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Exceptional sets with respect to Lebesgue differentiation of functions in Sobolev spaces

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Exceptional Sets with Respect to Lebesgue Differentiation of Functions in Sobolev Spaces (*).

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1. – Introduction.

Let $W_{k,p}^{\text{loc}}(\boldsymbol{R}_n)$, ($k$ integer $\geqslant 1$, $1 \leqslant p < \infty$), denote the Sobolev space of functions belonging to $L_p^{\text{loc}}(\boldsymbol{R}_n)$ whose distribution derivatives up to order k belong to $L_p^{\text{loc}}(\boldsymbol{R}_n)$. If $1 \leqslant p < n/k$ then, by Sobolev's embedding theorem:

$$u \in W_{k,p}^{\text{loc}}(\boldsymbol{R}_n) \Rightarrow u \in L_s^{\text{loc}}(\boldsymbol{R}_n)\,, \qquad s = np/(n-kp)\,. \tag{1.1}$$

In a recent paper, Federer and Ziemer [5] proved the following result:

If $u \in W_{1,p}^{\text{loc}}(\boldsymbol{R}_n)$, ($1 \leqslant p < n$), then there exists a set E whose Hausdorff dimension is at most $n-p$ (i.e. $H_{(n-p+\varepsilon)}(E) = 0$ for every $\varepsilon > 0$) such that for every $x_0 \in \boldsymbol{R}_n \setminus E$ there exists a number $Z(x_0)$ for which:

$$\lim_{r\to 0} \frac{1}{r^n} \int\limits_{B(x_0;r)} |u - Z(x_0)|^s\, dx = 0\,, \qquad s = np/(n-p)\,. \tag{1.2}$$

Here $H_{(\alpha)}$ denotes the α-dimensional Hausdorff measure in $\boldsymbol{R}_n$ and $B(x_0; r)$ is the open ball with center x_0 and radius r.

More precisely, Federer and Ziemer have shown that $\Gamma_p(E) = 0$, where Γ_p is the functional capacity of degree p. For the definition of Γ_p and its relation to Hausdorff measure, see [5]. We mention that, for $E \subset \boldsymbol{R}_n$, if $1 < \alpha < n$ then $\Gamma_\alpha(E) = 0$ implies that $H_{(n-\alpha+\varepsilon)}(E) = 0$ for every $\varepsilon > 0$, and $H_{(n-\alpha)}(E) < \infty$

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implies that $\Gamma_\alpha(E)=0$. (Thus, a σ-finite set with respect to the measure $H_{(n-\alpha)}$ has Γ_α capacity zero.) If $\alpha=1$, then $\Gamma_1(E)=0$ if and only if $H_{(n-1)}(E)=0$.

In the case $p=1$, the result quoted above had been previously obtained by Federer; it was announced in [3] and the proof was given in [4, § 4·5·9].

In this paper we obtain the following extension of the result of Federer and Ziemer.

If $u \in W^{\text{loc}}_{p,k}(\boldsymbol{R})_n$, $1 \leqslant p \leqslant n/k$, then there exists a set E whose Hausdorff dimension is at most $n-kp$, s.t. for every $x_0 \in \boldsymbol{R}_n \setminus E$:

$$\lim_{r\to 0} \frac{1}{r^n} \int_{B(x_0;r)} u(x)\,dx = \tilde{u}(x_0) \tag{1.3}$$

is defined and

$$\lim_{r\to 0} \frac{1}{r^n} \int_{B(x_0;r)} |u(x)-\tilde{u}(x_0)|^s\,dx = 0\,, \qquad s = np/(n-kp), \quad \textit{if } p<n/k \tag{1.4}$$

$$s \in [1,\infty), \quad \textit{if } p = n/k\,.$$

More precisely, we show that $H_\tau(E)=0$, where H_τ is the Hausdorff measure defined by the function $\tau = r^{n-kp}|\log r|^{-q}$, $(p<q)$.

We also show that there exists a set F such that $H_{(n-kp)}(F)=0$ and such that for $x_0 \in \boldsymbol{R}_n \setminus F$:

$$\frac{1}{r^n} \int_{B(x_0;r)} |u|\,dx = o(1)|\log r|\,, \qquad \text{as } r\to 0\,. \tag{1.5}$$

Other, related results may be found in section 5 of the paper.

Our method of proof is entirely different from that of Federer and Ziemer [5], in the case $k=1$ where they may be compared.

The plan of the paper is this: in sections 2, 3, 4 we derive various auxiliary results; the main results are formulated and proved in section 5, (for the case $p<n/k$) and section 6 (for the case $p=n/k$).

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2. – In this section we derive certain estimates related to Poincaré's lemma.

LEMMA 2.1. *Suppose that $u \in W^{\text{loc}}_{k,p}(\boldsymbol{R}_n)$ and $1 \leqslant p < n/k$. Let x_0 be a point in $\boldsymbol{R}_n$. There exists a unique polynomial of degree $\leqslant k-1$, which we shall* *denote by* $P_{k,r}(x; x_0)$, $(0 < r)$, *such that*

$$\int_{B(x_0,r)} D^\alpha P_{k,r}\, dx = \int_{B(x_0,r)} D^\alpha u\, dx\,, \qquad 0 \leqslant |\alpha| \leqslant k-1\,, \qquad 0 < r\,. \tag{2.1}$$

Also there exists a constant $C = C(n, k, p)$ *such that*

$$\left(\frac{1}{r^n}\int_{B(x_0,r)} |u - P_{k,r}|^s dx\right)^{1/s} \leqslant C\left(\frac{1}{r^{n-kp}}\int_{B(x_0,r)} |\nabla^k u|^p dx\right)^{1/p}, \tag{2.2}$$

(*for* $r > 0$), *where* $s = np/(n-kp)$. *The notation* $|\nabla^k u|$ *means* $\sum_{|\alpha|=k} |D_\alpha u|$.

Proof. The existence and uniqueness of a polynomial $P_{k,r}$ satisfying (2.1) is known (see e.g. [8; p. 85]). In fact it is easily proved by induction.

