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Introduction.

Classically, the multiplicative Cousin problem gives as data an open covering $\mathcal{V} = \{V_\alpha\}_{\alpha \in I}$ of a complex manifold M and a family $\{f_\alpha\}_{\alpha \in I}$ of functions, f_α holomorphic on V_α, such that $f_\alpha f_\beta^{-1}$ is holomorphic and zero-free on $V_\alpha \cap V_\beta$, and it asks whether there is a function F holomorphic on M such that for every α, $F f_\alpha^{-1}$ is holomorphic and zero-free on V_α. In an earlier paper [9], we showed that it is possible to solve the multiplicative Cousin problem, on the unit polydisc in $\mathbb{C}^N$, under the supplementary restrictions that the functions $f_\alpha, f_\alpha f_\beta^{-1}$ and F be bounded, provided the covering $\mathcal{V}$ is required to satisfy an additional geometric condition.

Once we know the result of [9], we are led naturally to inquire into the solvability of multiplicative Cousin problems, with bounded data, on other domains. The case of the unit ball comes to mind immediately. The methods of [9], the one dimensional Cauchy integral and a pinch of harmonic analysis, shed no light whatsoever on the ball case. In this note we shall show that the multiplicative Cousin problem is solvable on the ball in $\mathbb{C}^N$ and, indeed, on a somewhat more general class of domains. We denote by $\mathfrak{S}_N$ the class of bounded domains X in $\mathbb{C}^N$ with the property that there exists a real valued function ϱ of class $\mathcal{C}^2$ defined on a neighborhood W of ∂X which has the properties that $d\varrho \neq 0$ on ∂X, $X \cap W = \{\mathfrak{z} \in W : \varrho(\mathfrak{z}) < 1\}$, and H_ϱ^r, the real Hessian of ϱ, is positive definite on W. If we take coordinates $z_1, \dots, z_N$ on $\mathbb{C}^N$ with $z_j = x_j + i x_{N+j}$, $j = 1, \dots, N$, then given $t = (t_1, \dots, t_{2N})$, $s = (s_1, \dots, s_{2N}) \in \mathbb{R}^{2N}$, and given $\mathfrak{z}_0 \in W$,

$$H_\varrho^r(\mathfrak{z}_0)(t, s) = \sum_{j,k=1}^{2N} \frac{\partial^2 \varrho}{\partial x_j \partial x_k}(\mathfrak{z}_0)\, t_j s_k .$$

With this notation in mind, we may state our result:

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THEOREM. *The multiplicative Cousin problem with bounded data is solvable on every domain of class* $\mathfrak{S}_N$. *Explicitly, if* $X \in \mathfrak{S}_N$, *if* $\mathcal{V} = \{V_\alpha\}_{\alpha \in I}$ *is an open cover for* $\bar{X}$, *if for each* α, f_α *is a function holomorphic and bounded on* $V_\alpha \cap X$, *and if for all* $\alpha, \beta \in I$, $f_\alpha f_\beta^{-1}$ *is bounded on* $V_\alpha \cap V_\beta \cap X$, *then there is a bounded holomorphic function* F *on* X *such that for all* α, Ff_α^{-1} *is bounded and bounded away from zero on* $V_\alpha \cap X$.

The theorem admits a cohomological formulation. Define a sheaf $\mathcal{H}$ on $\bar{X}$ by requiring that if V is an open subset of $\bar{X}$, then

$$\mathcal{H}(V) = \{f \in \mathcal{O}(V \cap X): \text{ if } K \subset V \text{ is compact, then } f \text{ is bounded on } K \cap X\} \text{ [1]}.$$

Thus, $\mathcal{H}$ is the sheaf of germs of locally bounded holomorphic functions Its global sections are the bounded holomorphic functions on X, but for an arbitrary open set $V \subset \bar{X}$, the elements of $\mathcal{H}(V)$ need not be bounded on $V \cap X$. Note that the stalks $\mathcal{H}_\mathfrak{z}$, $\mathfrak{z} \in \partial X$, are neither local rings nor Noetherian. Also, $\mathcal{H}$ is not a subsheaf of the sheaf of germs of continuous complex valued functions on X (or $\bar{X}$). Denote by $\mathcal{E}$ the sheaf of multiplicative groups of invertible elements of $\mathcal{H}$: If V is open in $\bar{X}$,

$$\mathcal{E}(V) = \{f \in \mathcal{H}(V): f^{-1} \in \mathcal{H}(V)\}.$$

The theorem simply asserts the triviality of the cohomology group $H^1(\bar{X}, \mathcal{V}, \mathcal{E})$ for every open cover $\mathcal{V}$ of $\bar{X}$.

In this context, habit suggests that we consider the sequence of sheaves

$$0 \to \mathbb{Z} \xrightarrow{j} \mathcal{H} \xrightarrow{\exp} \mathcal{E} \to 0 \tag{1}$$

where $\mathbb{Z}$ is the constant sheaf of integers, j is the inclusion, and exp is the map $f \mapsto e^{2\pi i f}$. This kind of sequence is used traditionally to reduce the multiplicative Cousin problem to the union of an additive problem and a topological problem. This procedure is not applicable in the present situation, for the sequence 1) is not exact. Of course it is exact over every point of X, but it is not exact over boundary points. If $\mathfrak{z} \in \partial X$, not every element of $\mathcal{E}_\mathfrak{z}$ is of the form $e^{2\pi i f}$, as is seen already in one dimension. If u is a bounded harmonic function in the open unit disc with unbounded harmonic conjugate v, then $e^{2\pi i(v - iu)}$ is bounded and so is its reciprocal, but it is not of the form e^h for any bounded h.

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[1] As usual, $\mathcal{O}(S)$ denotes the space of holomorphic functions on the complex space S.

There is an alternative cohomological formulation of the result. Define the sheaf, $\mathcal{P}_b$, of locally bounded, pluriharmonic functions on $\bar{X}$ by requiring that if $V \subset \bar{X}$ is open, then

$$\mathcal{P}_b(V) = \{u : u \text{ is real and pluriharmonic on } V \cap X \text{ and } u \text{ is bounded on } K \cap X \text{ for every compact set } K \subset V\}$$

$$= \{u \in \mathcal{C}_{\mathbb{R}}(V \cap X) : e^{u+iv} \in \mathcal{H}(V) \text{ for some real } v\}.$$

Our theorem is equivalent to the vanishing of $H^1(\bar{X}, \mathcal{V}, \mathcal{P}_b)$ for every open cover $\mathcal{V}$.

The principal ingredient of our proof is the Cauchy-Fantappiè integral as developed by Leray [7] together with some H^p-space theory. At several points our discussion is strongly influenced by the work of Henkin [6].

I am indebted to my colleague Gunter Lumer for some very helpful discussions of H^p-theory and for the opportunity of seeing his *Comptes Rendus* note [8] in preprint.

