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ON THE REFLECTIONS IN BOUNDED SYMMETRIC DOMAINS

MAURO MESCHIARI

SUMMARY - This paper deals with the problem of existence of reflections in bounded symmetric domains. This problem was solved by E. Gottschling [4] for the domains of type **I** $\div$ **IV** using Cartan's realisation.

In this paper the same problem will be solved by an infinitesimal method, yielding as a by-product the solution of the problem for two exceptional domains.

After recalling some introductory material in § **1**, a condition which caracterizes the existence of reflections will be established in § **2**. Furthermore we will find a criterion for non-existence of reflections easier to handle and which, in § **3** and **4**, will be shown to be sufficient. The latter relation will be applied in § **3** to prove two exceptional domains have no reflections.

I thank here professor Soji Kaneyuki and professor Edoardo Vesentini for their invaluable help.

§ **1.** Let us fix some conventions that we shall use throughout this paper.

Let V be a finite dimensional real vector space and J a complex structure over V (i. e. a $\mathbf{R}$-linear endomorphism such that $J^2 + Id = 0$). We shall denote by ${}_JV$ the complex vector space obtained from V by defining the following scalar multiplication

$$(a + ib)\, X = aX + b\, JX.$$

V^c will denote the complex vector space ${}_J(V \times V)$ where J is the complex structure

$$J : (X, Y) \longmapsto (-Y, X).$$

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If $\mathfrak{g}$ is a real Lie algebre and J is a complex structure over $\mathfrak{g}$ (considered as a vector space) such that

$$[X, JY] = J[X, Y],$$

${}_J\mathfrak{g}$ will denote the complex Lie algebra obtained from the complex vector space ${}_J\mathfrak{g}$ with the bracket operation inherited from $\mathfrak{g}$. We shall denote by $\mathfrak{g}^c$ the complex Lie algebra obtained by defining on the complex vector space $\mathfrak{g}^c$ the bracket operation by $[X + iY, Z + iT] = [X, Z] - [Y, T] + + i([Y, Z] + [X, T])$.

Let D be a bounded symmetric domain. We shall use the following notation:

o is a fixed point of D.

$\mathrm{I}(D)$ Is the group of all biholomorphic map of D onto itself endowed with the Lie group structure compatible with the compact open topology.

G Is the connected component of the identity (e) of $I(D)$ and $\mathfrak{g}$ is its Lie algebra.

K Is the isotropy subgroup of G at o (i. e. $K = \{g \in G / g \cdot o = o\}$) and $\mathfrak{k}$ is its Lie algebra.

s Is the symmetry at o.

π Is the canonical projection $G \longrightarrow G/K$.

p Is the diffeomeomorphism $g \cdot K \longmapsto g \cdot o$ of G/K onto D (considered as a real manifold).

B Is the Killing form on $\mathfrak{g}$ or its extension to $\mathfrak{g}^c$.

Let $r: V \longrightarrow U$ be a C^∞ map between the two C^∞ manifolds. We shall denote by $T_p V$ the tangent space of V at p, and by $T_p r$ the differential of r at p.

Since the map $S: g \longmapsto sgs$ is an involutive automorphism of G, the map $T_e S: \mathfrak{g} \longrightarrow \mathfrak{g}$ is an involutive automorphism of $\mathfrak{g}$ (here we identify $T_e G$ and $\mathfrak{g}$). The 1-eigenspace of $T_e S$ is $\mathfrak{k}$; we shall denote by $\mathfrak{l}$ the (-1)-eigenspace of $T_e S$.

Let D be irreducible. We know that $\mathfrak{g}$ has a one-dimensional center $\mathfrak{z}$. In $\mathfrak{z}$ there exists an element Z such that

$$T_e(p\,\pi)\, adZ = J_0\, T_e(p\,\pi): \mathfrak{g} \longrightarrow T_0 D \simeq C^n.$$

Where J_p denotes the complex structure of $T_p D$ given by the usual identification $T_p D \simeq C_n$ (see [2] p. 136).

The holomorphic and antiholomorphic tangent spaces of D at o determine a decomposition of $\mathfrak{g}^c$ into the direct sum

$$\mathfrak{g}^c = \mathfrak{k}^c + \mathfrak{l}^- + \mathfrak{l}^-$$

where $\mathfrak{l}^+$ and $\mathfrak{l}^-$ are abelian $Ad(K)$ invariant sub-algebras of $\mathfrak{g}^c$.

Let $\mathfrak{c}$ denote a maximal abelian subalgebra of $\mathfrak{k}^c \cdot \mathfrak{c}^c$ determines the decomposition

$$\mathfrak{g}^c = \mathfrak{c}^c + \sum_{\alpha \in \Delta} \mathfrak{g}^\alpha,$$

where Δ is the set of roots and

$$\mathfrak{g}^\alpha = \{X \in \mathfrak{g}^c / [Y, X] = \alpha(Y) X \text{ for every } Y \in \mathfrak{c}^c\}.$$

The three sets:

$$\Phi^+ = \{\alpha \in \Delta / \mathfrak{g}^\alpha \subseteq \mathfrak{l}^+\}$$

$$\Phi^- = \{\alpha \in \Delta / \mathfrak{g}^\alpha \subseteq \mathfrak{l}^-\},$$

$$\Psi = \{\alpha \in \Delta / \mathfrak{g}^\alpha \subseteq \mathfrak{k}^c\},$$

determine a partition in Δ (ses [2] p. 140 ÷ 141).

There exist elements X_α and H_α in $\mathfrak{g}^c$ such that:

1) $X_\alpha \in \mathfrak{g}^\alpha$ and $X_\alpha \neq 0$ for every $\alpha \in \Delta$,

2) $H_\alpha = [X_\alpha, X_{-\alpha}] \in \mathfrak{c}^c$ for every $\alpha \in \Delta$,

$$[X_\alpha, X_\beta] = \begin{cases} 0 & \text{if } \alpha + \beta \notin \Delta,\ \alpha \neq \beta \\ N_{\alpha,\beta}\, X_{\alpha+\beta} & \text{if } \alpha + \beta \in \Delta,\ \alpha \neq \beta. \end{cases}$$

The reader can find a proof of the existence of such a set in [1] p. 151. We need later some property of the coefficients $N_{\alpha,\beta}$ stated in [1].

