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ON THE HOLDER-CONTINUITY OF SOLUTIONS OF A NONLINEAR PARABOLIC VARIATIONAL INEQUALITY

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Parabolic variational inequalities have been studied extensively by Brezis [2], Browder [4], Lions [6], Lions-Stampacchia [7] and others. The existence of a weak solution is shown and when the elliptic operator involved is strongly monotone, the solution is unique.

Using the penalisation method, Lions [6] has shown the regularity of solutions of some linear parabolic inequalities. For nonlinear parabolic inequalities, the regularity of solutions with respect to time has been obtained by Brezis [2] and the regularity with respect to both space and time by the writer in [8].

The purpose of this paper is to show the Holder-continuity of solutions u of:

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} - \sum_{j=1}^n \frac{\partial}{\partial x_j} (a_j(x, u, Du)) = f \text{ on the region where } u(x, t) \geq 0, \\ u(x, t) = 0 \text{ elsewhere, } u(x, t) = 0 \text{ on } \partial G \times [0, T], u(x, 0) = 0 \text{ and} \\ \text{« continuity » of } u, \partial u / \partial x_j \text{ at the two regions.} \end{array} \right.$$

Moreover, if $a_j(x, u, Du) = a_j(x) D_j u$ for $j = 1, \dots, n$, it will be shown that $u \in L^p(0, T; W^{2,p}(G))$ for any p , $2 \leq p < \infty$.

To prove the result, we use Lions' penalisation method, a time discretisation of the penalized equation and a nonlinear singular perturbation of the latter equation.

The notations and the main results of the paper are given in Section 1. Proofs are carried out in Section 2.

SECTION 1: Let G be a bounded open subset of R_n with a smooth boundary ∂G . Set: $D_j = i^{-1} \partial/\partial x_j$, $j = 1, \dots, n$ and for each n -tuple $\alpha = (\alpha_1, \dots, \alpha_n)$ of non-negative integers, we write:

$$D^\alpha \prod_{j=1}^n D_j^{\alpha_j} \quad \text{with } |\alpha| = \sum_{j=1}^n \alpha_j.$$

$W^{m,p}(G)$ is the real reflexive separable Banach space:

$$W^{m,p}(G) = \{u : u \text{ in } L^p(G), D^\alpha u \text{ in } L^p(G), |\alpha| \leq m\}$$

with the norm:

$$\|u\|_{m,p} = \left\{ \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^p(G)}^p \right\}^{1/p}, \quad 2 \leq p < \infty.$$

$W_0^{m,p}(G)$ is the completion of C_0^∞ , the family of all infinitely differentiable functions with compact support in G , in the $\|\cdot\|_{m,p}$ -norm. The pairing between $W_0^{m,p}(G)$ and its dual $W^{-m,q}(G)$ is denoted by (\cdot, \cdot) . Set: $H = L^2(G)$ and $\|u\|_s$ is the $L^s(G)$ -norm of u for $1 < s < \infty$.

$C^\lambda(G)$ is the space of all Holder-continuous functions of any compact subset of G , with Holder-exponent λ , $0 < \lambda < 1$.

Let $[0, T]$ be a compact interval of the real line E . The derivative of u with respect to t will be denoted by u' .

$L^p(0, T; W^{m,p}(G))$ is the space of all equivalence classes of functions $u(t)$ from $[0, T]$ to $W^{m,p}(G)$ which are L^p -integrals. It is a real reflexive separable Banach space with the norm:

$$\|u\|_{L^p(0, T; W^{m,p}(G))} = \left\{ \int_0^T \|u(t)\|_{m,p}^p dt \right\}^{1/p}.$$

$C^\lambda(0, T; C^{2\lambda}(G))$ is the space of all Holder-continuous functions on any compact subsets of $G \times [0, T]$ with Holder-exponent λ with respect to t and with exponent 2λ with respect to x . $0 < 2\lambda < 1$.

We consider nonlinear partial differential operators on G of the form:

$$A(u) = \sum_{|\alpha| \leq 1} D^\alpha A_\alpha(x, Du).$$

ASSUMPTION (I): (i) Let $\zeta = \{\zeta_\alpha : |\alpha| \leq 1\}$, then each $A_\alpha(x, \zeta)$ is continuously differentiable in x and in ζ .