I. THE GEOMETRY OF $\mathfrak{S}_N$ DOMAINS. We shall need certain elementary facts about the geometry of domains belonging to the class $\mathfrak{S}_N$. The first of these observations is the fact that if $\Omega \in \mathfrak{S}_N$, then Ω is convex. Suppose, toward this end, that Ω is defined by the function ϱ as in the definition of the class $\mathfrak{S}_N$. If $\mathfrak{z}_0 \in \partial\Omega$, and if B is a ball around $\mathfrak{z}_0$, then ϱ is a convex function on B provided B is small enough, for the Hessian of ϱ is supposed to be positive definite near $\partial\Omega$. (For all the information we require from the theory of convex functions, see [5].) It follows that $B \cap \Omega = \{\mathfrak{z} \in B : \varrho(\mathfrak{z}) < 0\}$ is a convex set. A set which is convex is a neighborhood of every boundary point is necessarily convex, so Ω is a convex set as we asserted. In fact, we see that $\bar{\Omega}$ is strictly convex.

Secondly, we shall need to know that an $\Omega \in \mathfrak{S}_N$ can be defined by a globally defined function. Consider therefore an $\Omega \in \mathfrak{S}_N$ which is defined by the $\mathcal{C}^2$ function ϱ with domain W, a neighborhood of $\partial\Omega$. We have just seen that Ω is convex. Let us suppose, in order to simplify notation, that $0 \in \Omega$. Let Q be the *Minkowski functional* associated with Ω so that Q is the nonnegative functional on $\mathbb{C}^N$ defined by

$$Q(\mathfrak{z}) = \lambda \text{ if and only if } \mathfrak{z} \in \lambda\partial\Omega.$$

The boundary of Ω is a $\mathcal{C}^2$-manifold, so it follows that Q is a $\mathcal{C}^2$ function on $\mathbb{C}^N \setminus \{0\}$. That this is so is an exercise in calculus. We will take up the details in a brief appendix at the end of this paper. The functional Q

is positive homogeneous, *i.e.*, $Q(t\mathfrak{z}) = tQ(\mathfrak{z})$ if $t \geq 0$, so since $Q(\mathfrak{z}) = 0$ only if $\mathfrak{z} = 0$, we find that dQ is zero at no point of $\mathbb{C}^N \setminus \{0\}$. Consequently, there is a positive function h such that

$$Q - 1 = h\varrho \tag{2}$$

on a neighborhood of $\partial\Omega$. (That h is positive follows from the fact that $\Omega = \{\mathfrak{z} : Q(\mathfrak{z}) < 1\}$.) Consequently, $dQ(\mathfrak{z}) = h(\mathfrak{z})\, d\varrho(\mathfrak{z})$ if $\mathfrak{z} \in \partial\Omega$, and from this we find that for every $\mathfrak{z} \in \partial\Omega$, the linear transformations $dQ(\mathfrak{z})$ and $d\varrho(\mathfrak{z})$ have the same null spaces. It follows from 2) that if $t = (t_1, \dots, t_{2N}) \in \mathbb{R}^{2N}$, then

$$\sum_{m,n=1}^{2N} \frac{\partial^2 Q}{\partial x_m\, \partial x_n}(\mathfrak{z})\, t_m t_n = \varrho(\mathfrak{z}) \sum_{m,n=1}^{2N} \frac{\partial^2 h}{\partial x_m\, \partial x_n}(\mathfrak{z})\, t_m t_n$$

$$+ 2\left(\sum_{m=1}^{2N} \frac{\partial \varrho}{\partial x_m}(\mathfrak{z})\, t_m\right)\left(\sum_{m=1}^{2N} \frac{\partial h}{\partial x_m}(\mathfrak{z})\, t_m\right)$$

$$+ h(\mathfrak{z}) \sum_{m,n=1}^{2N} \frac{\partial^2 \varrho}{\partial x_m\, \partial x_n}(\mathfrak{z})\, t_m t_n .$$

The fact that ϱ vanishes on $\partial\Omega$, the equality of the null spaces of $dQ(\mathfrak{z})$ and $d\varrho(\mathfrak{z})$ for $\mathfrak{z} \in \partial\Omega$, and the fact that the Hessian of ϱ is positive definite all taken together show that if

$$\sum_{m=1}^{2N} \frac{\partial Q}{\partial x_m}(\mathfrak{z})\, t_m = 0, \quad \mathfrak{z} \in \partial\Omega, \qquad \text{then} \qquad \sum_{m,n=1}^{2N} \frac{\partial^2 Q}{\partial x_m\, \partial x_n}(\mathfrak{z})\, t_m t_n > 0$$

provided not all the t_m's are zero. It follows that $H_Q^r(\mathfrak{z})$, the real Hessian of Q at $\mathfrak{z}$, satisfies $H_Q^r(\mathfrak{z})(t, t) > 0$ for all $t \in \mathbb{R}^{2N} \setminus \{0\}$ such that $dQ(\mathfrak{z})(t) \neq 0$, no matter what $\mathfrak{z} \in \mathbb{C}^N \setminus \{0\}$ we consider, for Q is positive homogeneous. It $t \in \mathbb{R}^{2N}$, we have

$$\sum_{m,n=1}^{2N} \frac{\partial^2 (Qe^Q - 1)}{\partial x_m\, \partial x_n}\, t_m t_n =$$

$$= e^Q(1 + Q)\left\{\sum_{m,n=1}^{2N} \frac{\partial^2 Q}{\partial x_m\, \partial x_n}\, t_m t_n + \left(\sum_{m=1}^{2N} \frac{\partial Q}{\partial x_m}\, t_m\right)^2\right\} + e^Q\left(\sum_{m=1}^{2N} \frac{\partial Q}{\partial x_m}\, t_m\right)^2,$$

so the Hessian H_ψ^r of $\psi = Qe^Q - 1$ is positive definite on $\mathbb{C}^N \setminus \{0\}$. The function ψ defines Ω in the sense of the definition of the class $\mathfrak{S}_N$ and is of class $\mathcal{C}^2$ on $\mathbb{C}^N \setminus \{0\}$.

II. The Cauchy-Fantappiè integral. The basic tool for what we do is an integral formula valid in convex domains. We recall in the present section the form of this integral; its derivation may be found in Leray's paper [7] (2).

The space Ξ^* of all complex hyperplanes (of codimension one) in $\mathbb{C}^N$ may be identified in a natural way with the N-dimensional projective space $\mathbb{P}_N(\mathbb{C})$. This correspondence is established by associating to a point $\mathfrak{w} \in \mathbb{P}_N(\mathbb{C})$ with projective coordinates $(\xi_0 : \xi_1 : \dots : \xi_N)$ the nullspace of the affine form $\xi_0 + \xi_1 z_1 + \dots + \xi_N z_N$. Denote by $\cdot$ the pairing of $\mathbb{C}^{N+1}$ and $\mathbb{C}^N$ given by $\mathfrak{z} \cdot \mathfrak{w} = z_0 + z_1 w_1 + \dots + z_N w_N$ for all $\mathfrak{z} = (z_0, \dots, z_N) \in \mathbb{C}^{N+1}$ and all $\mathfrak{w} = (w_1, \dots, w_N) \in \mathbb{C}^N$. For a fixed $\mathfrak{w} \in \mathbb{C}^N$, the map $\mathfrak{z} \mapsto \mathfrak{z} \cdot \mathfrak{w}$ is a linear functional on $\mathbb{C}^{N+1}$; in particular, it is homogeneous. If $\xi^* \in \Xi^*$ and $\mathfrak{w} \in \mathbb{C}^N$, the condition $\xi^* \cdot \mathfrak{w} = 0$ (or $\xi^* \cdot \mathfrak{w} \neq 0$) admits a well defined meaning: If ξ^* has projective coordinates $(\xi_0 : \xi_1 : \dots : \xi_N)$, then the vanishing of the number $(\xi_0, \xi_1, \dots, \xi_N) \cdot \mathfrak{w}$ is independent of the particular choice of projective coordinates.