If $v \in W_{k,p}(B(x_0, R))$, for some $R > 0$, then by Sobolev's embedding theorem:

$$\|v\|_{L_s(B(x_0,r))} \leqslant c_0(n, k, p, r)\|v\|_{W_{k,p}(B(x_0,r))}\,, \qquad 0 < r < R\,. \tag{2.3}$$

Further, by a standard homogeneity argument one obtains:

$$\|v\|_{L_s(B(x_0,r))} \leqslant c_1(n, k, p) r^{-n/p} \sum_{j=0}^{k} r^j \|\nabla^j v\|_{L_p(B(x_0,r))}\,. \tag{2.4}$$

Hence, taking $v = u - P_{k,r}$, and applying an extension of Poincaré's lemma [8; p. 85] we obtain:

$$\begin{aligned} \|u - P_{k,r}\|_{L_s(B(x_0,r))} &\leqslant c_1(n, k, p) r^{-n/p} \sum_{j=0}^{k} r^j \|\nabla^j (u - P_{k,r})\|_{L_p(B(x_0,r))} \leqslant \\ &\leqslant c_2(n, k, p) r^{k-n/p} \|\nabla^k u\|_{L_p(B(x_0,r))}\,. \end{aligned} \tag{2.5}$$

This is precisely (2.2).

In the next lemma we discuss an additional property of the polynomials $P_{k,r}$.

Lemma 2.2. *With the notation and assumptions of the previous lemma but without restriction on* p, *except that* $1 \leqslant p < n$, *we have*:

$$|P_{k,r}(x; x_0) - u_a(r; x_0)| \leqslant \text{const} \sum_{1 \leqslant |\gamma| \leqslant k-1} |(D^\gamma u)_a(r; x_0)| r^{|\gamma|} \tag{2.6}$$

for every $x \in B(x_0, r)$. *The constant depends only on* k *and* n. *The notation* $f_a(r; x_0)$ *means the average of the function* f *over* $B(x_0, r)$.

PROOF. The result is trivial for $k = 1$ and $k = 2$, since we have:

$$(2.7)\qquad P_{1,r} = u_a(r; x_0)\,; \qquad P_{2,r} = u_a(r; x_0) + \sum_{i=1}^{n} (D_{x_i} u)_a(r; x_0)(x_i - x_{0,i})\,.$$

Now suppose that (2.6) is proved for $k = 0, 1, \ldots, j-1$, $(j > 1)$; we proceed to prove it for $k = j$.

Let $Q_r = \sum_{|\beta|=j-1} (D^\beta u)_a(r; x_0)(x - x_0)^\beta$ and let S_r be the unique polynomial of order $j-2$ which satisfies:

$$(2.8)\qquad \int_{B(x_0,r)} D^\alpha Q_r\, dx = \int_{B(x_0,r)} D^\alpha S_r\, dx\,; \qquad 0 \leqslant |\alpha| \leqslant j-2\,.$$

Then,

$$(2.9)\qquad P_{j,r} = P_{j-1,r} + Q_r - S_r\,,$$

because the right hand side is a polynomial of order $j-1$ which satisfies (2.1) for $k = j$.

By assumption, the following inequality holds in $B(x_0, r)$:

$$|S_r - (Q_r)_a(r; x_0)| \leqslant \text{const} \sum_{1 \leqslant |\gamma| \leqslant j-2} |(D^\gamma Q_r)_a(r; x_0)| r^{|\gamma|},$$

the constant depending only on j and n.

But:

$$(D^\gamma Q_r)_a(r; x_0) = \sum_{\substack{\gamma \leqslant \beta \\ |\beta| = j-1}} c_{\beta,\gamma} (D^\beta u)_a(r; x_0)(x - x_0)^{\beta-\gamma}\,, \qquad 0 \leqslant |\gamma| \leqslant j-2$$

where $\gamma \leqslant \beta$ means $\gamma_i \leqslant \beta_i$ for $i = 1, \ldots, n$ and $c_{\beta,\gamma}$ is a number depending only on β, γ.

Thus:

$$(2.10)\qquad |S_r - (Q_r)_a(r; x_0)| \leqslant \text{const} \sum_{|\beta|=j-1} |(D^\beta u)_a(r; x_0)| r^{|\beta|}, \qquad x \in B(x_0, r)\,,$$

with the constant depending only on j and n. Denote the sum on the right-hand side of (2.10) by M_r. Clearly:

$$(2.11)\qquad \begin{cases} |Q_r| \leqslant M_r\,, \\ |(Q_r)_a(r; x_0)| \leqslant M_r\,. \end{cases}$$

By assumption $P_{j-1,r}$ satisfies (2.6) (with $k=j-1$). Hence, by (2.9), (2.10), (2.11) we obtain (2.6) for $k=j$.

3. – In this section we prove some properties of functions in $W_{1,1}^{\text{loc}}(\boldsymbol{R}_n)$.

Let F be a function in $L_1^{\text{loc}}(\boldsymbol{R}_n)$ and let x_0 be a point in R_n. We denote

$$F_a(r; x_0)=\frac{1}{V_n r^n}\int\limits_{B(r;x_0)} F\,dx\,, \qquad r>0\,, \tag{3.1}$$

where V_n is the volume of the unit ball in $\boldsymbol{R}_n$.

Let $(\varrho, \theta)=(\varrho, \theta_1, \dots, \theta_{n-1})$ denote a set of spherical coordinates with center at x_0. Thus $\varrho=|x-x_0|$; $x_1=\varrho\cos\theta_1$, $x_2=\varrho\sin\theta_1\cos\theta_2$, $x_3=\varrho\sin\theta_1\cdot\sin\theta_2\cos\theta_3, \dots, x_n=\varrho\sin\theta_1\dots\sin\theta_{n-2}\cos\theta_{n-1}$. The range of θ is:

$$A=\{\theta\colon\ 0\leqslant\theta_i\leqslant\pi \text{ for } i=1, \dots, n-2;\ -\pi<\theta_{n-1}<\pi\}\,.$$

Let $d\omega=\Phi(\theta)\,d\theta$ denote the surface element of the unit sphere in $\boldsymbol{R}_n$. Thus:

$$\int\limits_{B(r;x_0)} F\,dx=\int\limits_0^r\Big[\int\limits_A F\big(x(\varrho, \theta)\big)\,d\omega\Big]\varrho^{n-1}\,d\varrho\,.$$

We denote:

$$F^*(r, x_0)=\int\limits_A F\big(x(r, \theta)\big)\,d\omega\,, \qquad r>0\,. \tag{3.2}$$

By Fubini's theorem $F^*(r; x_0)$ is defined for almost every $r>0$ and belongs to L_1^{loc} for $r>0$.