(ii) There exists a positive constant C such that :

$$|A_\alpha(x, \zeta)| + |D_x A_\alpha(x, \zeta)| + (1 + |\zeta|) \sum_{|\beta| \leq 1} |A_{\alpha\beta}(x, \zeta)| \leq C |\zeta|$$

where $A_{\alpha\beta} = \partial A_\alpha / \partial \zeta_\beta$.

$$(iii) \quad \sum_{|\alpha|, |\beta| \leq 1} A_{\alpha\beta}(x, \zeta) \eta_\alpha \eta_\beta \geq c \sum_{|\alpha| \leq 1} \eta_\alpha^2.$$

c is a positive constant.

$$(iv) \quad \sum_{|\alpha| \leq 1} A_\alpha(x, \zeta) \zeta_\alpha \geq 0.$$

Let $K = \{u : u \text{ in } L^2(G), u \geq 0 \text{ a. e. on } G\}$. It is clear that K is a closed convex subset of both H and $W_0^{1,2}(G)$.

The main results of the paper are the following two theorems.

THEOREM 1: Let A be an elliptic operator satisfying Assumption (I). Suppose that $f \in L^\infty(0, T; L^\infty(G))$, $f' \in L^2(0, T; W^{-1,2}(G))$ with $f(0) = 0$. Then there exists a unique solution u in $L^2(0, T; W_0^{1,2}(G)) \cap L^2(0, T; W^{2,2}(G))$ with u' in $L^2(0, T; W^{1,2}(G))$, $u(t)$ in K a. e. and $u(0) = 0$ such that :

$$\int_0^T (u' + Au - f, v - u) dt \leq 0$$

for all v in $L^2(0, T; W^{1,2}(G))$, v' in $L^2(0, T; W^{-1,2}(G))$, $v(t)$ in K and $v(0) = 0$.

Moreover : $u \in C^1([0, T]; C^{2\lambda}(\text{cl}G))$, $D_j u \in C^\gamma(0, T; C^{2\gamma}(G))$, for any j , with $0 < 2\lambda, 2\gamma < 1$.

When A is a linear elliptic operator, we have a stronger result.

THEOREM 2: Let $Au = \sum_{|\alpha|, |\beta| \leq 1} D^\alpha (a_{\alpha\beta}(x) D^\beta u)$ be a positively strongly uniformly elliptic operator on G with coefficients $a_{\alpha\beta}(x)$ in $C^1(\text{cl}G)$. Suppose that $f \in L^\infty(0, T; L^\infty(G))$, $f' \in L^2(0, T; W^{-1,2}(G))$ and $f(0) = 0$. Then there exists a unique $u \in L^2(0, T; W_0^{1,2}(G)) \cap L^p(0, T; W^{2,p}(G))$, $1 < p < \infty$, $u(t)$ in K a. e., $u' \in L^p(0, T; L^p(G)) \cap L^2(0, T; W^{1,2}(G))$ and $u(0) = 0$ such that :

$$\int_0^T (u' + Au - f, v - u) dt \geq 0$$

for all v in $L^2(0, T; W_0^{1,2}(G))$, v' in $L^2(0, T; W^{-1,2}(G))$, $v(t)$ in K and $v(0) = 0$.

Moreover : $u \in C^{\lambda}([0, T]; C^{2\lambda}(\text{cl}G))$, $D_j u \in C^{\gamma}([0, T]; C^{2\gamma}(G))$ for any j , with $0 < 2\lambda, 2\gamma < 1$.

Theorem 1 is a consequence of Theorem 3 which will be proved in Section 2.

THEOREM 3 : *Suppose all the hypotheses of Theorem 1 are satisfied. Then for each ε , $0 < \varepsilon < 1$, there exists a unique solution u_{ε} of the equation :*

$$\varepsilon (u_{\varepsilon}' + Au_{\varepsilon}) - u_{\varepsilon}^{-} = \varepsilon f, \quad u_{\varepsilon}(x, t) = 0 \text{ on } \partial G \times [0, T], \quad u_{\varepsilon}(x, 0) = 0.$$

Moreover

$$\|u_{\varepsilon}\|_{L^{\infty}(0, T; L^{\infty}(G))} + \|u_{\varepsilon}\|_{L^2(0, T; W_0^{1,2}(G))} + \|u_{\varepsilon}'\|_{L^2(0, T; W^{1,2}(G))} \leq C.$$

C is a constant independent of ε .