§ 2. Def. 1. A biholomorphic map $\omega \in I(D)$ is a reflection at $p \in D$ if it satisfies the conditions:

1) ω has finite order and $\omega(p) = p$

2) The Jacobian matrix of ω at p has just one eigenvalue different from 1.

Since a bounded symmetric domain is homogeneous, it will be sufficient to consider reflections at a fixed point of D.

Let D be a bounded symmetric domain and

$$D \simeq D_1 \times D_2 \times \dots \times D_r$$

a decomposition of D as a product of symmetric domains. We have a canonical analytic isomorphism of $I(D_1) \times \dots \times I(D_r)$ onto a normal subgroup of $I(D)$, which we denote again by

$$I(D_1) \times \dots \times I(D_r).$$

Gottschling ([4] p. 702) has proved the following theorem.

THEOR. 1. D has reflections which are not in $I(D_1) \times \dots \times I(D_r)$ if and only if at least two of the factors $D_1, \dots, D_r$ have dimension 1.

In view of this result we can restrict our research to irreducible bounded symmetric domains.

In the following, (except in theorem 11) every bounded symmetric domain is, unless othervise stated, assumed to be irreducible.

PROP. 2. If D is an irreducible bounded symmetric domain there exists a 1 to 1 correspondence between the set $H = \{g \in I(D)/g \cdot o = o\}$ and the set of all the automorphisms $\sigma : \mathfrak{g}^c \longrightarrow \mathfrak{g}^c$ such that:

i) $\sigma(\mathfrak{g}) = \mathfrak{g}$

ii) $\sigma(\mathfrak{k}^c) = \mathfrak{k}^c$, $\sigma(\mathfrak{l}^{\pm}) = \mathfrak{l}^{\pm}$.

Proof. 1) Since $h \in H$, consider the inner automorphism of $I(D)$ determined by h^{-1} (i. e. $A_{h^{-1}} : g \longrightarrow hgh^{-1}$).

$A_{h^{-1}}$ induces on $T_e G$ (identified with $\mathfrak{g}$) the automorphism $T_e A_{h^{-1}} = Ad(h^{-1})$; we shall denote by the same symbol, $Ad(h^{-1})$, its extension to $\mathfrak{g}^c$.

We shall show that $Ad(h^{-1})$ satisfies i) and ii). Since $A_h^{-1} K = K$, implies $Ad(h^{-1})\mathfrak{g} = \mathfrak{g}$ and therefore $Ad(h^{-1})\mathfrak{g}^c = \mathfrak{g}^c$. Thus i) is proved.

α) We prove now that $(Ad(h^{-1}) - Id)\mathfrak{z} = 0$. As

$$p\pi(hgh^{-1}) = hgh^{-1} \cdot o = hg \cdot o = h \cdot p\pi(g),$$

passing to the differentials we have the commutative diagram

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{T_e(p\pi)} & T_0 D \\ \downarrow{\scriptstyle Ad(h^{-1})} & & \downarrow{\scriptstyle T_0 h} \\ \mathfrak{g} & \xrightarrow{T_e(p\pi)} & T_0 D. \end{array}$$

Being $T_0 h J_0 = J_0 T_0 h$, it follows from

$$J_0 T_0 h T_e(p\pi) = T_0 h J_0 T_e(p\pi)$$

that

$$T_e(p\pi)(Ad(h^{-1})\, adZ - adZ\, Ad(h^{-1}))\,\mathfrak{g} = 0.$$

From $Ad(h^{-1})\,\mathfrak{k} = \mathfrak{k}$, it follows that $Ad(h^{-1})\,\mathfrak{z} = \mathfrak{z}$. Since $\mathfrak{z}$ is one-dimensional we have $(Ad(h^{-1}) - \lambda\, Id)\,\mathfrak{z} = 0$ for a certain λ.

It follows from

$$T_e(p\pi)(adZ\, Ad(h^{-1}) - Ad(h^{-1})\, adZ)\,\mathfrak{g} = (1-\lambda)\, T_e(p\pi)\cdot$$

$$\cdot\, adZ\, Ad(h^{-1})\,\mathfrak{g} = (1-\lambda)\, J_0 \cdot T_0 h \cdot T_0 D = 0,$$

that $\lambda = 1$ and then $(Ad(h^{-1}) - Id)\,\mathfrak{z} = 0$.

β) From $(adZ \mp i\, Id)\,\mathfrak{l}^{\pm} = 0$ we have:

$$0 = Ad(h^{-1})(adZ \mp i\, Id)\,\mathfrak{l}^{\pm} = (adZ \mp i\, Id)\, Ad(h^{-1})\,\mathfrak{l}^{\pm},$$

and this implies $Ad(h^{-1})\,\mathfrak{l}^{\pm} = \mathfrak{l}^{\pm}$.

2) To prove the converse, suppose that σ is an automorphism of $\mathfrak{g}^c$ satisfying i) and ii). We shall show that there exists a unique $h \in H$ for which $\sigma = Ad(h^{-1})$.

α) From $\sigma(\mathfrak{k}) = \sigma(\mathfrak{k}^c \cap \mathfrak{g}) \subseteq \sigma(\mathfrak{k}^c) \cap \sigma(\mathfrak{g}) = \mathfrak{k}^c \cap \mathfrak{g} = \mathfrak{k}$

it follows that $(\sigma - \lambda\, Id)\,\mathfrak{z} = 0$ for a convenient λ.

