < 0\} \) or \( \mathbb{R} \cap \{t: 0 < t < r\} \). We use Case 1 with \( M, E, y \) replaced by \( M', E', 0 \) to choose a suitable analytic chain \( T' \) and open ball \( U' \) in \( M' \). Applying, for each \( G \in \mathcal{J}, [3, 4.1.31 (2)] \) to the component \( C_G \) of \( U' \cap \mu(G) \) whose closure contains 0, we obtain an orienting 1 vectorfield \( \xi_G \) of \( G \) such that

\[
\text{spt}(T' - \mu_y[(\mathcal{E}^1 \setminus G)/\xi_G]) \subset M' \sim C_G.
\]

Letting \( T = \sum_{\mathcal{J} \in \mathcal{J}}(\mathcal{E}^1 \setminus G)/\xi_G \) and \( U \) be an open ball about 0 in \( \mathcal{M} \cap W \) such that \( U \cap (\{0\} \cup \cup \mathcal{J}) \) equals \( U \cap E \), we see that \( U \cap \text{spt } T = U \cap \text{Clos } \text{Reg } E \) and \( \Theta^1(\|T\|, a) = 1 \) for \( a \in \cup \mathcal{J} \). Noting that \( \partial T|U = i\delta_{y}|U \) for some integer \( i \), we compute

\[
i\delta_{\mu(0)}|U' = \mu_y(\partial T)|U' = \partial T'|U' = 0
\]
by Case 1, hence \( i = 0 \). Finally if \( 0 \in \text{Reg } E \), then \( \Theta(T, 0) = 1 \) by [3, 4.1.31 (2)] as before.

**Example 7.2.** Letting \( g: \mathbb{R}^6 \to \mathbb{R}^6 \) be given by

\[
g(x_1, x_2, x_3, x_4, x_5, x_6) = (x_1^2 - 2x_1x_2, x_2^2 - 2x_2x_3, x_3^2 - 2x_3x_4, \sqrt{2}x_4 - x_3x_5, \sqrt{2}x_5 - x_4x_6, \sqrt{2}x_6 - x_5x_6)
\]

for \((x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6\), we compute that \( g(0) = 0 \) and that \( \dim Dg(x) (\mathbb{R}^6) = 3 \) whenever \( 0 \neq x \in \mathbb{R}^6 \). Thus \( C = g^{-1}(0) \) is, by [3, 3.1.18] and 2.1, a three dimensional real analytic subset of \( \mathbb{R}^6 \). Being the double cone over a real projective plane, \( C \) is not locally orientable at 0. Specifically we show that

if \( U \subset \mathbb{R}^6 \) is open, \( T \) is a 3 dimensional analytic chain in \( \mathbb{R}^6 \) and \( 0 \in U \cap \text{spt } T \subset U \cap C \), then \( 0 \in \text{spt } \epsilon T \) [hence \( H_3(C, C \sim \{0\}) \simeq H_3(C, C \sim \{0\}) \simeq 0 \)].

In fact suppose \( 0 \notin \text{spt } \epsilon T \). Letting \( u: \mathbb{R}^6 \to \mathbb{R}, u(x) = |x| \) for \( x \in \mathbb{R}^6 \), we note that \( (\text{grad } u)(x) \in \text{Tan}(C, x) \) whenever \( 0 \neq x \in C \) because \( C \) is a cone. Since the 3 vector \( \nabla^3 u(x) \) is associated with \( \text{Tan}(C, x) \) for every nonzero regular point \( x \) of \( \text{spt } T \sim \text{spt } \epsilon T \), and we may, by [3, 4.3.2 (1)] and [6, 2.2 (7), 4.3] choose a positive \( r > \text{dist}[0, (\text{Fr } U) \cup \text{spt } \epsilon T] \) so that \( \langle T, u, r \rangle \) is a nonzero two dimensional analytic cycle.