If we define forms

$$\omega(z) = dz_1 \wedge \dots \wedge dz_N$$

and

$$\omega'(\xi) = \sum_{k=1}^{N} (-1)^k \xi_K \, d\xi_1 \wedge \dots \wedge \widehat{d\xi_K} \wedge \dots \wedge d\xi_N$$

then

$$\Omega = \frac{\omega(z) \wedge \omega'(\xi)}{(\xi \cdot y)^N}$$

is a well defined holomorphic $(2N-1)$ form on $(\mathbb{C}^N \times \Xi) \setminus P$ where P denotes the variety

$$\{(\mathfrak{z}, \xi^*) \in \mathbb{C}^N \times \Xi^* : \xi^* \cdot \mathfrak{z} = 0\}.$$

Assume given a relatively compact open convex set Y in $\mathbb{C}^N$. Let $\mathfrak{y}$ be a point Y, and let $\xi^* : \partial X \to \Xi$ be a continuous map which satisfies

$$\xi^*(\mathfrak{x}) \cdot \mathfrak{x} = 0 \quad \text{and} \quad \xi^*(\mathfrak{x}) \cdot \mathfrak{y} \neq 0$$

for all $\mathfrak{x} \in \partial X$. The map $\mathfrak{x} \mapsto (\mathfrak{x}, \xi^*(\mathfrak{x}))$, call it Φ, is a continuous map from ∂X into $(\mathbb{C}^N \times \Xi^*) \setminus Q_{\mathfrak{y}}$, $Q_{\mathfrak{y}} = \{(\mathfrak{z}, \eta^*) \in \mathbb{C}^N \times \Xi^* : \eta^* \cdot \mathfrak{y} = 0\}$, and as ∂X is, topologically, a sphere, it represents a certain $(2N-1)$-dimensional homology class, h, in $\mathbb{C}^N \times \Xi^* \setminus Q_{\mathfrak{y}}$.

If f is a function holomorphic on a neighborhood of Y, then $f(\mathfrak{z})\,\Omega$ is

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(2) Added in proof. Another derivation of the integral formula for convex domains has been given by L. A. Aĭzenberg, Integral representations of functions which are holomorphic in convex regions of $\mathbb{C}^n$ space, Soviet Math. Dokl. 4 (1963), 1149.

a well defined $(2N-1)$-form, and the Cauchy-Fantappiè formula asserts that

(3) $$f(\mathfrak{y}) = \frac{(N-1)!}{(2\pi i)^N} \int_h f(\mathfrak{z})\, \Omega.$$

We are interested in a somewhat more special situation. We impose on Y the hypothesis that its boundary be manifold of class $\mathcal{C}^2$ so that for some real valued function ϱ defined and twice continuously differentiable on $\mathbb{C}^N$, $\partial Y = \{\mathfrak{z} \in \mathbb{C}^N : \varrho(\mathfrak{z}) = 0\}$ and $Y = \{\mathfrak{z} \in \mathbb{C}^N : \varrho(\mathfrak{z}) > 0\}$. Moreover, $d\varrho$ is to vanish at no point $\mathfrak{z} \in \partial Y$. Thus ∂Y has a well defined (real) tangent plane, $T_\mathfrak{z}$, at each point $\mathfrak{z} \in \partial Y$. At a point $\mathfrak{z}_0 = (x_1^{(0)} + i x_{N+1}^{(0)}, \dots, x_N^{(0)} + i x_{2N}^{(0)}) \in \partial Y$, the equation of $T_{\mathfrak{z}_0}$ is

$$\sum_{j=1}^{2N} \frac{\partial \varrho}{\partial x_j}(\mathfrak{z}_0)\,(x_j - x_j^0) = 0.$$

We define a map $\xi^* : \partial Y \to \Xi^*$ such that for every $\mathfrak{z} \in \partial X$, $\xi^*(\mathfrak{z})$ is tangent to ∂X, *i.e.*, $\xi^*(\mathfrak{z}) \subset T_{\mathfrak{z}_0}$. For this purpose, let $\langle\, , \rangle$ denote the customary inner product on $\mathbb{C}^N$:

$$\langle \mathfrak{z}, \mathfrak{w} \rangle = \sum_{j=1}^{N} z_j \bar{w}_j,$$

and let $\bar{\nabla} \varrho(\mathfrak{z})$ be the vector

$$\left(\frac{1}{2}\left(\frac{\partial \varrho}{\partial x_1}(\mathfrak{z}) + i \frac{\partial \varrho}{\partial x_{N+1}}(\mathfrak{z})\right), \dots, \frac{1}{2}\left(\frac{\partial \varrho}{\partial x_N}(\mathfrak{z}) + i \frac{\partial \varrho}{\partial x_{2N}}(\mathfrak{z})\right)\right)$$
$$= \left(\frac{\partial \varrho}{\partial \bar{z}_1}(\mathfrak{z}), \dots, \frac{\partial \varrho}{\partial \bar{z}_N}(\mathfrak{z})\right).$$

We define $\xi^*(\mathfrak{z})$, $\mathfrak{z} \in \partial Y$, to be the $(N-1)$ dimensional plane with equation

(4) $$\langle \mathfrak{w} - \mathfrak{z}, \bar{\nabla} \varrho(\mathfrak{z}) \rangle = 0.$$

A direct calculation shows that $\xi^*(\mathfrak{z})$ is contained in the real tangent plane $T_\mathfrak{z}$; it is the maximal complex hyperplane contained in $T_\mathfrak{z}$. The definition of ξ^* shows that the projective coordinates of $\xi^*(\mathfrak{z})$ are

$$\left(-\langle \mathfrak{z}, \bar{\nabla} \varrho(\mathfrak{z}) \rangle : \overline{\frac{\partial \varrho}{\partial \bar{z}_1}}(\mathfrak{z}) : \dots : \overline{\frac{\partial \varrho}{\partial \bar{z}_N}}(\mathfrak{z})\right).$$

$\left(\text{Note that } \overline{\frac{\partial \varrho}{\partial \bar{z}_j}} = \frac{\partial \varrho}{\partial z_j}\right)$. Also, we have that $\xi^*(\mathfrak{z}) \cdot \mathfrak{z} = 0$ for all $\mathfrak{z} \in \partial Y$, and $\xi^*(\mathfrak{z}) \cdot \mathfrak{w} = 0$ for no $\mathfrak{w} \in Y$. The second of these is clear, for $\xi^*(\mathfrak{z})$ lies in $T(\mathfrak{z})$, and the convexity of Y shows that Y and $T_\mathfrak{z}$ are disjoint.

