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REFLECTIONS IN BOUNDED SYMMETRIC DOMAINS (*)

KLAUS POMMERENING

Introduction.

On determining the reflections in bounded symmetric domains E. Gottchling omitted the two exceptional domains. M. Meschiari [10] recently showed how to fill in this gap. In this paper another independent way is given.

The first section contains a general theory of reflections in bounded symmetric domains of tube type. The main tool is the description of these domains by formally real Jordan algebras. We shall obtain new proofs of Theorem 2 and 3 of [3] restrained to the tube domains, and the further result that the 27-dimensional exceptional domain permits no reflections.

In the second section we shall prove the same result for the 16-dimensional domain. We shall obtain this result by a straightforward calculation after having determined the stability group of this domain at the origin following M. Ise [6], [7].

For the sake of brevity we denote the bounded symmetric domains as BS-domains. The irreducible BS-domains are denoted according to [3]:

Type $I_{p,q} = M_{p,q}$, type $II_r = T_r$, type $III_r = S_r$, type $IV_n = L_n$,

type $V = R_{16}$, type $VI = R_{27}$.

We have among others the analytic isomorphisms $L_3 \cong S_2$, $L_4 \cong M_{2,2}$. The letter $\Omega(D)$ always denotes the group of analytic automorphisms of a

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domain D. $\mathbf{U}(1)$ denotes the unitary group of order 1, that is the complex unit circle. A BS domain D in a complex vector space V is called *well-shaped,* if (i) $O \in D$, (ii) D is equal (not only isomorphic) to a cartesian product of irreducible domains, (iii) the group $\Omega(D)$ contains all mappings $z \longmapsto \lambda z$ where $\lambda \in \mathbf{U}(1)$, (iv) the stability group at 0, $\Sigma(D)$, entirely consists of linear mappings (that is $\Sigma(D) \subseteq GL(V)$).

I. Reflections in tube domains.

1. Some remarks on Jordan algebras. The main reference is [1]. Let A be a Jordan algebra over a fied K. $L(x) \in \mathrm{End}_K A$ is the multiplication by $x \in A$, $P(x) = 2L(x)^2 - L(x^2)$ the quadratic representation. Let $\mathbf{H}$ be the $\mathbf{R}$-algebra of quaternions, $\mathbf{O}$ the $\mathbf{R}$-algebra of octonions (Cayley numbers), $\mathbf{H}^\sim$ and $\mathbf{O}^\sim$ their complexifications. Let $r \geq 3$, $K = \mathbf{R}$ or $\mathbf{C}$, $M_r^+(K)$ the Jordan algebra of $r \times r$-matrices over K, $S_r(K)$ the Jordan algebra of symmetric $r \times r$-matrices over K, $H_r(C)$ the Jordan algebra of hermitean $r \times r$-matrices over the K-algebra $C = \mathbf{C}$, $\mathbf{H}$, $\mathbf{O}$ (if $K = \mathbf{R}$), $C = \mathbf{H}^\sim$, $\mathbf{O}^\sim$ (if $K = \mathbf{C}$). Let further $n \geq 3$ and $[K^n, \mu, e]$ be the Jordan algebra over K, where multiplication is defined by

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 y_1 + \dots + x_n y_n \\ x_1 y_2 + x_2 y_1 \\ \vdots \\ x_1 y_n + x_n y_1 \end{bmatrix}.$$

Every simple formally real Jordan algebra is isomorphic to one of the algebras $\mathbf{R}$, $[\mathbf{R}^n, \mu, e]$, $S_r(\mathbf{R})$, $H_r(\mathbf{C})$, $H_r(\mathbf{H})$, $H_3(\mathbf{O})$. Every simple complex Jordan algebra is isomorphic to one of the algebras $\mathbf{C}$, $[\mathbf{C}^n, \mu, e]$, $S_r(\mathbf{C})$, $M_r^+(\mathbf{C})$, $H_r(\mathbf{H}^\sim)$, $H_3(\mathbf{O}^\sim)$ [1; p. 331, 309].

If A is formally real, then

$$\sigma: A \times A \longrightarrow \mathbf{R}, \quad \sigma(x, y) := \mathrm{trace}\ L(xy),$$

is a positive definite symmetric associative bilinear form on A. All $L(x)$, $x \in A$, are σ-selfadjoint, all algebra automorphisms of A are σ-orthogonal.

2. Tube domains. Now the main references are [4] and [9]. Let A be a formally real Jordan algebra, $A^\sim$ its complexification. Then

$$D := \{z \in A^\sim \mid 1_{A^\sim} - P(z) \circ P(\bar{z}) \text{ positive definite}\}$$

is a BS-domain in $A^{\sim}$. These domains are called *BS-domains of tube type* or *tube domains.* D is irreducible if and only if A is simple; more generally, the decomposition of A into simple ideals corresponds to the decomposition of D into irreducible domains. A BS-domain is of tube type if and only if all of its irreducible components are of tube type. The following list gives the correspondence between the simple formally real Jordan algebras (column A), their complexifications (column $A^{\sim}$) and the irreducible tube domains (column D). We note that some of the listed domains are not identical but only analytically isomorphic to the usual representatives of BS domains.

A	$A^{\sim}$	dim	degree	D
$\mathbf{R}$	$\mathbf{C}$	1	1	$M_{1,1}$
$[\mathbf{R}^n, \mu, e]$	$[\mathbf{C}^n, \mu, e]$	$n \geq 3$	2	L_n
$S_r(\mathbf{R})$	$S_r(\mathbf{C})$	$\frac{1}{2} r(r+1)$	$r \geq 3$	S_r
$H_r(\mathbf{C})$	$M_r^+(\mathbf{C})$ (1)	r^2	$r \geq 3$	$M_{r,r}$
$H_r(\mathbf{H})$	$H_r(\mathbf{H}^{\sim})$	$r(2r-1)$	$r \geq 3$	T_{2r}
$H_3(\mathbf{O})$	$H_3(\mathbf{O}^{\sim})$	27	3	R_{27}

The listed domains are mutually non-isomorphic. The significance of the degree of the algebra will soon become clear.

THEOREM 1 (U. Hirzebruch). Let A be a formally real Jordan algebra, D the corresponding tube domain,

$$U := \{u \in A^{\sim} \mid u \text{ invertible, } u^{-1} = \bar{u}\}.$$

Then the stability group $\Sigma(D)$ at $0 \in D$ in $\Omega(D)$ exactly consists of the maps $P(u) \circ W$, where $u \in U$ and W is a real automorphism of the algebra $A^{\sim}$ (that is, the extension to $A^{\sim}$ of an automorphism of A). [4; p. 408], [1; p. 327].

REMARKS. 1) If $A = \mathbf{R}$, then $U = \mathrm{U}(1)$.

(1) only up to isomorphism, see [1; p. 331].

2) According to [4], U may be characterized as the set of all linear combinations $u = \eta_1 c_1 + \dots + \eta_s c_s$, where $\eta_i \in \mathrm{U}(1)$ and $\{c_1, \dots, c_s\}$ is any complete orthogonal system of idempotents of A.