Using the map \( f: S^2 \to C \) given by

\[
f(w_1, w_2, w_3) = r(w_1^2, w_2^2, w_3^2, \sqrt{2}w_2w_3, \sqrt{2}w_3w_1, \sqrt{2}w_1w_2) \quad \text{for } (w_1, w_2, w_3) \in S^2,
\]

we compute that \( \dim Df(w)[\text{Tan}(S^2, w)] = 2 \) for \( w \in S^2 \) and \( f(v) = f(w) \) if and only if \( v = \pm w \) for \( v, w \in S^2 \); thus, by [3, 3.1.18, 3.1.24], \( f(S^2) \) is a compact, connected analytic submanifold of \( \mathbb{R}^6 \). Moreover \( V \cap u^{-1} \{r \} \) is the disjoint union of \( f(S^2) \) and \( (-f)(S^2) \). In fact, if \( x = (x_1, x_2, x_3, x_4, x_5, x_6) \in V \cap u^{-1} \{r \} \), then \( (x_1, x_2, x_3) \neq 0 \). Let \( \rho = 1/\sqrt{r} \). In case \( x_1 > 0 \), \( f(w) = x \) where \( w = \rho(\sqrt{x_1}, x_4/\sqrt{2x_1}, x_6/\sqrt{2x_1}) \) belongs to \( S^2 \) because

\[
|w|^2 = \rho^4(x_1 + x_2 + x_3)^2 = (x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2)/r^2 = 1.
\]

In case \( x_1 < 0 \), \( (-f)(v) = x \) where \( v = \rho(\sqrt{-x_1}, x_4/\sqrt{-2x_1}, x_6/\sqrt{-2x_1}) \in S^2 \).

The remaining cases—\( x_2 > 0, x_3 < 0, x_4 > 0, x_5 < 0 \)—are similarly treated.

Since \( 0 \neq \text{spt } \langle T, u, r \rangle \subset f(S^2) \cup (-f)(S^2) \) and \( \partial \langle T, u, r \rangle = 0 \), we infer from [3, 4.1.31 (2)] that one, and hence both of the components \( f(S^2) \) and \( (-f)(S^2) \) are orientable. If \( \varphi \) is an orienting 2 form for \( f(S^2) \) then \( \psi = f^\# \varphi/|f^\# \varphi| \) is one.
of the two standard unit orienting 2 forms for $S^2$; hence for $v \in S^2$, $\psi(v) = -\psi(-v) = -\psi(v)$ or $\psi(v) = 0$, a contradiction.

**Theorem 7.3.** There exists a unique analytic chain modulo two $Z_E$ in $M$ with $\text{spt}^2 Z_E = \text{Clos Reg } E$; moreover $\partial Z_E = 0$ for $k \geq 1$ and hence ([7, 3.1]) the homology class of $Z_E$ generates $H_k(E, E \sim \{x\}; \mathbb{Z}_2)$ for every $x \in \text{Reg } E$.

**Proof.** If $Q$ is any analytic chain modulo two in $M$ with $\text{spt}^2 Q = \text{Clos Reg } E$ and $W'$ is a relatively open semianalytic subset of $\text{Reg } E'$ with orienting $k$ vectorfield $\xi$, then [7, 3.1], applied to each component of $W \sim \text{spt}^2 \partial Q$, shows that $W \cap \text{spt}^2 (Q - [(\mathcal{K}^k \sqcap W) \wedge \xi]^2) \cap \text{spt}^2 \partial Q$; since $\mathcal{K}^k(\text{spt}^2 \partial Q) = 0$, it follows from [3, 4.2.6 (4.1.14)] that $(Q - [(\mathcal{K}^k \sqcap W) \wedge \xi]^2)|W = 0$. Thus $\text{spt}^2 \partial Q \subset \text{Fr Reg } E$, and uniqueness follows from [7, 3.1] and [3, 4.2.6 (4.1.14)].