Eliminating from A the surfaces $\theta_i=0$, $\theta_i=\pm\pi/2$, $\theta_i=\pm\pi$ we obtain a partition of A (minus these surfaces) into 2^n domains in each of which $\Phi(\theta)\neq 0$ and thus has a fixed sign. We shall denote these domains by A^j, $j=1, \dots, 2^n$ and the corresponding open cones in R_n (with vertix at x_0) by B^j.

Suppose that $u\in W_{1,1}^{\text{loc}}(\boldsymbol{R}_n)$. The transformation $x\to(\varrho, \theta)$ and its inverse are smooth in each of the domains $B^j=\{(\varrho, \theta)\colon 0<\varrho, \theta\in A^j\}$. Thus $u\big(x(\varrho, \theta)\big)\in W_{1,1}^{\text{loc}}$ in each of these domains and the distribution derivative $D_\varrho u$ is given by the usual formula:

$$D_\varrho u=\frac{1}{\varrho}\sum_{i=1}^n (x_i-x_{0,i})D_{x_i}u\,, \tag{3.3}$$

(see e.g. [8; p. 64]).

In the next lemmas we describe certain properties of the functions $u^*(r; x_0)$ and $u_a(r; x_0)$.

LEMMA 3.1. *Let* $u \in W_{1,1}^{\text{loc}}(\boldsymbol{R}_n)$ *and let* x_0 *be a point in* $\boldsymbol{R}_n$. *Then* $u^*(r) = u^*(r; x_0)$ *coincides a.e. with a locally absolutely continuous function of* r, *say* $\mu(r)$, *for* $r > 0$. *Moreover*:

$$\mu'(r) = \int_A D_r u(x(r, \theta))\, d\omega \qquad \text{a.e. in } (0, \infty)\,. \tag{3.4}$$

PROOF. First we note that by (3.3) and the remarks preceeding it, $\Phi(\theta) D_\varrho u(x(\varrho, \theta)) \in L_1$ in the domain $\{(\varrho, \theta) : r_1 < \varrho < r_2, \theta \in A\}$, for every r_1, r_2 such that $0 < r_1 < r_2$. Thus, by Fubini's theorem, the function defined by the righthand side of (3.4)—which we shall denote for the moment by $\nu(r)$—belongs to $L_1^{\text{loc}}(0, \infty)$ and:

$$\int_{r_1}^{r_2} \left[\int_A D_r(u(r, \theta))\, d\omega\right] dr = \int_A \left[\int_{r_1}^{r_2} D_r u(x(r, \theta))\, dr\right] d\omega\,. \tag{3.5}$$

Since $u(x(\varrho, \theta)) \in W_{1,1}^{\text{loc}}(B^j)$ it follows that there exists a function $v(\varrho, \theta)$ which coincides with u a.e. in the entire space, such that $v(\cdot, \theta)$ is locally absolutely continuous in $(0, \infty)$, for almost every θ, and such that $\partial v / \partial \varrho = D_\varrho u$ a.e. (see Morrey [7] and Gagliardo [6]). (Here $\partial v / \partial \varrho$ denotes the classical derivative of v with respect to ϱ, which exists a.e. in the space.) Thus from (3.5) we obtain:

$$\begin{aligned} \int_{r_1}^{r_2} \nu(r)\, dr &= \int_A \left[\int_{r_1}^{r_2} \frac{\partial v}{\partial \varrho}\, d\varrho\right] d\omega \\ &= \int_A \big(v(r_2, \theta) - v(r_1, \theta)\big)\, d\omega\,. \end{aligned} \tag{3.6}$$

But $v(r, \cdot) = u(x(r, \cdot))$ a.e. in A, for a.e. $r > 0$. If r_1 is a value for which this is the case, we finally obtain:

$$\int_{r_1}^{r} \nu(r)\, dr = u^*(r) - u^*(r_1) \qquad \text{for a.e. } r > 0\,. \tag{3.7}$$

Thus the function $\mu(r) = \int_{r_1}^{r} \nu(r)\, dr$ has the properties stated in the lemma.

LEMMA 3.2. *With the assumptions and notations of the previous lemma we have*:

$$n\int_0^r \mu(\varrho)\varrho^{n-1}\,d\varrho - \mu(r)r^n = -\int_0^r \mu'(\varrho)\varrho^n\,d\varrho\,. \tag{3.8}$$

If we denote the righthand side of (3.8) *by* $-f(r)$ *we have*:

$$u_a(r) = u_a(r;x_0) = V_n^{-1}\Big(c + \int_1^r f(\varrho)\varrho^{-n-1}d\varrho\Big)\,, \qquad r>0\,, \tag{3.9}$$

where $c = c(x_0)$ *is independent of* r.

PROOF. As $\mu(r)$ is locally absolutely continuous in $(0,\infty)$ we have:

$$\int_\varepsilon^r (\mu(\varrho)\varrho^n)'\,d\varrho = \mu(r)r^n - \mu(\varepsilon)\varepsilon^n\,, \qquad (0<\varepsilon<r)\,. \tag{3.10}$$

The left hand side of (3.10) converges when $\varepsilon\to 0$. In fact $\mu(\varrho)\cdot\varrho^{n-1}$ belongs to $L_1[0,r]$ for any $r>0$ and

$$\int_0^r \mu(\varrho)\varrho^{n-1}d\varrho = \int_{B(r;x_0)} u\,dx\,; \qquad \int_0^r |\mu(\varrho)|\varrho^{n-1}\,d\varrho \leqslant \int_{B(r;x_0)} |u|\,dx\,. \tag{3.11}$$

Similarly, by the previous lemma,

$$\begin{cases} \displaystyle\int_0^r \mu'(\varrho)\varrho^n\,d\varrho = \int_{B(r;x_0)} \varrho D_\varrho u\,dx = \sum_{i=1}^n \int_{B(r;x_0)} (x_i - x_{0,i})D_{x_i}u\,dx \\ \displaystyle\int_0^r |\mu'(\varrho)|\varrho^n\,d\varrho \leqslant \sum_{i=1}^n \int_{B(r;x_0)} |x_0 - x_{0,1}|\,|D_{x_i}u|\,dx\,. \end{cases} \tag{3.12}$$

Hence $\varepsilon^n\mu(\varepsilon)$ converges to some limit, say b, when $\varepsilon\to 0$. Suppose that $b\neq 0$. Then, for sufficiently small ϱ,

$$\frac{|b|}{2\varrho} \leqslant |\mu(\varrho)|\varrho^{n-1}\,.$$

But this contradicts the fact that $\mu(\varrho)\varrho^{n-1}\in L_1[0,r]$. Thus $b=0$ and so (3.8) is obtained from (3.10) by taking the limit when ε tends to zero.