It follows that in our case, the Cauchy-Fantappiè integral assumes the form

$$(5) \qquad f(\mathfrak{w}) = \frac{(N-1)!}{(2\pi i)^N} \int_{\partial Y} f(\mathfrak{z}) \cdot$$

$$\frac{dz_1 \wedge \ldots \wedge dz_N \left\{ \Sigma_{k=1}^N (-1)^k \frac{\partial \varrho}{\partial z_k} d\xi_1(\mathfrak{z}) \wedge \ldots \wedge \widehat{d\xi_k(\xi)} \wedge \ldots \wedge d\xi_N(\mathfrak{z}) \right\}}{\langle \mathfrak{w} - \mathfrak{z}, \overline{\nabla}\varrho(\mathfrak{z}) \rangle^N}.$$

Let us denote the numerator of (5) by Ψ. Recalling that $d = \partial + \bar{\partial}$ and that $\xi_k(\mathfrak{z}) = \frac{\partial \varrho}{\partial z_k}(\mathfrak{z}) = \varrho_{z_k}(\mathfrak{z})$, we have

$$d\xi_k = \sum_{m=1}^N \varrho_{z_k z_m}(\mathfrak{z})\, dz_m + \sum_{m=1}^N \varrho_{z_k \bar{z}_m}(\mathfrak{z})\, d\bar{z}_m .$$

It follows that

$$\Psi = \sum_{k=1}^N \eta_k(\mathfrak{z})\, dz_1 \wedge \ldots \wedge dz_N \wedge d\bar{z}_1 \wedge \ldots \wedge \widehat{d\bar{z}_k} \wedge \ldots \wedge d\bar{z}_N ,$$

where the coefficients η_k are combinations of the first and seccnd order derivatives of ϱ. They are, therefore, continuous. If we write $z_k = x_k + i x_{k+N}$ then

$$dz_1 \wedge \ldots \wedge dz_N \wedge d\bar{z}_1 \wedge \ldots \wedge \widehat{d\bar{z}_k} \wedge \ldots \wedge d\bar{z}_N$$

is except for a constant factor,

$$dx_1 \wedge dx_{n+1} \wedge \ldots \wedge (dx_k \wedge d_{N+k})^{\wedge} \wedge \ldots \wedge (dx_N \wedge dx_{2N}) \wedge (dx_k + i dx_{N+k}).$$

If we impose the additional hypothesis that the set Y be strictly convex, then each of the forms

$$dx_1 \wedge dx_{N+1} \wedge \ldots \wedge (dx_k \wedge dx_{N+k})^{\wedge} \wedge \ldots \wedge (dx_N \wedge dx_{2N}) \wedge dx_k$$

and

$$dx_1 \wedge dx_{N+1} \wedge \ldots \wedge (dx_k \wedge dx_{N+k})^{\wedge} \wedge \ldots \wedge (dx_N \wedge dx_{2N}) \wedge dx_{N+k}$$

is of the form $\chi\, d\sigma$, $d\sigma$ the element of surface area on ∂Y, χ a continuous function. Combining this observation with the preceding paragraph we obtain an integral representation

$$(6) \qquad f(\mathfrak{w}) = \frac{(N-1)!}{(2\pi i)^N} \int_{\partial Y} f(\mathfrak{z}) \frac{k(\mathfrak{z})}{\langle \mathfrak{w} - \mathfrak{z}, \overline{\nabla}\varrho(\mathfrak{z}) \rangle^N}\, d\sigma(\mathfrak{z}),$$

k a continuous function.

The formula (6) is valid, so far, under the hypothesis that f be holomorphic on a neighborhood of $\bar{Y}$. We need it for functions which satisfy less stringent boundary conditions. Because our geometric configuration is so simple, it extends immediately to functions $f \in \mathcal{A}(Y) = \{g \in \mathcal{C}(\bar{Y}) :\ g$ is holomorphic in $Y\}$, for every $f \in \mathcal{A}(Y)$ is uniformly approximable on $\bar{Y}$ by functions holomorphic on a neighborhood of Y: Fix $\mathfrak{w}_0 \in Y$, and if $f \in \mathcal{A}(Y)$, define, for $r \in (0, 1)$, $f_r(\mathfrak{z})$ by

$$f_r(\mathfrak{z}) = f(\mathfrak{w}_0 + r(\mathfrak{z} - \mathfrak{w}_0)).$$

For each r, $f_r \in \mathcal{O}(\bar{Y})$, *i.e.*, f is holomorphic on a neighborhood of Y, and $f_r \to f$ uniformly as $r \to 1$. It follows that (6) holds for functions in $\mathcal{A}(Y)$.

To extend further the class of functions for which the formula (6) holds, we need a result of the Fatou type, a result affirming the existence of nontangential limits for certain classes of functions. The smoothness condition we have imposed suffices for us to conclude, by way of results of Aronszjan and Smith [1, Section 10] that if $f \in \mathcal{O}(Y)$ and $|f|^p$ admits a harmonic majorant, $p \geq 1$, then f has nontangential limits at almost every point of ∂Y, *almost every* understood in the sense of the measure σ, and, moreover, these limits belong to $L^p(d\sigma)$. This holds in particular, for all functions f for which $|f|^p$ admits a *pluriharmonic* majorant (3).

Next we establish, with an argument shown us by Lumer, that certain functions necessarily admit pluriharmonic majorants. For this purpose, we shall say that a domain Ω in $\mathbb{C}^N$ is *star shaped* with *star center* $\mathfrak{z}_0$ if for all $\mathfrak{w} \in \Omega \setminus \{\mathfrak{z}_0\}$, the ray $\mathfrak{z}_0 + r(\mathfrak{w} - \mathfrak{z}_0)$, $r \geq 0$, meets $\partial\Omega$ in one point.

LEMMA 1. *If Ω is a bounded star shaped domain in $\mathbb{C}^N$, and if $f \in \mathcal{O}(Y)$ has bounded real part, then $|f|^p$ has a pluriharmonic majorant for every $p > 0$.*

PROOF. We suppose without loss of generality, that $0 \in \Omega$, and indeed, that 0 is star center for Ω. Let $\mathcal{M}$ denote the family of all probability measures on the Shilov boundary Γ of Ω (with respect to $\mathcal{A}(\Omega)$) which represent 0: $\mu \in \mathcal{M}$ if $f(0) = \int_\Gamma f d\mu$ for all $f \in \mathcal{A}(\Omega)$. According to Bochner's general version of the Riesz conjugation theorem [3], there are constants C_n depending only on $n = 1, 2, \ldots$ such that for each $\mu \in \mathcal{M}$ and each $f = u + iv \in \mathcal{A}(\Omega)$ with $v(0) = 0$,

$$\left(\int |v|^{2n} d\mu\right)^{1/2n} \leq C_n \left(\int |u|^{2n} d\mu\right)^{1/2n}.$$

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(3) Added in proof. Since this paper was written, Stein's book *Boundary Behavior of Holomorphic Functions of Several Complex Variables*, Princeton, 1972, has appared. It contains all the results we need on the boundary properties of functions in the Hardy classes.