To prove existence let $\{V_1, V_2, \ldots\}$ be a cover of $\text{Reg } E$ consisting of relatively open semianalytic subsets of $\text{Reg } E$ with orienting $k$ vectorfields $\xi_1, \xi_2, \ldots$. Also let $W_i = V_i, W_i = V_i \sim \text{Clos}(V_1 \cup \ldots \cup V_{i-1})$ for $i \in \{2, 3, \ldots\}$, $F = (\text{Reg } E) \sim (W_1 \cup W_2 \cup \ldots)$, and

$$Z_E = \left( \sum_{i=1}^{\infty} (\mathcal{K}^k \sqcap W_i) \wedge \xi_i \right)^2 \in \mathcal{R}^\text{loc,2}_k(M);$$

hence $\mathcal{K}^k(F) = 0$ and $\text{spt}^2 Z_E = \text{Clos Reg } E$. By the argument of the previous paragraph (Reg $E) \cap \text{spt}^2 \partial Z_E$, being contained in $F$ by [7, 3.1], is empty by [3, 4.2.6 (4.1.14)]. Since $\dim \text{spt}^2 \partial Z_E < \dim \text{Fr Reg } E < k - 1$, $Z_E$ is an analytic chain modulo two in $M$.

Assuming now $k > 1$ we verify that $\partial Z_E = 0$ in two cases.

**Case 1,** $k = 1$. Here we assume $y \in \text{spt}^2 \partial Z_E$. Choosing $T, U$ as in 7.1, we infer from uniqueness, with $M$ and $E$ replaced by $U$ and $U \cap E$, that $(T)^2|U = Z_E$ hence,

$$U \cap \text{spt}^2 \partial Z_E = U \cap \text{spt}^2 \partial T \subset U \cap \text{spt} \partial T = \emptyset,$$

a contradiction.

**Case 2,** $k > 1$. We assume $M$ is an open subset of $\mathbb{R}^n$, $x$ is a regular point for $B = \text{spt}^2 \partial Z_E$, and $\dim[\text{Tan}(B, x)] = k - 1$. The remainder of the proof will consist of choosing an orthogonal projection $\mu: \mathbb{R}^n \to \mathbb{R}^{k-1}$, a neighborhood $U$ of $x$, and a point $b \in U \cap B$ satisfying the conditions:

$$\dim[\text{E} \cap \mu^{-1}\{\mu(b)\}] = 1, \quad U \cap B \cap \mu^{-1}\{\mu(b)\} = \{b\},$$

$$(\delta_b)^2 = \langle \partial Z_E, \mu, \mu(b) \rangle^2|U = \partial[\langle Z_E, \mu, \mu(b) \rangle^2|U] = \partial[\text{Z}_{E^\mu}(\mu(b))]|U,$$

which contradict Case 1 with $M$ and $E$ replaced by $U$ and $U \cap E \cap \mu^{-1}\{\mu(b)\}$.
Let $D$ be a countable dense subset of $\text{Reg } E$, $\mu: \mathbb{R}^m \to \mathbb{R}^{k-1}$ be an orthogonal projection with

$$\dim \mu[\text{Tan}(B, x)] = k - 1 = \dim \mu[\text{Tan}(E, d)] \quad \text{for all } d \in D$$

and $U$ be an open neighborhood of $x$ such that $G = U \cap B$ is a non-empty connected analytic submanifold of $M$ and $\mu|G$ is an analytic isomorphism. Recalling the proof of 3.2, we see that the real analytic dimension of $F = (\text{Reg } E) \cap \{a: \dim \mu[\text{Tan}(E, a)] < k - 1\}$
is at most $k - 1$ because every component of $\text{Reg } E$ meets $D$. Fixing, by [7, 4.1], an analytic chain $T$ in $M$ with $(T)^2 = Z_E$ and $\text{spt } T = \text{spt}^2 T = \text{Clos } \text{Reg } E$ and choosing, by [6, 2.2 (7)], $b \in G$ so that

$$\dim(S \cap \mu^{-1}\{\mu(b)\}) < 1, \quad \dim((S \cup \text{spt } \partial T) \cap \mu^{-1}\{\mu(b)\}) < 0,$$

