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EXISTENCE AND APPROXIMATION OF WEAK SOLUTIONS OF ABSTRACT DIFFERENTIAL EQUATIONS

by M. DJEDOUR

Let $(,)$ and $\| \cdot \|$ denote the scalar product and the norm in the Hilbert space H .

In the following we will be concerned with the differential equation :

$$(1) \quad u^{(n)}(t) + A_1 u^{(n-1)}(t) + A_2 u^{(n-2)}(t) + \dots + A_n u(t) = f(t)$$

where $A_1, A_2, \dots, A_n$ are linear operators defined on the Hilbert space H . The operators $A_1, \dots, A_n$ are supposed to be continuous on H except for one of them, A_{k_0} which is generally (an unbounded operator) a closed operator defined on a dense subset $\mathcal{D}_{A_{k_0}}$ of H .

The function $f(t)$ belongs to $L^2_{loc}(R; H)$, the space of all H -valued strongly measurable functions such that the norm $\|g(t)\|$ is square integrable on every compact subset of R .

In Lemma 2, we show that (1) has a local solution then, with Lemma 6 (Density) and 7 (approximation) we are able to prove the existence of a solution of (1) in the sense of Definition I below.

For convenience, let :

$$K^*(a, b) = \{ \Phi : \Phi(t) \in C_0^n(a, b); \mathcal{D}_{A_{k_0}^*}, A_j^* \Phi \in C^{n-j}((a, b); H), \quad j = 1, \dots, n \}$$

$$K^*(a, b) = \{ \Phi : \Phi(t) \in C_0^n(a, b); \mathcal{D}_{A_{k_0}}, A_j \Phi \in C^{n-j}((a, b); H), \quad j = 1, \dots, n \}$$

$$K^* = \{ \Phi : \Phi(t) \in C_0^n(R; \mathcal{D}_{A_{k_0}^*}); A_j^* \Phi \in C^{n-j}(R; H), \quad j = 1, \dots, n \}$$

$$K = \{ \Phi : \Phi(t) \in C_0^n(R, \mathcal{D}_{A_{k_0}}); A_j \Phi \in C^{n-j}(R; H); \quad j = 1, \dots, n \}.$$

DEFINITION I

a) For a given function $f(t) \in L^2((a, b); H)$ we say that $u(t) \in L^2((a, b); H)$ is a weak solution of (1) on (a, b) if the following hold :

$$(2) \quad \int_a^b (u(t), (-1)^n \Phi^{(n)} + \sum_{j=1}^n (-1)^{n-j} (A_j^* \Phi)^{(n-j)}) dt = \int_a^b (f(t), \Phi(t)) dt$$

for all $\Phi \in K^*(a, b)$.

b) Similarly we define $u(t) \in L_{loc}^2(R; H)$ as a weak solution of (1) on R_1 if :

$$(3) \quad \int_{-\infty}^{\infty} (u(t), (-1)^n \Phi^{(n)}(t) + \sum_{j=1}^n (-1)^{n-j} (A_j^* \Phi)^{(n-j)}) dt = \int_{-\infty}^{\infty} (f(t), \Phi(t)) dt$$

hold for all $\Phi \in K^*$ where $f(t)$ is given in $L_{loc}^2(R; H)$.

In [1] and [2], S. Zaidman considered the following equations :

$$(4) \quad u'(t) + A u(t) = f(t)$$

$$(5) \quad u'' + A u(t) = f(t)$$

with A a closed operator with dense domain $\mathcal{D}_A$ in H . Upon certain condition on A , S. Zaidman has shown that a weak solution $u(t)$ in $L_{loc}^2(R; H)$ exists in the sense of (3) for every given function $f(t)$ in $L_{loc}^2(R; H)$.

The purpose of this paper is to generalize the method of S. Zaidman to get a weak solution of (1) in the sense of (3) for every given function $f(t)$ in $L_{loc}^2(R; H)$.

DEFINITION II : [3] Let j be a positive integer and s a positive real and F a family of vertical lines of the complex plane given by $\operatorname{Re} \lambda = \sigma_n$ and $\operatorname{Re} \lambda = \sigma'_n$, $\sigma_n \rightarrow +\infty$, $\sigma'_n \rightarrow -\infty$.

We shall say that the operators : $A_1, A_2, \dots, A_{k_0}, \dots, A_n$ satisfy the condition S on F if :

$$(6) \quad \left\| \left[(-1)^n \lambda^n I + \sum_{j=0}^{n-1} (-1)^j \lambda^j A_{n-j}^* \right]^{-1} \right\| \leq M$$

hold on every line of F except possibly for j intervals of length s .

We will say that $\{A_1, A_2, \dots, A_n\}$ are (j, s) bounded on F .

We will prove the following theorem :

THEOREM: Let the equation (1), with $A_1, A_2, \dots, A_n$ continuous except for A_{k_0} which is a (generally) unbounded closed operator with dense domain and suppose moreover $\{A_1, A_2, \dots, A_n\}$ satisfying the condition S above, then for any given $f(t) \in L^2_{\text{loc}}(R; H)$ there exist a $u(t) \in L^2_{\text{loc}}(R; H)$ solution of (3).