3) Obviously D is well-shaped.

4) Let $u = \eta_1 c_1 + \dots + \eta_s c_s \in U$, $A^\sim = \bigoplus_{i \le j} A_{ij}^\sim$ the Peirce decomposition relative to $\{c_1, \dots, c_s\}$ and C_{ij} the corresponding projections [1; p. 239]. Then

$$P(u) = \sum_{i \le j} \eta_i \eta_j C_{ij}.$$

3. **Classification of reflections in tube domains.** Hence-forth let A be a formally real Jordan algebra, e its unit element, D the corresponding tube domain, $S = P(u) \circ W$ (notation as in Theorem 1) a reflection in $\Sigma(D)$ at $O \in D$. We know that S is diagonalizable and has exactly one eigenvalue $\neq 1$, namely a root of unity λ. In the rest of section I, the final theorems excepted, the adjective «real» always refers to the fixed real form A of $A^\sim$, a «reflection» is a reflection in D at O. The following three propositions give a classification of all possible reflections (case (*a*)-(*d*)).

PROPOSITION 1. If $S = P(u) \circ W$ is a reflection, then either $u^2 = e$ or $u^2 - e$ is an eigenvector belonging to λ.

(ii) If $u^2 = e$, then S is a real automorphism of $A^\sim$.

Proof. (ii) $\bar{u} = u^{-1} = u$, hence u is real and $P(u)$ is real. Since $u^2 = e$, $P(u)$ is an automorphism [1; p. 158].

(i) Let $u^2 \neq e$. Then $Se = P(u)\,We = P(u)\,e = u^2 \neq e$, hence e is not in the eigenspace $A_1^\sim$ of S belonging to the eigenvalue 1. Therefore $A^\sim = \mathbf{C}e \oplus A_1^\sim$. Now let w be an eigenvector belonging to λ. We may assume $w = e + z_1$, where $z_1 \in A_1^\sim$. Since

$$u^2 - e + w = u^2 + z_1 = Se + Sz_1 = Sw = \lambda w,$$

$u^2 - e = (\lambda - 1)\,w$ is an eigenvector belonging to λ. Q. E. D.

At first we treat the case $u^2 = e$ separately:

PROPOSITION 2. If a real automorphism S of $A^\sim$ is a reflection, then $\lambda = -1$, and one of the following statements is true:

(*a*) There is a complete orthogonal system $\{d_1, d_2, d_3\}$ of idempotents of A such that $Sd_1 = d_2$, $Sd_2 = d_1$, $Sd_3 = d_3$.

(*b*) There is a complete orthogonal system $\{d_1, d_2\}$ of idempotents of A such that $Sd_1 = d_2$, $Sd_2 = d_1$.

Proof. If A has degree 1, then $A = \mathbf{R}$, and there exists only the real automorphism $1_{\mathbf{C}}$ of $A^{\sim} = \mathbf{C}$ which is not a reflection. Thus we may assume that A has at least two orthogonal idempotents. Since $S|_A$ is σ-orthogonal, $1/\lambda$ must be an eigenvalue of S together with λ. Since S is a reflection, only $\lambda = -1$ is possible; then in particular $S^2 = 1_{A^\sim}$.

The real eigenvalue -1 has a real eigenvector $x \in A$, that is $Sx = -x$. Since there are no nilpotents in the formally real algebra A, we conclude that x^k is an eigenvector with eigenvalue $(-1)^k$. Therefore $\dim_{\mathbf{R}} \mathbf{R}[x] \leq 3$ ($\mathbf{R}[x]$ is spanned by e, x, x^2), that is, $\mathbf{R}[x]$ has at most three primitive idempotents $d_1, \ldots,$ and the minimal decomposition of x is $x = \xi_1 d_1 + \ldots$ $\ldots + \xi_s d_s$, $s = 2$ or 3. The minimal decomposition of $Sx = -x$ is

$$-\xi_1 d_1 - \ldots - \xi_s d_s = -x = Sx = \xi_1 Sd_1 + \ldots + \xi_s Sd_s,$$

where $Sd_1, \ldots, Sd_s$ again represent the primitive idempotents of the subalgebra $\mathbf{R}[x] = \mathbf{R}[-x]$. By properly renumbering we conclude

$$Sd_1 = d_2, \quad \xi_1 = -\xi_2, \quad Sd_2 = d_1 \quad (\text{and } Sd_3 = d_3 \text{ if } s = 3).$$

Q. E. D.

In order to treat the case $u^2 \neq e$, we need the following two lemmas. As before $S = P(u) \circ W$ is assumed to be a reflection with eigenvalue $\lambda \neq 1$.

LEMMA 1. Let $u^2 \neq e$. Then $W(u^2 - e) = \lambda(e - \overline{u^2})$, and W maps the associative subalgebra $\mathbf{C}[u^2]$ onto itself.

Proof. Application of $P(\overline{u}) = P(u^{-1}) = P(u)^{-1}$ to the equation $P(u) \cdot$ $\cdot W(u^2 - e) = \lambda(u^2 - e)$ (Proposition 1 (i)) yields

$$W(u^2 - e) = \lambda P(\overline{u})(u^2 - e) = \lambda(e - \overline{u^2}).$$

Now $\overline{u^2} = (u^2)^{-1}$, hence $\mathbf{C}[u^2] = \mathbf{C}[\overline{u^2}] = \mathbf{C}[u^2 - e] = \mathbf{C}[\overline{u^2} - e]$ and the second assertion follows (compare [1; p. 142, Satz. 2.1]). Q. E. D..

Now let $\{c_1, \ldots, c_r\}$ be a complete orthogonal system of idempotents of A such that $u = \eta_1 c_1 + \ldots + \eta_r c_r$, where $\eta_1, \ldots, \eta_r \in \mathbf{U}(1)$, according to Remark 2 following Theorem 1. Since r is the degree of A, the c_i are primitive, the η_i not necessarily distinct. We have

$$u^2 = \eta_1^2 c_1 + \ldots + \eta_r^2 c_r .$$

Let $\eta_1^2, \ldots, \eta_s^2$ ($s \leq r$) be the distinct values among the η_j^2, $N_j := \{k \mid 1 \leq k \leq r,$ $\eta_k^2 = \eta_j^2\}$, $j = 1, \ldots, s$. The elements

$$d_j := \sum_{k \in N_j} c_k, \qquad j = 1, \ldots, s,$$

are the primitive idempotents of the subalgebra $\mathbb{C}[u^2]$ of $A^{\sim}$.