and the slice $\langle T, \mu, \mu(b) \rangle$ is determined by integration along the fiber as in [3, 4.3.8 (2)], we conclude first by [7, 3.2 (6)(2)] that $\langle \partial Z_E, \mu, \mu(b) \rangle^2 U = (S_b)^2$, hence $\dim(S \cap \mu^{-1}\{\mu(b)\}) = 1$, and second, by [3, 4.3.8 (2)], that

$$U \cap \text{spt}^2 \langle Z_E, \mu, \mu(b) \rangle^2 = U \cap \text{spt}^2 \langle T, \mu, \mu(b) \rangle = U \cap \mu^{-1}\{\mu(b)\} \cap \text{Clos}[\text{Reg } E \sim F] = U \cap \text{Clos Reg } (E \cap \mu^{-1}\{\mu(b)\});$$

hence $\langle Z_E, \mu, \mu(b) \rangle^2 U = Z_E \cap \mu^{-1}(\mu(b))) U$ by uniqueness.

**Remark 7.4.** If

$$E = \mathbb{R}^3 \cap \{(x, y, z): z(x^2 + y^2) = x^3\}, \quad F = \mathbb{R}^3 \cap \{(x, y, z): x = 0\},$$

$$Y = \mathbb{R}^3 \cap \{(x, y, z): x = 0 = z\}, \quad Z = \mathbb{R}^3 \cap \{(x, y, z): x = 0 = y\},$$

then the closed semianalytic set $\text{spt}^2 Z_E = \text{Clos}(E \sim Z)$ is not analytic ([15, p. 106]). Moreover

$$Z_E \cap F = Z_Y \cup Z = Z_Y \neq Z_Y = Z_E \cap Z_F$$

([7, 4.4]) even though $\dim(E \cap F) = 1 = \dim E + \dim F - 3$.

8. – The Real Part of a Holomorphic Chain.

In this section we assume that $\mathbb{R}^m$ and $\mathbb{R}^n$ are embedded in $\mathbb{C}^m$ and $\mathbb{C}^n$ in the usual fashion and the $U \subset \mathbb{C}^m$ and $V \subset \mathbb{C}^n$ are open sets. For any map
f: U \rightarrow V with f(R^n \cap U) \subset R^n, we let \mathcal{R}f: R^n \cap U \rightarrow R^n \cap V, \mathcal{R}f(x) = f(x) for x \in R^n \cap U.

We first observe that if D is a complex j dimensional holomorphic submanifold of C^n and R^n \cap D \neq \emptyset, then R^n \cap D is a j dimensional real analytic submanifold of R^n. In fact for x \in R^n \cap D there is an open neighborhood W of x in C^n and a multi-index \lambda \in A(m, j) such that

\pi_1: W \cap D \rightarrow C^j, \quad \pi_2(z_1, ..., z_m) = (z_{j+1}, ..., z_{j+m}) \quad for (z_1, ..., z_m) \in W \cap D,

is a holomorphic isomorphism; then \mathcal{R}(\pi_2|W \cap D)^{-1} is a real analytic isomorphism mapping R^j \cap \pi_2(W \cap D) onto R^m \cap W \cap D.

Next, recalling [3, 3.4.12] and [6, 2.2], we define, for \emptyset \neq E \subset C^w, the complex dimension of E, denoted dim_C E, as

\sup_{x \in C^w} \inf \{\dim_C \beta: \beta \text{ is the germ of a holomorphic subvariety at } x \text{ and } \beta \text{ contains the germ of } E \text{ at } x\};

in addition, we let dim_C \emptyset = -1. We obtain the inequality

\dim(R^n \cap E) \leq \dim_C E.

In fact, if x \in R^n \cap E, \beta is the germ of a holomorphic subvariety at x, \beta contains the germ of E at x, and \varepsilon is the complexification of \gamma_x(R^n) \cap \beta ([15, p. 91]), then, by [15, p. 93],

\dim \gamma_x(R^n \cap E) \leq \dim[\gamma_x(R^n) \cap \beta] = \dim_C \varepsilon \leq \dim_C \beta.