PROOF OF THEOREM 1 USING THEOREM 3 : We shall make use of the of the following crucial estimate of Theorem 3 :

$$\|u_{\varepsilon}\|_{L^{\infty}(0, T; L^{\infty}(G))} \leq C.$$

1) Since $u_{\varepsilon} \in L^{\infty}(0, T; L^{\infty}(G)) \cap L^2(0, T; W_0^{1,2}(G))$, we have :

$$u_{\varepsilon}^{-} \in L^{\infty}(0, T; L^{\infty}(G)) \cap L^2(0, T; W_0^{1,2}(G)).$$

Thus : $|u_{\varepsilon}^{-}|^{2s-2} u_{\varepsilon}^{-} \in L^2(0, T; W_0^{1,2}(G))$ for any positive integer s . It follows from conditions (ii) and (iii) of Assumption (I) that :

$$\int_0^T (Au_{\varepsilon}^{-}, -|u_{\varepsilon}^{-}|^{2s-2} u_{\varepsilon}^{-}) dt = - \int_0^T (Au_{\varepsilon}^{-}, |u_{\varepsilon}^{-}|^{2s-2} u_{\varepsilon}^{-}) dt \geq 0.$$

Since $u_{\varepsilon}' \in L^2(0, T; H)$ and $u_{\varepsilon}^{-} \in L^{\infty}(0, T; L^{\infty}(G))$ with $u_{\varepsilon}(0) = 0$, we have :

$$2s \int_0^T (u_{\varepsilon}', -|u_{\varepsilon}^{-}|^{2s-2} u_{\varepsilon}^{-})_H dt = \int_0^T \int_{\substack{G \\ u_{\varepsilon} \geq 0}} \frac{d}{dt} (u_{\varepsilon}^{2s}) dx dt = \int_{\substack{G \\ u_{\varepsilon} \geq 0}} |u_{\varepsilon}(x, T)|^{2s} dx \geq 0.$$

Hence :

$$\varepsilon^{-1} \|u_{\varepsilon}^{-}\|_{L^{2s}(0, T; L^{2s}(G))}^{2s} \leq C \|f\|_{L^{\infty}(0, T; L^{\infty}(G))} \|u_{\varepsilon}^{-}\|_{L^{2s}(0, T; L^{2s}(G))}^{2s-1}.$$

Therefore :

$$\varepsilon^{-1} \|u_\varepsilon^-\|_{L^{2s}(0, T; L^{2s}(G))} \leq C \|f\|_{L^\infty(0, T; L^\infty(G))}.$$

C is a constant independent of s and of ε .

Since u_ε^- lies in $L^\infty(0, T; L^\infty(G))$, we may let $s \rightarrow +\infty$ and the above inequality gives :

$$\varepsilon^{-1} \|u_\varepsilon^-\|_{L^\infty(0, T; L^\infty(G))} \leq C \|f\|_{L^\infty(0, T; L^\infty(G))}.$$

2) From the weak compactness of the unit ball in a reflexive Banach space, we obtain by taking subsequences if necessary: $u_\varepsilon \rightarrow u$ weakly in $L^2(0, T; W_0^{1,2}(G))$, $u'_\varepsilon \rightarrow u'$ weakly in $L^2(0, T; W^{1,2}(G))$, $Au_\varepsilon \rightarrow h$ weakly in $L^2(0, T; W^{-1,2}(G))$, $\varepsilon^{-1} u_\varepsilon^- \rightarrow g$ in the weak*-topology of $L^\infty(0, T; L^\infty(G))$ and $u_\varepsilon^- \rightarrow 0$ in $L^2(0, T; H)$.

Thus: $u' + h + g = f$, $u(0) = 0$ and $u \in K$.