LEMMA 2. W operates on the set $\{d_1, \dots, d_s\}$ as permutation of order 2. More precisely

$$Wd_j = d_k \Longleftrightarrow \eta_j^2 - 1 = \lambda(1 - \overline{\eta_k^2}).$$

Proof. The automorphism $W|_{\mathbb{C}[u^2]}$ of $\mathbb{C}[u^2]$ preserves the property of being a primitive idempotent. Therefore W permutes $d_1, \dots, d_s$. Now

$$(\eta_1^2 - 1)\, Wd_1 + \dots + (\eta_s^2 - 1)\, Wd_s = W(u^2 - e)$$

$$= \lambda(e - \overline{u^2}) = \lambda(1 - \overline{\eta_1^2})\, d_1 + \dots + \lambda(1 - \overline{\eta_s^2})\, d_s.$$

Again this is a unique minimal decomposition [1; p. 22, Satz 4.2], hence the second assertion follows. Since

$$\eta_j^2 - 1 = \lambda(1 - \overline{\eta_k^2}) \Longleftrightarrow \eta_k^2 - 1 = \lambda(1 - \overline{\eta_j^2}),$$

our permutation has order 2. Q. E. D..

REMARKS. 1) $\eta_j^2 = 1$ can happen at most once, since $\eta_1^2, \dots, \eta_s^2$ are assumed distinct. Of course $\eta_j^2 = 1$ implies $Wd_j = d_j$.

2) Clearly $P(u)\, d_j = \eta_j^2\, d_j$ for $j = 1, \dots, s$.

PROPOSITION 3. Let $S = P(u) \circ W$ be a reflection in D. Then one of the following four statements (*a*)-(*d*) holds.

(*a*) There is a complete orthogonal system $\{d_1, d_2, d_3\}$ of idempotents of A such that $u^2 = \varrho\, d_1 + \overline{\varrho}\, d_2 + d_3$, where $\varrho \in \mathrm{U}(1)$, and $Wd_1 = d_2$, $Wd_2 = d_1$, $Wd_3 = d_3$.

(*b*) There is a complete ortogonal system $\{d_1, d_2\}$ of idempotents of A such that $u^2 = \varrho\, d_1 + \overline{\varrho}\, d_2$, where $\varrho \in \mathrm{U}(1)$, and $Wd_1 = d_2$, $Wd_2 = d_1$.

If (*a*) or (*b*) is true, then $\lambda = -1$.

(*c*) There is a complete orthogonal system $\{d_1, d_2\}$ of idempotents of A such that $u^2 = \lambda d_1 + d_2$ and $Wd_1 = d_1$, $Wd_2 = d_2$.

(*d*) $u^2 = \lambda e$, that is, $P(u) = \lambda 1_{A^{\sim}}$, $S = \lambda W$.

Proof. As was shown in Proposition 2, the assumption $u^2 = e$ implies (*a*) or (*b*) (with $\varrho = 1$). Now let $u^2 \neq e$. We go back to Lemma 2 and distinguish two cases:

I. There are indices $j \neq k$ such that $Wd_j = d_k$, in particular $s \geq 2$. Then

$$P(u)\, W(\eta_j d_j \pm \eta_k d_k) = P(u)(\eta_j d_k \pm \eta_k d_j)$$
$$= \eta_j \eta_k^2 d_k \pm \eta_k \eta_j^2 d_j = \pm \eta_j \eta_k (\eta_j d_j \pm \eta_k d_k).$$

Hence $\pm \eta_j \eta_k$ are eigenvalues of S and we see that $Wd_j = d_k$, where $j \neq k$, happens at most once, and then $\lambda = -1$, $\eta_j \eta_k = \pm 1$, $\eta_j^2 \eta_k^2 = 1$. All other d_i are mapped onto themselves by W, hence

$$P(u)\, Wd_i = P(u)\, d_i = \eta_i^2 d_i \quad \text{for} \quad i \neq j, k,$$

therefore $\eta_i^2 = 1$ because the eigenvalue $+1$ is already consumed. According to Remark 1 following Lemma 2, this case also can occur at most once. Hence $s = 2$ or 3, and by renumbering we get

$$u^2 = \varrho d_1 + \bar{\varrho} d_2 + d_3 \quad \text{or} \quad u^2 = \varrho d_1 + \bar{\varrho} d_2,$$

where $\varrho = \eta_1^2 \neq 1$.

II. We have $Wd_j = d_j$ for all $j = 1, \ldots, s$. Since $P(u)\, Wd_j = \eta_j^2 d_j$, all η_j^2 are eigenvalues of S. Since there are only two distinct eigenvalues and $u^2 \neq e$, (c) or (d) follows. Q. E. D..

4. Explicit determination of all reflections in tube domains. In the following four propositions we treat the cases (a) (d) separately. Finally the results are summarized by two theorems.

PROPOSITION 4. (a) Let A be a formally real Jordan algebra, D the corresponding BS-domain, W a real automorphism of $A^\sim$. Further let $u \in A^\sim$, $u^2 = \varrho d_1 + \bar{\varrho} d_2 + d_3$, where $\varrho \in \mathrm{U}(1)$ and $\{d_1, d_2, d_3\}$ is a complete orthogonal system of idempotents of A such that $Wd_1 = d_2$, $Wd_2 = d_1$, $Wd_3 = d_3$. Let $S = P(u) \circ W$ be a reflection.

Then D is reducible, $D = D_1 \times D_2$, where S is the identity on D_2 and the case (b) occurs on D_1.

Proof. Let $A^\sim = \bigoplus_{i \leq j} A_{ij}^\sim$ be the Peirce decomposition of $A^\sim$ relative to $\{d_1, d_2, d_3\}$. By the definition of the $A_{ij}^\sim$ we get

$$W(A_{11}^\sim) = A_{22}^\sim, \quad W(A_{22}^\sim) = A_{11}^\sim, \quad W(A_{33}^\sim) = A_{33}^\sim,$$
$$W(A_{12}^\sim) = A_{12}^\sim, \quad W(A_{13}^\sim) = A_{23}^\sim, \quad W(A_{23}^\sim) = A_{13}^\sim,$$

and $P(u)(A_{ij}^\sim) = A_{ij}^\sim$ for all i, j by Remark 4 following Theorem 1.

Hence S leaves fixed $A_0^\sim(d_3) = A_{11}^\sim \oplus A_{12}^\sim \oplus A_{22}^\sim$, $A_{1/2}^\sim(d_3) = A_{13}^\sim \oplus A_{23}^\sim$ and $A_1^\sim(d_3) = A_{33}^\sim$, the eigenvalue -1 being consumed by $A_0^\sim(d_3)$. Therefore on $A_{1/2}^\sim(d_3)$ all eigenvalues of S are $= 1$, hence S is the identity there (since S is diagonalizable). Now $S(A_{13}^\sim) = A_{23}^\sim$ implies $A_{13}^\sim = A_{23}^\sim = 0$, hence $A_{1/2}^\sim(d_3) = 0$. Therefore $A^\sim$ splits into two ideals, $A^\sim = A_0^\sim(d_3) \oplus A_1^\sim(d_3)$, where S is the identity on $A_1^\sim(d_3)$ and (*b*) is valid on $A_0^\sim(d_3)$. Q. E. D..