8.1. Complex holomorphic chains. Let H be a complex j dimensional holomorphic chain in U ([3, 4.2.29], [6, §6]). From [3, 4.2.29] and [15, pp. 67-68] we recall that H is a locally finite sum of integral multiples of chains corresponding to integration over the global irreducible components of the holomorphic set spt H; hence, spt^2 H, being the union of those irreducible components occurring with odd multiplicity, is a (pure) complex j dimensional holomorphic subset of U. We now define a j dimensional (real) analytic chain modulo two in R^n \cap U, \mathcal{R}H, called the real part of H, by

\mathcal{R}H = Z_{R^n \cap \text{spt}^2 H} \text{ in case } \dim(R^n \cap \text{spt}^2 H) = j,

\mathcal{R}H = 0 \text{ in case } \dim(R^n \cap \text{spt}^2 H) < j.

If I is a complex holomorphic chain in V, then \mathcal{R}(H \times I) = (\mathcal{R}H) \times (\mathcal{R}I).
8.2. Proper Mapping Formula. If \( f \) maps \( U \) holomorphically into \( V \), 
\( f(\mathbb{R}^n \cap U) \subset \mathbb{R}^n \), \( H \) is a complex \( j \) dimensional holomorphic chain in \( U \), and 
\( f|\text{spt} \, H \) is proper, then \( \mathcal{R}(f_H) = (\mathcal{R}f)_H(\mathcal{R}H) \).

Proof. By the proper mapping theorem ([15, p. 129]), \( f(\text{spt} \, H) \) is
holomorphic in \( V \) with \( \dim_C f(\text{spt} \, H) = j \); hence, by the argument of
[3, 4.2.28], \( f_H \) is a holomorphic chain in \( V \). Let

\[
A = (\text{spt} \, H) \sim \{ x : x \text{ is a regular point of spt } H \} \quad \text{and} \quad \dim_C Df(x)[\text{Tan}(\text{spt} \, H, x)] = j \},
\]

\[
B = f(\text{spt} \, H) \sim \{ y : y \text{ is a regular point of } f(\text{spt} \, H) \} \quad \text{and} \quad \dim_C \text{Tan}[f(\text{spt} \, H), y] = j \},
\]

and observe, by the real and complex rank theorems, that the restriction
of \( f \) induces a holomorphic covering map of holomorphic submanifolds

\[
(\text{spt} \, H) \sim f^{-1}[f(A) \cup B] \to f(\text{spt} \, H) \sim [f(A) \cup B]
\]

and a real analytic covering map of real analytic submanifolds

\[
\mathbb{R}^n \cap (\text{spt} \, H) \sim f^{-1}[f(A) \cup B] \to \mathbb{R}^n \cap f(\text{spt} \, H) \sim [f(A) \cup B] .
\]

For any connected component \( C \) of \( \mathbb{R}^n \cap f(\text{spt} \, H) \sim [f(A) \cup B] \) and \( y \in C \)
we compute

\[
\Theta^j(\| \mathcal{R}(f_H) \|^2, y) = \Theta^j(\| f_H \|^2, y) = \sum_{x \in f^{-1}(y) \cap \text{spt} \, H} \Theta^j(\| H \|^2, x) = \Theta^j(\| (\mathcal{R}f)_H(\mathcal{R}H) \|^2, x) \mod 2 ,
\]

and observe that

\[
\partial \mathcal{R}(f_H) = 0 = (\mathcal{R}f)_H \partial(\mathcal{R}H) = (\partial(\mathcal{R}f)_H(\mathcal{R}H)
\]

by 7.3, and deduce from [7, 3.1] that

\[
C \cap \text{spt}^d[\mathcal{R}(f_H) - (\mathcal{R}f)_H(\mathcal{R}H)] = \emptyset .
\]

