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# ON RECURRENT SPACES OF FIRST ORDER

by RANJAN KUMAR GARAI

1. We denote by  $R_{ijk}^h$ ,  $R_{ij}$  and  $R$  the curvature tensor, the Ricci tensor and the scalar curvature of a Riemannian space  $V_n$  respectively and put

$$(1.1) \quad Z_{ijk}^h = R_{ijk}^h - \frac{R}{n(n-1)} (\delta_k^h g_{ij} - \delta_j^h g_{ik}),$$

$$(1.2) \quad W_{ijk}^h = R_{ijk}^h - \frac{1}{n-1} (\delta_k^h R_{ij} - \delta_j^h R_{ik})$$

and

$$(1.3) \quad C_{ijk}^h = R_{ijk}^h - \frac{1}{n-2} (R_k^h g_{ij} - R_j^h g_{ik} + R_{ij} \delta_k^h - R_{ik} \delta_j^h) + \\ + \frac{R}{(n-1)(n-2)} (\delta_k^h g_{ij} - \delta_j^h g_{ik}).$$

These tensors  $Z_{ijk}^h$ ,  $W_{ijk}^h$  and  $C_{ijk}^h$  are known as the concircular curvature tensor, the projective curvature tensor and the conformal curvature tensor of  $V_n$  respectively. A tensor  $L_{ijk}^h$  is called first order recurrent if

$$(1.4) \quad L_{ijk,l}^h = \lambda_l L_{ijk}^h,$$

for a non-zero vector  $\lambda_l$ , where comma denotes covariant differentiation with respect to the metric tensor  $g_{ij}$ . The vector  $\lambda_l$  in (1.4) is called the vector of recurrence.

A non-flat Riemannian space is called a recurrent space of first order if  $R_{ijk}^h$  is first order recurrent. A Riemannian space  $V_n$  ( $n \geq 3$ ) is called a projective recurrent space of first order if  $W_{ijk}^h$  is first order recurrent. A Riemannian space  $V_n$  ( $n > 3$ ) is called conformally recurrent space of first

order if  $C_{ijk}^h$  is first order recurrent. A Riemannian space  $V_n$  ( $n \geq 3$ ) for which  $R_{ij}$  is non zero and recurrent of first order is called a Ricci recurrent space of first order. Recurrent spaces, projective recurrent spaces, conformally recurrent spaces and Ricci recurrent spaces, all of first order, will be denoted by  $K_n$ ,  $P_n$ ,  $C_n$  and  $R_n$  respectively.

A Riemannian Space  $V_n$  is called symmetric, projective symmetric, conformally symmetric or Ricci-symmetric according as

$$(1.5) \quad R_{ijk, l}^h = 0,$$

$$(1.6) \quad W_{ijk, l}^h = 0,$$

$$(1.7) \quad C_{ijk, l}^h = 0,$$

or

$$(1.8) \quad R_{ij, l} = 0$$

is satisfied in  $V_n$ .

These spaces have been studied by many authors including Walker [1], Roy-chowdhury [2] and Matsumoto [3]. In this paper some properties of these spaces are obtained. It is believed that some of the results are new while others are extensions of results obtained by previous authors.

## 2. Some properties of a $P_n$ .

(i) Let a Riemannian  $V_n$  satisfy the relation

$$(2.1) \quad Z_{ijk, l}^h = \lambda_l Z_{ijk}^h,$$

where  $\lambda_l$  is a non-zero vector and  $Z_{ijk}^h$  is given by (1.1). From (1.1) and (2.1), we have

$$(2.2) \quad R_{ijk, l}^h = \lambda_l R_{ijk}^h - \frac{R \lambda_l}{n(n-1)} (\delta_k^h g_{ij} - \delta_j^h g_{ik}) + \frac{R_{, l}}{n(n-1)} (\delta_k^h g_{ij} - \delta_j^h g_{ik}).$$

Contraction over the indices  $h$  and  $k$  in (2.2) gives

$$(2.3) \quad R_{ij, l} = \lambda_l R_{ij} - \frac{R}{n} \lambda_l g_{ij} + \frac{R_{, l}}{n} g_{ij}.$$

From (1.2), it follows that

$$(2.4) \quad W_{ijk, l}^h = R_{ijk, l}^h - \frac{1}{n-1} (\delta_k^h R_{ij, l} - \delta_j^h R_{ik, l}).$$

Using (2.2) and (2.3) in (2.4), we get

$$(2.5) \quad W_{ijk, l}^h = \lambda_l W_{ijk}^h.$$

Conversely, let a  $V_n$  satisfy (2.5), where  $\lambda_l$  is some non-zero vector and  $W_{ijk}^h$  is given by (1.2); then

$$(2.6) \quad R_{ijk, l}^h - \frac{1}{n-1} (\delta_k^h R_{ij, l} - \delta_j^h R_{ik, l}) = \lambda_l \left\{ R_{ijk}^h - \frac{1}{n-1} (\delta_k^h R_{ij} - \delta_j^h R_{ik}) \right\}.$$

Transvecting (2.6) by  $g^{ij}$ , we obtain

$$R_{k, l}^h - \frac{1}{n-1} (\delta_k^h R_{, l} - R_{k, l}^h) = \lambda_l \left\{ R_k^h - \frac{1}{n-1} (\delta_k^h R - R_k^h) \right\}$$

or,

$$(2.7) \quad R_{k, l}^h = \lambda_l R_k^h - \frac{1}{n} (\lambda_l R - R_{, l}) \delta_k^h.$$

Therefore,

$$(2.8) \quad R_{ij, l} = \lambda_l R_{ij} - \frac{1}{n} (\lambda_l R - R_{, l}) g_{ij}.$$