Condition (iii) of Assumption (I) implies that A is monotone. Moreover :

$$\int_0^T (Au_\varepsilon, u_\varepsilon - u) dt = \int_0^T (f + \varepsilon^{-1} u_\varepsilon^- - u'_\varepsilon, u_\varepsilon - u)_H dt.$$

Aubin's theorem [1] gives :

$$\limsup \int_0^T (Au_\varepsilon, u_\varepsilon - u) dt \leq 0.$$

By a standard argument of the theory of monotone operators, we get $h = Au$ and

$$\int_0^T (Au, u) dt \leq \liminf \int_0^T (Au_\varepsilon, u_\varepsilon) dt.$$

We have :

$$\int_0^T (u'_\varepsilon + Au_\varepsilon - f - \varepsilon^{-1} u_\varepsilon^-, v - u_\varepsilon) dt = 0.$$

Let v be an element of $L^2(0, T; W_0^{1,2}(G))$, v' in $L^2(0, T; W^{-1,2}(G))$, with $v(t)$ in K and $v(0) = 0$. Then $v^- = 0$. and we have :

$$\int_0^T (u'_\varepsilon + Au_\varepsilon - f, v - u_\varepsilon) dt \geq 0.$$

Let $\varepsilon \rightarrow 0$ and we get :

$$\int_0^T (u' + Au - f, v - u) dt \geq 0.$$

3) So : $u' + Au = f - g \in L^\infty(0, T; L^\infty(G))$, $u(x, t) = 0$ on $\partial G \times [0, T]$ and $u(x, 0) = 0$. It follows from Theorem 6.4 of Ladyzenskaya Solonnikov and Uralceva [5] (page 460), that $u \in C^k[0, T; C^{2k}(clG))$, $D_j u \in C^\gamma(0, T; C^{2\gamma}(G))$. $0 < 2\lambda, 2\gamma < 1$.

4) All the other assertions of Theorem 1 have been proved in [8],

PROOF OF THEOREM 2: From the proof of Theorem 1, we know that there exists a unique u in $L^2(0, T; W_0^{1,2}(G))$ with u' in $L^2(0, T; W^{-1,2}(G))$, $u(t)$ in K and $u(0) = 0$ such that :

$$\int_0^T (u' + Au - f, v - u) dt \geq 0$$

for all v in $L^2(0, T; W_0^{1,2}(G))$, v' in $L^2(0, T; W^{-1,2}(G))$, $v(t)$ in K and $v(0) = 0$.

Moreover u satisfies the equation : $u' + Au = f - g \in L^\infty(0, T; L^\infty(G))$. By a well-known result of the theory of linear parabolic equations of order 2 (Cf. e. g. Theorem 9.1 of [5], p. 341-342), we have :

$$u \in L^p(0, T; W^{2,p}(G)), u' \in L^p(0, T; L^p(G)) \text{ for any } p, 1 < p < \infty.$$

SECTION 2: The proof of Theorem 3 is long and will be carried out in this section. We shall give an outline of the proof before going into the details.

Consider the equation :

$$u_{h\varepsilon}^k - u_{\varepsilon h}^{k-1} + hAu_{h\varepsilon}^k - h\varepsilon^{-1}(u_{\varepsilon h}^k)^- = hf^k, u_{\varepsilon h}^0 = 0.$$

It is obtained by a discretisation of the time-variable of the equation of Theorem 3.

Let $A_2 v$ be the nonlinear elliptic operator :

$$A_2 v = \sum_{j=1}^n D_j (|D_j v|^{p-2} D_j v) \quad \text{with } p > n.$$

1) First, we shall consider the equation :

$$u_{\varepsilon h \eta}^k - u_{\varepsilon h \eta}^{k-1} + \eta A_2 u_{\varepsilon h \eta}^k + h A u_{\varepsilon h \eta}^k - \varepsilon^{-1} h (u_{\varepsilon h \eta}^k)^- = h f^k, u_{\varepsilon h \eta}^0 = 0; \eta > 0.$$

It has a unique solution $u_{\varepsilon h \eta}^k$ in $W_0^{1,p}(G)$ and since $p > n$, $u_{\varepsilon h \eta}^k$ is in $C(\text{cl}G)$.

2) $\|u_{\varepsilon h \eta}^k\|_{L^\infty(G)} \leq C$. C is a constant independent of ε, h, k and η .

Then let $\eta \rightarrow 0$.

3) The final step is standard.

Set : $Bv = -v^-$. Denote by $f^k = h^{-1} \int_{kh}^{(k+1)h} f(t) dt$ with $h > 0$.