Thus \( \text{spt}^d[\mathcal{R}(f_H) - (\mathcal{R}f)_H(\mathcal{R}H)] \subset \mathbb{R}^n \cap [f(A) \cup B] \). Since, by [15, p. 65],
\( \dim_C B < j \), to complete the proof it suffices by [3, 4.2.26 (4.2.14)'] to show
that \( \dim_C A < j \); hence \( \dim(\mathbb{R}^n \cap A) < j \) and \( \dim(\mathbb{R}^n \cap B) < j \) by 8.0 and

\[
\mathcal{J}(\mathbb{R}^n \cap [f(A) \cup B]) = \mathcal{J}([f(\mathbb{R}^n \cap A) \cup (\mathbb{R}^n \cap B)] = 0 .
\]
By the reasoning of \([6, 2.9]\), \(A\) is holomorphic in \(U\). If \(\dim_C A = j\), then

\[
D = A \cap \{x: x \text{ is a regular point of } A \text{ and } \dim \Tan(A, x) = j\}
\]

would be nonempty. Choosing a point \(d \in D\) so that \(\dim_C Df(d)\) is maximal, we infer from the complex rank theorem that \(\dim_C\{f^{-1}(d)\} > 1\). But this is impossible because \(f^{-1}\{d\} \cap \spt H\), being a compact holomorphic subset of \(C^n\), is finite by \([15, \text{p. 52}]\).

**THEOREM 8.3.** If \(f\) maps \(U\) holomorphically into \(C^n\), \(j(\mathbb{R}^n \cap U) \subset \mathbb{R}^n\), \(H\) is a complex \(j\)-dimensional holomorphic chain in \(U\), \(j \geq n\), \(y \in \mathbb{R}^n\), and \(\dim_C\{f^{-1}(y) \cap \spt H\} < j - n\), then \(\mathcal{R} \langle H, j, y \rangle = \langle \mathcal{R}H, \mathcal{R}f, y \rangle^2\).

**PROOF.** Both sides are defined because \(\partial H = 0\) and

\[
\dim([\mathcal{R}f^{-1}\{y\} \cap \spt \mathcal{R}H]) < \dim(\mathbb{R}^n \cap f^{-1}\{y\} \cap \spt H) < j - n.
\]

**Case 1, \(j = n\).** Here we assume \(x \in \mathbb{R}^n \cap f^{-1}\{y\} \cap H\) and select \(0 < \varrho < \text{dist}(x, \Fr U)\) and \(a, b \in \{0, 1\}\) such that

\[
B(x, \varrho) \cap f^{-1}\{y\} \cap \spt H = \{y\},
\]

\[
[\mathcal{R} \langle H, f, y \rangle - (a\delta^2)] \cdot U(x, \varrho) = 0 = [\langle \mathcal{R}H, \mathcal{R}f, y \rangle - (b\delta^2)] \cdot U(x, \varrho).
\]

Recalling \([7, 3.2 (7)]\) we also choose

\[
0 < \sigma < \frac{1}{2} \inf \{|y - f(z)|: z \in \Fr U(x, \varrho) \cap \spt H\},
\]

\[
W = U(x, \varrho) \cap f^{-1}U(y, \sigma),
\]

so that \((bE^n|\mathbb{R}^n \cap U(y, \sigma))^2 = [\mathcal{R}(f|W)]_y[(\mathcal{R}H)|\mathbb{R}^n \cap W]\). Moreover \(f|W \cap \spt H\) is proper and

\[
(aE^n|\mathbb{R}^n \cap U(y, \sigma))^2 = \mathcal{R}[aE^n|U(y, \sigma)] = \mathcal{R}[(f|W)_y(H|W)] = [\mathcal{R}(f|W)]_y[(\mathcal{R}H)|\mathbb{R}^n \cap W]
\]

by \([6, 3.6 (1)(6)(8)]\) and 8.2; hence \(a = b\).