Furthermore the relation

$$0 = \sigma(adZ - iId)\,\mathfrak{l}^{+} = (ad\,\sigma Z - iId)\,\sigma\mathfrak{l}^{+} = \lambda\,(adZ - \frac{i}{\lambda}\, Id)\,\mathfrak{l}^{+}$$

implies $\lambda = 1$ and therefore

$$(\sigma - Id)\,\mathfrak{z} = 0.$$

β) The relations

$$[\mathfrak{z}, \mathfrak{g}] = [\mathfrak{z}, \mathfrak{l}] = \mathfrak{l}$$

imply

$$\sigma(\mathfrak{l}) = \sigma([\mathfrak{z}, \mathfrak{g}]) = [\mathfrak{z}, \sigma(\mathfrak{g})] = \mathfrak{l}.$$

γ) Let G' be the simply connected Lie group which Lie algebra is $\mathfrak{g}$, and let K' be the analytic subgroup for the subalgebra $\mathfrak{k}$. We can extend σ to an automorphism $S: G' \to G'$ such that $S(K') = K'$. Since we have a diffeomorphism $p': G'/K' \to D$, let us define a differentiable map $S': D \to D$

such that

$$S' \, p' \, \pi' = p' \, \pi' \, S$$

(where $\pi' : G' \longrightarrow G'/K'$ is the canonical map). Passing to differentials we have:

$$T_{p' \pi' (g)} \, S' \cdot T_g \, (p' \, \pi') = T_{S(g)} \, (p' \, \pi') \cdot T_g \, S.$$

Since $T_e (p' \, \pi')$ is an homomorphism of $\mathfrak{g}$ on $T_0 \, D$ the equality

$$T_0 \, S' \cdot J_0 \cdot T_e (p' \, \pi') = T_0 \, S' \cdot T_e (p' \, \pi') \cdot adZ = T_e (p' \, \pi') \, \sigma \, adZ =$$

$$= T_e (p' \, \pi') \, adZ \, \sigma = J_0 \, T_0 \, S' \, T_e (p' \, \pi')$$

implies

$$T_0 \, S' \, J_0 = J_0 \, T_0 \, S'.$$

Let now $p = g \cdot o \in D$ with $g \in G'$. We have

$$T_p \, S' \cdot J_p \cdot T_0 \, g = T_p \, S' \cdot T_0 g \cdot J_0 = T_0 \, S(g) \cdot T_0 \, S' \cdot J_0 = T_0 \, S(g) \cdot$$

$$\cdot J_0 \cdot T_0 \, S' = J_{S'(p)} \cdot T_0 \, S(g) \cdot T_0 \, S' = J_{S'(p)} \cdot T_p \, S' \cdot T_0 \, g,$$

and then

$$T_p \, S' \cdot J_p = J_{S'(p)} \cdot T_p \, S'.$$

It follows that $S' \in I(D)$.

δ) Let us show that $Ad(S'^{-1}) = \sigma$. Since $Ad(S'^{-1})$ satisfies i) and ii), the equality

$$T_e (p' \, \pi') \, (Ad(S'^{-1}) - \sigma) = T_0 \, S' \, T_e (p' \, \pi') - T_e (p' \, \pi') \, \sigma = 0,$$

implies

$$(Ad(S'^{-1}) - \sigma) \, \mathfrak{l} = 0.$$

As $\mathfrak{k} = [\mathfrak{l}, \mathfrak{l}]$, it follows

$$(Ad(S'^{-1}) - \sigma) \, \mathfrak{g} = (Ad(S'^{-1}) \, \sigma -) \cdot ([\mathfrak{l}, \mathfrak{l}] + \mathfrak{l}) = 0$$

and then $Ad(S'^{-1}) = \sigma : \mathfrak{g}^c \longrightarrow \mathfrak{g}^c$.

ε) To conclude the proof it is enough to show that $Ad(h) = Ad(k)$ implies $h = k$ for all $h, k \in H$.

Let $Ad(h) = Ad(k)$. Being G' connected and simply connected, $A_h = A_k$ and then $A_{hk^{-1}} = Id$.

It follows that hk^{-1} acts over D like the identity and therefore $hk^{-1} = e$.

LEMMA 3. For every $h \in H$ the complex linear maps

$$Ad(h^{-1})/_{\mathfrak{l}^+} : \mathfrak{l}^+ \longrightarrow \mathfrak{l}^+$$

$$T_0 h : {}_{J_0}T_0 D \longrightarrow {}_{J_0}T_0 D$$

have the same eigenvalues.

Proof. We know $\{X_\alpha / \alpha \in \Phi^+\}$ is a basis of $\mathfrak{l}^+$ and $\{X_\alpha + X_{-\alpha}, i(X_\alpha - X_{-\alpha}) / \alpha \in \Phi^+\}$ is a basis of $\mathfrak{l}$.

Thus we can define a $\mathbf{R}$ isomorphism $\tau : \mathfrak{l}^+ \longrightarrow \mathfrak{l}$ snch that

$$\tau(X_\alpha) = X_\alpha + X_{-\alpha} \text{ and } \tau(iX_\alpha) = i(X_\alpha - X_{-\alpha})$$

for every $\alpha \in \Phi^+$.

It is immediate to verify that τ is also a $\mathbf{C}$ isomorphism

$$\tau : \mathfrak{l}^+ \longrightarrow {}_{adZ}\mathfrak{l}.$$

Recalling that $T_e(p\,\pi) \cdot adZ = J_0 . T_e(p\,\pi)$ we find that $T_e(p\,\pi)\,\tau : \mathfrak{l}^+ \longrightarrow {}_{J_0}T_0 D$ is a $\mathbf{C}$ isomorphism.

To prove the lemma we have only to show that

$$T_e(p\,\pi) \cdot \tau \cdot Ad(h^{-1}) = T_0 h \cdot T_e(p\,\pi) \cdot \tau.$$

From $(\tau - Id)\,\mathfrak{l}^+ \subseteq \mathfrak{l}^-$ we have

$$\mathfrak{l} \supseteq (Ad(h^{-1})\,\tau - \tau\, Ad(h^{-1}))\,\mathfrak{l}^+ = (Ad(h^{-1})(\tau - Id) - (\tau - Id)\, Ad(h^{-1}))\,\mathfrak{l}^+ \subseteq$$

$$\subseteq Ad(h^{-1})(\tau - Id)\,\mathfrak{l}^+ \cup (\tau - Id)\, Ad(h^{-1})\,\mathfrak{l}^+ \subseteq \mathfrak{l}^-.$$

As $\mathfrak{l} \cap \mathfrak{l}^- = \{0\}$, it follows that $Ad(h^{-1})\,\tau = \tau\, Ad(h^{-1})$ and $T_e(p\,\pi)\, Ad(h^{-1})\,\tau = T_e(p\,\pi)\,\tau\, Ad(h^{-1})$.