LEMMA 1 : Let $h = T/N$ and suppose all the hypotheses of Theorem 3 are satisfied. Then for each $k, 1 \leq k \leq N$, there exists a unique solution $u_{\varepsilon h \eta}^k$ in $W_0^{1,p}(G)$ of the equation :

$$u_{\varepsilon h \eta}^k - u_{\varepsilon h \eta}^{k-1} + \eta h A_2 u_{\varepsilon h \eta}^k + h A u_{\varepsilon h \eta}^k + \varepsilon^{-1} h B u_{\varepsilon h \eta}^k = h f^k, u_{\varepsilon h \eta}^0 = 0.$$

Moreover :

$$\|u_{\varepsilon h \eta}^n\|_{L^\infty(G)} + \eta h \sum_{k=1}^n \|u_{\varepsilon h \eta}^k\|_{1,p}^p + h \sum_{k=1}^n \|u_{\varepsilon h \eta}^k\|_{1,2}^2 \leq C.$$

C is a constant independent of h, ε, η and n .

PROOF : It is clear that $u + \eta h A_2 u + h \varepsilon^{-1} B u$ is a monotone hemi-continuous, coercive operator mapping bounded sets of $W_0^{1,p}(G)$ into bounded sets of $W^{-1,q}(G)$. It follows from the theory of monotone operators that for each k , there exists a unique solution $u_{\varepsilon h \eta}^k$ of :

$$u_{\varepsilon h \eta}^k - u_{\varepsilon h \eta}^{k-1} + \eta h A_2 u_{\varepsilon h \eta}^k + h A u_{\varepsilon h \eta}^k + h \varepsilon^{-1} B u_{\varepsilon h \eta}^k = h f^k, u_{\varepsilon h \eta}^0 = 0, \quad k = 1, \dots, N.$$

So :

$$\frac{1}{2} \|u_{\varepsilon h \eta}^k\|_H^2 + \eta h \|u_{\varepsilon h \eta}^k\|_{1,p}^p + h \|u_{\varepsilon h \eta}^k\|_{1,2}^2 \leq Ch \|f^k\|_{L^\infty(G)} + \frac{1}{2} \|u_{\varepsilon h \eta}^{k-1}\|_H^2.$$

Taking the summation from $k = 1$ to n , we obtain :

$$\sum_{k=1}^n \eta h \|u_{\varepsilon h \eta}^k\|_{1,p}^p + h \|u_{\varepsilon h \eta}^k\|_{1,2}^2 \leq Ch \sum_{k=1}^n \|f^k\|_{L^\infty(G)} \leq C_2 T.$$

C_2 is a constant independent of ε, η, h and n .

2) We show the crucial estimate : $\|u_{\varepsilon h \eta}^n\|_{L^\infty(G)}$.

Since $p > n$, the Sobolev imbedding theorem gives : $W_0^{1,p}(G) \subset C(\text{cl}G)$. Thus $|u_{\varepsilon h \eta}^k|^{s-2} u_{\varepsilon h \eta}^k$ is in $W_0^{1,p}(G)$ for any positive integer $s \geq 2$. Therefore :

$$\begin{aligned} & \|u_{\varepsilon h \eta}^k\|_s^s + \eta h (A_2 u_{\varepsilon h \eta}^k, |u_{\varepsilon h \eta}^k|^{s-2} u_{\varepsilon h \eta}^k) + h (A u_{\varepsilon h \eta}^k, |u_{\varepsilon h \eta}^k|^{s-2} u_{\varepsilon h \eta}^k) \\ & + h \varepsilon^{-1} (B u_{\varepsilon h \eta}^k, |u_{\varepsilon h \eta}^k|^{s-2} u_{\varepsilon h \eta}^k) \leq Ch \|f^k\|_{L^\infty(G)} \|u_{\varepsilon h \eta}^k\|_s^{s-1} + \|u_{\varepsilon h \eta}^{k-1}\|_s \|u_{\varepsilon h \eta}^k\|_s^{s-1}. \end{aligned}$$

Consider the second term of the left hand side of the inequality :

$$(A_2 u_{\varepsilon h \eta}^k, |u_{\varepsilon h \eta}^k|^{s-2} u_{\varepsilon h \eta}^k) = \sum_j (s-1) \int_G |u_{\varepsilon h \eta}^k|^{s-2} |D_j u_{\varepsilon h \eta}^k|^p dx \geq 0.$$

On the other hand :

$$(A u_{\varepsilon h \eta}^k, |u_{\varepsilon h \eta}^k|^{s-2} u_{\varepsilon h \eta}^k) = \sum_{|\alpha| \leq 1} (s-1) \int_G A_\alpha(x, u_{\varepsilon h \eta}^k, D u_{\varepsilon h \eta}^k) |u_{\varepsilon h \eta}^k|^{s-2} D^\alpha(u_{\varepsilon h \eta}^k) dx$$

It follows from condition (iv) of Assumption (I) that the above expression is positive.