**Case 2, \(j > n\).** Here we assume the theorem false. Noting that the complex dimension of

\[
X = (f^{-1}\{y\} \cap \spt H) \sim \{x: x \text{ is a regular point of } f^{-1}\{y\} \cap \spt H \text{ and } \dim_C \Tan(f^{-1}\{y\} \cap \spt H, x) = j - n\}
\]
does not exceed \( j - n - 1 \), we choose first by \([3, 4.2.26 (4.1.14)]\), \([6, 2.2 (4)]\), and 4.1 a point
\[
x \in \text{spt}^2(\mathcal{R}\langle H, f, y \rangle - \mathcal{R}f, y^2) \sim X,
\]
then a neighborhood \( W \) of \( x \) along with a projection \( \mu : \mathcal{C}^n \to \mathcal{C}^{i-n} \) whose restriction to \( W \cap f^{-1}\{y\} \cap \text{spt} H \) is a holomorphic isomorphism. Letting \( I = H|W, g = f|W, h = \mu|W \) and using \([7, 3.1, 3.3, 4.3 (4.5)]\), \([6, 4.5]\), and Case 1 twice we obtain the contradiction
\[
0 \neq \mathcal{R}\langle I, g, y \rangle - \mathcal{R}\langle I, \mathcal{R}g, y^2, \mathcal{R}h, h(x) \rangle^2
= \mathcal{R}\langle I, g, y, h, h(x) \rangle - \mathcal{R}\langle I, \mathcal{R}g \square (\mathcal{R}h), (y, h(x)) \rangle^2
= \mathcal{R}\langle I, g \square h, (y, h(x)) \rangle - \mathcal{R}\langle I, g \square h, (y, h(x)) \rangle = 0.
\]

8.4. BOREL-HAEFLIGER FORMULA. If \( I \) and \( J \) are complex \( i \) and \( j \) dimensional holomorphic chains in \( U \), \( i + j > m \), and \( \dim_c(\text{spt} I \cap \text{spt} J) < i + j - m \), then
\[
\mathcal{R}(I \cap J) = (\mathcal{R}I) \cap^2 (\mathcal{R}J).
\]

**Proof.** Using the two maps \( f : U \times U \to \mathcal{C}^m \) and \( \mu : U \times U \to U, f(z, w) = z - w \) and \( \mu(z, w) = z \) for \((z, w) \in U \times U\), we recall the definitions \([6, \S 5], [7, 4.3 (\S 5')]\)
\[
I \cap J = \mu_\# \langle I \times J, f, 0 \rangle,
(\mathcal{R}I) \cap^2 (\mathcal{R}J) = (\mathcal{R}\mu)_\# \langle (\mathcal{R}I) \times (\mathcal{R}J), \mathcal{R}f, 0 \rangle^2,
\]
note that \( \mu|f^{-1}\{0\} \) is proper, and then apply 8.2 and 8.3.

*Added in proof.*

Here we mention some recent results relevant to the present paper. Dennis Sullivan's theorem \([23]\) on the oddness of the local Euler characteristic \( \chi(A, A \sim \{a\}) \) for a point \( a \) in a real analytic set \( A \) has been established in \([24]\) and \([25]\). Many of the properties enjoyed by the class of semianalytic sets have now been obtained for the larger class of subanalytic sets consisting of all proper analytic images of semianalytic sets. That subanalytic sets admit stratifications into subanalytic, real analytic submanifolds was proven independently in \([21]\) (first) and in \([8]\) (where they are called semianalytic shadows). Using his desingularization theorems to represent locally a subanalytic set as the finite union of proper analytic images of quadrants in Euclidean spaces, H. Hironaka has, in \([22]\), generalized to subanalytic sets many of the results of \([13]\) including the Lojasiewicz inequalities. The stratification of subanalytic sets leads in \([26]\) to subanalytic \( CW \) decomposition and in \([27]\) to triangulation by homeomorphisms with subanalytic graphs. In \([26]\), where the subanalytic analogues of 4.6, \( \S 5 \) and \( \S 6 \) of the present paper are obtained, the homology of subanalytic pairs is represented by subanalytic chains whereas, here, for semianalytic pairs we use the smaller group of analytic chains.
REFERENCES