From PROP. 2 and LEMMA 3 it follows immediately

PROP. 4. Let D be an irreducible bounded symmetric domain. D has reflections if, and only if, there exists an automorphism $\sigma : \mathfrak{g}^c \longrightarrow \mathfrak{g}^c$ such that

i) $\sigma(\mathfrak{g}) = \mathfrak{g}$.

ii) $\sigma(\mathfrak{k}^c) = \mathfrak{k}^c$, $\sigma(\mathfrak{l}^\pm) = \mathfrak{l}^\pm$.

iii) The $\mathbf{C}$ linear map $\sigma/\mathfrak{l}^+$ has exactly one eigenvalue $\lambda \neq 1$ which is a root of 1.

Now, let us derive from PROP. 4 a condition easier to handle. To do this we need some more lemmas and definitions.

LEMMA 5. Let $\sigma : \mathfrak{g}^C \longrightarrow \mathfrak{g}^C$ be an automorphism that satisfies i) $\div$ iii) of PROP. 4. We have

$$\sigma(X_\alpha) = \sum_{\beta \in \Phi^+} (c\, \varrho_\alpha \bar{\varrho}_\beta + \delta_\alpha^\beta) X_\beta$$

$$\sigma(X_{-\alpha}) = \sum_{\beta \in \Phi^+} (\bar{c}\, \bar{\varrho}_\alpha \varrho_\beta + \delta_\alpha^\beta) X_{-\beta},$$

for every $\alpha \in \Phi^+$, where ϱ_α are complex quantities and c is an absolute constant.

Proof. From ii) we have

$$\sigma(X_\alpha) = \sum_{\beta \in \Phi^+} a_\alpha^\beta X_\beta \qquad \sigma(X_{-\alpha}) = \sum_{\beta \in \Phi^+} a_\alpha^{\prime\beta} X_{-\beta}$$

and then, by i), it follows that $\bar{a}_\alpha^\beta = a_\alpha^{\prime\beta}$.

To prove the lemma it is then enough to prove the first of the two relations.

From iii), it follows that the matrix $\| a_\alpha^\beta - \delta_\alpha^\beta \|$ must have rank 1. Thus we have $a_\alpha^\beta = \varrho_\alpha \varrho^\beta + \delta_\alpha^\beta$. Now, let B denote the extension to $\mathfrak{g}^C$ of the Killing form of $\mathfrak{g}$. As σ is an automorphism of $\mathfrak{g}^C$ we have

$$B(X_\alpha, X_{-\beta}) = B(\sigma(X_\alpha), \ \sigma(X_{-\beta})) \text{ for every } \alpha, \beta \in \Phi^+ \cdot$$

and then

$$\delta_\alpha^\beta = \sum_{\varphi \in \Phi^+} (\varrho_\alpha \varrho^\varphi + \delta_\alpha^\varphi)(\bar{\varrho}_\beta \bar{\varrho}^\varphi + \delta_\beta^\varphi) \text{ for every } \alpha, \beta \in \Phi^+$$

that is

$$\varrho_\alpha \bar{\varrho}_\beta \Big(\sum_{\varphi \in \Phi^+} |\varrho^\varphi|^2 \Big) + \varrho_\alpha \varrho^\beta + \bar{\varrho}_\beta \bar{\varrho}^\alpha = 0.$$

By iii) there exists a $\gamma \in \Phi^+$ such that $\varrho_\gamma \neq 0$.

We have therefore

$$\varrho^\beta = \bar{\varrho}_\beta \left(- \sum_{\varphi \in \Phi^+} |\varrho^\varphi|^2 - \frac{\bar{\varrho}^\gamma}{\varrho_\gamma} \right) \text{ for every } \beta \in \Phi^+,$$

and this proves the lemma.

LEMMA 6. Let σ be the automorphism considered in PROP. 4 and $\alpha, \beta, \gamma \in \Phi^+$ three roots such that

i) $\sigma(X_\alpha) = X_\alpha$.

ii) $\beta - \alpha \in \psi$ and $\gamma - \alpha \notin \psi$.

Then either $\sigma(X_\beta) = X_\beta$ or $\sigma(X_\gamma) = X_\gamma$.

Proof. Let $\{\varrho_\varphi\}$ be the set of complex numbers of lemma 5. We have

$$\sigma(\varrho_\delta X_\varepsilon - \varrho_\varepsilon X_\delta) = \varrho_\delta X_\varepsilon - \varrho_\varepsilon X_\delta$$

for every $\delta, \varepsilon \in \Phi^+$. Then, since

$$[\varrho_\gamma X_\beta - \varrho_\beta X_\gamma, \ X_{-\alpha}] = \varrho_\gamma N_{\beta,-\alpha} X_{\beta-\alpha},$$

it follows from i) and from the hypotheses that σ is an automorphism, that

$$\sigma(\varrho_\gamma N_{\beta,-\alpha} X_{\beta-\alpha}) = \varrho_\gamma N_{\beta,-\alpha} X_{\beta-\alpha}.$$

Let us suppose $\sigma(X_\gamma) \neq X_\gamma$. We have $\varrho_\gamma \neq 0$ and then $\sigma(X_{\beta-\alpha}) = X_{\beta-\alpha}$. It follows from

$$\sigma[X_\alpha, \ X_{\beta-\alpha}] = [X_\alpha, X_{\beta-\alpha}] \text{ that } \sigma(X_\beta) = X_\beta.$$

The proof is completed.