In view of (2.8),

$$(2.9) \quad \delta_k^h R_{ij, l} - \delta_j^h R_{ik, l} = \lambda_l (\delta_k^h R_{ij} - \delta_j^h R_{ik}) - \frac{R}{n} \lambda_l (\delta_k^h g_{ij} - \delta_j^h g_{ik}) + \\ + \frac{R_{, l}}{n} (\delta_k^h g_{ij} - \delta_j^h g_{ik}).$$

Substituting (2.9) in (2.6), we get

$$R_{ijk, l}^h - \frac{R_{, l}}{n(n-1)} (\delta_k^h g_{ij} - \delta_j^h g_{ik}) = \lambda_l \left\{ R_{ijk}^h - \frac{R}{n(n-1)} (\delta_k^h g_{ij} - \delta_j^h g_{ik}) \right\}$$

or

$$Z_{ijk, l}^h = \lambda_l Z_{ijk}^h.$$

We, therefore, state the following theorem:

**THEOREM 1.** *A necessary and sufficient condition that a Riemannian space  $V_n$  be a  $P_n$  with  $\lambda_i$  as its vector of recurrence is that the concircular curvature tensor  $Z_{ijk}^h$  is first order recurrent, the vector of recurrence being the same  $\lambda_i$ .*

When  $R = 0$ ,  $Z_{ijk}^h = R_{ijk}^h$ . Hence applying theorem 1, we have

THEOREM 2. *Every  $P_n$  with zero scalar curvature is a  $K_n$  with the same vector of recurrence.*

(ii) Consider  $C_{ijk}^h$  in a  $P_n$  ( $n > 3$ ). Differentiating (1.3) covariantly with respect to  $x^l$ , we have

$$(2.10) \quad C_{ijk, l}^h = R_{ijk, l}^h - \frac{1}{n-2} (R_{k, l}^h g_{ij} - R_{j, l}^h g_{ik} + R_{ij, l} \delta_k^h - R_{ik, l} \delta_j^h) + \\ + \frac{R_{, l}}{(n-1)(n-2)} (\delta_k^h g_{ij} - \delta_j^h g_{ik}).$$

Using (2.2), (2.3) and (2.7) in (2.10) and simplifying, we get

$$C_{ijk, l}^h = \lambda_l C_{ijk}^h.$$

Hence we have the result:

THEOREM 3. *A  $P_n$  ( $n > 3$ ) is a  $C_n$  with the same vector of recurrence.*

(iii) The relation (2.2) which holds in a  $P_n$ , can be rewritten as

$$R_{ijk, l}^h - \lambda_l R_{ijk}^h = \frac{1}{n(n-1)} (\delta_k^h g_{ij} - \delta_j^h g_{ik}) (R_{, l} - \lambda_l R).$$

If  $R = 0$ , the space is a  $K_n$  (theorem 2). Again, if  $R \neq 0$  and  $R_{, l} = \lambda_l R$  i. e.  $\lambda_l = \frac{\partial}{\partial x^l} (\log R)$ , the space is a  $K_n$ . Hence the following theorem:

THEOREM 4. *A  $P_n$  of non-zero scalar curvature is a  $K_n$  if its vector of recurrence  $\lambda_i$  is given by  $\lambda_i = \frac{\partial}{\partial x^i} (\log R)$ .*

From (1.1), it follows that

$$Z_{hijk, lm} - Z_{hijk, ml} = R_{hijk, lm} - R_{hijk, ml},$$

where  $Z_{hijk} = g_{ht} Z_{ijk}^t$ . By Walker's Lemma 1 [1],

$$R_{hijk, lm} - R_{hijk, ml} + R_{jklm, hi} - R_{jklm, ih} + R_{lmhi, jk} - R_{lmhi, kj} = 0.$$

Hence

$$(2.11) \quad Z_{hijk, lm} - Z_{hijk, ml} + Z_{jklm, hi} - Z_{jklm, ih} + Z_{lmhi, jk} - Z_{lmhi, kj} = 0.$$

In a  $P_n$  with  $\lambda_i$  as its vector of recurrence,

$$Z_{hijk, lm} - Z_{hijk, ml} = (\lambda_{l, m} - \lambda_{m, l}) Z_{hijk} = b_{lm} Z_{hijk},$$

where  $b_{lm} = \lambda_{l,m} - \lambda_{m,l}$ . Hence, in a  $P_n$ , (2.11) reduces to

$$b_{lm} Z_{hijk} + b_{hi} Z_{jklm} + b_{jk} Z_{lmhi} = 0.$$

Since  $Z_{hijk} = Z_{jklhi}$ , Walker's Lemma 2 [1] gives either  $b_{lm} = 0$  or  $Z_{hijk} = 0$ . Hence the theorem:

**THEOREM 5.** *In a  $P_n$ , either the vector of recurrence is a gradient or the space is of constant curvature.*

It is known that the vector of recurrence of an Einstein  $K_n$  ( $n > 2$ ) is null [1]. Hence, in view of theorem 2, we may state the following result:

**THEOREM 6.** *In a non-flat Einstein  $P_n$  with zero scalar curvature the vector of recurrence is null.*

(iv) If a vector  $v^i$  in a Riemannian space satisfies

$$v^i_{;j} = \varrho \delta^i_j,$$

where  $\varrho$  is a non-zero constant,  $v^i$  is said to be a concurrent vector field [4]. Let a  $P_n$  with  $\lambda_i$  as its vector of recurrence admit a concurrent vector field  $v^i$ . Then

$$(2.12) \quad v^i_{;j} = \varrho g_{ij}.$$

Consequently

$$v^i_{;jk} - v^i_{;kj} = 0.$$

Hence, by Ricci identity,

$$(2.13) \quad v^h R_{ijk} = 0.$$

Differentiating (2.13) covariantly with respect to  $x^l$  and using (2.12), we get

$$(2.14) \quad \varrho R_{ijk} + v^h R_{ijk;l} = 0.$$

Transvecting (2.14) by  $g^{ij}$ , we obtain

$$(2.15) \quad \varrho R_{ik} + v^h R_{hk;l} = 0.$$

Transvecting (2.15) again by  $g^{ik}$ , we have

$$(2.16) \quad \varrho R + \frac{1}{2} v^h R_{;h} = 0.$$

Now transvecting (2.2) by  $v_h$  and using (2.13) and (2.14), we get

$$(2.17) \quad \frac{1}{n(n-1)} (v_k g_{ij} - v_j g_{ik}) (R \lambda_l - R_{;l}) = \varrho R_{ikj}.$$