LEMMA I. Let the operators $A_1, A_2, \dots, A_{k_0}, \dots, A_j, \dots, A_n$ be continuous except for A_{k_0} which is closed with dense domain $\mathcal{D}_{A_{k_0}}$ and $\{A_1, A_2, \dots, A_n\}$ (j, s) -bounded on the line $\text{Re } \lambda = \sigma$.

Then for every bounded interval $(a, b) \subset R$ and for every $u(t) \in K^*(a, b)$ we have:

$$(7) \quad \int_a^b \|u(t)\|^2 dt \leq C e^{2\sigma(b-a)} \int_a^b \left\| (-1)^n u^{(n)}(t) + \sum_{j=1}^n (-1)^{n-j} (A_j^* u)^{(n-j)} \right\|^2 dt.$$

PROOF: Let $V(t) = e^{\sigma t} u(t)$.

From:

$$(-1)^n u^{(n)}(t) + \sum_{j=1}^n (-1)^{n-j} (A_j^* u)^{(n-j)} = f(t)$$

we deduce:

$$\begin{aligned} & (-1)^n \left[\sum_{l=0}^n C_l^n (-1)^l \sigma^l V^{(n-l)}(t) \right] \\ & + \sum_{j=1}^n (-1)^{n-j} \left[\sum_{l=0}^{n-j} C_l^{n-j} (-1)^l \sigma^l (A_j^* V)^{(n-j-l)} \right] = f(t) e^{\sigma t} = g(t) \end{aligned}$$

which can be written as:

$$(-1)^n \left[\sum_{l=0}^n C_l^n (-1)^l \sigma^l V^{(n-l)}(t) \right] + \sum_{j=0}^{n-1} (-1)^j \left[\sum_{l=0}^j C_l^j (-1)^l \sigma^l (A_{n-j}^* V)^{(j-l)} \right] = g(t).$$

Let us take the Fourier transform on both sides: we obtain:

$$(-1)^n \left[\sum_{l=0}^n C_l^n (-1)^l \sigma^l (i\tau)^{n-l} \widehat{V}(\tau) \right] + \sum_{j=0}^{n-1} (-1)^j \left[C_l^j (-1)^l \sigma^l (i\tau)^{j-l} A_{n-j}^* \widehat{V}(\tau) \right] = \widehat{g}(\tau)$$

i. e., if we set: $\lambda = -\sigma + i\tau$

$$\left[(-1)^n \lambda^n I + \sum_{j=0}^{n-1} (-1)^j \lambda^j A_{n-j}^* \right] \widehat{V}(\tau) = \widehat{g}(\tau)$$

where $\widehat{V}(\tau)$ and $\widehat{g}(\tau)$ are the Fourier transform of $V(t)$ and $g(t)$.

Let Γ be the real axis $-\infty < \tau < +\infty$, from which we delete j intervals of length s .

Then for $\tau \in \Gamma$; by hypothesis :

$$\left\| \left[(-1)^n \lambda^n I + \sum_{j=0}^{n-1} (-1)^j \lambda^j A_{n-j}^* \right]^{-1} \right\| \leq M \quad \tau \in \Gamma \quad \operatorname{Re} \lambda = -\sigma$$

$$\Rightarrow \|\widehat{V}(\tau)\| \leq M \|\widehat{g}(\tau)\|, \quad \tau \in \Gamma$$

$$\Rightarrow \int_{\Gamma} \|\widehat{V}(\tau)\|^2 d\tau \leq M^2 \int_{-\infty}^{\infty} \|\widehat{g}(\tau)\|^2 d\tau.$$

Since $V(t)$ has compact support in (a, b) by a result of S. Agmon-L. Nirenberg [3], there exist $k = k(j, s)$ such that :

$$\int_{-\infty}^{\infty} \|\widehat{V}(\tau)\|^2 d\tau \leq k \int_{\Gamma} \|\widehat{V}(\tau)\|^2 d\tau \Rightarrow \int_{-\infty}^{\infty} \|\widehat{V}(\tau)\|^2 d\tau \leq k M^2 \int_{-\infty}^{\infty} \|\widehat{g}(\tau)\|^2 d\tau.$$

Using the vector form of Parseval's Theorem we get :

$$\int_{-\infty}^{\infty} \|V(t)\|^2 dt = \int_a^b e^{2\sigma t} \|u(t)\|^2 dt \leq k M^2 \int_a^b \|f(t)\|^2 e^{2\sigma t} dt$$

If we suppose $\sigma < 0$, we have,

$$e^{2\sigma b} \int_a^b \|u(t)\|^2 dt \leq k M^2 e^{2\sigma a} \int_a^b \left\| (-1)^n u^{(n)}(t) + \sum_{j=1}^n (-1)^{n-j} (A_j^* u(t))^{(n-j)} \right\|^2 dt.$$

Hence (7) with $C = k M^2$.