Thus, if $f = u + iv$ with u bounded, and if $f_r(\mathfrak{z}) = f(r\mathfrak{z})$, $0 < r < 1$, then for given $n = 1, 2, \ldots$,

$$\sup \left\{ \int |f_r|^{2n} \, d\mu : 0 < r < 1, \mu \in \mathcal{M} \right\} < \infty.$$

It follows now from Theorem 2 of [8] that for every n, $|f|^{2n}$ has a pluriharmonic majorant, and as this holds for all n, the lemma is proved.

It follows from this lemma, from the quoted result on the existence of nontangential boundary values and from Theorem 10 of [8] that the integral formula (6) retains its validity for all functions $f \in \mathcal{O}(Y)$ which have bounded real parts provided we understand by the $f(\mathfrak{z})$ which appear on the right the boundary function of f. It is in this generality that we will use the formula.

III. Proof of the theorem. As the first step in our proof of the theorem, we need a lemma on integrals of Cauchy type. We fix attention on a $Y \in \mathfrak{S}_N$ which is defined by a function ϱ as in the definition of the class $\mathfrak{S}_N$ so that $d\varrho$ is not zero at any point of ∂Y and the Hessian H_ϱ^r is positive definite on a neighborhood of ∂Y. As we have noted above, Y is strictly convex, so the Cauchy-Fantappiè integral formula is valid on it.

Lemma 2. *With Y and ϱ as just given, let $f \in \mathcal{O}(Y)$ have bouuded real part, and let φ be a real function on ∂Y which satisfies a Lipschitz condition. If*

$$f_\varphi(\mathfrak{w}) = \frac{(N-1)!}{(2\pi i)^N} \int_{\partial Y} \frac{f(\mathfrak{z})\,\varphi(\mathfrak{z})\,k(\mathfrak{z})}{\langle \mathfrak{w} - \mathfrak{z}, \overline{\nabla}\varrho(\mathfrak{z})\rangle^N} \, d\sigma(\mathfrak{z})$$

then f_φ has bounded real part.

Proof. We assume, as we may without loss of generality, that $0 \in Y$. Under this hypothesis, every point $\mathfrak{w} \neq 0$ of Y is of the form $r\,\mathfrak{w}_1$ for some unique $r \in [0, 1)$ and some unique point $\mathfrak{w}_1 \in \partial Y$. If $\mathfrak{w}_1 \in \partial Y$, $r \in [0, 1)$, we have

$$f_\varphi(r\mathfrak{w}_1) = \frac{(N-1)!}{(2\pi i)^N} \int_{\partial Y} \frac{f(\mathfrak{z})\,\{\varphi(\mathfrak{z}) - \varphi(\mathfrak{w}_1)\}\,k(\mathfrak{z})}{\langle r\mathfrak{w}_1 - \mathfrak{z}, \overline{\nabla}\varrho(\mathfrak{z})\rangle^N} \, d\sigma(\mathfrak{z}) + \varphi(\mathfrak{w}_1) f(r\mathfrak{w}_1). \tag{7}$$

As φ is bounded and real, the boundedness of $\operatorname{Re} f_\varphi$ can be proved by proving that the integral on the right in (7), call it $I(r, \mathfrak{w}_1)$, is bounded uniformly in r and $\mathfrak{w}_1$. According to Lemma 1, $|f|^p$ has a pluriharmonic majorant for every $p > 0$, so the nontangential limits of f belong to $L^p(d\sigma)$

(at least if $p \geq 1$). Consequently, if p and q are conjugate indices $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$, Hölders' inequality and the boundedness of k imply

$$|I(r, \mathfrak{w}_1)|^p \leq C\, \|f\|_q^p \int_{\partial Y} \frac{|\varphi(\mathfrak{z}) - \varphi(\mathfrak{w}_1)|^p}{|\langle r\, \mathfrak{w}_1 - \mathfrak{z}, \overline{V}\varrho(\mathfrak{z})\rangle|^{Np}}\, d\sigma(\mathfrak{z}),$$

C some constant. Recalling that φ satisfies a Lipschitz condition, we see that to prove I bounded, it suffices to show that if

$$J(r, \mathfrak{w}_1) = \int_{\partial Y} \frac{\|\mathfrak{z} - \mathfrak{w}_1\|^p}{|\langle r\mathfrak{w}_1 - \mathfrak{z}, \overline{V}\varrho(\mathfrak{z})\rangle|^{Np}}\, d\sigma(\mathfrak{z}),$$

then J is bounded uniformly in r and $\mathfrak{w}_1$. This we can do, at least for certain values of p, values which depend on N.

In proving J bounded, we find it necessarily to discuss in some detail the function $\langle \mathfrak{w} - \mathfrak{z}, \overline{V}\varrho(\mathfrak{z})\rangle$. This function differs from the function $F(z, \zeta)$ considered by Henkin [6, pp. 603-4] only by a certain second order term, so much of Henkin's analysis applies, *mutatis mutandis*, to our situation. If $\mathfrak{w} = (w_1, \dots, w_N)$, $\mathfrak{z} = (z_1, \dots, z_N)$, $\mathfrak{z}$ near ∂Y so that $\varrho(\mathfrak{z})$ is defined, we have

$$\langle \mathfrak{w} - \mathfrak{z}, \overline{V}\varrho(\mathfrak{z})\rangle = \sum_{j=1}^{N} (w_j - z_j) \frac{\partial \varrho}{\partial z_j}(\mathfrak{z}).$$

We apply Taylor's theorem to the function ϱ, expanding about the point $\mathfrak{z}$, to find

$$\varrho(\mathfrak{w}) - \varrho(\mathfrak{z}) = \sum_{j=1}^{N} (w_j - z_j) \frac{\partial \varrho}{\partial z_j}(\mathfrak{z}) + \sum_{j=1}^{N} \overline{(w_j - z_j)} \frac{\partial \varrho}{\partial \bar{z}_j}$$
$$+ Q_{\mathfrak{z}}(\mathfrak{w} - \mathfrak{z}) + 0(\|\mathfrak{w} - \mathfrak{z}\|^3).$$

If we write $z_j = x_j + i x_{N+j}$, $w_j = \xi_j + i \xi_{N+j}$, then

$$Q_{\mathfrak{z}}(\mathfrak{w} - \mathfrak{z}) = \frac{1}{2} \sum_{j,k=1}^{2N} \frac{\partial^2 \varrho}{\partial x_j\, \partial x_k}(\mathfrak{z})\, (\xi_j - x_j)(\xi_k - x_k).$$

We have assumed that the Hessian H_ϱ^r is positive definite, so there exists an absolute constant $\gamma_1 > 0$ such that $|Q_{\mathfrak{z}}(\mathfrak{w} - \mathfrak{z})| \geq \gamma_1 \|\mathfrak{w} - \mathfrak{z}\|^2$ if $\mathfrak{z} \in \partial X$. Thus, for some absolute constant $\gamma > 0$, we have

$$-2\,\mathrm{Re}\,\langle \mathfrak{w} - \mathfrak{z}, \overline{V}\varrho(\mathfrak{z})\rangle > \varrho(\mathfrak{z}) - \varrho(\mathfrak{w}) + \gamma\, \|\mathfrak{w} - \mathfrak{z}\|^2. \tag{8}$$

For $\mathfrak{z} \in \partial Y$ and $\mathfrak{w} \in Y$, $\mathfrak{w}$ in the domain of ϱ, the right side of (8) is strictly positive, for $\varrho(\mathfrak{z}) = 0$ and $\varrho(\mathfrak{w}) < 0$.