As  $\varrho \neq 0$ ,  $R = 0$  implies

$$(2.18) \quad R_{ijk} = 0.$$

Again, transvecting (2.17) by  $g^{ij}g^{lk}$  and using (2.16), we have

$$(2.19) \quad R [\nu^l \lambda_l - (n-2)\varrho] = 0.$$

Therefore, either

$$R = 0$$

or,

$$(2.20) \quad \nu^l \lambda_l = (n-2)\varrho.$$

But, transvecting (2.17) by  $\nu^l g^{ij}$  and using (2.16), we get

$$(2.21) \quad \frac{R}{n} \nu_k (\nu^l \lambda_l + 2\varrho) = \varrho \nu^l R_{lk}.$$

In view of (2.13), (2.21) reduces to

$$R \nu_k (\nu^l \lambda_l + 2\varrho) = 0.$$

Since  $\nu_k$  cannot be zero, either

$$R = 0$$

or

$$(2.22) \quad \nu^l \lambda_l = -2\varrho.$$

But (2.20) and (2.22) give  $\varrho = 0$ , which is impossible. Therefore  $R = 0$ .

Hence, from (2.18), we have the result:

**THEOREM 7.** *If a  $P_n$  admits a concurrent vector field, then the scalar curvature  $R$  is zero and the space is flat.*

Putting  $\varrho = 0$  in (2.17) and transvecting the resulting equation by  $g^{ij}$ , we get

$$\nu_k (R \lambda_l - R_{,l}) = 0$$

If  $\nu_k \neq 0$ ,  $\lambda_l = \frac{\partial}{\partial x^l} (\log R)$ , for  $R \neq 0$ . Thus we have the result:

**THEOREM 8.** *If a  $P_n$  with non-zero scalar curvature  $R$  admits a non-zero parallel vector field; the vector of recurrence  $\lambda_i$  is given by  $\lambda_i = \frac{\partial}{\partial x^i} (\log R)$ .*

In view of Theorems 2, 4 and 8 we have the result:

THEOREM 9. A  $P_n$  admitting a non-zero parallel vector field is a  $K_n$ .  
The equation (2.19) reduces to

$$R \nu^l \lambda_l = 0 \quad \text{if} \quad \varrho = 0.$$

Hence the result :

THEOREM 10. If a  $P_n$  admits a parallel vector field  $\nu^i$ , then either the scalar curvature is zero or  $\nu^i$  is orthogonal to the vector of recurrence of the space.

Roy-chowdhury [2] proved that if a  $P_n$  admits a parallel vector field  $\nu^i$ , then either the space is of constant curvature or  $\nu^i$  is orthogonal to the vector of recurrence of the space. Again, a space of constant curvature with zero scalar curvature is flat. Hence, in view of theorem 10 and Roy-chowdhury's result, we may state the following theorem.

THEOREM 11. If a  $P_n$  admits a parallel vector field  $\nu^i$ , then either the space is flat or  $\nu^i$  is orthogonal to the vector of recurrence of the space.

(vi) If there exists a vector field  $\nu^i$  such that

$$(2.23) \quad \nu^i_{,j} = \varrho \delta^i_j,$$

$\varrho$  being a non-constant scalar, then  $\nu^i$  is said to be concircular vector field [5]. Let us assume that  $P_n$  admits a concircular vector field  $\nu^i$ . From (2.23) and Ricci identity, we have

$$(2.24) \quad \nu^h R_{ijk} = \varrho_k g_{ij} - \varrho_j g_{ik},$$

where  $\varrho_k = \frac{\partial \varrho}{\partial x^k}$ . From (2.24) one gets

$$(2.25) \quad \nu^h R_{hjk}^i = \varrho_j \delta_k^i - \varrho_k \delta_j^i.$$

Contraction over the indices  $i$  and  $k$  in (2.25), gives

$$(2.26) \quad \nu^h R_{hj} = (n-1) \varrho_j.$$

In consequence of (2.25) and (2.26), we have

$$(2.27) \quad \nu^h W_{hjk}^i = 0.$$

Differentiating (2.27) covariantly with respect to  $x^l$  and using (2.23), we get

$$(2.28) \quad \varrho W_{ijk}^i + \nu^h W_{hjk,l}^i = 0.$$



In virtue of (2.5) and (2.27), (2.28) reduces to

$$\varrho W_{ijk}^i = 0.$$

Since  $\varrho \neq 0$ ,  $W_{ijk}^i = 0$ , i. e. the space is of constant curvature. Again, in a space of constant curvature, we have

$$(2.29) \quad R_{hjk}^i = \frac{R}{n(n-1)} (\delta_k^i g_{hj} - \delta_j^i g_{hk}).$$

Transvecting (2.29) by  $v^h$  and using (2.25), we get

$$(2.30) \quad \varrho_j \delta_k^i - \varrho_k \delta_j^i = \frac{R}{n(n-1)} (\delta_k^i v_j - \delta_j^i v_k).$$

Contraction over the indices  $i$  and  $k$  in (2.30), gives

$$(2.31) \quad \varrho_j = \frac{R}{n(n-1)} v_j.$$

Now  $R = 0$  implies  $\varrho_j = 0$ , for every  $j$ , which is not possible. Therefore  $R$  is a non-zero constant. Hence we have the result:

**THEOREM 12.** *A  $P_n$  admitting a concircular vector field is a space of non-zero constant curvature.*

Differentiating (2.31) covariantly with respect to  $x^l$ , we get

$$\varrho_{j,l} = \frac{R}{n(n-1)} v_{j,l}.$$

In view of (2.23), the above equation reduces to

$$(2.32) \quad \varrho_{j,l} = \frac{R\varrho}{n(n-1)} g_{jl}.$$

From (2.31) and (2.32), we can state the result:

**THEOREM 13.** *If a  $P_n$  admits a concircular vector field  $v^i$  such that  $v_{,j}^i = \varrho \delta_j^i$ ,  $\varrho$  being a non-constant scalar, then (i)  $v^i$  is a gradient proportional to  $\varrho_i$  and (ii)  $\varrho_i$  itself is a concircular vector field.*

(vii) Suppose that the scalar curvature  $R$  of  $P_n$  is constant. Since  $R$  is constant, we have, from (1.1) and Bianchi identity,

$$(2.33) \quad Z_{hijk,l} + Z_{hikl,j} + Z_{hilj,k} = 0,$$

where  $Z_{hijk} = g_{hi} Z_{ijk}^t$ . Since the space is a  $P_n$ , we have from (2.33)

$$(2.34) \quad \lambda_i Z_{hijk} + \lambda_j Z_{hikl} + \lambda_k Z_{hilj} = 0,$$

$\lambda_i$  being the vector of recurrence. It can be verified that  $Z_{hijk}$  satisfies the following relations

$$(2.35) \quad Z_{hijk} = -Z_{ihjk} = Z_{jkhi} = -Z_{hikj} = Z_{ihkj}.$$

Now we proceed as in Walker [1]. We choose  $u^i$  so that  $u^i \lambda_i = 1$  and put

$$(2.36) \quad S_{ij} = -u^h u^k Z_{hijk} = -u^h u^k Z_{jkhi} = -u^h u^k Z_{kjih} = S_{ji}.$$

Transvecting (2.34) by  $u^i u^h$  and using (2.35) and (2.36), we get

$$(2.37) \quad u^h Z_{hijk} = \lambda_j S_{ik} - \lambda_k S_{ij}.$$

Transvecting (2.34) by  $u^l$  and applying (2.37), one obtains

$$(2.38) \quad Z_{hijk} = S_{hj} \lambda_i \lambda_k + S_{ik} \lambda_h \lambda_j - S_{hk} \lambda_i \lambda_j - S_{ij} \lambda_h \lambda_k.$$

Hence we can state the following theorem:

**THEOREM 14.** *In a  $P_n$  the scalar curvature of which is constant, the tensor  $Z_{hijk}$  can be expressed in the form*

$$Z_{hijk} = S_{hj} \lambda_i \lambda_k + S_{ik} \lambda_h \lambda_j - S_{hk} \lambda_i \lambda_j - S_{ij} \lambda_h \lambda_k,$$

where  $S_{ij}$  is a symmetric tensor and  $\lambda_i$  the vector recurrence of the space.

It should be noted here that  $S_{ij}$  can be modified by the addition of  $\lambda_i \alpha_j + \alpha_i \lambda_j$  for any  $\alpha_i$ .

In an Einstein space  $V_n$  ( $n > 2$ ), the scalar curvature is constant and the tensors  $W_{hijk}$  and  $Z_{hijk}$  are identical. Therefore we can state the theorem:

**THEOREM 15.** *In an Einstein  $P_n$  ( $n > 2$ ) the tensor  $W_{hijk}$  can be expressed in the form*

$$W_{hijk} = S_{hj} \lambda_i \lambda_k + S_{ik} \lambda_h \lambda_j - S_{hk} \lambda_i \lambda_j - S_{ij} \lambda_h \lambda_k,$$

where  $S_{ij}$  is a symmetric tensor and  $\lambda_i$  the vector of recurrence of the space.

Differentiating (2.34) covariantly with respect to  $x^m$  and using it again, we get

$$(2.39) \quad \lambda_{i,m} Z_{hijk} + \lambda_{j,m} Z_{hikl} + \lambda_{k,m} Z_{hilj} = 0.$$

Proceeding as in Walker [1], we choose  $u^i$  so that  $u^i \lambda_i = 1$ . Putting  $\theta_j = u^i \lambda_{i,j}$ , we get, from (2.39) after transvecting by  $u^l u^h$ ,

$$u^h \theta_m Z_{hijk} + \lambda_{j,m} (u^l u^h Z_{hikl}) - \lambda_{k,m} (u^h u^l Z_{hijl}) = 0.$$

Applying (2.36) and (2.37), the above equation reduces to

$$(2.40) \quad S_{ik} (\lambda_{j,m} - \theta_m \lambda_j) = S_{ij} (\lambda_{k,m} - \theta_m \lambda_k).$$

Since  $S_{ij}$  is a symmetric tensor, (2.40) shows that either

$$(2.41) \quad \lambda_{j,m} = \theta_m \lambda_j$$

or

$$S_{ij} \text{ is of the form } \varrho_i \varrho_j.$$

Now, if  $Z_{hijk} \neq 0$ , since  $\lambda_i$  is a gradient (theorem 5), then (2.41) gives

$$\lambda_{j,m} = \Phi \lambda_j \lambda_m,$$

$\Phi$  being a scalar function.

Also, if  $S_{ij} = \varrho_i \varrho_j$ , (2.38) reduces to

$$Z_{hijk} = m_{jk} m_{hi},$$

where  $m_{ij} = \varrho_i \lambda_j - \varrho_j \lambda_i$  is a skew symmetric tensor. Thus we may state the following theorems:

**THEOREM 16.** *In a  $P_n$  the scalar curvature of which is constant and  $Z_{hijk} \neq 0$ , either*

$$\lambda_{i,j} = \Phi \lambda_i \lambda_j$$

or

$$Z_{hijk} = m_{jk} m_{hi},$$

where  $\lambda_i$  is the vector of recurrence of the space,  $\Phi$  a scalar function and  $m_{ij}$  a skew symmetric tensor.