To prove (3.9) we set

$$\sigma(r) = \int_0^r \mu(\varrho)\,\varrho^{n-1}\,d\varrho\,, \tag{3.13}$$

and rewrite (3.8) in the form:

$$r\sigma'(r) - n\sigma(r) = f(r)\,. \tag{3.14}$$

It follows from the above that:

$$\sigma(r) = cr^n + r^n\int_1^r f(\varrho)\,\varrho^{-n-1}\,d\varrho\,, \tag{3.15}$$

for some constant c. But this implies (3.9) since

$$u_a(r) = (V_n r^n)^{-1}\sigma(r)\,.$$

REMARK. The equality (3.8) may be obtained also by an application of Green's theorem which is valid also for functions in $W_{1,1}$, in domains with smooth boundary.

COROLLARY 3.3.

(i) *If* $|f(r)| \leqslant \text{con.}\, r^n|\log r|^{-1-\delta}$ *for some* $\delta > 0$ *then* $u_a(r)$ *converges when* $r \to 0$.

(ii) *If* $f(r) = o(1)r^n h(r)^{-1}$, *where* $h(r) = r^b|\log r|^c$, *(b and c constants,* $b > 0$*) then* $h(r)u_a(r) \to 0$ *when* $r \to 0$.

(iii) *If* $f(r) = o(1)|\log r|^{-c}$, $(c < 1)$, *then* $|\log r|^{c-1}u_a(r) \to 0$ *when* $r \to 0$.

PROOF. (i) is an immediate consequence of (3.9). To prove (ii) we have to show that:

$$h(r)\int_1^r f(\varrho)\,\varrho^{-n-1}\,d\varrho \to 0 \qquad \text{when } r \to 0\,. \tag{3.16}$$

Given $\varepsilon > 0$, let $r_0 < 1$ be sufficiently small so that $|f(r)| \leqslant \varepsilon r^n h(r)^{-1}$ for $0 < r < r_0$ and so that $|c/b|\,|\log r_0|^{-1} \leqslant \frac{1}{2}$. Then, for $0 < r < r_0$:

$$h(r)\int_r^1 f(\varrho)\,\varrho^{-n-1}\,d\varrho \leqslant h(r)\int_{r_0}^1 f(\varrho)\,\varrho^{-n-1}\,d\varrho + \varepsilon h(r)\int_r^{r_0} \varrho^{-1-b}|\log \varrho|^{-c}\,d\varrho\,. \tag{3.17}$$

Now:

$$\int_r^{r_0} \varrho^{-1-b}|\log\varrho|^{-c}\,d\varrho \leqslant \left|\frac{c}{b}\right|\left|\int_r^{r_0} b\varrho^{-b-1}|\log\varrho|^{-c-1}\,d\varrho\right| + \frac{1}{b}\,\big|r^{-b}|\log r|^{-c} - r_0^{-b}|\log r_0|^{-c}\big|$$

$$\leqslant \left|\frac{c}{b}\right|\,|\log r_0|^{-1}\int_r^{r_0} \varrho^{-b-1}|\log\varrho|^{-c}\,d\varrho + \frac{1}{b}\,\big|r^{-b}|\log r|^{-c} - r_0^{-b}|\log r_0|^{-c}\big|.$$

Since $|c/b|\,|\log r_0|^{-1} \leqslant \frac{1}{2}$ we obtain:

$$h(r)\int_r^{r_0} \varrho^{-1-b}|\log\varrho|^{-c}\,d\varrho \leqslant \frac{2}{b}\,h(r)\big|r^{-b}|\log r|^{-c} - r_0^{-b}|\log r_0|^{-c}\big| \tag{3.18}$$

$$\leqslant \frac{2}{b}\,|1 - h(r)/h(r_0)|\,.$$

By (3.17) and (3.18): $\overline{\lim}_{r\to 0} h(r)\int_r^1 f(\varrho)\,\varrho^{-n-1}\,d\varrho \leqslant (2/b)\,\varepsilon$. This implies (3.16).

Assertion (iii) is verified by a similar (but simpler) computation.

4. – In this section we consider the Hausdorff measure of certain exceptional sets related to functions in $L_p^{\mathrm{loc}}(\boldsymbol{R}_n)$.

Let $\tau = \tau(r)$ be a continuous, monotonic increasing function in an interval $[0, d]$, such that $\tau(0) = 0$ and $\tau(r) > 0$ for $r > 0$. Suppose also that $\lim_{r\to 0} r^n/\tau(r) = 0$.

We recall the definition of the Hausdorff measure H_τ on $\boldsymbol{R}_n$, defined by the function τ.

Let E be a set in $\boldsymbol{R}_n$. Given $0 < \varrho < d$, let $\{S_\nu\}_1^\infty$ be a cover of E by balls such that $r_\nu = \text{radius}\,(S_\nu) \leqslant \varrho$. Set:

$$H_\tau^\varrho = \inf \sum_1^\infty \tau(r_\nu)\,, \tag{4.1}$$

where the infinum is taken over all covers $\{S_\nu\}$ as above. (The balls may be open or closed; it is clear that this does not affect H_τ^ϱ.) Obviously, H_τ^ϱ increases as ϱ decreases. We denote:

$$H_\tau = \lim_{\varrho\to 0} H_\tau^\varrho = \sup_{0<\varrho<d} H_\tau^\varrho\,. \tag{4.2}$$

For $\tau = r^a$, $(0 < a < n)$, H_τ will also be denoted by $H_{(a)}$.