At this point in our argument, we can invoke the analysis given by Henkin [6, p. 608] to find that there is $\delta_1 > 0$, such that if $\mathfrak{z} \in \partial Y$, then it is possible to introduce real coordinates $t_1, t_2, \dots, t_{2N}$ on the ball $S(\mathfrak{z}, \delta_1)$ centered at the point $\mathfrak{z}$ such that $t_j(\mathfrak{z}) = 0$, $j = 1, \dots, 2N$, $t_1(\mathfrak{w}) = \varrho(\mathfrak{w})$, and $t_2 = \operatorname{Im} \langle \mathfrak{w} - \mathfrak{z}, \overline{\nabla}\varrho(\mathfrak{z}) \rangle$. The number δ_1, if chosen small enough, will work for every choice of $\mathfrak{z} \in \partial Y$.

Fix attention on a $\mathfrak{w}_1 \in \partial Y$, and let $\Sigma_+ = \{\mathfrak{z} \in \partial Y : |\mathfrak{w}_1 - \mathfrak{z}| \leq \delta_1\}$, $\Sigma_- = \partial Y \setminus \Sigma_+$. We write the integral $J(r, \mathfrak{w}_1)$ as the sum of two integrals, $J_+(r, \mathfrak{w}_1)$ and $J_-(r, \mathfrak{w}_1)$ which correspond, respectively, to integration over Σ_+ and Σ_-. The first of these integrals is bounded, independently of the choice of $\mathfrak{w}_1$ because its integrand is. To estimate the integral $J_-(r, \mathfrak{w}_1)$, notice that on Σ_-, $d\sigma \leq \text{Const}\, t_2 \dots dt_{2N}$. This leads to the estimate, valid for some absolute constant C and uniformly in $\mathfrak{w}_1$,

$$(9) \qquad J_-(r, \mathfrak{w}_1) \leq C \int\limits_{\|t\| < \sigma_1} \frac{(t_2^2 + \dots + t_{2N}^2)^{p/2}\, dt_2 \dots dt_{2N}}{\{(|\varrho(r\mathfrak{w}_1)| + t_2^2 + \dots + t_{2N}^2)^2 + t_2^2\}^{Np/2}}.$$

Here, for $A = \langle r\mathfrak{w}_1 - \mathfrak{z}, \nabla\varrho(\mathfrak{z}) \rangle$, we write $|A| = \{(\operatorname{Re} A)^2 + (\operatorname{Im} A)^2\}^{1/2}$, and then use the estimate (8) and the choice of t_2 in the t-coordinate system In (9) we replace $|\varrho(r\mathfrak{w}_1)|$ by zero and we pass to spherical coordinates in the $t_2, t_3, \dots, t_{2n}$-space. These coordinates are given by

$$t_2 = s \cos\varphi_1$$

$$t_{j+1} = s \sin\varphi_1 \dots \sin\varphi_{j-1} \cos\varphi_j \qquad 1 \leq j \leq 2n-2$$

$$t_{2n} = \sin\varphi_1 \dots \sin\varphi_{2n-3} \sin\varphi_{2n-2}$$

with $0 < s < \infty$, $0 \leq \varphi_j \leq \pi$ for $j = 1, 2, \dots, 2n-3$, $0 \leq \varphi_{2n-2} \leq 2\pi$. Thus, (9) implies

$$(10) \qquad J_-(r, \mathfrak{w}_1) \leq C' \int_0^C ds \int_0^\pi \frac{s^{p+2N-2}\, d\cos\varphi_1}{(s^4 + s^2\cos^2\varphi_1)^{1/2Np}}$$

$$= C' \int_0^C ds \int_0^1 \frac{s^{p+2N-2-2Np}}{\left(1 + \left(\frac{\alpha}{s}\right)^2\right)^{1/2Np}}\, d\alpha$$

$$= C' \int_0^C ds \int_0^\infty \frac{s^{p+2N-2Np-1}}{(1+\beta^2)^{1/2Np}}\, d\beta \leq C'\, C(p) \int_0^C s^{p+2N-2Np-1}\, ds$$

where $C(p)$ is a constant which depends on p : $C(p) = \int_0^\infty (1+\beta^2)^{-1/2Np}\, d\beta$.

This is finite for all $p > 1$ because $N \geq 2$. The last integral in the string of inequalities (10) is finite if $p + 2N - 2Np - 1 > -1$, and for fixed $N \geq 2$, this is equivalent to $p < \dfrac{2N}{2N-1}$.

Our argument shows that if we choose p at the outset to satisfy $1 < p < \dfrac{2N}{2N-1}$, then the integral $J(r, \mathfrak{w}_1)$ is bounded uniformly in r and $\mathfrak{w}_1$. Thus, Re f_φ is bounded, as we were to prove.

In the proof of our theorem, we shall need a decomposition lemma.

Let us suppose given a domain $Y \in \mathfrak{S}_N$, and let L be a real hyperplane which intersects Y, say the hyperplane $x_{2N} = A$ where we take complex coordinates $x_1 + ix_{N+1}, \dots, x_N + ix_{2N}$ on $\mathbb{C}^N$. Denote by $Y_{L;\varepsilon}$ the *ε-slab of Y about L, i.e.*, the set $\{\mathfrak{z} \in Y : |x_{2N} - A| < \varepsilon\}$. We fix an $\varepsilon > 0$ so small that $Y \setminus Y_{L;\varepsilon}$ consists of two components.

Assume Y_+, Y_- to be two members of $\mathfrak{S}_N$ which satisfy the following conditions:

i) $Y_+ \supset \{\mathfrak{z} \in Y : x_{2N} > A - \varepsilon\}$, $\quad Y_- \supset \{\mathfrak{z} \in Y : x_{2N} < A + \varepsilon\}$.

ii) $Y_+ \cap Y_-$ contains $U \cap Y$ where U is some neighborhood in $\mathbb{C}^N$ of the compact set $\overline{Y}_{L;\varepsilon_0}$.

iii) The set $Y_{+-} = Y_+ \cap Y_-$ belongs to $\mathfrak{S}_N$.