**THEOREM 17.** *In an Einstein  $P_n$  for which  $W_{hijk} \neq 0$ , either*

$$\lambda_{i,j} = \Phi \lambda_i \lambda_j$$

or

$$W_{hijk} = m_{hi} m_{jk},$$

where  $\lambda_i$  is the vector of recurrence of the space,  $\Phi$  a scalar function and  $m_{ij}$  a skew symmetric tensor.

### 3. Some properties of a $C_n$ .

(i) In an Einstein space, we have

$$(3.1) \quad R_{ij} = \frac{R}{n} g_{ij}.$$

In consequence of (3.1), the tensor  $C_{ijk}^h$  reduces to  $Z_{ijk}^h$ . Hence we have the result :

**THEOREM 18.** *An Einstein  $C_n$  is a  $P_n$  with the same vector of recurrence.*

Since an Einstein  $C_n$  is a  $P_n$ , the results similar to the theorems 15 and 17 will hold for  $C_{hijk}$  in an Einstein  $C_n$ .

(ii) Let us assume that a  $C_n$  admits a concurrent vector field  $\nu^i$ . In a  $C_n$  the relation

$$(3.2) \quad C_{ijk, l}^h = \lambda_l C_{ijk}^h$$

holds for a non-zero vector  $\lambda_l$ . Contracting the indices  $h$  and  $l$  in (2.10) and simplifying, we obtain

$$(3.3) \quad C_{ijk, h}^h = \frac{n-3}{n-2} \left[ (R_{ij, k} - R_{ik, j}) + \frac{1}{2(n-1)} (R_{, j} g_{ik} - R_{, k} g_{ij}) \right].$$

Again, contracting  $h$  and  $l$  in (3.2) and applying (3.3), we get

$$(3.4) \quad \frac{n-3}{n-2} \left[ (R_{ij, k} - R_{ik, j}) + \frac{1}{2(n-1)} (R_{, j} g_{ik} - R_{, k} g_{ij}) \right] = \\ = \lambda_h \left[ R_{ijk}^h - \frac{1}{n-2} (R_k^h g_{ij} - R_j^h g_{ik} + R_{ij} \delta_k^h - R_{ik} \delta_j^h) + \frac{R}{(n-1)(n-2)} (\delta_k^h g_{ij} - \delta_j^h g_{ik}) \right].$$

In view of (2.13), we have

$$(3.5) \quad \begin{cases} \nu^i R_{ijk}^h = 0, \\ \nu^i R_{ij} = 0. \end{cases}$$

Transvecting (3.4) by  $\nu^i$  and applying (2.15) and (3.5), we obtain

$$(3.6) \quad \frac{n-3}{2(n-1)} (R_{, j} \nu_k - R_{, k} \nu_j) = -\lambda_h (R_k^h \nu_j - R_j^h \nu_k) + \frac{R \lambda_h}{(n-1)} (\nu_j \delta_k^h - \nu_k \delta_j^h).$$

Again, transvecting (3.6) by  $\nu^j$  and writing  $\nu^i \nu_i = \nu$ , we get

$$(3.7) \quad \frac{n-3}{2(n-1)} (\nu^j R_{,j} \nu_k - R_{,k} \nu) = \\ = -\lambda_h (R_k^h \nu - \nu^j R_j^h \nu_k) + \frac{R \lambda_h}{(n-1)} (\nu \delta_k^h - \nu_k \nu^h).$$

In consequence of (2.16) and (3.5), (3.7) reduces to

$$(3.8) \quad \frac{3-n}{2(n-1)} (2R \varrho \nu_k + \nu R_{,k}) = -\lambda_h R_k^h \nu + \frac{R}{n-1} (\nu \lambda_k - \nu_k \lambda^h \nu_h).$$

If  $R = 0$ , (3.8) gives

$$(3.9) \quad \lambda_h R_k^h \nu = 0.$$

Since  $\nu^i$  cannot be null vector (3.9) reduces to

$$\lambda_h R_k^h = 0.$$

Thus we have the theorem:

**THEOREM 19.** *If a  $C_n$  of zero scalar curvature admits a concurrent vector field, then  $\lambda_h R_k^h = 0$ ,  $\lambda_h$  being the vector of recurrence of the space.*

(iii) Putting  $\varrho = 0$ , in (3.8) one gets

$$\frac{n-3}{2(n-1)} \nu R_{,k} = \lambda_h R_k^h \nu - \frac{R}{n-1} (\nu \lambda_k - \nu_k \lambda^h \nu_h).$$

Hence we have the theorem:

**THEOREM 20.** *If a  $C_n$  of zero scalar curvature admits a non-null parallel vector field, then  $\lambda_h R_k^h = 0$ ,  $\lambda_h$  being the vector of recurrence of the space.*

#### 4. Some properties of a $R_n$ .

(i) In a  $R_n$  the relation

$$(4.1) \quad R_{ij, l} = \lambda_l R_{ij},$$

holds for a non zero vector  $\lambda_l$ . From Bianchi identity, we have

$$(4.2) \quad R_{hjk, l}^l = R_{hj, k} - R_{hk, j}.$$

Since the space is  $R_n$ , (4.2) reduces to

$$R_{hjk, l}^l = \lambda_k R_{hj} - \lambda_j R_{hk}.$$

Transvecting the above equation with  $g^{hj}$  and simplifying, we have

$$(4.3) \quad \lambda_h R_k^h = \frac{1}{2} R \lambda_k$$

Hence we have the result :

**THEOREM 21.** *In a  $R_n$  with  $\lambda_i$  as its vector of recurrence,*

$$\lambda_h R_i^h = \frac{1}{2} R \lambda_i,$$

$R$  being the scalar curvature of the space.