It is trivial to check that : $(B u_{\varepsilon h \eta}^k, |u_{\varepsilon h \eta}^k|^{s-2} u_{\varepsilon h \eta}^k) \geq 0$.

Therefore :

$$\|u_{\varepsilon h \eta}^k\|_s \leq Ch \|f^k\|_{L^\infty(G)} + \|u_{\varepsilon h \eta}^{k-1}\|_s.$$

Taking the summation from $k = 1$ to n , we obtain :

$$\|u_{\varepsilon h \eta}^n\|_s \leq Ch \sum_{k=1}^n \|f^k\|_{L^\infty(G)} \leq C_2 T.$$

We know that $u_{\varepsilon h \eta}^n$ is in $C(\text{cl}G)$, thus letting $s \rightarrow +\infty$, we have :

$$\|u_{\varepsilon h \eta}^n\|_{L^\infty(G)} \leq C_2 T.$$

The lemma is proved.

LEMMA 2: Let $h = T/N$ and suppose all the hypotheses of Theorem 3 are satisfied. Then for each k , $1 \leq k \leq N$, there exists a unique solution $u_{\varepsilon h}^k$ in $W_0^{1,2}(G)$ of:

$$u_{\varepsilon h}^k - u_{\varepsilon h}^{k-1} + hAu_{\varepsilon h}^k + h\varepsilon^{-1}Bu_{\varepsilon h}^k = hf^k, u_{\varepsilon h}^0 = 0.$$

Moreover:

$$\|u_{\varepsilon h}^n\|_{L^\infty(G)} + \sum_{k=1}^n h \|u_{\varepsilon h}^k\|_{1,2}^2 \leq C.$$

PROOF: From Lemma 1, we know that for each k , $k = 1, \dots, N$, there exists a unique solution $u_{\varepsilon h \eta}^k$ in $W_0^{1,p}(G)$ of the equation:

$$u_{\varepsilon h \eta}^k - u_{\varepsilon h \eta}^{k-1} + \eta h A_2 u_{\varepsilon h \eta}^k + hAu_{\varepsilon h \eta}^k + h\varepsilon^{-1}Bu_{\varepsilon h \eta}^k = hf^k, u_{\varepsilon h \eta}^0 = 0.$$

Moreover:

$$\|u_{\varepsilon h \eta}^n\|_{L^\infty(G)} + \sum_{k=1}^n \eta h \|u_{\varepsilon h \eta}^k\|_{1,p}^p + h \sum_{k=1}^n \|u_{\varepsilon h \eta}^k\|_{1,2}^2 \leq C.$$

C is a constant independent of ε, η, h and n .

Let $\eta \rightarrow 0$. The weak compactness of the unit ball in a reflexive Banach space gives: $h^{\frac{1}{2}}u_{\varepsilon h \eta}^k \rightarrow h^{\frac{1}{2}}u_{\varepsilon h}^k$ weakly in $W_0^{1,2}(G)$, $(h\eta)^{1/p}u_{\varepsilon h \eta}^k \rightarrow 0$ weakly in $W_0^{1,p}(G)$, $Au_{\varepsilon h \eta}^k \rightarrow g_{\varepsilon h}^k$ weakly in $W^{-1,2}(G)$ and $u_{\varepsilon h \eta}^k \rightarrow u_{\varepsilon h}^k$ in the weak*-topology of $L^\infty(0, T; L^\infty(G))$.

It follows from the Sobolev imbedding theorem that $u_{\varepsilon h \eta}^k \rightarrow u_{\varepsilon h}^k$ in $L^2(G)$ and thus $Bu_{\varepsilon h \eta}^k \rightarrow Bu_{\varepsilon h}^k$ weakly in $L^2(G)$.

We obtain:

$$u_{\varepsilon h}^k - u_{\varepsilon h}^{k-1} + h g_{\varepsilon h}^k + h\varepsilon^{-1}Bu_{\varepsilon h}^k = hf^k, u_{\varepsilon h}^0 = 0.$$

Since A is monotone, it is easy to show that $g_{\varepsilon h}^k = Au_{\varepsilon h}^k$.