LEMMA 3. *In the geometric configuration just described, if $f = u + iv$ is a holomorphic function on Y_{+-} with u bounded, then $f = f_+ + f_-$ where f_+ and f_- are holomorphic on X_+ and X_- respectively and both have bounded real part.*

PROOF. Let ψ be a function on ∂Y_{+-} which satisfies a Lipschitz condition and which has the properties that

$$\psi = 0 \ \text{ on} \left\{ \mathfrak{z} \in \partial Y_{+-} : x_{2N} < A - \frac{1}{2}\varepsilon_0 \right\}$$

$$\psi = 1 \ \text{ on} \left\{ \mathfrak{z} \in \partial Y_{+-} : x_{2N} > A + \frac{1}{2}\varepsilon_0 \right\}.$$

Let ϱ be a $\mathcal{C}^2$ function on a neighborhood of Y_{+-} which defines this set as in the first section of the paper. Write f as a Cauchy-Fantappiè integral:

$$f(\mathfrak{w}) = \frac{(N-1)!}{(2\pi i)^N} \int\limits_{Y_{+-}} \frac{f(\mathfrak{z})\, k(\mathfrak{z})}{\langle \mathfrak{w} - \mathfrak{z}, \bar{\nabla}\varrho(\mathfrak{z}) \rangle^N}\, d\sigma(\mathfrak{z}) = f_+ + f_-$$

where

$$f_-(\mathfrak{w}) = \frac{(N-1)!}{(2\pi i)^N} \int\limits_{\partial Y_{+-}} \frac{f(\mathfrak{z})\{1 - \psi(\mathfrak{z})\}\, k(\mathfrak{z})}{\langle \mathfrak{w} - \mathfrak{z}, \bar{\nabla}\varrho(\mathfrak{z}) \rangle^N}\, d\sigma(\mathfrak{z})$$

and

$$f_+(\mathfrak{w}) = \frac{(N-1)!}{(2\pi i)^N} \int\limits_{\partial Y_{+-}} \frac{f(\mathfrak{z})\, \psi(\mathfrak{z})\, k(\mathfrak{z})}{\langle \mathfrak{w} - \mathfrak{z}, \bar{\nabla}\varrho(\mathfrak{z}) \rangle^N}\, d\sigma(\mathfrak{z}).$$

The functions f_+ and f_- are holomorphic and have bounded real parts on Y_{+-}. Moreover, f_+ and f_- are holomorphic on substantially larger sets. The function f_+ is holomorphic on a neighborhood of the set $E_- = \{\mathfrak{z} \in Y : x_{2N} \leq A - \varepsilon_0\}$. This is immediate, for since $\psi(\mathfrak{z}) = 0$ if $x_{2N} < A - \frac{1}{2}\varepsilon_0$, we have

$$f_+(\mathfrak{w}) = \frac{(N-1)!}{(2\pi i)^N} \int\limits_{\Sigma} \frac{f(\mathfrak{z})\, \psi(\mathfrak{z})\, k(\mathfrak{z})}{\langle \mathfrak{w} - \mathfrak{z}, \bar{\nabla}\varrho(\mathfrak{z}) \rangle^N}\, d\sigma(\mathfrak{z}),$$

$\Sigma = \left\{\mathfrak{z} \in \partial Y_{+-} : x_{2N} \geq A_0 - \frac{1}{2}\varepsilon_0\right\}$. For $\mathfrak{z} \in \partial Y_{+-}$, $\langle \mathfrak{w} - \mathfrak{z}, \bar{\nabla}\varrho(\mathfrak{z}) \rangle$ as a function of $\mathfrak{w}$ vanishes only on the complex tangent to Y_{+-} at $\mathfrak{z}$, and the strict convexity of $\bar{Y}_{+-}$ implies that these tangents are at a strictly positive distance from the set E_- if $\mathfrak{z}$ is constrained to lie in Σ. It follows that f_+ is holomorphic on a neighborhood of E_-. Similarly, if we define E_+ to be $\{\mathfrak{z} \in \bar{Y} : x_{2N} \geq A - \varepsilon_0\}$, then f_- is holomorphic on a neighborhood of E_+. The lemma is proved.

The following lemma records an obvious geometric fact, a fact which will enable us to apply the lemma just proved.

LEMMA 4. *Let $Y \in \mathfrak{S}_N$, and let Π and Π' be real hyperplanes in $\mathbb{C}^N$ which are parallel and which meet Y. Let $Y \setminus \Pi = Y_1 \cup Y_2$ and $Y \setminus \Pi' = Y_1' \cup Y_2'$ with $Y_1 \subset Y_1'$, $Y_1' \cap \Pi \neq \varnothing$. Let H be the real half space of $\mathbb{C}^N$ determined by Π which contains Y_1. There is X, an element of $\mathfrak{S}_N$ such that $\bar{X} \cap \Pi' = \varnothing$ and $\bar{X} \cap H = Y \cap H$.*

Roughly speaking, we can flatten Y toward Π without altering the part of Y in H and still stay within the class $\mathfrak{S}_N$.

Proof. We can choose real coordinates $x_1, x_2, \dots, x_{2N}$ on $\mathbb{C}^N$ in such a way that Π and Π' are the hyperplanes $x_{2N} = 0$ and $x_{2N} = 1$ respectively, and so that, moreover, the origin lies in Y.

Let Q be the Minkowski functional associated with Y. As we saw in Section I, the function $\psi = Qe^Q - 1$ is a $\mathcal{C}^2$ function on $\mathbb{C}^N \setminus \{0\}$ which defines Y as an element of $\mathfrak{S}$ and which has positive definite real Hessian at every point of $\mathbb{C}^N \setminus \{0\}$. Moreover, $d\psi$ never vanishes on $\mathbb{C}^N \setminus \{0\}$.

Let φ be a real valued function of a real variable such that (1) φ is of class $\mathcal{C}^2$, (2) $\varphi(t) = 0$ if $t \leq \frac{1}{2}$, (3) $\varphi(t) \geq 2$ if $t \geq 1$, and (4) $\varphi''(t) > 0$ if $t > \frac{1}{2}$. The function P defined by

$$P(\mathfrak{z}) = \psi(\mathfrak{z}) + \varphi(x_{2N})$$

is of class $\mathcal{C}^2$ on $\mathbb{C}^N \setminus \{0\}$ and its Hessian is positive definite. Since dP never vanishes on $\mathbb{C}^N \setminus \{0\}$, the set

$$X = \{\mathfrak{z} : P(\mathfrak{z}) - 1 < 0\}$$

is an element of $\mathfrak{S}_N$ with the properties we seek.

Finally, we turn to the proof of the theorem.

The proof is achieved by a patching argument familiar in this sort of context. Assume given X, $\mathcal{V} = \{V_\alpha\}$, and $\{f_\alpha\}$ as in the statement of the theorem, but suppose, in order to derive a contradiction, that no F with the stated properties exists.

Let $M = \max\{x_{2N}:$ for some $\mathfrak{z} \in \bar{X}$, $\mathfrak{z} = (z_1, \dots, z_N)$. $x_{2N} = \operatorname{Im} z_N\}$, and let m be the corresponding minimum. Let ε satisfy $0 < \varepsilon < \frac{1}{6}(M-m)$. Apply Lemma 4 with X in place of Y and with the hyperplanes $x_{2N} = \frac{1}{2}(M+m)+2\varepsilon$, $x_{2N} = \frac{1}{2}(M+m)+\varepsilon$ and the hyperplanes $x_{2N} = \frac{1}{2}(M+m)-\varepsilon$, $x_{2N} = \frac{1}{2}(M+m) - 2\varepsilon$ in place of Π and Π' to obtain sets X^+ and X^- as in the conclusion of that lemma. By construction the sets X^+ and X^- satisfy the geometric hypotheses of Lemma 3.