DEF. 2. Let A be any subset of Φ^+, we call closure of A the set

$$\bar{A} = \bigcup_{i \in \mathbf{N}} A_i$$

(where $\mathbf{N}$ is the set of naturals) by definition, where $A_0 = A$ and

$$A_{i+1} = A_i \cup \{\alpha \in \Phi^+ / \alpha = \alpha_1 + \alpha_2 - \alpha_3 : \alpha_1, \alpha_2, \alpha_3 \in A_i : \alpha_2 - \alpha_3 \in \psi\}.$$

For every $\alpha \in \Phi^+$ let

$$\mathcal{N}_\alpha = \{\beta \in \Phi^+ / \beta - \alpha \in \psi \text{ or } \beta = \alpha\},$$

and

$$\mathcal{T}_\alpha = \{\beta \in \Phi^+ / \beta - \alpha \notin \psi \text{ or } \beta = \alpha\}.$$

DEF. 3. Let A be any subset of Φ^+, we define the set

$$A^\wedge = \bigcup_{i \in \mathbf{N}} A_i^\wedge$$

where we have defined $A_0^\wedge = A$ and

$$A_{i+1}^\wedge = \bigcup_{\alpha \in A_i^\wedge} \overline{(A_i^\wedge \cup \mathcal{N}_\alpha)} \cap \overline{(A_i^\wedge \cup \mathcal{F}_\alpha)}$$

For the sets defined above the following lemma holds.

LEMMA 7. For every $A \subseteq \Phi^+$

i) $A_i^\wedge \subseteq A_{i+1}^\wedge \subseteq A^\wedge$.

ii) If $A \subseteq B$ then $A_i^\wedge \subseteq B_i^\wedge$ and $A^\wedge \subseteq B^\wedge$.

iii) $B_i^\wedge \cup A_i^\wedge \subseteq (B \cup A)_i^\wedge$, $B^\wedge \cup A^\wedge \subseteq (B \cup A)^\wedge$.

LEMMA 8. Let $\sigma : \mathfrak{g}^c \to \mathfrak{g}^c$ be an automorphism that satisfies i) $\div$ iii) of PROP. 4. If $\sigma(X_\alpha) = X_\alpha$ is verified for every $\alpha \in A$ then it is also verified for every $\alpha \in A^\wedge$.

Proof. By the hypotesis of the lemma, $\sigma(X_\varepsilon) = X_\varepsilon$ for every $\varepsilon \in A_0$. Let us suppose that $\sigma(X_\varepsilon) = X_\varepsilon$ for every $\varepsilon \in A_i$.

If $\alpha = \alpha_1 + \alpha_2 - \alpha_3 \in A_{i+1}$ we have

$$\sigma[X_{\alpha_1}, [X_{\alpha_2}, X_{-\alpha_3}]] = [X_{\alpha_1}, [X_{\alpha_2}, X_{-\alpha_3}]],$$

that is $\sigma(X_\alpha) = X_\alpha$.

By induction it follows that $\sigma(X_\alpha) = X_\alpha$ for every $\alpha \in A_i$, where i is any natural number.

This relation is also verified for every $\alpha \in \overline{A}$.

Let us suppose that $\sigma(X_\varepsilon) = X_\varepsilon$ holds for every $\varepsilon \in A_i^\wedge$. If $\alpha \in A_{i+1}^\wedge$, then there exists necessarily $\beta \in A_i^\wedge$ such that

$$\alpha \in \overline{(A_i^\wedge \cup \mathcal{N}_\beta)} \cap \overline{(A_i^\wedge \cup \mathcal{F}_\beta)}$$

By lemma 6 and $\sigma(X_\beta) = X_\beta$ it follows that $\sigma(X_\varepsilon) = X_\varepsilon$ for every $\varepsilon \in \mathcal{N}_\beta$ or for every $\varepsilon \in \mathcal{F}_\beta$.

In both cases we have, by the first part of this proof, $\sigma(X_\alpha) = X_\alpha$. Our lemma follows by induction starting from $A_0^\wedge = A$.

LEMMA 9. Let $\beta \in \Phi^+$ and $\alpha \in \psi$ be roots such that the α-chain containing β has only two roots β and $\beta + \alpha$. If $\omega : \mathfrak{l}^+ \to \mathbf{C}$ is a $\mathbf{C}$ linear map, there exists an element $h \in K$ such that

$$\omega(Ad(h) X_\beta) = 0.$$

Proof. For every ϱ, $\theta \in \mathbf{R}$, $\varrho \geq 0$, we have

$$\varrho\,(e^{i\theta} X_\alpha - e^{-i\theta} X_{-\alpha}) \in \mathfrak{k}$$

and then

$$h_{\varrho,\theta} = \exp\ \varrho\,(e^{i\theta} X_\alpha - e^{-i\theta} X_{-\alpha}) \in K.$$

From the relation

$$Ad\,(h_{\varrho\,\theta}) = e^{ad\,\varrho\,(e^{i\theta} X_\alpha - e^{-i\theta} X_{-\alpha})}$$

remembering that

$$N_{\alpha,\beta} = N_{\alpha+\beta,-\alpha} = \pm\sqrt{\frac{\alpha\,(H_\alpha)}{2}}$$

after some computation we arrive to

$$Ad\,(h_{\varrho,\theta})\,X_\beta = X_\beta \cos\varrho\sqrt{\frac{\alpha\,(H_\alpha)}{2}} \pm X_{\alpha+\beta}\,e^{i\theta} \sin\varrho\sqrt{\frac{\alpha\,(H_\alpha)}{2}}$$

It follows that

$$\omega\,(Ad\,(h_{\varrho,\theta})\,X_\beta) = \omega\,(X_\beta)\cos\varrho\sqrt{\frac{\alpha\,(H_\alpha)}{2}} \pm e^{i\theta}\,\omega\,(X_{\alpha+\beta})\sin\varrho\sqrt{\frac{\alpha\,(H_\alpha)}{2}}\,.$$

It is now obvious that there exist two real numbers $\bar{\varrho}, \bar{\theta}$ such that

$$\omega\,(Ad\,(h_{\bar{\varrho},\bar{\theta}})\,X_\beta) = 0.$$

and the proof is complete.

Now we are able to prove

PROP. 10. Let D be an irreducible bounded symmetric domain such that its Lie algebra $\mathfrak{g}^c$ (where $\mathfrak{g}$ is the Lie algebra of G) has two roots $\alpha \in \Phi^+$ and $\beta \in \psi$ such that

i) the β-chain containing α consists of two roots α and $\alpha + \beta$.

ii) $\{\alpha\}^\wedge = \Phi^+$.

Then D has no reflections.