PROPOSITION 4. (*b*) Let A be a formally real Jordan algebra of degree r, D the corresponding BS-domain, W a real automorphism of $A^\sim$. Further let $u \in A^\sim$, $u^2 = \varrho d_1 + \bar{\varrho} d_2$, where $\varrho \in \mathrm{U}(1)$ and $\{d_1, d_2\}$ is a complete orthogonal system of idempotents of A such that $Wd_1 = d_2$, $Wd_2 = d_1$. Let $S = P(u) \circ W$ be a reflection.

Then d_1, d_2 are primitive idempotents, that is, $r = 2$, $D \cong M_{1,1} \times M_{1,1}$ or $D \cong L_n$, and S is given by

$$Sd_1 = \bar{\varrho} d_2, \quad Sd_2 = \varrho d_1, \quad S|_{A_{1/2}^\sim(d_1)} = 1_{A_{1/2}^\sim(d_1)}.$$

This map is in fact a reflection in $\Sigma(D)$.

Proof. Let $A^\sim = A_{11}^\sim \oplus A_{12}^\sim \oplus A_{22}^\sim$ be the Peirce decomposition of $A^\sim$ relative to $\{d_1, d_2\}$. Again it is clear that

$$W(A_{11}^\sim) = A_{22}^\sim, \quad W(A_{22}^\sim) = A_{11}^\sim, \quad W(A_{12}^\sim) = A_{12}^\sim, \quad P(u)(A_{ij}^\sim) = A_{ij}^\sim \quad \text{for all } i, j.$$

Since the eigenvalue -1 is consumed by $A_{11}^\sim \oplus A_{22}^\sim$, S is the identity on $A_{12}^\sim$. If we assume $\dim_{\mathbf{C}} A_{11}^\sim \geq 2$, the eigenspace belonging to -1 becomes too big. Therefore $A_{11}^\sim = \mathbf{C}\, d_1$ and likewise $A_{22}^\sim = \mathbf{C}\, d_2$. Hence d_1, d_2 are primitive and $r = 2$.

It remains to show that the given map is in $\Sigma(D)$: Choose $\eta \in \mathrm{U}(1)$ such that $\eta^2 = \varrho$, $u = \eta d_1 \pm \bar{\eta} d_2$, and take the real automorphism W of $A^\sim$ given by $Wd_1 = d_2$, $Wd_2 = d_1$, $W|_{A_{12}^\sim} = 1_{A_{12}^\sim}$. The fact that this is an automorphism is obvious, since $A \cong \mathbf{R} \oplus \mathbf{R}$ or $[\mathbf{R}^n, \mu, e]$. Q. E. D..

PROPOSITION 4. (*c*) Let A be a formally real Jordan algebra, D the corresponding BS-domain, W a real automorphism of $A^\sim$, $u \in A^\sim$. Let $\{d_1, d_2\}$ be a complete orthogonal system of idempotents of A such that $Wd_1 = d_1$, $Wd_2 = d_2$, $S = P(u) \circ W$ a reflection with the eigenvalue $\lambda \neq 1$, and $u^2 = \lambda d_1 + d_2$.

Then D is reducible, $D = D_1 \times D_2$, where S is the identity on D_2 and case (*d*) occurs on D_1.

Proof. Let $A^{\sim} = A_{11}^{\sim} \oplus A_{12}^{\sim} \oplus A_{22}^{\sim}$ be the Peirce decomposition relative to $\{d_1, d_2\}$. Since $S(A_{ij}^{\sim}) = A_{ij}^{\sim}$ for all i, j, and $d_1 \in A_{11}^{\sim}$ is an eigenvector belonging to λ, it suffices to show $A_{12}^{\sim} = 0$.

Now $u = \eta c_1 - \eta c_2 + c_3 - c_4$, where $\eta^2 = \lambda$, the $c_i \in A$ are idempotent or 0 and $c_1 + c_2 = d_1$, $c_3 + c_4 = d_2$ (proof: According to [1; p. 22], the minimal decomposition of u arises from Remark 2 following Theorem 1. By taking the square, $u^2 = \lambda d_1 + d_2$ must result). Hence $P(u) = \lambda C_1 + C_2 + \eta C_3$, where

$$C_1 := P(c_1) + P(c_2) - 4L(c_1) \circ L(c_2),$$

$$C_2 := P(c_3) + P(c_4) - 4L(c_3) \circ L(c_4),$$

$$1/4\, C_3 := L(c_1) \circ L(c_3) - L(c_1) \circ L(c_4) - L(c_2) \circ L(c_3) + L(c_2) \circ L(c_4),$$

in particular

$$C_1 |_{A_{12}^{\sim}} = C_2 |_{A_{12}^{\sim}} = 0, \quad \eta C_3 |_{A_{12}^{\sim}} = P(u) |_{A_{12}^{\sim}}.$$

Since c_1, c_2, c_3, c_4 are real, C_3 is real too. Moreover W is real and $S|_{A_{12}^{\sim}} = 1_{A_{12}^{\sim}}$. Since $S|_{A_{12}^{\sim}} = \eta C_3 \circ W|_{A_{12}^{\sim}}$, we must have either $A_{12}^{\sim} = 0$ or η real. But η real, that is $\eta = \pm 1$, contradicts $\lambda \neq 1$. Therefore $A_{12}^{\sim} = 0$.

Q. E. D..

PROPOSITION 4. (*d*) Let A be a formally real Jordan algebra of degree r, D the corresponding BS-domain, W a real automorphism of $A^{\sim}$, $u \in A^{\sim}$. Let $S = P(u) \circ W$ be a reflection, λ the eigenvalue $\neq 1$ of S, and $u^2 = \lambda e$. Then either

(i) $r = 1$, $M \cong M_{1,1}$, and S is given by $z \longmapsto \lambda z$, or

(ii) $r = 2$, $\lambda = -1$, and we have case (*b*), where $\varrho = -1$.

In every case S is in fact a reflection in $\Sigma(D)$.

Proof. Let $\lambda = \eta^2$. Hence $(1/\eta \cdot u)^2 = e$, and $P(1/\eta \cdot u) = 1/\lambda \cdot P(u)$ is an automorphism of $A^{\sim}$. Since $1/\eta \cdot u \in U$ (see Theorem 1), $1/\eta \cdot u$ is real, hence $1/\lambda \cdot P(u)$ is real. Therefore $S = P(u) \circ W = \lambda W_0$, where W_0 is a real automorphism of $A^{\sim}$ with the simple eigenvalue 1; all other eigenvalues of W_0 must be $= 1/\lambda = \bar{\lambda}$.