We have assumed that the theorem in false, and in particular, that the function F does not exist. This entails the conclusion that our problem is not solvable on both X^+ and X^-: There cannot exist F_+ and F_- in $\mathcal{H}(X^+)$ and $\mathcal{H}(X^-)$ respectively such that for all α, $F_+ f_\alpha^{-1}$ and $F_- f_\alpha^{-1}$ are bounded and bounded away from zero on $V_\alpha \cap X^+$ and $V_\alpha \cap X^-$ respectively. Suppose, to the contrary, that such functions F_+ and F_- exist. The function $f_0 = F_- F_+^{-1}$ is bounded and bounded away from zero on $X^+ \cap X^-$, so $f_0 = e^f$ with Ref bounded. Decompose f according to Lemma 3 : $f = f_+ + f_-$. Then $f_0 = e^{f_+} e^{f_-}$ so we have that on $X^+ \cap X^-$, $e^{f_+} e^{f_-} = F_- F_+^{-1}$ whence

$$F_+ e^{f_+} = F_- e^{-f_-} .$$

It follows that we can define G on X by $G = F_+ e^{f_+}$ on X^+, $G = F_- e^{-f_-}$ on X^-. The function G is a well defined bounded holomorphic function on X, and its definition shows that on $V_\alpha \cap X$, Gf_α^{-1} is bounded and bounded away from zero. We have supposed that no such function G exists, so either F_+ or F_- does not exist: The insolvability of our problem on X forces its insolvability on one of X^+, X^-, say on X^+.

The x_{2N}-width of X^+, *i. e.*, the number max $| x'_{2N} - x''_{2N} |$, the maximum taken over all pairs of points $\mathfrak{z}'$, $\mathfrak{z}''$ in X^+, is not more than two thirds of the x_{2N}-width of X. We now treat X^+ as we treated X, using the coordinate x_{2N-1} rather than x_{2N}, and we find a smaller set $X^{++} \subset X^+$ on which the problem is not solvable and which has the property that the x_{2N-1}-width of X^{++} is not more than two thirds that of X^+.

We iterate this process, running cyclically through the real coordinate of $\mathbb{C}^N$, and we obtain a shrinking sequence of sets on which our problem is not solvable. Since $\mathcal{V}$ is an open cover, the sets we obtain eventually lie in some element, say V_α, of $\mathcal{V}$, and on V_α, the function f_α is a solution to the induced problem. Thus, we have a contradiction and the theorem is proved.

APPENDIX. ON THE SMOOTHNESS OF MINKOWSKI FUNCTIONALS In this brief appendix we take up a geometric matter which should be well known but for which we are unable to cite a reference.

PROPOSITION. *Let $\Omega \subset \mathbb{R}^m$ be an open convex set with $\partial\Omega$ a manifold of class $\mathcal{C}^p$, $p \geq 1$, so that for some neighborhood U of $\partial\Omega$ and some real valued function ϱ defined and possessing continuous derivatives of order p in U, $\partial\Omega = \{x \in U : \varrho(x) = 0\}$ and, moreover, $d\varrho$ does not vanish on $\partial\Omega$. Assume $0 \in \Omega$. Then Minkowski functional, Q, associated with Ω is of class $\mathcal{C}^p$ on $\mathbb{R}^m \setminus \{0\}$.*

PROOF. The function Q is continuous.

Without loss of generality, we can suppose that ϱ is defined and of class $\mathcal{C}^p$ on the whole of $\mathbb{R}^m$ and that it vanishes only on $\partial\Omega$.

Define a function $\Phi : \mathbb{R}^m \times (\mathbb{R} \setminus \{0\})$ by

$$\Phi(x, t) = \varrho(t^{-1} x).$$

As ϱ is of class $\mathcal{C}^p$ on $\mathbb{R}^m$, Φ is of class $\mathcal{C}^p$. Moreover, for all $x \in \mathbb{R}^m \setminus \{0\}$, $\Phi(x, Q(x)) = \varrho(Q(x)^{-1} x) = 0$, because by the definition $Q(x)^{-1} x$ lies in $\partial\Omega$ if $x \neq 0$.

We have

$$\frac{\partial \Phi}{\partial t}(x, t) = \frac{-1}{t^2} \sum_{j=1}^{m} x_j \frac{\partial \varrho}{\partial x_j}(t^{-1} x).$$

We see, therefore, that if $t^{-1} x$ is near $\partial\Omega$, $\frac{\partial \Phi}{\partial t}$ is not zero: If $x_0 = (x_1^0, \ldots, x_m^0) \in \partial\Omega$, then the tangent plane, T_{x_0}, to $\partial\Omega$ at x_0 has the equation

$$\sum_{j=1}^{m} (x_j - x_j^0) \frac{\partial \varrho}{\partial x_j}(x_j^0) = 0.$$

We have assumed that $0 \in \Omega$, so since Ω lies on one side of T_{x_0}, $0 \notin T_{x_0}$. This means that for no x_0 in $\partial\Omega$ is $\sum_{j=1}^{m} x_j^0 \frac{\partial \varrho}{\partial x_j}(x_i^0) = 0$, and it follows that $\frac{\partial \Phi}{\partial t}(x, t) \neq 0$ if $t^{-1} x$ is close enough to $\partial\Omega$.

We now apply the implicit function theorem [2]. If $x_0 \in \mathbb{R}^m$, then $\Phi(x_0, Q(x_0)) = 0$, and according to the implicit function theorem, there is a neighborhood V of x_0 in which there exists a unique continuous map $\psi: V \to \mathbb{R}$ with $\psi(x_0) = Q(x_0)$ and $\Phi(x, \psi(x)) = 0$ if x is in V. The p fold continuous differentiability of Φ implies that of ψ. Since Q is continuous and $\Phi(x, Q(x)) = 0$, we find that Q has p continuous derivatives near x_0. As x_0 is an arbitrary chosen point of $\mathbb{R}^m \setminus \{0\}$, we have the stated result.

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REFERENCES

1. ARONSZAJN, N., and K. T. SMITH, *Functional Spaces and Functional Completion*, Ann. Inst. Fourier (Grenoble) VI (1955-56), 125-185.

2. DIEUDONNÉ, J., *Foundations of Modern Analysis*, Academic Press, New York, 1960.

3. GAMELIN, T. W., *Uniform Algebras*, Prentice-Hall, Englewood Cliffs, 1970.

4. GUNNING, R. C., and H. ROSSI, *Analytic Functions of Several Complex Variables*, Prentice-Hall, Englewood Cliffs, 1965.

5. HARDY, G. H., J. E. LITTLEWOOD, and G. POLYA, *Inequalities*, 2nd edition, Cambridge Univ. Press, London, 1959.

6. HENKIN, G. M., *Integral representations of functions holomorphic in strictly pseudo-convex domains and some applications.* Math. USSR Sbornik 7 (1969), 597-616. (Mat. Sbornik 78 (1969))

7. LERAY, J., *Le calcul différential et intégral sur une variété analytique complexe* (Problème de Cauchy, III), Bull. Soc. Math. France 87 (1959), 81-180.

8. LUMER, G., *Éspaces de Hardy en plusieur variables complexes, to appear in the Comptes Rendus* (Paris).

9. STOUT E. L., *The Second Cousin Problem with Bounded Data*, Pacific J. Math, 26 (1968), 379-387.