Proof. If Σ is a reflection for D, then by PROP. 2 and PROP 3 $Ad(\Sigma^{-1})$ is an automorphism of $\mathfrak{g}^c$ which satisfies i) $\div$ iii) PROP. 4.

Let $\omega : \mathfrak{l}^+ \to \mathbf{C}$ be any linear map such that $\operatorname{Ker}\omega = \{X \in \mathfrak{l}^+ / Ad(\Sigma^{-1})X = X\}$. By lemma 9, there is $h \in K$ such that $\omega(Ad(h)X_a) = 0$.

The automorphism $h^{-1}\Sigma h : D \to D$ is a reflection which induces on $\mathfrak{g}^c$ the automorphism $\sigma' = Ad(h^{-1}\Sigma^{-1}h)$. As $\sigma'(X_a) = X_a$, we have $(\sigma' - Id)\mathfrak{l}^+ = 0$ by lemma 8, and this is a contradiction.

In the next section we shall apply PROP. 10 to the six types of irreducible bounded symmetric domains.

§ 3. A)

$$D = SU(m, n)/S(U(m) \times U(n)), \quad 1 \leq m \leq n.$$

This irreducible bounded symmetric domain has Lie algebras:

$$\mathfrak{g} = \mathfrak{su}(m, n) = \left\{ \begin{pmatrix} A & C \\ {}^t\bar{C} & B \end{pmatrix} \middle| \begin{array}{c} A \text{ skew Herm. } m \times m \text{ matrix} \\ B \text{ skew Herm. } n \times n \text{ matrix} \\ C \text{ complex } m \times n \text{ matrix} \\ TrA + TrB = 0 \end{array} \right\}$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \middle| \begin{array}{c} A \text{ skew Herm. } m \times m \text{ matrix} \\ B \text{ skew Herm. } n \times n \text{ matrix} \\ TrA + TrB = 0 \end{array} \right\}$$

(See [1] p. 348.)

The center of $\mathfrak{k}$ is given by $\mathfrak{z} = i\mathbf{R} \begin{pmatrix} \frac{1}{m} I & 0 \\ 0 & -\frac{1}{n} I \end{pmatrix}$.

We have:

$$\mathfrak{g}^c = \mathfrak{sl}(m + n, G).$$

$\mathfrak{g}^c$ can be decomposed in the three subalgebras:

$$\mathfrak{k}^c = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \middle| \begin{array}{c} A \text{ complex } m \times m \text{ matrix} \\ B \text{ complex } n \times n \text{ matrix} \\ TrA + TrB = 0 \end{array} \right\}$$

$$\mathfrak{l}^+ = \left\{ \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix} \middle| C \text{ compley } m \times n \text{ matrix} \right\}$$

$$\mathfrak{l}^- = \left\{ \begin{pmatrix} 0 & 0 \\ {}^t\bar{C} & 0 \end{pmatrix} \middle| C \text{ complex } m \times n \text{ matrix} \right\}.$$

A maximal abelian subalgebra of $\mathfrak{k}$ is given by.

$$\mathfrak{c} = \left\{ \begin{array}{l} \text{set of purely immaginary diagonal matrices} \\ \text{of order } m+n \text{ with trace} = 0 \end{array} \right\}$$

$\mathfrak{c}^c$ is a Cartan subalgebra of $\mathfrak{g}^c$. Roots of $\mathfrak{g}^c$ with respect to $\mathfrak{c}^c$ are the linear maps $\alpha_r - \alpha_s : \mathfrak{c}^c \to \mathbf{C}$, for every $1 \leq r, s \leq m+n$, $r \neq s$, (where $\alpha_r(a_q^p) = a_r^r$). Root spaces are $\mathfrak{g}^{\alpha_r - \alpha_s} = \mathbf{C}\, E_s^r$ (where E_s^r is a $(m+n) \times (m+n)$ matrix whose elements are zero except (r, s)-element which is one)

We have the following sets of roots:

$$\Phi^+ = \{\alpha_r - \alpha_s / r \leq m < s \leq m+n\}$$

$$\Phi^- = \{\alpha_r - \alpha_s / s \leq m < r \leq m+n\}$$

$$\psi \quad = \{\alpha_r - \alpha_s / r \neq s : r,\ s \leq m \text{ or } m < r,\ s \leq m+n\}.$$

Now let us see for which values of m and n the hypoteses of PROP. 10 are satisfied.

For $r \leq m < s \leq m+n$, we have

$$\mathcal{N}_{\alpha_r - \alpha_s} \{\alpha_r - \alpha_p,\ \alpha_q - \alpha_s / m < p \leq m+n,\ 1 \leq q \leq m\},$$

$$\mathcal{F}_{\alpha_r - \alpha_s} = \{\alpha_r - \alpha_s, \alpha_p - \alpha_q / 1 \leq q \leq m < p \leq m+n,\ p \neq s,\ q \neq r\}$$

We find

$$\overline{\mathcal{N}_{\alpha_r - \alpha_s}} = \Phi^+,\ \overline{\mathcal{F}_{\alpha_r - \alpha_s}} = \mathcal{F}_{\alpha_r - \alpha_s},$$

and then

$$\{\alpha_r - \alpha_s\}_1^\wedge = \mathcal{F}_{\alpha_r - \alpha_s}.$$

If $m = 1$, we have $\{\alpha_r - \alpha_s\}_1^\wedge = \{\alpha_r - \alpha_s\}$. Thus $\{\alpha_r - \alpha_s\}^\wedge = \{\alpha_r - \alpha_s\}$, and the hypotheses of PROP. 10 are never satisfied because either Φ^+ has only a root or $\{\alpha_r - \alpha_s\}^\wedge \neq \Phi^+$.