LEMME 2 (Local existence):

Under the same hypothesis as in Lemma 1, for every $f(t) \in L^2((a, b); H)$ there exist a function $u(t) \in L^2((a, b); H)$ satisfying (2).

PROOF: Consider the linear subspace

$$\left[(-1)^n \frac{d^n}{dt^n} + \sum_{j=0}^{n-1} (-1)^j \frac{d^j}{dt^j} 0 A_{n-j}^* \right] (K^*(a, b))$$

in $L^2((a, b); H)$. We can define a linear form F by

$$F \left[(-1)^n \Phi^{(n)}(t) + \sum_{j=0}^{n-1} (-1)^j (A_{n-j}^* \Phi)^{(j)} \right] = \int_a^b (f, \Phi)_H dt, \quad \Phi \in K^*(a, b)$$

which is well defined by (7),

F is continuous since:

$$\begin{aligned} \left| \int_a^b (f, \Phi) dt \right| &\leq C \left\{ \int_a^b \|\Phi\|^2 dt \right\}^{\frac{1}{2}} \leq \\ &\leq C_1 \left\{ \int_a^b \left\| (-1)^n \Phi^{(n)} + \sum_{j=0}^{n-1} (-1)^j (A_{n-j}^* \Phi)^{(j)} \right\|^2 dt \right\}^{\frac{1}{2}} \\ &= C_1 \left\| (-1)^n \Phi^{(n)}(t) + \sum_{j=0}^{n-1} (-1)^j (A_{n-j}^* \Phi)^{(j)} \right\|_{L^2(a, b); H}. \end{aligned}$$

Hence by the Hahn-Banach theorem F has an extension to $L^2((a, b); H)$ and there exist $u(t) \in L^2(a, b); H$ such that:

$$\begin{aligned} F \left((-1)^n \Phi^{(n)}(t) + \sum_{j=0}^{n-1} (-1)^j (A_{n-j}^* \Phi)^{(j)} \right) &= \\ &= \int_a^b (u(t), (-1)^n \Phi^{(n)}(t) + (-1)^j (A_{n-j}^* \Phi)^{(j)}) dt = \int_a^b (f, \Phi) dt. \end{aligned}$$

for every $\Phi \in K^*(a, b)$.

LEMME 3 (Unicity): Let $\{A_1, A_2, \dots, A_n\}$ satisfy the condition S and $u(t)$ defined on (a, b) with values in $D_{A_{k_0}}^*$ such that $u^{(n-k_0)} \in \mathcal{D}_{A_{k_0}^*}$ and:

$$(8) \quad (-1)^n u^{(n)}(t) + \sum_{j=0}^{n-1} (-1)^j (A_{n-j}^* u)^{(j)} = 0 \quad t \in (a, b),$$

and: $0 \leq j \leq n-1 : u^{(j)}(c) = 0$ for each $j = 0, \dots, n-1$, then $u \equiv 0$ in (a, b) .

PROOF: We prove that $u \equiv 0$ for $c \leq t < b$.

For this, let:

$$\zeta(t) \in C^\infty(c, b)$$

such that:

$$\zeta(t) = \begin{cases} 1 & c \leq t \leq \alpha < b \\ 0 & \alpha + \delta \leq t \leq b \end{cases}$$

and set:

$$V(t) = \begin{cases} 0 & t < c, t > b \\ e^{\sigma t} \zeta(t) u(t) & t \in (c, b) \end{cases}$$

$$\implies V(t) e^{-\sigma t} = \zeta(t) u(t).$$

Then equation (8) on this function gives:

$$\begin{aligned} (9) \quad & (-1)^n \sum_{k=0}^n C_k^n (-1)^k \sigma^k V^{(n-k)} e^{-\sigma t} + \sum_{j=0}^{n-1} \left[\sum_{k=0}^j C_k^j (-1)^{k+1} \sigma^k (A_{n-j}^* V(t))^{(j-k)} e^{-\sigma t} \right] \\ & = (-1)^n \sum_{k=0}^n C_k^n \zeta^{(k)} u^{(n-k)} + \sum_{j=0}^{n-1} \left[\sum_{k=0}^j C_k^j (-1)^j \zeta^{(k)} (A_{n-j}^* u)^{(j-k)} \right]. \end{aligned}$$

The right hand side of (9) can be rewritten as:

$$\begin{aligned} & \left[(-1)^n u^{(n)} + \sum_{j=0}^{n-1} (-1)^j (A_{n-j}^* u)^{(j)} \right] \zeta + \dots + \\ & + \zeta^{(k)} \left[(-1)^n u^{(n-k)} C_k^n + \sum_{j \geq k}^{n-1} C_k^j (-1)^j (A_{n-j}^* u)^{(j-k)} \right] + \dots \end{aligned}$$

And in behalf of (8), the right hand side of (9) can be written in the form:

$$(10) \quad \sum_{k=1}^{n-1} \zeta^{(k)} \left[(-1)^n C_k^n u^{(n-k)} + \sum_{j \geq k}^{n-1} C_k^j (-1)^j (A_{n-j}^* u)^{(j-k)} \right] + \zeta^{(n)} u.$$