LEMMA 4.1. *Let $f \in L_1^{\text{loc}}(\boldsymbol{R}_n)$ and let τ be a function possessing the properties described above. In addition, suppose that for every $\beta > 0$ there exists a constant $c(\beta)$ such that*

$$\tau(\beta r) \leqslant c(\beta)\,\tau(r)\,, \qquad 0 < r < d/\beta\,. \tag{4.3}$$

Let:

$$E = \left\{x_0 \in \boldsymbol{R}_n;\ \varlimsup_{r\to 0} \frac{1}{\tau(r)} \int\limits_{B(x_0,r)} |f|\,dx > 0\right\}. \tag{4.4}$$

Then $H_\tau(E) = 0$.

REMARK. The lemma is known in the case $\tau = r^\alpha$ (see [1], Lemma 1). For the sake of completeness we give a proof.

PROOF. Without loss of generality we may and shall assume that f has compact support.

Given $\alpha > 0$, we denote:

$$E_\alpha = \left\{x_0 \in \boldsymbol{R}_n\colon\ \varlimsup_{r\to 0} \frac{1}{\tau(r)} \int\limits_{B(x_0,r)} |f|\,dx > \alpha\right\}. \tag{4.5}$$

Clearly, it is sufficient to prove that $H_\tau(E_\alpha) = 0$ for every $\alpha > 0$.

Let $\varepsilon > 0$. Since $\mathfrak{L}_n(E) = 0$ (where $\mathfrak{L}_n$ is the Lebesgue measure), there exists an open set U such that $U \supset E_\alpha$, $\mathfrak{L}_n(U) < \infty$ and $\int\limits_U |f|\,dx < \varepsilon$.

Let $\bar{\varrho}$ be a fixed number, $0 < \bar{\varrho} < d$. Let $\mathfrak{B}$ be the family of all closed balls $\bar{B}(x_0, r)$ which satisfy the following requirements:

$$\begin{cases} x_0 \in E_\alpha\,; \qquad 0 < r < \bar{\varrho}/7\,; \qquad B(x_0, r) \subseteq U\,; \\ \int\limits_{B(x_0,r)} |f|\,dx > \alpha\tau(r)\,. \end{cases} \tag{4.6}$$

Clearly, if $x_0 \in E_\alpha$, there are balls in $\mathfrak{B}$, of arbitrarily small diameter, which contain x_0. We note also that E is bounded (because f has compact support), and so the union of the balls in $\mathfrak{B}$ is bounded.

By a result related to the Vitali covering theorem (see e.g. [2; p. 210]), it follows that there exists a finite or countable family $\{\bar{B}(x_k, r_k)\} \subset \mathfrak{B}$, consisting of disjoint balls, such that $\mathfrak{B}^* = \{\bar{B}(x_k, 7r_k)\}$ is a cover of E_α.

Since this cover consists of balls of radius less than $\bar{\varrho}$, we have:

$$
\begin{aligned}
H_\tau^{\bar{\varrho}}(E_\alpha) &\leqslant \sum_k \tau(7r_k) \\
&\leqslant c(7)\sum_k \tau(r_k) && \text{(by (4.3))} \\
&\leqslant \frac{c(7)}{\alpha}\sum_k \int_{B(x_k,r_k)} |f|\,dx && \text{(by (4.6))} \\
&\leqslant \frac{c(7)}{\alpha}\int_U |f|\,dx \leqslant \frac{c(7)}{\alpha}\varepsilon .
\end{aligned}
\tag{4.7}
$$

Thus $H_\tau(E_\alpha) \leqslant (c(7)/\alpha)\varepsilon$. Finally, since ε is arbitrary, $H_\tau(E_\alpha^{*}) = 0$.

COROLLARY 4.2. *Let $f \in L_p^{\mathrm{loc}}(\boldsymbol{R}_n)$, $1 \leqslant p < \infty$. Let g be a positive continuous function in the interval $(0, d]$, such that $g(r) \to \infty$ when $r \to 0$. Suppose that, for every $\beta > 0$, there exists a constant $c_1(\beta)$ such that $g(\beta r) \leqslant c_1(\beta) g(r)$ for $r \in (0, d]$. Set $\tau_p(r) = r^n g^p(r)$ for $0 < r \leqslant d$. Suppose that $\lim_{r\to 0} \tau_p(r) = 0$ and that τ_p is monotonic increasing. Let E be defined as in (4.4) with $\tau = r^n g(r)$. Then $H_{\tau_p}(E) = 0$.*

PROOF. We note that τ_p satisfies all the assumptions made with respect to τ in the statement of the lemma. Hence, if we set

$$
E_p = \left\{ x_0 \in \boldsymbol{R}_n\colon\ \overline{\lim_{r\to 0}} \frac{1}{\tau_p(r)} \int_{B(x_0;r)} |f|^p\,dx > 0 \right\},
$$

it follows from the lemma that $H_{\tau_p}(E_p) = 0$.

But, by Hölder's inequality, if $p > 1$:

$$
\frac{1}{\tau_1(r)} \int_{B(x_0;r)} |f|\,dx \leqslant \mathrm{const} \left(\frac{1}{\tau_p(r)} \int_{B(x_0;r)} |f|^p\,dx \right)^{1/p}, \tag{4.8}
$$

the constant being independent of r.

Thus $E = E_1 \subseteq E_p$ and so $H_{\tau_p}(E) = 0$.

REMARK. Suppose that $g = r^{-b}|\log r|^{-c}$ where $0 < b < n/p$ and c is an arbitrary constant, or $b = 0$ and $c < 0$, or $b = n/p$ and $c > 0$. Then all the assumptions of the corollary, with respect to g and τ_p, are satisfied.

5. – We are ready now to prove our main results, whose formulation is given below.

TH. I. *Let* $u \in W^{\mathrm{loc}}_{k,p}(\boldsymbol{R}_n)$ *where* $1 \leqslant k$ *is an integer and* $1 \leqslant p < n/k$. *Let* $\tau_\delta(r) = r^{n-kp}|\log r|^{-(1+\delta)p}$, $\delta > 0$. *Then there exists a set* E *such that* $H_{\tau_\delta}(E) = 0$ *for every* $\delta > 0$ *and such that the following statements hold*:

(i) *For every* $x_0 \in \boldsymbol{R}_n \setminus E$, $u_a(r; x_0)$ *converges when* $r \to 0$.