(ii) Let us assume that  $R_n$  admits a concircular vector field  $v^i$ . Differentiating (2.24) covariantly with respect to  $x^l$  and using (2.23), we obtain

$$(4.4) \quad \varrho R_{ijk} + v_h R_{ijk, l}^h = \varrho_{k, l} g_{ij} - \varrho_{j, l} g_{ik}.$$

Transvecting (4.4) by  $g^{il}$  and simplifying, we get

$$v^h R_{hjk, l}^l = 0$$

or

$$(4.5) \quad v^h (R_{hj, k} - R_{hk, j}) = 0.$$

Again, transvecting (4.4) by  $g^{ij}$ , one gets

$$(4.6) \quad \varrho R_{lk} + v_h R_{k, l}^h = (n - 1) \varrho_{k, l}.$$

Transvecting (4.6) once again with  $g^{lk}$ , we obtain

$$(4.7) \quad \varrho R + \frac{1}{2} v^h R_{, h} = (n - 1) g^{lk} \varrho_{k, l}.$$

Transvecting (4.4) with  $g^{lk}$ , we get

$$\varrho R_{ij} - v^h R_{jhi, l}^l = (g^{lk} \varrho_{k, l}) g_{ij} - \varrho_{i, j}$$

or,

$$(4.8) \quad \varrho R_{ij} - v^h R_{hj, i} + v^h R_{ij, h} = (g^{lk} \varrho_{k, l}) g_{ij} - \varrho_{i, j}.$$

In virtue of (4.6), (4.8) reduces to

$$(4.9) \quad 2\varrho R_{ij} - (n-2)\varrho_{i,j} + \nu^h R_{ij,h} = (g^{ik}\varrho_{i,k})g_{ij}.$$

From (4.1) and (4.5), we get

$$\lambda_i(\nu^i R_{ij}) = \lambda_j(\nu^i R_{il}).$$

Therefore,

$$(4.10) \quad \nu^h R_{hj} = \bar{\psi} \lambda_j,$$

where  $\bar{\psi}$  is a scalar function of  $x$ 's. In consequence of (2.26), (4.10) gives

$$(4.11) \quad \varrho_i = \psi \lambda_i,$$

where  $\psi = \frac{\bar{\psi}}{n-1}$ . Thus we can state the following theorem:

**THEOREM 22.** *If a  $R_n$  admits a concircular vector field  $\nu^i$  such that  $\nu^i_{,j} = \varrho \delta^i_j$ ,  $\varrho$  being a non constant scalar, then both  $\nu^h R_{hi}$  and  $\varrho_i$  are proportional to  $\lambda_i$ , where  $\lambda_i$  is the vector of recurrence of the space and  $\varrho_i = \frac{\partial \varrho}{\partial x^i}$ .*

(iii) In a  $R_n$  the scalar curvature  $R$  cannot be a non-zero constant. Suppose that  $R$  is zero. Then (4.7) gives

$$(4.12) \quad g^{ik}\varrho_{k,i} = 0.$$

Hence (4.8) reduces to

$$(4.13) \quad \varrho R_{ij} - \lambda_i \nu^h R_{hj} + \nu^h \lambda_h R_{ij} = -\varrho_{i,j}.$$

In view of (4.10), (4.13) finally reduces to

$$(4.14) \quad (\varrho + \nu^h \lambda_h) R_{ij} = \bar{\psi} \lambda_i \lambda_j - \varrho_{i,j}.$$

Transvecting (4.14) with  $g^{ij}$  and using (4.12), we get

$$(4.15) \quad \bar{\psi} g^{ij} \lambda_i \lambda_j = 0,$$

since  $R$  is zero. Now  $\bar{\psi} = 0$  implies  $\psi$  of (4.11) is zero which gives  $\varrho_i = 0$ . But  $\varrho_i$  cannot be zero. Hence  $\bar{\psi} \neq 0$ . Therefore  $\lambda_i$  is null and consequently, by (4.11),  $\varrho_i$  is also null. So transvecting (4.11) by  $\lambda^i$ , we get

$$\varrho_i \lambda^i = 0.$$

Thus,  $\lambda_i$  is orthogonal to  $\varrho_i$ . Since  $R = 0$ , (4.3) gives

$$(4.16) \quad \lambda_h R_k^h = 0.$$

In consequence of (4.11), (4.16) reduces to

$$\varrho_h R_k^h = 0.$$

Hence we have the theorem:

**THEOREM 23.** *If a  $R_n$  of zero scalar curvature admits a concircular vector field  $v^i$  such that  $v^i_{,j} = \varrho \delta_j^i$ ,  $\varrho$  being a non-constant scalar, then (i) both  $\varrho_i$  and  $\lambda_i$  are null, (ii)  $\lambda_i$  is orthogonal to  $\varrho_i$  and (iii)  $\varrho_h R_i^h = 0$ , where  $\lambda_i$  is the vector of recurrence and  $\varrho_i = \frac{\partial \varrho}{\partial x^i}$ .*

## 5. Some properties of a $K_n$ .

Since every  $K_n$  ( $n \geq 3$ ) for which  $R_{ij} \neq 0$  is a  $R_n$ , results similar to the theorems 21, 22 and 23 will hold in a  $K_n$  ( $n \geq 3$ ) with  $R_{ij} \neq 0$ .