So (9) becomes:

$$\begin{aligned} (11) \quad & (-1)^n \left[\sum_{k=0}^n C_k^n (-1)^k \sigma^k V^{(n-k)} \right] + \sum_{j=0}^{n-1} \left[(-1)^j \sum_{k=0}^j C_k^j (-1)^k \sigma^k (A_{n-j}^* V)^{(j-k)} \right] \\ & = e^{\sigma t} \left\{ \sum_{k=1}^n \zeta^{(k)} \left[(-1)^n C_k^n u^{(n-k)} + \sum_{j \geq k}^{n-1} C_k^j (-1)^j (A_{n-j}^* u)^{(j-k)} \right] + \zeta^{(n)} u \right\} \\ & = f(t) \text{ for } c \leq t \leq b. \end{aligned}$$

If we set $f(t) = 0$ outside (c, b) . The hypothesis on u and $u^{(j)}$ imply that the equation (11) is valid for all $-\infty < t < +\infty$. We can then consider the Fourier transform of $f(t)$.

So we get :

$$(12) \quad (-1)^n \left[\sum_{k=0}^n C_k^n (-1)^k \sigma^k (+i\tau)^{n-k} \right] \widehat{V}(\tau) +$$

$$\sum_{j=0}^{n-1} \left[(-1)^j \sum_{k=0}^j (-1)^k \sigma^k (i\tau)^{j-k} A_{n-j}^* \right] \widehat{V}(\tau) = \widehat{f}(\tau)$$

$$\Rightarrow \left[(-1)^n \lambda^n I + \sum_{j=0}^{n-1} (-1)^j \lambda^j A_{n-j}^* \right] \widehat{V}(\tau) = \widehat{f}(\tau), \quad \text{with } \lambda = -\sigma + i\tau.$$

Γ being as in lemma 1, the condition S on $\{A_1, \dots, A_n\}$ gives for $\tau \in \Gamma$

$$\|\widehat{V}(\tau)\| \leq M \|\widehat{f}(\tau)\| \Rightarrow \int_{\Gamma} \|\widehat{V}(\tau)\|^2 d\tau \leq M^2 \int_{-\infty}^{\infty} \|\widehat{f}(\tau)\|^2 d\tau.$$

And with the same argument as in Lemma 1 :

$$\int_{-\infty}^{\infty} \|V(t)\|^2 dt \leq k M^2 \int_{-\infty}^{\infty} \|f(t)\|^2 dt$$

$$\Rightarrow \int_c^a e^{2\sigma t} \|u(t)\|^2 dt \leq k M^2 \int_a^{a+\delta} \sum_{k=1}^n \zeta^{(k)} \left[(-1)^n C_k^n u^{(n-k)} \right.$$

$$\left. + \sum_{j \geq k}^{n-1} C_k^j (-1)^j (A_{n-j}^* u)^{(j-k)} \right] + \zeta^{(n)} u \|^2.$$

If we suppose : $\sigma < 0$ and take $\beta < \alpha$, we get :

$$e^{2\sigma\beta} \int_c^\beta \|u(t)\|^2 dt \leq c_u k M^2 e^{2\sigma\alpha} \quad \forall \sigma < 0.$$

In particular for : $\sigma = \sigma'_n \rightarrow -\infty$.

$$\Rightarrow \int_c^\beta \|u(t)\|^2 dt \leq c_u k M^2 e^{2\sigma'_n(\alpha-\beta)} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

$\implies u(t) \equiv 0$ for $c \leq t \leq \beta$. As $\beta < \alpha$ and β and α where arbitrary it follows that

$$u(t) = 0 \quad c \leq t < b.$$

A similar method using the sequence $\sigma_n \rightarrow \infty$, gives

$$u(t) = 0 \quad a < t \leq c \implies u(t) = 0 \quad t \in (a, b).$$

COROLLARY. If $\{A_1, A_2, \dots, A_n\}$ satisfy condition S , $u(t) \in K^*$ and

$$(-1)^n u^{(n)}(t) + \sum_{j=0}^{n-1} (-1)^j (A_{n-j}^* u)^{(j)} = 0 \quad \text{outside } [a, b].$$

Then $u(t) = 0$ outside $[a, b]$.

LEMMA 4 (regularisation): Let $A_1, A_2, \dots, A_n$ be all continuous on H except for some A_{k_0} which is a closed operator of dense domain in H .

Then for given $f(t) \in L_{\text{loc}}^2(R; H)$ and $u(t) \in L_{\text{loc}}^2(R; H)$ satisfying (3), i. e.:

$$\int_{-\infty}^{\infty} (u(t), (-1)^n \Phi^{(n)}(t) + \sum_{j=1}^n (-1)^{n-j} (A_j^* \Phi)^{(n-j)}) dt = \int_{-\infty}^{\infty} (f(t), \Phi(t))^{(n-j)} dt.$$

for all $\Phi \in K^*$.