(ii) *Let* $x_0 \in \boldsymbol{R}_n \setminus E$ *and set* $\tilde{u}(x_0) = \lim_{r\to 0} u_a(r; x_0)$. *Then*:

$$\lim_{r\to 0} \frac{1}{r^n} \int_{B(x_0;r)} |u - \tilde{u}(x_0)|^s dx = 0 , \qquad (s = np/(n-kp)) . \tag{5.1}$$

Note that $\tilde{u} = u$ $\mathfrak{L}_n$ *a.e. in* $\boldsymbol{R}_n$.

TH. II. *Let* u *be as in the previous theorem. Let* h *be a positive continuous function in the interval* $(0, 1)$. *Set*:

$$\begin{aligned} E^q_h &= \left\{x_0 \in \boldsymbol{R}_n \colon \overline{\lim_{r\to 0}} \frac{h^q(r)}{r^n} \int_{B(x_0;r)} |u|^q dx > 0\right\}, \\ E'_h &= \{x_0 \in \boldsymbol{R}_n \colon \overline{\lim_{r\to 0}} h(r)|\mu(r; x_0)| > 0\} . \end{aligned} \tag{5.2}$$

Here, $\mu(r; x_0)$ *is the locally absolutely continuous function which coincides* $\mathfrak{L}_1$ *a.e. in* $(0, \infty)$ *with* $u^*(r; x_0)$, (*see Lemma* 3.1). *Finally, let* $\tau_h(r) = r^{n-kp}h(r)^{-p}$. *Then the following statements hold*:

(i) *If* $h(r) = r^b|\log r|^c$, $(0 < b < n/p - k$ *and* c *arbitrary, or* $b = n/p - k$ *and* $c > 0)$ *then* $H_{\tau_h}(E^q_h) = 0$ $(1 \leqslant q \leqslant s)$ *and* $H_{\tau_h}(E'_h) = 0$.

(ii) *If* $h(r) = |\log r|^c$, $(c < 1)$, *then* $H_{\tau_h}(E^q_{h_1}) = 0$ $(1 \leqslant q \leqslant s)$ *and* $H_{\tau_h}(E'_{h_1}) = 0$, *where* $h_1 = |\log r|^{c-1}$. *Here* $s = np/(n-kp)$.

The theorems will be proved by induction on k. Each step of the induction will be proved for both theorems together, because their proofs are interdependent.

We shall use the abbreviation « $I - (i)_j$ » for « Theorem I(i) for $k = j$ » and similarly for other parts of the theorems.

The case $k = 1$. Let

$$f(r; x_0) = \int_0^r \frac{d\mu}{d\varrho}(\varrho; x_0)\varrho^n d\varrho , \tag{5.3}$$

as in Lemma 3.2. As a consequence of Lemma 3.1:

$$f(r;x_0)=\int_{B(x_0;r)}(\varrho D_\varrho u)\,dx=\int_{B(x_0;r)}\sum_{i=1}^{n}(x_i-x_{0,i})D_{x_i}u\,dx\,. \tag{5.4}$$

Let

$$F_h=\left\{x_0\in \boldsymbol{R}_n\colon\ \overline{\lim_{r\to 0}}\ \frac{h}{r^n}f(r;x_0)>0\right\}. \tag{5.5}$$

If $h=r^b|\log r|^c$, ($0\leqslant b<n/p-1$ and c arbitrary, or $b=n/p-1$ and $c>0$), then by Corollary 4.2, applied to $D_{x_i}u$ with $g=r^{-1}h(r)^{-1}$, we have

$$H_{\tau_h}(F_h)=0\,,\qquad \tau_h=r^{n-p}h(r)^{-p}\,. \tag{5.6}$$

Let $h_\delta=|\log r|^{1+\delta}$ and $F=\bigcap_{0<\delta}F_{h_\delta}$. If $x_0\in\boldsymbol{R}_n\setminus F$ then, by Corollary 3.3(i), $u_a(r;x_0)$ converges when r tends to zero. Thus I-(i)$_1$ follows from (5.6) with $h=h_\delta$.

By Lemmas 2.1 and 4.1:

$$\lim_{r\to 0}r^{-n}\int_{B(x_0;r)}|u-u_a(r;x_0)|^s\,dx=0\,,\qquad H_{(n-p)}\ \text{a.e. in } \boldsymbol{R}_n\,, \tag{5.7}$$

where $s=np/(n-p)$. This together with I-(i)$_1$ implies I-(ii)$_1$.

Now let $h=r^b|\log r|^c$, ($0<b<n/p-1$ or $b=n/p-1$ and $c>0$) and let $\tau_h=r^{n-p}h(r)^{-p}$. By (5.6) and Corollary 3.3(ii) we have:

$$\lim_{r\to 0}h(r)u_a(r;x_0)=0\,,\qquad H_{\tau_h}\ \text{a.e. in } \boldsymbol{R}_n\,. \tag{5.8}$$

Also, by Lemmas 2.1 and 4.1:

$$\lim_{r\to 0}\frac{h(r)^s}{r^n}\int_{B(x_0;r)}|u-u_a(r;x_0)|^s\,dx=0\,,\qquad H_{\tau_h}\ \text{a.e. in } \boldsymbol{R}_n\,. \tag{5.9}$$

By (5.8) and (5.9):

$$\lim_{r\to 0}\frac{h(r)^s}{r^n}\int_{B(x_0;r)}|u|^s\,dx=0\,,\qquad H_{\tau_h}\ \text{a.e. in } \boldsymbol{R}_n\,. \tag{5.10}$$

But by Hölder's inequality:

$$\left(r^{-n}\int\limits_{B(x_0;r)}|u|^q dx\right)^{1/q} \leqslant \text{const}\left(r^{-n}\int\limits_{B(x_0;r)}|u|^s dx\right)^{1/s}, \qquad (1\leqslant q\leqslant s)\,.$$

Thus (5.10) implies that $H_{\tau_h}(E_h^q)=0$.