Now suppose that a  $K_n$  admits a concurrent vector field  $v^i$ . In a  $K_n$ , we have

$$(5.1) \quad R_{ijk,l}^h = \lambda_l R_{ijk}^h,$$

for a non-zero vector  $\lambda_l$ . In view of (5.1), (2.14) reduces to

$$(5.2) \quad \varrho R_{ijk} + \lambda_l v_h R_{ijk}^h = 0.$$

In consequence of (2.13), (5.2) finally reduces to

$$\varrho R_{ijk} = 0.$$

But neither  $\varrho$  nor  $R_{ijk}$  can be zero. Hence the result:

**THEOREM 24.** *A  $K_n$  cannot admit a concurrent vector field.*

## 6. Some properties of a symmetric space.

In a symmetric space  $V_n$ , we have

$$(6.1) \quad R_{ijk,l}^h = 0.$$



Let the symmetric space  $V_n$  admit a concircular vector field  $\nu^i$ . In virtue of (6.1), (4.6) and (4.9) reduce to

$$(6.2) \quad \varrho R_{ik} = (n-1) \varrho_{i,k}$$

and

$$(6.3) \quad 2\varrho R_{ij} = (n-2) \varrho_{i,j} + (g^{lk} \varrho_{l,k}) g_{ij}$$

respectively. From (6.2) and (6.3), one gets

$$(6.4) \quad \varrho_{i,j} = \Phi g_{ij},$$

where  $\Phi = \frac{1}{n} g^{ik} \varrho_{i,k}$ . Transvecting (6.2) with  $g^{ik}$ , we have

$$(6.5) \quad \varrho R = (n-1) g^{ik} \varrho_{i,k} = n(n-1) \Phi.$$

If  $R \neq 0$ , then  $\Phi$  is a non-constant scalar. Hence we have the theorem:

**THEOREM 25.** *If a symmetric space  $V_n$  of non zero scalar curvature admits a concircular vector field  $\nu^i$  such that  $\nu^i_{,j} = \varrho \delta^i_j$ ,  $\varrho$  being a non-constant scalar, then  $\varrho_i = \frac{\partial \varrho}{\partial x^i}$  itself a concircular vector field of  $V_n$ .*

If  $R = 0$ , (6.5) gives  $\Phi = 0$  and consequently (6.4) reduces to  $\varrho_{i,j} = 0$ . Thus we have the result:

**THEOREM 26.** *If a symmetric space  $V_n$  of zero scalar curvature admits a concircular vector field  $\nu^i$  such that  $\nu^i_{,j} = \varrho \delta^i_j$ ,  $\varrho$  being a non-constant scalar, then  $\varrho_i = \frac{\partial \varrho}{\partial x^i}$  is parallel in  $V_n$ .*

## 7. Some properties of a projective symmetric space.

In a projective symmetric space, we have

$$(7.1) \quad \begin{cases} W^h_{ijk,i} = 0 \\ \text{i. e. } R^h_{ijk,i} = \frac{1}{n-1} (\delta^h_k R_{ij,i} - \delta^h_j R_{ik,i}). \end{cases}$$

Transvecting by  $g^{ij}$ , we have

$$(7.2) \quad R^h_{k,i} = \frac{1}{n} R_{,i} \delta^h_k.$$

Hence

$$(7.3) \quad R_{ij, \iota} = \frac{1}{n} R_{, \iota} g_{ij}.$$

From (7.1) and (7.3), we have

$$(7.4) \quad \begin{cases} R_{ijk, \iota}^h = \frac{R_{, \iota}}{n(n-1)} (\delta_k^h g_{ij} - \delta_j^h g_{ik}) \\ \text{i. e. } Z_{ijk, \iota}^h = 0, \end{cases}$$

where  $Z_{ijk}^h$  is given by (1.1).

Conversely (7.4) implies (7.3), and (7.3) together with (7.4) implies (7.1). Thus we have the theorem :

**THEOREM 27.** *A Riemannian  $V_n$  is projective symmetric if and only if  $Z_{ijk, \iota}^h = 0$ .*

Contracting  $h$  and  $l$  in (7.2), we get  $(n-2)R_{, k} = 0$ . Thus we can state the theorem :

**THEOREM 28.** *A projective symmetric space  $V_n (n > 2)$  is of constant scalar curvature.*

In view of theorems 27 and 28, we get the following result obtained by Matsumoto [3].

**THEOREM 29** *A projective symmetric space  $V_n (n > 2)$  is a symmetric space.*

With the help of the theorem 29 it can be verified that

$$(R^{hijk} R_{hijk}),_{\iota} = 0.$$

Hence we have the theorem :

**THEOREM 30.** *In a projective symmetric space  $R^{hijk} R_{hijk}$  is a constant.*

Every symmetric space is conformally symmetric. Hence, in view of theorem 29, we can state the following result :

**THEOREM 31.** *Every projective symmetric  $V_n (n > 3)$  is conformally symmetric.*

Next we suppose that a projective symmetric space admits a concurrent vector field  $\nu^i$ . In view of theorem 29, (2.14) reduces to

$$\varrho R_{ijk} = 0.$$

Since  $\varrho \neq 0$ ,  $R_{ijk} = 0$ . Hence the result :

**THEOREM 32.** *Every projective symmetric  $V_n (n > 2)$  admitting a concurrent vector field is flat.*

### 8. Some properties of a conformally symmetric space.

In a conformally symmetric space (1.7) holds. Therefore, from (3.3), we get

$$(8.1) \quad \frac{n-3}{n-2} \left[ (R_{ij,k} - R_{ik,j}) + \frac{1}{2(n-1)} (R_{,j} g_{ik} - R_{,k} g_{ij}) \right] = 0.$$

For  $n > 3$ , (8.1) reduces to

$$(8.2) \quad (R_{ij,k} - R_{ik,j}) + \frac{1}{2(n-1)} (R_{,j} g_{ik} - R_{,k} g_{ij}) = 0.$$

(i) Now suppose that the space admits a concircular vector field  $\nu^i$ . Transvecting (8.2) by  $\nu^i$  and using (4.5), we get

$$R_{,j} \nu_k = R_{,k} \nu_j$$

which shows that

$$\nu_i = \Phi R_{,i}.$$

where  $\Phi$  is a scalar function of  $x$ 's. Hence we have the result:

**THEOREM 33.** *If a conformally symmetric space  $V_n$  ( $n > 3$ ) of non-constant scalar curvature  $R$  admits a concircular vector field  $\nu^i$ , then  $\nu^i$  will necessarily be a scalar multiple of  $\frac{\partial}{\partial x^i}(R)$ .*

(ii) Next we suppose that the space admits a concurrent vector field  $\nu^i$ .