We have for every $\alpha(t) \in C_0^\infty(R)$, if: $u * \alpha(t) = \int_{-\infty}^{\infty} u(\tau) \alpha(t - \tau) d\tau$, then

$$(13) \quad (-1)^n \frac{d^n}{dt^n} (u_* \alpha) + A_1 (u_* \alpha)^{(n-1)} + A_2 (u_* \alpha)^{(n-2)} + \dots + A_n (u_* \alpha) = f_* \alpha.$$

with

$$(u_* \alpha)^{n-k_0} \in \mathcal{D}_{A_{k_0}}.$$

PROOF: We have by hypothesis:

$$\int_{-\infty}^{\infty} (u(t), (-1)^n \Phi^{(n)} + \sum_{j=0}^{n-1} (-1)^j (A_{n-j}^* \Phi)^{(j)}) dt = \int_{-\infty}^{\infty} (f(t), \Phi(t)) dt \quad \Phi \in K^*.$$

Denote by $\check{*}$ the operation:

$$\Phi(t) \rightarrow \int_{-\infty}^{\infty} \alpha(\zeta) \Phi(t + \zeta) d\zeta \quad \text{with } \alpha \in C_0^\infty(R).$$

So we have :

$$\begin{aligned}
 & \int_{-\infty}^{\infty} \left((u * \alpha)(t), (-1)^n \Phi^{(n)} + \sum_{j=0}^{n-1} (-1)^j (A_{n-j}^* \Phi)^{(j)} \right) dt \\
 &= \int_{-\infty}^{\infty} \left(u(t), \left[(-1)^n \Phi^{(n)} + \sum_{j=0}^{n-1} (-1)^j (A_{n-j}^* \Phi)^{(j)} \right] * \alpha \right) dt \\
 &= \int_{-\infty}^{\infty} \left(u, (-1)^n \Phi^{(n)} * \alpha + \sum_{j=0}^{n-1} (-1)^j (A_{n-j}^* \Phi)^{(j)} * \alpha \right) dt \\
 &= \int_{-\infty}^{\infty} \left(u, (-1)^n (\Phi * \alpha)^{(n)} + \sum_{j=0}^{n-1} (-1)^j (A_{n-j}^* \Phi * \alpha)^{(j)} \right) dt \\
 &= \int_{-\infty}^{\infty} (f(t), \Phi * \alpha) dt
 \end{aligned}$$

since $\Phi \rightarrow \Phi * \alpha \in K^*$

$$\Rightarrow \int_{-\infty}^{\infty} \left(u * \alpha, (-1)^n \Phi^{(n)} + \sum_{j=0}^{n-1} (-1)^j (A_{n-j}^* \Phi)^{(j)} \right) dt = \int_{-\infty}^{\infty} (f * \alpha, \Phi) dt.$$

And since $u * \alpha$ is infinitely differentiable in H , we get :

$$\int_{-\infty}^{\infty} (u * \alpha, (-1)^n \Phi^{(n)}) dt = \int_{-\infty}^{\infty} ((u * \alpha)^{(n)}, \Phi(t)) dt.$$

So we have :

$$\begin{aligned}
 \sum_{j=0}^{n-1} \int_{-\infty}^{\infty} (u * \alpha, (-1)^j (A_{n-j}^* \Phi)^{(j)}) dt &= \int_{-\infty}^{\infty} (f * \alpha, \Phi) dt - \int_{-\infty}^{\infty} ((u * \alpha)^{(n)}, \Phi) dt \\
 \sum_{j=0}^{n-1} \int_{-\infty}^{\infty} ((u * \alpha)^{(j)}, A_{n-j}^* \Phi) dt &= \int_{-\infty}^{\infty} (f * \alpha, \Phi) dt - \int_{-\infty}^{\infty} ((u * \alpha)^{(n)}, \Phi) dt.
 \end{aligned}$$

As the operator $A_1, A_2, \dots, A_n$ are continuous except for one of them A_{k_0} which is supposed to be a closed operator, we have :

$$\begin{aligned} \int_{-\infty}^{\infty} ((u * \alpha)^{(n-k_0)}, A_{k_0}^* \Phi) dt &= \int_{-\infty}^{\infty} (f * \alpha, \Phi) dt - \int_{-\infty}^{\infty} ((u * \alpha)^{(n)}, \Phi) dt \\ &\quad - \sum_{n-j \neq k_0} \int_{-\infty}^{\infty} ((u * \alpha)^{(j)}, A_{n-j}^* \Phi) dt. \end{aligned}$$

Let now $\Phi(t) = \nu(t) V$ where $\nu(t) \in C_0^\infty(R)$ and $V \in \mathcal{D}_{A_{k_0}^*}$, so we obtain :

$$\begin{aligned} \int_{-\infty}^{\infty} ((u * \alpha)^{(n-k_0)}, A_{k_0}^* \nu(t) V) dt &= \int_{-\infty}^{\infty} (f * \alpha, \nu(t) V) dt - \int_{-\infty}^{\infty} ((u * \alpha)^{(n)}, \nu(t) V) dt \\ &\quad - \sum_{n-j \neq k_0} \int_{-\infty}^{\infty} ((u * \alpha)^{(j)}, A_{n-j}^* \nu(t) V) dt \\ (14) \quad &\Rightarrow \left(\int_{-\infty}^{\infty} \nu(t) (u * \alpha)^{(n-k_0)}(t) dt, A_{k_0}^* V \right) \\ &= \left(\int_{-\infty}^{\infty} \nu(t) \left[(f * \alpha) - (u * \alpha)^{(n)} - \sum_{n-j \neq k_0} A_{n-j} (u * \alpha)^{(j)} \right] dt, V \right). \end{aligned}$$

As the $A_j, j \neq k_0$ are continuous.