If $r = 1$, we have (i) at once; otherwise the eigenvalue $\bar{\lambda}$ of W_0 occurs in fact. Since W_0 is an orthogonal map, $\lambda = -1$ follows; in particular $W_0^2 = 1_{A^{\sim}}$. The eigenspace belonging to the simple eigenvalue 1 of W_0 is spanned by e. Since W_0 is real, there is an $x \in A$ such that $W_0 x = -x$. Let $x = \xi_1 d_1 + \ldots + \xi_s d_s$ be the minimal decomposition. The minimal decomposition of $W_0 x = -x$ is given by

$$-\xi_1 d_1 - \dots - \xi_s d_s = -x = W_0 x = \xi_1 W_0 d_1 + \dots + \xi_s W_0 d_s .$$

Since $x \neq 0$, we may assume $\xi_1 \neq 0$. Hence $W_0 d_1 \neq d_1$. Let $W_0 d_1 = d_2$, hence $W_0 d_2 = d_1$. It follows that $W_0 (d_1 + d_2) = d_1 + d_2$, hence $d_1 + d_2 \in A$ is an eigenvector belonging to 1 and therefore

$$d_1 + d_2 = \mu e = \mu (d_1 + \dots + d_s), \quad \text{where } \mu \in \mathbf{R}.$$

We obtain $s = 2$ (and $\mu = 1$). It follows that $u^2 = \lambda e = -d_1 - d_2$, hence we obtain (b), in particular $r = 2$. Q. E. D..

Before summarizing our results we reformulate the statements concerning the case $r = 2$.

LEMMA 3. Within $\Sigma(D)$ the reflections of Proposition 4 (b) are conjugate to the map S_0 given by

$$S_0 d_1 = d_2 , \quad S_0 d_2 = d_1 , \quad S_0 |_{\tilde{A_{12}}} = 1_{\tilde{A_{12}}} .$$

Proof. Choose $u_0 \in U$ such that $u_0^2 = d_1 + \varrho d_2$. Then $P(u_0)(\tilde{A_{12}}) = \tilde{A_{12}}$ and $P(u_0) \circ S \circ P(u_0)^{-1} |_{\tilde{A_{12}}} = 1_{\tilde{A_{12}}}$. Furthermore $P(u_0)^{-1} = P(u_0^{-1})$, $u_0^{-2} = d_1 + \bar{\varrho} d_2$, hence

$$P(u_0) \circ S \circ P(u_0)^{-1} d_1 = P(u_0) S d_1 = P(u_0) \bar{\varrho} d_2 = \varrho \bar{\varrho} d_2 = d_2 ,$$

likewise $P(u_0) \circ S \circ P(u_0)^{-1} d_2 = d_1$, hence $P(u_0) \circ S \circ P(u_0)^{-1} = S_0$. Q. E. D..

COROLLARY. (i) Within $\Sigma(M_{1,1} \times M_{1,1})$ all the reflections of type (b) are conjugate to

$$(z_1 , z_2)^t \longmapsto (z_2 , z_1)^t \text{ (}^2\text{)}.$$

(ii) Every reflection in L_n is conjugate to

$$(+) \qquad (z_1 , z_2 , \dots , z_n)^t \longmapsto (-z_1 , z_2 , \dots , z_n)^t .$$

Proof. (i) Obvious, since $(1,0)^t$ and $(0,1)^t$ are the only proper idempotents of $\mathbf{R} \oplus \mathbf{R}$.

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(2) $(z_1 , \dots , z_n)^t$ is the column vector with components $z_1 , \dots , z_n$.

(ii) Let $c_1 = (1/2, 1/2, 0, \dots, 0)^t$, $c_2 = (1/2, -1/2, 0, \dots, 0)^t$. Take a real automorphism W_0 of $[\mathbf{C}^n, \mu, e]$ which sends $\{d_1, d_2\}$ into the ortogonal system $\{c_1, c_2\}$ [1 ; p. 329]. Then $W_0 \circ S_0 \circ W_0^{-1}$ is the map

$$(++) \qquad (z_1, z_2, z_3, \dots, z_n)^t \longmapsto (z_1, -z_2, z_3, \dots, z_n)^t.$$

Since the map $(+)$ is a reflection in $\Sigma(L_n)$ (namely $P(u) \circ W$, where $u = ie$ and W is the canonical involution of $[\mathbf{C}^n, \mu, e]$), $(+)$ and $(++)$ must be in the same conjugacy class. Q. E. D..

THEOREM 2. (The reflections in the irreducible tube domains). Let A be a simple formally real Jordan algebra of degree r, D the corresponding BS-domain. Then

(i) if $r \geq 3$, D permits no reflections.

(ii) If $r = 2$ (that is, $D \cong L_n$, $n \geq 3$), D permits reflections only of order 2, which are all conjugate. The reflections in $\Sigma(D)$ are exactly the maps described in Proposition 4 (*b*).

(iii) If $r = 1$ (that is $D \cong M_{1,1}$), D permite reflections of any order; the reflections in $\Sigma(D)$ are the maps $z \longrightarrow \lambda z$, where λ is a root of unity.

COROLLARY. There are no reflections in the domains

$$S_r\,(r \geq 3), \quad M_{r,r}\,(r \geq 3), \quad T_{2r}\,(r \geq 3), \quad R_{27}.$$

Identifying the group $\Omega'(D) = \Omega(D_1) \times \dots \times \Omega(D_s)$ canonically as a subgroup of $\Omega(D)$, where $D = D_1 \times \dots \times D_s$, we get

THEOREM 3. (The reflections in the reducible tube domains). Let D be a tube domain, $D = D_1 \times \dots \times D_s$ the decomposition into irreducible tube domains.

(i) There are reflections in $\Omega(D) - \Omega'(D)$ if and only if D contains at least two factors $\cong M_{1,1}$, say $D_1 = D_2 = M_{1,1}$. Within $\Omega(D)$ all these reflections are conjugate to the map

$$(z_1, z_2, z_3, \dots, z_n)^t \longmapsto (z_2, z_1, z_3, \dots, z_n)^t.$$

(ii) There are reflections in $\Omega'(D)$ if and only if D contains factors $\cong M_{1,1}$ or L_n. All these reflections are given by a reflection on one of these factors and the identity on the other factors.

(iii) D permits no reflections, if there is no factor $\cong M_{1,1}$ or L_n.

II. The non-existence of reflections in the 16-dimensional exceptional domain.

As for results on spin groups and spin representations we refer to [2].