If $m \geq 2$, let us compute $\{\alpha_r - \alpha_s\}_2^\wedge$. For every $\alpha_p - \alpha_q \in \{\alpha_r - \alpha_s\}_1^\wedge$ we have:

$$\overline{\mathcal{N}_{\alpha_p - \alpha_q} \cup \{\alpha_r - \alpha_s\}_1^\wedge} \supseteq \overline{\mathcal{N}_{\alpha_p - \alpha_q}} = \Phi^+$$

$$\overline{\mathcal{F}_{\alpha_p - \alpha_q} \cup \{\alpha_r - \alpha_s\}_1^\wedge} = \begin{cases} \{\alpha_r - \alpha_s\}_1^\wedge & \text{if } m = n = 2 \\ \Phi^+ & \text{if } m \geq 2,\ n > 2. \end{cases}$$

It follows that, if $m = n = 2$, hypotheses of PROP. 10 are never satisfied. If $m \geq 2$, $n > 2$, it is $\{\alpha_r - \alpha_s\}^\wedge = \Phi^+$. Observing that only $\alpha_r - \alpha_s$ and $\alpha_r - \alpha_p$ are the roots in the $(\alpha_r - \alpha_p)$-chain containing $\alpha_r - \alpha_s$ we conclude that our domain has no reflections.

We know, from [4] p. 703, that D has reflections for $m = 1$ and for $m = n = 2$.

B) $$D = Sp(n, \mathbf{R})/U(n), \quad n \geq 2.$$

The Lie algebras of this irreducible bounded domains are (see [1] p. 350)

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \middle| \begin{array}{l} A,\ B,\ C \text{ real } n \times n \text{ matrices} \\ B,\ C \text{ symmetric} \end{array} \right\},$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \middle| \begin{array}{l} A,\ B \text{ real } n \times n \text{ matrices} \\ A \text{ skew symmetric, } B \text{ symmetric} \end{array} \right\},$$

$$\mathfrak{z} = \mathbf{R} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

We have:

$$\mathfrak{g}^c = \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \middle| \begin{array}{l} A,\ B,\ C \text{ complex } n \times n \text{ matrices} \\ \mathrm{B},\ C \text{ symmetric} \end{array} \right\}$$

$$\mathfrak{k}^c = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \middle| \begin{array}{l} A,\ B \text{ complex } n \times n \text{ matrices} \\ A \text{ skew symmetric, } B \text{ symmetric} \end{array} \right\}$$

$$\mathfrak{l}^+ = \left\{ \begin{pmatrix} A & -iA \\ -iA & -A \end{pmatrix} \middle| A \text{ symmetric compley } n \times n \text{ matrix} \right.$$

$$\mathfrak{l}^- = \left\{ \begin{pmatrix} A & iA \\ iA & -A \end{pmatrix} \middle| A \text{ symmetric complex } n \times n \text{ matrix} \right\}.$$

$$\mathfrak{c} = \left\{ \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix} \middle| A \text{ real diagonal } n \times n \text{ matrix} \right\}$$

is a maximal abelian subalgebra of $\mathfrak{k}$, so $\mathfrak{c}^c$ is a Cartan subalgebra of $\mathfrak{g}^c$.

Let $\alpha_r : \mathfrak{c}^c \to \mathbf{C}$ be a linear map defined by

$$\alpha_r \begin{pmatrix} 0 & (a^i_j) \\ -(a^i_j) & 0 \end{pmatrix} = i a^r_r$$

It is easy to verify that Φ^+, Φ^- and ψ are given by:

$$\Phi^+ = \{\alpha_r + \alpha_s / 1 \leq r \leq s \leq n\},$$

$$\Phi^- = \{-\alpha_r - \alpha_s / 1 \leq r \leq s \leq n\}$$

$$\Psi = \{\alpha_r - \alpha_s / r \neq s,\ 1 \leq r,\ s \leq n\}.$$

If $n = 2$, then $\Phi^+ = \{2\alpha_1,\ \alpha_1 + \alpha_2,\ 2\alpha_2\}$, and every chain containing two roots of Φ^+ contains all of them and the hypotheses of PROP. 10 can never be satisfied.

If $n > 2$, we have the following sets of roots:

$$\mathcal{N}_{\alpha_r+\alpha_s} = \{\alpha_r + \alpha_p,\ \alpha_q + \alpha_s / 1 \leq p,\ q \leq n\},$$

$$\mathcal{F}_{\alpha_r+\alpha_s} = \{\alpha_r + \alpha_s,\ \alpha_p + \alpha_q / p \neq r,\ q \neq s,\ 1 \leq p,\ q \leq n\}$$

and then

$$\overline{\mathcal{N}}_{\alpha_r+\alpha_s} = \Phi^+ \quad \text{and} \quad \overline{\mathcal{F}}_{\alpha_r+\alpha_s} = \mathcal{F}_{\alpha_r+\alpha_s}.$$

We have

$$\{\alpha_r + \alpha_s\}_1^\wedge = \mathcal{F}_{\alpha_r+\alpha_s}.$$

and by lemma 7

$$\{\alpha_r + \alpha_s\}_2^\wedge = \bigcup_{\alpha_p+\alpha_q \in \{\alpha_r+\alpha_s\}_1^\wedge} \overline{\{\alpha_r + \alpha_s\}_1^\wedge \cup \mathcal{F}_{\alpha_p+\alpha_q}} \geq \{2\alpha_t / 1 \leq t \leq n\}.$$

It follows

$$\{\alpha_r + \alpha_s\}_3^\wedge \supseteq \bigcup_{t=1}^{n} \{2\alpha_t\}_1^\wedge = \Phi^+$$

and then, noting that the $(\alpha_2 - \alpha_1)$-chain containing $\alpha_1 + \alpha_3$ has only two roots, the hypotheses of PROP. 10 are satisfied and our domain D has not reflections.

For $n = 2$ D has reflections. See [4] p. 703.