Since (14) is valid for all $V \in \mathcal{D}_{A_{k_0}^*}$ it follows that

$$\int_{-\infty}^{\infty} \nu(t) (u * \alpha)^{(n-k_0)}(t) dt \mathcal{D}_{A_{k_0}^{**}} = \mathcal{D}_{A_{k_0}} \quad \forall \nu(t) \in C_0^\infty(R).$$

And since $(u * \alpha)^{(n-k_0)}$ is continuous there exists a sequence $v_p(t)$ such that :

$$\int_{-\infty}^{\infty} v_p(t) (u * \alpha)^{(n-k_0)}(t) dt \rightarrow (u * \alpha)^{(n-k_0)}(t) \quad \forall t \in R.$$

From (14) we have :

$$A_k \int_{-\infty}^{\infty} \nu(t) (u * \alpha)^{(n-k)}(t) dt = \int_{-\infty}^{\infty} \nu(t) \left[(f * \alpha) - (u * \alpha)^{(n)} - \sum_{n-j \neq k_0} A_{n-j} (u * \alpha)^{(j)} \right] dt.$$

And since A_{k_0} is closed :

$$A_{k_0} (u * \alpha)^{(n-k_0)} = (f * \alpha) - (u * \alpha)^{(n)}(t) - \sum_{n-j \neq k_0} A_{n-j} (u * \alpha)^{(j)}(t) \quad \forall t \in R$$

so the relation (13)

$$(u * \alpha)^{(n)} + \sum_{j=0}^{n-1} A_{n-j} (u * \alpha)^{(j)} = f * \alpha \quad \forall t \in R.$$

LEMMA 5 (Unicity): Let $A_1^*, \dots, A_n^*$ satisfy condition S and let $u(t) \in L_{loc}^2(R; H)$ with compact support in R be such that :

$$\int_R \left(u(t), (-1)^n + \sum_{j=0}^{n-1} (-1)^j (A_{n-j} \Phi^{(j)}) \right) dt = \int_R (f(t), \Phi(t)) dt, \quad \Phi \in K$$

with $f \in L_{loc}^2(R; H)$ and $\text{supp } f \subset [a, b]$.

Then $\text{supp } u \subset [a, b]$.

PROOF: Let $\alpha_n(t) \in C_0^\infty$ such that $\alpha_n(t) \rightarrow \delta$ (the Dirac function) with $\alpha_n(t) = 0, \quad |t| > \frac{1}{n}$.

Then by the preceding Lemma 4 we have for

$$u_k = u * \alpha_k, \quad k = 1, 2, \dots$$

$$(u * \alpha_k)^{(n)} + \sum_{j=0}^{n-1} A_{n-j}^* (u * \alpha_k)^{(j)} = 0 \quad \text{for } t \notin \left(a - \frac{1}{k}, b + \frac{1}{k} \right)$$

by corollary of lemma: $u_k(t) = 0$ for $t \notin \left(a - \frac{1}{k}, b + \frac{1}{k} \right)$.

Since $u(t)$ has compact support, $u_k(t)$ has compact support :

And $u_k(t) \rightarrow u(t)$ in $L_{loc}^2(R; H)$ implies

$$u(t) = 0 \text{ outside } [a, b] \text{ a. e.}$$

DEFINITION. For $T > 0$, let V_T be the set of functions $u(t) \in L^2(-T, T; H)$ such that :

$$\int_{-T}^T (u(t), (-1)^n \Phi^{(n)} + \sum_{j=0}^{n-1} (A_{n-j}^* \Phi)^{(j)}) dt = 0$$

or $\forall \Phi \in K^*(-T, T)$.

LEMMA 6 (Density): Let $\{A_1, \dots, A_n\}$ satisfy condition S and $0 < T_1 < T_2 < T_3$ three arbitrary positive numbers.

Then V_{T_3} is dense in V_{T_2} for the $L^2(-T_1, T_1; H)$ topology.

PROOF: Let $\psi(t) \in L^2(-T_1, T_1; H)$ such that $\int_{-T_1}^{T_1} (\psi, V) dt = 0$ for all $v \in V_{T_3}$. We shall show that:

$$\int_{-T}^{T_1} (\psi, h) dt = 0 \text{ for all } h(t) \in V_{T_2}.$$

For this, let $\psi(t) = 0$ outside $[-T_1, T_1]$.