1. Some remarks on the canonical realization of R_{16} by M. Ise. The main references are [6] and [7]. Let D be an irreducible BS-domain, $\mathfrak{g}$ the Lie algebra of the Lie group $\Omega(D)$. The simply connected Lie group G with Lie algebra $\mathfrak{g}$ acts on D via the covering map $G \longrightarrow \Omega(D)$ (whose image is the connected component of 1_D). Let $\mathfrak{g}^\sim$ be the complexification of $\mathfrak{g}$, ϱ a non-trivial simple representation of $\mathfrak{g}^\sim$ in a complex vector space V of lowest dimension. ϱ induces a representation of G denoted also by ϱ. Let K be the stability group in G of some point of D, $\mathfrak{k} \subseteq \mathfrak{g}$ the corresponding Lie algebra, $\mathfrak{k}^\sim \subseteq \mathfrak{g}^\sim$ the complexification of $\mathfrak{k}$. Then $\varrho|_{\mathfrak{k}^\sim}$ is semisimple, say $\varrho|_{\mathfrak{k}^\sim} = \varrho_1 \oplus \dots \oplus \varrho_s$. We decompose V into the corresponding $\mathfrak{k}^\sim$-submodules, $V = V_1 \oplus \dots \oplus V_s$. The sequence $V_1, \dots, V_s$ is fixed in a special way, see [6; p. 119]. Let $p := \dim_{\mathbf{C}} V_1$, $r := \dim_{\mathbf{C}} V_2$, $q := \dim_{\mathbf{C}}(V_2 \oplus \dots \oplus V_s)$. The space of complex $k \times l$-matrices is denoted by $\mathfrak{M}_{k,l}(\mathrm{C})$. $\mathfrak{M}_{p,r}(\mathrm{C}) \subseteq \mathfrak{M}_{p,q}(\mathrm{C})$ is identified by

$$\begin{bmatrix} z_{11} & \dots & z_{1r} \\ \vdots & & \vdots \\ z_{p1} & \dots & z_{pr} \end{bmatrix} \longmapsto \begin{bmatrix} z_{11} & \dots & z_{1r} & 0 & \dots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ z_{p1} & \dots & z_{pr} & 0 & \dots & 0 \end{bmatrix}.$$

For $\omega \in G$, $\varrho(\omega) \in \mathfrak{M}_{p+q,\,p+q}(\mathrm{C}) = \mathrm{End}_{\mathbf{C}}\, V$ is given by the matrix $\varrho(\omega) = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$, where $A_1 \in \mathfrak{M}_{p,p}(\mathrm{C}) = \mathrm{End}_{\mathbf{C}}\, V_1$, $A_2 \in \mathfrak{M}_{p,q}(\mathrm{C})$, $A_3 \in \mathfrak{M}_{q,p}(\mathrm{C})$, $A_4 \in \mathfrak{M}_{q,q}(\mathrm{C}) = \mathrm{End}_{\mathbf{C}}(V_2 \oplus \dots \oplus V_s)$. Thus we get

PROPOSITION 5. (Ise-Yokonuma). The irreducible BS-domain D is analytically isomorphic to a domain D' in $\mathfrak{M}_{p,r}(\mathrm{C})$ on which G acts as follows: If $\omega \in G$ and $Z \in D' \subseteq \mathfrak{M}_{p,r}(\mathrm{C}) \subseteq \mathfrak{M}_{p,q}(\mathrm{C})$, then

$$\omega(Z) = (A_1 Z + A_2)(A_3 Z + A_4)^{-1}.$$

If $\omega \in K$, then $A_2 = 0$, $A_3 = 0$, $A_1 = \varrho_1(\omega)$, $A_4 = \begin{bmatrix} \varrho_2(\omega) & & 0 \\ & \ddots & \\ 0 & & \varrho_s(\omega) \end{bmatrix}$,

where $\varrho_2(\omega) \in \mathfrak{M}_{r,r}(\mathrm{C}) = \mathrm{End}_{\mathbf{C}}\, V_2$.

COROLLARY. The subgroup K acts as stability group in G of $0 \in D'$ in the following way: For $\omega \in K$, $Z \in D' \subseteq \mathfrak{M}_{p,r}(\mathbf{C})$ we have $\omega(Z) = \varrho_1(\omega) \cdot Z \cdot \varrho_2(\omega)^{-1}$.

According to M. Ise we call D' the canonical realization of D. But we note that this definition is by no means canonical nor even unique.

Now let $D = D' = R_{16}$. In [11; p. 121] it is proved that $\Omega(R_{16})$ is connected. Hence application of the above Corollary yields the full stability group $\Sigma(R_{16})$.

$\mathfrak{g}^{\sim}$ is a certain Lie algebra of transformations of the exceptional complex Jordan algebra $A^{\sim} = H_3(\mathbf{O}^{\sim})$, ϱ is the identity representation of $\mathfrak{g}^{\sim}$ on $A^{\sim}$. Let c be the primitive idempotent $c = (1, 0, 0; 0, 0, 0)$ of $A^{\sim}$ (notation of [1; p. 227]) and consider the subalgebra $\mathfrak{D}^{\sim} := \{\Delta \in \operatorname{Der} A^{\sim} \mid \Delta c = 0\}$ of the derivation algebra $\operatorname{Der} A^{\sim}$, $W^{\sim} := \{x \in A_1^{\sim}(c) \oplus A_0^{\sim}(c) \mid \operatorname{trace} x = 0\}$. Then the complexification $\mathfrak{k}^{\sim} \subseteq \mathfrak{g}^{\sim} \subseteq \mathfrak{gl}\, A^{\sim}$ of $\mathfrak{k}$ is $\mathfrak{k}^{\sim} = \mathfrak{D}^{\sim} \oplus L(W^{\sim}) \cong \mathbf{C} \oplus \mathfrak{so}(10, \mathbf{C})$. We need a result of M. Ise [7; Proposition 4, p. 235], whose exact formulation is as follows:

PROPOSITION 6 (M. Ise). The decomposition of $A^{\sim}$ into simple $\mathfrak{k}^{\sim}$-modules is the Peirce decomposition $A^{\sim} = A_1^{\sim}(c) \oplus A_{1/2}^{\sim}(c) \oplus A_0^{\sim}(c)$. The corresponding representations $\varrho_1, \varrho_2, \varrho_3$ of $\mathfrak{k}^{\sim} \cong \mathbf{C} \oplus \mathfrak{so}(10, \mathbf{C})$ are given by

$$\varrho_1 : \mathbf{C} \oplus \mathfrak{so}(10, \mathbf{C}) \to \mathfrak{gl}\, A_1^{\sim}(c) \cong \mathbf{C}, \qquad (\lambda, T) \mapsto 2\lambda,$$

$$\varrho_2 : \mathbf{C} \oplus \mathfrak{so}(10, \mathbf{C}) \to \mathfrak{gl}\, A_{1/2}^{\sim}(c) \cong \mathfrak{gl}(16, \mathbf{C}), \quad (\lambda, T) \mapsto \lambda/2 \cdot 1_{16} + \varrho_h^*(T),$$

$$\varrho_3 : \mathbf{C} \oplus \mathfrak{so}(10, \mathbf{C}) \to \mathfrak{gl}\, A_0^{\sim}(c) \cong \mathfrak{gl}(10, \mathbf{C}), \quad (\lambda, T) \mapsto -\lambda 1_{10} + T,$$

where 1_m denotes the $m \times m$ unit matrix and ϱ_h^* is one of the half-spin representations of $\mathfrak{so}(10, \mathbf{C})$.