All the other assertions of the lemma follow trivially from the above arguments.

PROOF OF THEOREM 3: Let $u_{\varepsilon h}^k$, $k = 1, \dots, N$, be the solution of:

$$u_{\varepsilon h}^k - u_{\varepsilon h}^{k-1} + hAu_{\varepsilon h}^k + h\varepsilon^{-1}Bu_{\varepsilon h}^k = hf^k \text{ with } u_{\varepsilon h}^0 = 0.$$

1) Set: $u_{\varepsilon h}(t) = u_{\varepsilon h}^k$ when $kh \leq t < (k+1)h$, $k = 0, \dots, N-1$ and $h = T/N$. Then from Lemma 2, we obtain:

$$\|u_{\varepsilon h}\|_{L^\infty(0, T; L^\infty(G))} + \|u_{\varepsilon h}\|_{L^2(0, T; W_0^{1,2}(G))} \leq C$$

C is a constant independent of both ε and h .

It is easy to show that:

$$\sum_{k=1}^n \|h^{-1}(u_{\varepsilon h}^k - u_{\varepsilon h}^{k-1})\|_{W^{-1,2}(G)}^2 \leq M(\varepsilon).$$

$M(\varepsilon)$ is independent of h and n .

2) From the weak compactness of the unit ball in a reflexive Banach space, we get by taking subsequences if necessary: $u_{\varepsilon k} \rightarrow u_\varepsilon$ weakly in $L^2(0, T; W_0^{1,2}(G))$, $u_{\varepsilon h} \rightarrow u_\varepsilon$ in the weak*-topology of $L^\infty(0, T; L^\infty(G))$, $Au_{\varepsilon h} \rightarrow g_\varepsilon$ weakly in $L^2(0, T; W^{-1,2}(G))$ and $h^{-1}(u_{\varepsilon h}(t) - u_{\varepsilon h}(t-h)) \rightarrow u_\varepsilon'$ weakly in $L^2(0, T; W^{-1,2}(G))$.

Since the injection mapping of $W^{1,2}(G)$ into $L^2(G)$ is compact, the discrete analogue of Aubin's theorem [1] gives: $u_{\varepsilon h} \rightarrow u_\varepsilon$ in $L^2(0, T; H)$. Hence: $Bu_{\varepsilon h} \rightarrow Bu_\varepsilon$ weakly in $L^2(0, T; H)$.

Thus:

$$u_\varepsilon' + g_\varepsilon = \varepsilon^{-1} Bu_\varepsilon = f.$$

3) We show that $u_\varepsilon(0) = 0$.

Let $v \in W_0^{1,2}(G)$ and $\varphi \in C([0, T])$. Set: $\varphi_h(t) = \varphi(nh)$ with $nh \leq t < (n+1)h$. Then:

$$h^{-1}(u_{\varepsilon h}^n - u_{\varepsilon h}^{n-1}, v) \varphi_h(t) + (Au_{\varepsilon h}^n, v) \varphi_h(t) + \varepsilon^{-1}(Bu_{\varepsilon h}^n, v) \varphi_h(t) = (f^n, v) \varphi_h(t)$$

Let $h \rightarrow 0$ and we get:

$$\int_0^T (u_\varepsilon' + g_\varepsilon + \varepsilon^{-1} Bu_\varepsilon - f, v) \varphi dt = 0$$

for all v in $W_0^{1,2}(G)$ and all φ in $C([0, T])$.

A standard argument gives:

$$\int_0^T (u_\varepsilon' + g_\varepsilon + \varepsilon^{-1} Bu_\varepsilon - f, \varphi) dt = 0$$

for all φ in $L^2(0, T; W_0^{1,2}(G))$.