Let

$$M = \left\{ (-1)^{n_k(n)} + \sum_{j=0}^{n-1} (-1)^{(j)} (A_{n-j}^* k)^{(j)} \right\}; \quad k(t) \in K^*(-T_3, T_3).$$

Then $\psi(t) \in \bar{M}$ [the closure in $L^2(-T_3, T_3; H)$]

For if $U(t) \in L^2(-T_3, T_3; H)$ satisfies

$$\int_{-T_3}^{T_3} (U, (-1)^{n_k(n)} + \sum_{j=0}^{n-1} (-1)^{(j)} (A_{n-j}^* k)^{(j)}) dt = 0:$$

$$k(t) \in K^*(-T_3, T_3)$$

then $U(t) \in V_{T_3}$ and

$$\int_{-T_3}^{T_3} (\psi, U) dt = \int_{-T_1}^{T_1} (\psi, U) dt = 0.$$

It follows that there exist a sequence $\{k_m\} \in K^*(-T_3, T_3)$ such that:

$$\lim_{m \rightarrow \infty} \left((-1)^n k_m^{(n)} + \sum_{j=0}^{n-1} (-1)^{(j)} (A_{n-j}^* k_m)^{(j)} \right) = \psi \text{ in } L^2(-T_3, T_3; H) \text{ and since } \{k_m\} \text{ and } \psi \text{ have their support in } [-T_3, T_3], \text{ the limit is valid in } L^2(-\infty, \infty; H).$$

But by the Lemma 1, the sequence $\{k_m\}$ is also convergent in $L^2(-T_3, T_3; H)$ (and hence in $L^2(R; H)$).

Let

$$\lim_{m \rightarrow \infty} k_m = \chi(t)$$

Furthermore for any $\Phi(t) \in K$, we have

$$\begin{aligned} \int_{-\infty}^{\infty} \left(x(t), \Phi^{(n)} + \sum_{j=0}^{n-1} A_{n-j} \Phi^{(j)} \right) dt &= \lim_{m \rightarrow \infty} \int_{-\infty}^{\infty} \left(k_m, \Phi^{(n)} + \sum_{j=0}^{n-1} A_{n-j} \Phi^{(j)} \right) dt \\ &= \lim_{m \rightarrow \infty} \int_{-\infty}^{\infty} \left((-1)^n k_m^{(n)} + \sum_{j=0}^{n-1} (-1)^j (A_{n-j}^* k_m)^{(j)}, \Phi \right) dt. \end{aligned}$$

Hence there exists $\chi(t) \in L^2(R; H)$ such that

$$(15) \quad \int_{-\infty}^{\infty} \left(\chi(t), \Phi^{(n)} + \sum_{j=0}^{n-1} A_{n-j} \Phi^{(j)} \right) dt = \int_{-\infty}^{\infty} (\psi, \Phi) dt \quad \forall \Phi \in K.$$

And since χ has compact support, it follows by Lemma 5 that χ has support in $[-T_1, T_1]$. Hence there exists $\chi(t) \in L^2_{\text{loc}}(R; H)$ and $\chi(t)$ satisfies (15):

To complete the proof, it remains to show that for $h(t) \in V_{T_2}$ we have:

$$\int_{-T_1}^{T_1} (\psi, h) dt = 0.$$

For this let $\{\alpha_n\} \rightarrow \delta$, $\alpha_n \in C_0^\infty(R)$. Then the function $\psi * \alpha_n$ has its support contained in $(-T_2, T_2)$ for sufficiently large n . We have for large n :

$$\begin{aligned} \int_{-T_2}^{T_2} (\psi * \alpha_n, h) dt &= 0. \\ \int_{-\infty}^{\infty} \left(\chi, \Phi^{(n)} + \sum_{j=0}^{n-1} A_{n-j} \Phi^{(j)} \right) dt &= \int_{-\infty}^{\infty} (\psi, \Phi) dt \quad \forall \Phi \in K. \end{aligned}$$

By Lemma 4:

$$(-1)^n (\chi * \alpha_m)^{(n)} + \sum_{j=0}^{n-1} (-1)^j (A_{n-j}^* \chi * \alpha_m)^{(j)} = \psi * \alpha_m \quad \forall t \in R.$$

Then:

$$\begin{aligned} \int_{-\infty}^{\infty} \left((-1)^n (\chi * \alpha_m)^{(n)} + \sum_{j=0}^{n-1} (-1)^j (A_{n-j}^* \chi * \alpha_m)^{(j)}, h \right) dt &= \int_{-\infty}^{\infty} (\chi * \alpha_m, h) dt \\ &= 0 \quad \text{for } h \in V_{T_2}. \end{aligned}$$

And since $\chi * \alpha_m \in K^*(-T_2, T_2)$, $(\text{supp } \chi \subset [-T_1, T_1])$ for large m .
It follows that for large m :

$$\int_{-T_2}^{T_2} (\psi * \alpha_m, h) dt = 0.$$

Hence :

$$\int_{-T_2}^{T_2} (\psi, h) dt = \int_{-T_1}^{T_1} (\Phi, h) dt = 0.$$

And the Lemma is proved.