By Lemma 3.2:

$$\mu(r;x_0)=nV_n u_a(r;x_0)+r^{-n}f(r;x_0)\,. \tag{5.11}$$

Therefore, (5.6) and (5.8) imply that $H_{\tau_h}(E'_h)=0$. This completes the proof of II-(i)$_1$ Assertion II-(ii)$_1$ is proved in the same way.

The case $k>1$. We assume that the assertions of the two theorems are proved for $k=1,\ldots,j-1$ and we proceed to prove them for $k=j$.

Let $h=r^b|\log r|^c$, $(-1<b<n/p-j$, or $b=n/p-j$ and $c>0)$, and $\tau_h=r^{n-jp}h^{-p}$. By II-(i)$_{j-1}$ applied to $D_{x_i}u$:

$$H_{\tau_h}\left\{x_0\in \boldsymbol{R}_n\colon\ \overline{\lim_{r\to 0}}\,\frac{rh(r)}{r^n}\int\limits_{B(x_0;r)}|D_{x_i}u|\,dx>0\right\}=0\,.$$

Hence, (see (5.4) and (5.5)):

$$H_{\tau_h}(F_h)=0\,. \tag{5.12}$$

Assertion I-(i)$_j$ follows from (5.12), (with $h=|\log r|^{1+\delta}$, $\delta>0$), and Corollary (3.3)(i).

By Lemmas 2.1 and 4.1:

$$\lim_{r\to 0} r^{-n}\int\limits_{B(x_0;r)}|u(x)-P_{j,r}(x;x_0)|^s dx=0\,, \qquad H_{(n-jp)}\text{ a.e. in } \boldsymbol{R}_n\,, \tag{5.13}$$

where $s=np/(n-kp)$. By Lemma 2.2:

$$T_r(x_0)=\sup_{x\in B(x_0;r)}|P_{j,r}(x;x_0)-u_a(r;x_0)|\leqslant \text{const}\sum_{1\leqslant|\gamma|\leqslant j-1}|(D^\gamma u)_a(r;x_0)|\,r^{|\gamma|}\,. \tag{5.14}$$

Hence by II-(i)$_\nu$, $(\nu=1,\ldots,j-1)$:

$$\lim_{r\to 0}T_r(x_0)=0\,, \qquad H_{(n-jp)}\text{ a.e. in } \boldsymbol{R}_n\,.$$

Thus, by (5.13):

$$\lim_{r\to 0} r^{-n}\int_{B(x_0;r)} |u-u_a(r;x_0)|^s\,dx = 0\,, \qquad H_{(n-jp)} \text{ a.e. in } \boldsymbol{R}_n\,. \tag{5.15}$$

This, together with I-(i)$_j$, implies I-(ii)$_j$.

Now let h and τ_h be as before, except that we assume also that $b>0$.

By (5.12) and Corollary 3.3(ii):

$$\lim_{r\to 0} h u_a(r;x_0) = 0\,, \qquad H_{\tau_h} \text{ a.e. in } \boldsymbol{R}_n\,. \tag{5.16}$$

By (5.14) and II-(i)$_\nu$, $(\nu = 1, \ldots, j-1)$:

$$\lim_{r\to 0} h T_\nu(x_0) = 0\,, \qquad H_{\tau_h} \text{ a.e. in } \boldsymbol{R}_n\,.$$

Hence,

$$\lim_{r\to 0} h(r)\Big(\sup_{x\in B(x_0;r)} P_{j,r}(x;x_0)\Big) = 0\,, \qquad H_{\tau_h} \text{ a.e. in } \boldsymbol{R}_n\,.$$

By Lemmas 2.1 and 4.1:

$$\lim_{r\to 0} \frac{h^s}{r^n}\int_{B(x_0;r)} |u-P_{j,r}(x;x_0)|^s\,dx = 0\,, \qquad H_{\tau_h} \text{ a.e. in } \boldsymbol{R}_n\,.$$

Thus:

$$\lim_{r\to 0} \frac{h^s}{r^n}\int_{B(x_0;r)} |u|^s\,dx = 0\,, \qquad H_{\tau_h} \text{ a.e. in } \boldsymbol{R}_n\,. \tag{5.17}$$

By Hölder's inequality this implies that $H_{\tau_h}(E_h^q) = 0$, $(1 \leqslant q \leqslant s)$. Further, from (5.11), (5.12) and (5.16) it follows that $H_{\tau_h}(E_h') = 0$. Thus we proved II-(i)$_j$. Assertion II-(ii)$_j$ is proved in the same way. This completes the proof of the theorems.

6. – In the previous section we discussed the Lebesgue set of functions in $W_{k,p}$ when $1 \leqslant p < n/k$. If $u \in W_{k,p}^{\text{loc}}(\boldsymbol{R}_n)$ and $n/k < p$, then u coincides a.e. in $\boldsymbol{R}_n$ with a continuous function; hence, in this case, $\tilde{u}(x_0) = \lim_{r\to 0} u_a(r;x_0)$ is defined for every x_0 in $\boldsymbol{R}_n$, and the Lebesgue set of u is the entire space. If $u \in W_{k,p}^{\text{loc}}(\boldsymbol{R}_n)$, with $p = n/k$, then $u \in L_s^{\text{loc}}(\boldsymbol{R}_n)$ for every $s \in [1, \infty)$; but u

need not be locally bounded. Results of the type obtained in the previous section are valid also in this case. The main line of the proof is the same as in the case $p < n/k$; however, some modifications are necessary. This we shall discuss in the present section.

We start with an auxiliary result.

LEMMA 6.1. *Let* $u \in W^{\text{loc}}_{k,p}(\boldsymbol{R}_n)$, *(k integer* $\geqslant 1$, $p = n/k$*). Then:*

$$(6.1) \qquad \lim_{r\to 0} \frac{1}{r^n} \int\limits_{B(x_0;r)} |u - u_a(r; x_0)|^s dx = 0\,, \qquad \forall x_0 \in \boldsymbol{R}_n \text{ and } \forall s \in [1, \infty)\,.$$

PROOF. Let $P_{k,r}$ be the polynomial described in Lemma 2.1. Then by the argument given in the proof of Lemma 2.1 we have, (for $s \in [1, \infty)$):

$$(6.2) \qquad \left(\frac{1}{r^n} \int\limits_{B(x_0;r)} |u - P_{k,r}(\cdot\,; x_0)|^s dx\right)^{1/s} \leqslant \text{const}\, \|\nabla^k u\|_{L_p(B(x_0;r))}\,,$$

the constant depending on s, k, n.