Transvecting (8.2) by  $\nu^i$  and applying (2.15), we get

$$(8.3) \quad \nu_k R_{,j} - \nu_j R_{,k} = 0.$$

Hence we have the result:

**THEOREM 34.** *If a conformally symmetric space  $V_n$  ( $n > 3$ ) of non-constant scalar curvature  $R$  admits a concurrent vector field  $\nu^i$ , then  $\nu^i$  will necessarily be a scalar multiple of  $\frac{\partial}{\partial x^i}(R)$ .*

Transvecting (8.3) by  $\nu^j$  and writing  $\nu^j \nu_j = \nu$ , we obtain

$$(8.4) \quad \nu_k R_{,j} \nu^j - \nu R_{,k} = 0.$$

In view of (2.16), (8.4) reduces to

$$(8.5) \quad 2\rho R \nu_k = -\nu R_{,k}.$$

Putting  $\rho = 0$  in (8.5), one gets  $\nu R_{,k} = 0$ . Hence the result:

**THEOREM 35.** *If a conformally symmetric space  $V_n$  ( $n > 3$ ) admits a non-null parallel vector field, then the scalar curvature of the space is constant.*

### 9. Some properties of a Ricci-symmetric space.

(i) In a Ricci-symmetric space, we have

$$(9.1) \quad R_{ij, \iota} = 0.$$

Consequently

$$(9.2) \quad R_{, \iota} = 0.$$

In virtue of (9.2),  $Z_{ijk, \iota}^h$  reduces to  $R_{ijk, \iota}^h$ , where  $Z_{ijk}^h$  is given by (1.1). If the Ricci-symmetric space is also a  $P_n$  with  $\lambda_i$  as its vector of recurrence, then we have

$$(9.3) \quad Z_{ijk, \iota}^h = \lambda_i Z_{ijk}^h.$$

In view of (1.1) and (9.2), (9.3) reduces to

$$(9.4) \quad R_{ijk, \iota}^h = \lambda_i \left[ R_{ijk}^h - \frac{R}{n(n-1)} (\delta_k^h g_{ij} - \delta_j^h g_{ik}) \right].$$

Contracting  $h$  and  $k$  in (9.4) and using (9.1), we get

$$(9.5) \quad \lambda_i \left[ R_{ij} - \frac{R}{n} g_{ij} \right] = 0.$$

Since  $\lambda_i$  is a non-zero vector, (9.5) gives

$$R_{ij} = \frac{R}{n} g_{ij}.$$

Hence we have the result :

**THEOREM 36.** *A Ricci-symmetric  $P_n$  is an Einstein space.*

Since a Ricci-symmetric  $P_n$  is an Einstein space, the results similar to theorems 15 and 17 will hold in a Ricci-symmetric  $P_n$ .

(ii) Next suppose that the Ricci-symmetric space is a  $C_n$ . In view of (9.1) and (9.2) we have

$$(9.6) \quad C_{hijk, \iota} = R_{hijk, \iota}.$$

Also  $C_{hijk}$  is seen to satisfy the relations

$$C_{hik} = -C_{ihjk} = -C_{hikj} = C_{jkhi}.$$

Therefore, proceeding as in section 2 (vii), one can show that

$$C_{hijk} = S_{hj} \lambda_i \lambda_k + S_{ik} \lambda_j \lambda_h - S_{hk} \lambda_i \lambda_j - S_{ij} \lambda_h \lambda_k ,$$

where  $S_{ij}$  is a symmetric tensor and  $\lambda_i$  the vector of recurrence of the space of the space. Thus, we have the result :

**THEOREM 37.** *In a Ricci-symmetric  $C_n$ , the tensor  $C_{hijk}$  can be expressed in the form*

$$C_{hijk} = S_{hj} \lambda_i \lambda_k + S_{ik} \lambda_j \lambda_h - S_{hk} \lambda_i \lambda_j - S_{ij} \lambda_h \lambda_k ,$$

where  $S_{ij}$  is a symmetric tensor and  $\lambda_i$  the vector of recurrence of the space.

Because of the relation (9.6), proceeding as in section 2 (iii), one can prove the following result :

**THEOREM 38.** *In a Ricci-symmetric  $C_n$ , either the vector of recurrence is a gradient or the space is conformally flat.*

Proceeding as in section 2 (vii) and using the relation (9.6) and the theorems 37 and 38, one can establish the following result :

**THEOREM 39.** *In a Ricci-symmetric  $C_n$  which is not conformally flat either*

$$\lambda_{i,j} = \Phi \lambda_i \lambda_j$$

or

$$C_{hijk} = m_{hi} m_{jk} ,$$

where  $\lambda_i$  is the vector of recurrence of the space,  $\Phi$  a scalar function and  $m_{ij}$  a skew symmetric tensor.

(iii) Let us assume that a Ricci symmetric space admits a concurrent vector field  $v^i$ . In a Ricci-symmetric space (2.15) reduces to

$$\varrho R_{ik} = 0.$$

As  $\varrho \neq 0$ ,  $R_{ik} = 0$ . Thus we have the result :

**THEOREM 40.** *A Ricci-symmetric space for which  $R_{ij}$  is non-zero does not admit a concurrent vector field.*

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