LEMMA 7. (approximation): $\{A_1, \dots, A_n\}$ satisfy condition S and $0 < T_1 < T_2$. Then, the set of functions:
 $u(t) \in L^2_{\text{loc}}(R; H)$ such that:

$$\int_{-\infty}^{\infty} \left(u, (-1)^n \Phi^{(n)} + \sum_{j=0}^{n-1} (-1)^j (A_{n-j}^* \Phi)^{(j)} \right) dt = 0, \Phi \in K^*$$

is dense in V_{T_2} for the $L^2(-T_1, T_1; H)$ topology.

PROOF: Let $u_0(t) \in V_{T_2}$ and $\varepsilon > 0$. By Lemma 6, there exists $u_1(t) \in V_{T_2+1}$ such that:

$$\int_{-T_1}^{T_1} \|u_0(t) - u_1(t)\|^2 dt < \frac{\varepsilon^2}{4}.$$

And there exist $u_2(t) \in V_{T_2+2}$ such that

$$\left\{ \int_{-T_2}^{T_2} \|u_2 - u_1\|^2 dt \right\}^{\frac{1}{2}} \leq \frac{\varepsilon}{2^2}$$

$$\Rightarrow \left\{ \int_{-T_1}^{T_1} \|u_2 - u_0\|^2 dt \right\}^{\frac{1}{2}} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} < \varepsilon.$$

So we can find a sequence $\{u_n\}$ $u_n(t) \in V_{T_2+n}$ such that

$$\left\{ \int_{-(T_2+n)}^{+T_2+n} \|u_{n+2} - u_{n+1}\|^2 dt \right\}^{\frac{1}{2}} \leq \frac{\varepsilon}{2^{n+2}},$$

and

$$\left\{ \int_{-T_1}^{T_1} \|u_{n+2} - u_0\|^2 dt \right\}^{\frac{1}{2}} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + \dots + \frac{\varepsilon}{2^{n+2}} < \varepsilon.$$

Consider the series

$$u_1 + (u_2 - u_1) + \dots$$

this series converges in $L_{loc}^2(R; H)$

$$\text{And so } \lim_{n \rightarrow \infty} u_n = u_\varepsilon \text{ in } L_{loc}^2(R; H).$$

and we have :

$$\int_{-T_1}^{T_1} \|u_\varepsilon - u_0\|^2 dt \leq \varepsilon^2.$$

Furthermore since : $u_k \in V_{T_2+k}$ satisfy :

$$\int_{-\infty}^{\infty} (u_k, (-1)^n \Phi^{(n)} + \sum_{j=0}^{n-1} (-1)^j (A_{n-j}^* \Phi)^{(j)}) dt = 0$$

$$\forall \Phi \in K^*(-T_2 - k, T_2 + k).$$

u_ε satisfy the same equation for all $\Phi \in K^*$.

PROOF OF THE THEOREM. $\{A_1, \dots, A_n\}$ satisfy condition S , and $f(t) \in L_{loc}^2(R; H)$. Let $f_n(t)$ be the restriction of $f(t)$ to $(-n, +n)$.

Then by Lemma 2, there exist a function $u_n(t) \in L^2(-n, n; H)$ such that :

$$\int_{-n}^n (u(t), (-1)^n \Phi^n + \sum_{j=0}^{n-1} (-1)^j (A_{n-j}^* \Phi)^{(j)}) dt = \int_{-n}^n (f_n(t), \Phi) dt; \forall \Phi \in K^*(-n, n)$$

Let us consider the series :

$$u_1 + (u_2 - u_1) + (u_3 - u_2) + \dots$$

The function : $u_n - u_{n-1} \in V_{n-1}$. So by lemma 6, there exist $h_n \in L_{loc}^2(R; H)$ such that :

$$\int_{-\infty}^{\infty} (h_n, (-1)^n \Phi^{(n)} + \sum_{j=0}^{n-1} (-1)^j (A_{n-j} \Phi)^{(j)}) dt = 0, \quad \forall \Phi \in K^*$$

and

$$\left\{ \int_{-n+2}^{n-2} \|u_n - u_{n-1} - h_{n-1}\|^2 dt \right\}^{\frac{1}{2}} < \frac{\varepsilon}{2^n}.$$

Then the series :

$$u_1 + (u_2 - u_1 - h_1) + \dots + (u_n - u_{n-1} - h_{n-1}) + \dots$$

is convergent in $L^2_{loc}(R; H)$ to a function $u(t)$ in $L^2_c(R; H)$ which satisfy :

$$\int_{-\infty}^{\infty} (u(t), (-1)^n \Phi^{(n)} + \sum (-1)^j (A_{n-j}^* \Phi)^{(j)}) dt = \int_{-\infty}^{\infty} (f(t), \Phi(t)) dt$$

for all $\Phi \in K^*$, i. e., $u(t)$ is a solution of (1) in the sense of (3).

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