Let $1 \leqslant j \leqslant k$. By the Sobolev embedding theorem:

$$(6.3) \qquad D^\gamma u \in L^{\text{loc}}_{n/j}(\boldsymbol{R}_n)\,, \qquad \text{for } |\gamma| = j\,.$$

By Hölder's inequality:

$$(6.4) \qquad \int\limits_{B(x_0;r)} |D^\gamma u| dx \leqslant \text{const}\, r^{n-j} \Big(\int\limits_{B(x_0;r)} |D^\gamma u|^{n/j} dx \Big)^{j/n}, \qquad |\gamma| = j\,.$$

Hence:

$$(6.5) \qquad \lim_{r\to 0} r^{|\gamma|} \left(\frac{1}{r^n} \int\limits_{B(x_0;r)} |D^\gamma u|\, dx\right) = 0\,, \qquad \forall x_0 \in \boldsymbol{R}_n\,,\ 1 \leqslant |\gamma| \leqslant k\,.$$

The conclusion of the lemma follows from (6.2), (6.5) and Lemma 2.2.

TH. III. *Let* $u \in W^{\text{loc}}_{k,p}(\boldsymbol{R}_n)$, $p = n/k$. *Let* $\tau_\delta(r) = |\log r|^{-(1+\delta)p}$, $\delta > 0$. *Then there exists a set E such that* $H_{\tau_\delta}(E) = 0$ *for every* $\delta > 0$ *and such that:*

$$(6.6) \qquad \tilde{u}(x_0) = \lim_{r\to 0} u_a(r; x_0) \qquad \textit{is defined for every } x_0 \in \boldsymbol{R}_n \setminus E\,.$$

$$(6.7) \qquad \lim_{r\to 0} \frac{1}{r^n} \int\limits_{B(x_0;r)} |u - \tilde{u}(x_0)|^s dx = 0\,, \qquad \forall x_0 \in \boldsymbol{R}_n \setminus E \textit{ and } \forall s \in [1, \infty)\,.$$

Furthermore:

$$(6.8)\qquad \lim_{r\to 0} \frac{|\log r|^{(c-1)s}}{r^n} \int_{B(x_0;r)} |u|^s dx = 0\,, \qquad H_\tau \text{ a.e. in } \boldsymbol{R}_n\,,$$

where $\tau = |\log r|^{-cp}$, $0 < c < 1$, *and* s *is any number in* $[1, \infty)$. *Also*:

$$(6.9)\qquad \lim_{r\to 0} \frac{|\log r|^{-s}}{r^n} \int_{B(x_0;r)} |u|^s dx = 0\,, \qquad \forall x_0 \in \boldsymbol{R}_n\,,\ \forall s \in [1, \infty)\,.$$

PROOF. Let F_h be as in (5.5). Then:

$$(6.10)\qquad H_{\tau_h}(F_h) = 0\,, \quad \text{for } h = |\log r|^c\,, \quad c > 0\,,\ \tau_h = h^{-p}\,.$$

In the case $k = 1$ this follows from Corollary 4.2 applied to $D_{x_i}u$ with $g = r^{-1}h(r)^{-1}$. In the case $k > 1$ this follows from Theorem II-(i)$_{k-1}$ (with $b = 1$, $c > 0$) applied to $D_{x_i}u$. (Note that we may use the results of Theorems I and II for $j < k$, because $p < n/j$.)

In addition we have (see (5.4) and (6.5)):

$$(6.11)\qquad \lim_{r\to 0} \frac{1}{r^n} f(r; x_0) = 0 \qquad \forall x_0 \in \boldsymbol{R}_n\,.$$

Assertion (6.6) follows from (6.10) (with $h = |\log r|^{1+\delta}$, $0 < \delta$) and Corollary 3.3(i). Then (6.7) follows from (6.6) and Lemma 6.1.

By (6.10) and Corollary 3.3(iii) we have:

$$(6.12)\qquad \lim_{r\to 0} |\log r|^{c-1} u_a(r; x_0) = 0\,, \qquad H_\tau \text{ a.e. in } \boldsymbol{R}_n\,,$$

where $\tau = |\log r|^{-cp}$, $0 < c < 1$. Hence, by Lemma 6.1, we obtain (6.8). In the same way, starting with (6.11), we obtain (6.9). This completes the proof of the theorem.

Note added in proofs.

Since the present paper was submitted, two papers whose results overlap with ours have appeared. The papers are, N. G. MEYERS, *Taylor expansion of Bessel potentials*, Ind. Univ. Math. J., **23** (1974), pp. 1043-1049; T. BAGBY - W. P. ZIEMER, *Pointwise differentiability and absolute continuity* (preprint). In both of these papers, the results apply to space $W_{\alpha,p}$ where α need not be an integer. However, both of them treat only the case where p is greater than one.

REFERENCES

[1] C. P. CALDERÒN - E. B. FABES - N. M. RIVIERE, *Maximal smoothing operators* Ind. Univ. Math. J., **23** (1974), pp. 889-898.

[2] N. DUNFORD - J. T. SCHWARTZ, *Linear operators*, **I**, Interscience, New York.

[3] H. FEDERER, *Some properties of distributions whose partial derivatives are representable by integration*, Bull. Am. Math. Soc., **74** (1968), pp. 183-186.

[4] H. FEDERER, *Geometric measure theory*, Springer-Verlag, Heidelberg and New York, 1969.

[5] H. FEDERER - W. P. ZIEMER, *The Lebesgμe set of a function whose distribution derivatives are p-th power summable*, Ind. Univ. Math. J., **22** (1972), pp. 139-158.

[6] E. GAGLIARDO, *Proprietà di alcune classi di funzioni in più variabili*, Richerche di Math., **7** (1958), pp. 102-137.

[7] C. B. MORREY, *Functions of several variables and absolute continuity II*, Duke Math. J., **6** (1940), pp. 187-215.

[8] C. B. MORREY, *Multiple integrals in the calculus of variations*, Springer-Verlag, Heidelberg and New York, 1966.