2. The stability group $\Sigma(R_{16})$. Let $A = H_3(\mathbf{O}) \subseteq A^{\sim}$ be the exceptional formally real Jordan algebra, $W := W^{\sim} \cap A$, $\mathfrak{D} := \{\Delta \in \mathfrak{D}^{\sim} \mid \Delta(A) \subseteq A\}$. Then the Lie algebra $\mathfrak{k} \subseteq \mathfrak{g}^{\sim}$ of $\Sigma(R_{16})$ is $\mathfrak{k} = \mathfrak{D} \oplus i \cdot L(W) \cong \mathbf{R} \oplus \mathfrak{so}(10, \mathbf{R})$; on the center $\mathbf{R}$ the isomorphism $\mathbf{R} \oplus \mathfrak{so}(10, \mathbf{R}) \to \mathfrak{k}$ is given by $t \mapsto it \cdot L(3c - e)$. We conclude

LEMMA 4. The restrictions of $\varrho_1, \varrho_2, \varrho_3$ to $\mathfrak{k}$ are given by

$$\varrho_1 : \mathbf{R} \oplus \mathfrak{so}(10, \mathbf{R}) \to \mathbf{C}, \qquad (t, T) \mapsto 2it,$$

$$\varrho_2 : \mathbf{R} \oplus \mathfrak{so}(10, \mathbf{R}) \to \mathfrak{gl}(16, \mathbf{C}), \quad (t, T) \mapsto it/2 \cdot 1_{16} + \varrho_h^*(T),$$

$$\varrho_3 : \mathbf{R} \oplus \mathfrak{so}(10, \mathbf{R}) \to \mathfrak{gl}(10, \mathbf{C}), \quad (t, T) \mapsto -it \cdot 1_{10} + T.$$

Now we can compute $\Sigma(R_{16})$: The simply connected Lie group of $\mathbf{R} \oplus \mathfrak{so}(10, \mathbf{R})$ is $(\mathbf{R}, +) \times \mathbf{Spin}(10, \mathbf{R})$. Thereon ϱ_1 and ϱ_2 induce the representations

$$\varrho_1' : \mathbf{R} \times \mathbf{Spin}(10, \mathbf{R}) \to \mathbf{C}^{\times}, \qquad (t, s) \longmapsto \exp(2it),$$

$$\varrho_2' : \mathbf{R} \times \mathbf{Spin}(10, \mathbf{R}) \to \mathbf{GL}(16, \mathbf{C}), \ (t, s) \longmapsto \exp(it/2) \cdot \varrho_h^*(s),$$

where ϱ_h^* is one of the half-spin representations of $\mathbf{Spin}(10, \mathbf{C}) \supseteq \mathbf{Spin}(10, \mathbf{R})$. ϱ_1' and ϱ_2' can be factored via $\mathbf{R} \to \mathbf{U}(1)$, $t \longmapsto \exp(it/2)$, hence induce

$$\varrho_1'' : \mathbf{U}(1) \times \mathbf{Spin}(10, \mathbf{R}) \to \mathbf{C}^{\times}, \qquad (\eta, s) \longmapsto \eta^4,$$

$$\varrho_2'' : \mathbf{U}(1) \times \mathbf{Spin}(10, \mathbf{R}) \to \mathbf{GL}(16, \mathbf{C}), \ (\eta, s) \longmapsto \eta \cdot \varrho_h^*(s).$$

Now the Corollary of Proposition 5 shows that every automorphism of $R_{16} \subseteq \mathfrak{M}_{1,16}(\mathbf{C})$ is given by $Z \longmapsto \eta \cdot Z \cdot \varrho_h^*(s)^{-1}$, where $\eta \in \mathbf{U}(1)$, $s \in \mathbf{Spin}(10, \mathbf{R})$. We identify $\mathfrak{M}_{1,16}(\mathbf{C}) = \mathbf{C}^{16}$ by transposing. Since the two half-spin representations ϱ_h and ϱ_h^* are contragredient, we get

Theorem 4. Let $R_{16} \subseteq \mathbf{C}^{16}$ be the canonical realization of the 16-dimensional exceptional domain. Then the stability group $\Sigma(R_{16})$ at $0 \in R_{16}$ exactly consists of the mappings

$$z \longmapsto \eta \cdot \varrho_h(s)\, z,$$

where $\eta \in \mathbf{U}(1)$, $s \in \mathbf{Spin}(10, \mathbf{R})$ and ϱ_h is one of the two half-spin representations of $\mathbf{Spin}(10, \mathbf{C})$. In particular R_{16} is well-shaped.

Remark. Both groups $\mathbf{U}(1)$ and $\mathbf{Spin}(10, \mathbf{R})$ contain the cyclic subgroup $Z_4 = \{i, -1, -i, 1\}$. Let Z_4^* be the diagonally imbedded subgroup of $\mathbf{U}(1) \times \mathbf{Spin}(10, \mathbf{R})$ over Z_4. Then one calculates that the action of $\mathbf{U}(1) \times \mathbf{Spin}(10, \mathbf{R})$ on R_{16} has kernel Z_4^*.

3. The reflections of R_{16}. Let always $\varrho : \mathbf{Spin}(10, \mathbf{C}) \to GL_{\mathbf{C}}(S)$ and $\varrho_i : \mathbf{Spin}(10, \mathbf{C}) \to GL_{\mathbf{C}}(S_i)$, $i = 1, 2$, be the spin representation and the two half-spin representations; we have $S = S_1 \oplus S_2$. We assume that one of the maps $\eta \cdot \varrho_i(s) \in GL_{\mathbf{C}}(S_i)$, $\eta \in \mathbf{U}(1)$, $s \in \mathbf{Spin}(10, \mathbf{R})$, is a reflection at 0 (we don't need to know whether $i = 1$ or 2). Hence $\varrho_i(s)$ must be diagonalizable and have the 15fold eigenvalue $1/\eta$ and the simple one λ/η, where λ is a root of unity $\neq 1$. Since ϱ_1 and ϱ_2 are contragredient, the following lemma holds:

LEMMA 5. If there is a reflection in $\Sigma(R_{16})$ with eigenvalue $\lambda \neq 1$, then there exists an $s \in \mathbf{Spin}(10, \mathbf{R})$ such that $\varrho(s)$ is diagonalizable and has the eigenvalues

$$\eta \text{ and } 1/\eta \text{ 15-fold}, \qquad \eta/\lambda \text{ and } \lambda/\eta \text{ simple}$$

(not necessarily all distinct).