On the other hand :

$$\sum_{n=1}^N - (u_{\varepsilon h}, v)_H (\varphi(nh) - \varphi(nh - h)) + h (Au_{\varepsilon h}, v) \varphi(nh - h) + h\varepsilon^{-1} (Bu_{\varepsilon h}, v) \varphi(nh - h) \\ - h (f^n, v) \varphi(nh - h) = - (u_{\varepsilon h}(T), v)_H \varphi(T).$$

Take $\varphi \in C([0, T])$ with $\varphi(T) = 0$ and let $h \rightarrow 0$. We obtain :

$$- \int_0^T (u_\varepsilon, v) \varphi' dt + \int_0^T (g_\varepsilon + \varepsilon^{-1} Bu_\varepsilon - f, v) \varphi dt = 0.$$

So :

$$- \int_0^T (u_\varepsilon, \varphi') dt + \int_0^T (g_\varepsilon + \varepsilon^{-1} Bu_\varepsilon - f, \varphi) dt = 0$$

for all φ in $L^2(0, T; W_0^{1,2}(G))$ with φ' in $L^2(0, T; W^{-1,2}(G))$ and $\varphi(T) = 0$. Therefore: $(u_\varepsilon(0), \varphi(0))_H = 0$ for all φ in $L^2(0, T; W^{1,2}(G))$ with φ' in $L^2(0, T; W^{-1,2}(G))$ and $\varphi(T) = 0$.

Since the set $\{\varphi(0) : \varphi \text{ in } L^2(0, T; W_0^{1,2}(G)), \varphi' \text{ in } L^2(0, T; W^{-1,2}(G)) \text{ and } \varphi(T) = 0\}$ is dense in H , we have: $u_\varepsilon(0) = 0$.

4) We show that $g_\varepsilon = Au_\varepsilon$.

An elementary computation gives :

$$\frac{1}{2} \|u_{\varepsilon h}^N(T)\|_H^2 + \int_0^T (Au_{\varepsilon h} + \varepsilon^{-1} Bu_{\varepsilon h} - f, u_{\varepsilon h}) dt \leq 0.$$

So :

$$\frac{1}{2} \|u_{\varepsilon h}(T)\|_H^2 + \limsup \int_0^T (Au_{\varepsilon h}, u_{\varepsilon h}) dt \leq \int_0^T (f - \varepsilon^{-1} Bu_\varepsilon, u_\varepsilon) dt.$$

On the other hand: $u'_\varepsilon + g_\varepsilon + \varepsilon^{-1} Bu_\varepsilon = f$.

Thus :

$$\frac{1}{2} \|u_\varepsilon(T)\|_H^2 + \limsup \int_0^T (Au_{\varepsilon h}, u_{\varepsilon h}) dt \leq \int_0^T (u'_\varepsilon + g_\varepsilon, u_\varepsilon) dt \\ \leq \frac{1}{2} \|u_\varepsilon(T)\|_H^2 + \int_0^T (g_\varepsilon, u_\varepsilon) dt.$$

Hence :

$$\limsup \int_0^T (Au_{\varepsilon h}, u_{\varepsilon h}) dt \leq \int_0^T (g_\varepsilon, u_\varepsilon) dt.$$

Since A is monotone, the above inequality implies that $g_\varepsilon = Au_\varepsilon$. It is clear that the solution is unique.

5) It remains to show that $\|u'_\varepsilon\|_{L^2(0,T;W^{1,2}(G))} \leq C$.
 C is a constant independent of ε .

The proof has been carried out in [8]. To show it, we note that u_ε is the restriction to $[0, T]$ of v_ε where v_ε is the unique solution of a global boundary-value problem :

$$v'_\varepsilon + Av_\varepsilon + \varepsilon^{-1}Bv_\varepsilon = \widehat{f} \quad \text{on } E \times G, \quad v_\varepsilon = 0 \quad \text{on } E \times \partial G.$$

$\widehat{f} = \zeta(t)f$ where $\zeta \in C_0^1(E)$, $\zeta(t) = 1$ for t in $[0, T]$, $\zeta(t) = 0$ for $t \leq -1$ and $t \geq 2T$. $f(t)$ is extended to E with $f(t) = 0$ for $t \leq 0$ and $f(t) = f(T)$ for $t \geq T$.

The method of difference quotients applied to v_ε gives the desired estimate.
 Since :

$$Au_\varepsilon + \varepsilon^{-1}Bu_\varepsilon = f - u'_\varepsilon \quad \text{is now in } L^2(0, T; L^2(G)),$$

by using again the method of difference quotients and some standard results of the theory of elliptic operators, it is not difficult to show that :

$$\|u_\varepsilon\|_{L^2(0,T;W^{2,2}(G))} \leq C.$$

C is independent of ε . Cf. [8].

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Added in proof:

H. BREZIS : *Problèmes unilatéraux*. J. Math. Pures et Appl. 51 (1972), 1-68.

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