Let σ be the usual bilinear form on $\mathbf{R}^{10}$, $\sigma^{\sim}$ the bilinear extension of σ to $\mathbf{C}^{10}$. Then $\sigma^{\sim}$ is a bilinear form of maximal index. Let $C(\sigma^{\sim})$ be the corresponding Clifford algebra, $C(\sigma^{\sim}) = C^{+} \oplus C^{-}$ the decompnsition into the even and odd part. C^{+} is a subalgebra and $\mathbf{Spin}(\sigma^{\sim}) = \mathbf{Spin}(10, \mathbf{C})$ is a multiplicative subgroup thereof. According to [2; p. 84, 95, 91] the representation $\varrho \otimes \varrho : \mathbf{Spin}(10, \mathbf{C}) \to GL_{\mathbf{C}}(S \otimes S)$ is equivalent to χ: $\mathbf{Spin}(10, \mathbf{C}) \to \mathrm{Aut}\, C(\sigma^{\sim})$, where $\chi(s)x = s\chi s^{-1}$. If we identify $S \otimes S = C(\sigma^{\sim})$, we get $C^{+} = (S_1 \otimes S_2) \oplus (S_2 \otimes S_1)$, $C^{-} = (S_1 \otimes S_1) \oplus (S_2 \otimes S_2)$. An application of this result yields

LEMMA 6. If there is a reflection in $\Sigma(R_{16})$ with eigenvalue $\lambda \neq 1$, then there is an $s \in \mathbf{Spin}(10, \mathbf{R})$ such that $\chi(s)$ is diagonalizable and has the eigenvalues

$$1 \text{ 452-fold}, \qquad \lambda \text{ 30-fold}, \qquad 1/\lambda \text{ 30-fold} \qquad \text{on } C^{+}$$

and (with $\zeta := \eta^2$)

$$\begin{array}{lll} \zeta \text{ 225-fold}, & 1/\zeta \text{ 225-fold}, & \\ \zeta/\lambda \text{ 30-fold}, & \lambda/\zeta \text{ 30-fold}, & \\ \zeta/\lambda^2 \text{ simple} & \lambda^2/\zeta \text{ simple} & \text{on } C^{-} \end{array}$$

(not necessarily all distinct).

Proof. Obvious by tensoring eigenbases belonging to $\varrho_1(s)$ and $\varrho_2(s)$.

Q. E. D..

Starting with an assumed reflection where $s \in \mathbf{Spin}(10, \mathbf{R})$, we have constructed an automorphism of $C(\sigma^{\sim})$, the eigenvalues of which are distributed in a certain manner.

But there is another way starting at s and leading to $\chi(s)$: via the vector representation and extension to $C(\sigma^{\sim})$; this extension is unique by the universal property of the Clifford algebra [2; p. 39]. We have $\mathbf{R}^{10} \subseteq \mathbf{C}^{10} \subseteq C^{-}$, $\chi(s)(\mathbf{R}^{10}) \subseteq \mathbf{R}^{10}$ and $\chi(s)|_{\mathbf{R}^{10}} \subseteq SO(\sigma) = \mathbf{SO}(10, \mathbf{R})$. Hence $\chi(s)|_{\mathbf{C}^{10}}$ has five (not necessarily distinct) pairs of mutually inverse eigenvalues

$\mu_1, \mu_2 = 1/\mu_1, \ldots, \mu_9, \mu_{10} = 1/\mu_9$. Let $\{x_1, \ldots, x_{10}\}$ be a corresponding eigenbasis of $\mathbf{C}^{10}$ and $x_M := x_{i_1} \cdot \ldots \cdot x_{i_m} \in C(\sigma^{\sim})$, where $M = \{i_1, \ldots, i_m\} \subseteq \subseteq \{1, \ldots, 10\}$ with $i_1 < \ldots < i_m$. Then $\{x_M \mid M \subseteq \{1, \ldots, 10\}\}$ is an eigenbasis of $C(\sigma^{\sim})$ relative to $\chi(s)$. Thus $\chi(s)$ has the eigenvalues $\mu_M := \mu_{i_1} \ldots \mu_{i_m}$, in particular $\mu_{\emptyset} = 1$. On C^+ or C^- resp. $\chi(s)$ has the eigenvalues μ_M, where m is even or odd resp.

Now we have to solve the elementary combinatorial problem of whether the distribution of eigenvalues given in Lemma 6 is compatible with any choice of $\mu_1, \mu_3, \mu_5, \mu_7, \mu_9$. For that purpose it suffices to allow $\mu_1, \mu_3, \mu_5, \mu_7, \mu_9$ to take the values

$$1/\zeta, \quad \lambda/\zeta, \quad \lambda^2/\zeta,$$

the last one at most once.

LEMMA 7. $\mu_1 = \mu_3 = \mu_5 = \mu_7 = \mu_9$ is impossible.

Proof. Let $\mu = \mu_1 = \ldots$. Then the eigenvalues μ_M on $C(\sigma^{\sim})$ are

$$\mu^{5-\nu} \quad \binom{10}{\nu}\text{-fold}, \quad \nu = 0, \ldots, 10,$$

and $\mu^{5-\nu}$ belongs to C^+ or C^- according to whether ν is odd or even. Thus on C^+ $\chi(s)$ has the eigenvalues

μ^4 10-fold, μ^2 120-fold, 1 252-fold, $1/\mu^2$ 120-fold, $1/\mu^4$ 10-fold.

The 1 has to be found 452 times, hence we must have $\mu^2 = 1$ or $1/\mu^2 = 1$. But then all values are $= 1$, and λ and $1/\lambda$ don't occur, a contradiction.

Q. E. D..

LEMMA 8. If the eigenvalue λ^2/ζ occurs among the μ_i, it must be equal to one of the other possible values on C^-, ζ, $1/\zeta$, λ/ζ, ζ/λ.

Proof. Say $\mu_9 = \lambda^2/\zeta$. Then λ^2/ζ moreover occurs on C^- as $\mu_{\{1,2,9\}}$ and $\mu_{\{3,4,9\}}$, that is, at least three times. Lemma 6 yields the desired result.

Q. E. D.

In view of Lemmas 7 and 8, we have only the following four cases, which are disposed of by brute force:

I. $\mu_1 = \mu_3 = \mu_5 = 1/\zeta$ and a) $\mu_7 = 1/\zeta$, $\mu_9 = \lambda/\zeta$,

b) $\mu_7 = \mu_9 = \lambda/\zeta$;

II. $\mu_1 = \mu_3 = \mu_5 = \lambda/\zeta$ and a) $\mu_7 = \lambda/\zeta$, $\mu_9 = 1/\zeta$,

b) $\mu_7 = \mu_9 = 1/\zeta$.

LEMMA 9. The case I is impossible.

LEMMA 10. The case II is impossible.

Proofs. Consider the μ_M where $M \subseteq \{1, \ldots, 6\}, \ldots, M \subseteq \{1, \ldots, 10\}$ succesively. Calculate that in every case the values λ and $1/\lambda$ occur too frequently. The execution is easy but requires some patience. Q. E. D..

The Lemmas 7-10 show that the distribution of eigenvalues required by Lemma 6 is impossible; thus:

THEOREM 5. The 16-dimensional irreducible exceptional BS-domain permits no reflections.

REMARK. By the results of I. Satake and S. Ihara on holomorphic imbeddings of BS-domains one sees that the maximal dimension of a domain imbedded in the 16 — or 27 — dimensional exceptional domain resp. is 10 or 16 resp. Now it is not difficult to prove that the set of fixed points of an analytic automorphism of a BS-domains is analytically isomorphic to a BS-domain of lower dimension. This gives a third proof of the fact that the exceptional domains permit no reflections. Moreover, the existence of reflections in a BS-domain is equivalent to the existence of a holomorphically imbedded BS-domain of codimension 1. At detailed proof will be given in a forthcoming article.

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