C) $$D = SO^*(2n)/U(n), \quad n \geq 5$$

D has the following Lie algebras:

$$\mathfrak{g} = \left\{ \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix} \middle| \begin{array}{l} A,\ B\ n \times n \text{ complex matrices} \\ A \text{ skew symmetric, } B \text{ Hermitian} \end{array} \right\}$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \middle| \begin{array}{l} A,\ B\ n \times n \text{ real matrices} \\ A \text{ skew symmetric, } B \text{ symmetric} \end{array} \right\}$$

$$\mathfrak{z} = \mathbf{R} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Then, we have:

$$\mathfrak{g}^C = \mathfrak{so}\,(2n, \mathbf{C}).$$

$$\mathfrak{k}^C = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \middle/ \begin{array}{l} A,\ B\ n \times n \text{ complex matrices} \\ A \text{ skew symmetric, } B \text{ symmetric} \end{array} \right\}$$

$$\mathfrak{l}^+ = \left\{ \begin{pmatrix} i\,(A - {}^tA) & 2A \\ -2\,{}^tA & -i(A - {}^tA) \end{pmatrix} \middle| A\ n \times n \text{ complex matrix} \right\}$$

$$\mathfrak{l}^- = \left\{ \begin{pmatrix} -i\,(A - {}^tA) & 2A \\ -2\,{}^tA & i\,(A - {}^tA) \end{pmatrix} \middle| A\ n \times n \text{ complex matrix} \right\}.$$

A maximal abelian subalgebra of $\mathfrak{k}$ is given by

$$\mathfrak{c} = \left\{ \begin{pmatrix} 0 & A \\ -A & 0 \end{pmatrix} \middle| A \text{ diagonal real } n \times n \text{ matrix} \right\}.$$

The maps $\alpha_s : \mathfrak{c}^C \to \mathbf{C}$ are roots (α_r are linear form defined as in the previous case) and we have the sets of roots:

$$\Phi^+ = \{\alpha_r + \alpha_s / 1 \leq r < s \leq n\},$$

$$\Phi^- = \{-\alpha_r - \alpha_s / 1 \leq r < s \leq n\}$$

$$\Psi = \{\alpha_r - \alpha_s / r \neq s,\ 1 \leq r,\ s \leq n\}.$$

Let us consider now, for $1 \leq r < s \leq n$,

$$\mathcal{N}_{\alpha_r+\alpha_s} = \{\alpha_r + \alpha_p,\ \alpha_s + \alpha_q / p \neq r,\ q \neq s,\ 1 \leq p,\ q \leq n\},$$

$$\mathcal{F}_{\alpha_r+\alpha_s} = \{\alpha_r + \alpha_s,\ \alpha_p + \alpha_q / p,\ q \neq r, s \quad 1 \leq p,\ q \leq n\}.$$

We have

$$\overline{\mathcal{N}}_{\alpha_r+\alpha_s} = \Phi^+, \quad \overline{\mathcal{F}}_{\alpha_r+\alpha_s} = \mathcal{F}_{\alpha_r+\alpha_s},$$

and

$$\{\alpha_r + \alpha_s\}_1^\wedge = \mathcal{F}_{\alpha_r+\alpha_s}.$$

In particular we have

$$\{\alpha_1 + \alpha_2\}_2^\wedge \geq \{\alpha_1 + \alpha_2,\ \alpha_3 + \alpha_4,\ \alpha_3 + \alpha_5,\ \alpha_4 + \alpha_5\}_1^\wedge \supseteq$$
$$\supseteq \{\alpha_1 + \alpha_2\}_1^\wedge \cup \{\alpha_3 + \alpha_4\}_1^\wedge \cup \{\alpha_3 + \alpha_5\}_1^\wedge \cup \{\alpha_4 + \alpha_5\}_1^\wedge = \Phi^+,$$

then $\{\alpha_1 + \alpha_2\}^\wedge = \Phi^+$, and, as the $(\alpha_3 - \alpha_2)$-chain containing $\alpha_1 + \alpha_2$ has only two roots, by PROP. 10 this domain has no reflections.

D) $$D = SO_0(n, 2)/SO(n) \times SO(2), \quad n \geq 5.$$

This type of domain has Lie algebras

$$\mathfrak{g} = \mathfrak{so}(n, 2)$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \middle| \begin{array}{l} A,\ B \text{ skew symmetric real matrices} \\ A\ n \times n,\ B\ 2 \times 2. \end{array} \right\}$$

$$\mathfrak{z} = \mathbb{R} \begin{pmatrix} 0 & 0 \\ 0 & J_2 \end{pmatrix} \left(\text{where } J_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right).$$

It follows that:

$$\mathfrak{k}^c \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \middle| \begin{array}{l} A,\ B \text{ skew symmetric complex matrices} \\ A\ n \times n,\ B\ 2 \times 2. \end{array} \right\},$$

$$\mathfrak{l}^- = \left\{ \begin{pmatrix} 0 & A & iA \\ {}^tA & 0 & 0 \\ i\,{}^tA & 0 & 0 \end{pmatrix} \middle| A \text{ complex } n \times 1 \text{ matrix} \right\},$$

$$\mathfrak{l}^+ = \left\{ \begin{pmatrix} 0 & A & -iA \\ {}^tA & 0 & 0 \\ -i\,{}^tA & 0 & 0 \end{pmatrix} \middle| A \text{ complex } n \times 1 \text{ matrix} \right\}.$$

$$\mathfrak{c} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \middle| \begin{array}{l} A,\ B \text{ skew symmetric real matrices, } B \text{ is } 2 \times 2. \\ A \text{ is a } n \times n \text{ matrix whose elements are zero} \\ \text{except the } (r, s)\text{-elements with } r + s = n + 1 \end{array} \right\}$$

is a maximal abelian subalgebra of $\mathfrak{k}$.

Let $\alpha_p : \mathfrak{c}^c \longrightarrow \mathbb{C}$ be a linear map defined by

$$\alpha_p : \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \longmapsto i\, a^p_{n-p+1} - ib.$$

$\left(\text{Where } A = (a^i_j) \text{ and } B = \begin{pmatrix} 0 & 0 \\ -b & 0 \end{pmatrix} \right).$

We have the root sets :

$$\Phi^+ = \{\alpha_r / 1 \leq r \leq n\},$$

$$\Phi^- = \{-\alpha_r / 1 \leq r \leq n\},$$

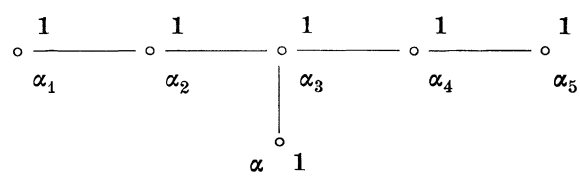
$$\Psi = \{\alpha_r - \alpha_s / r - s \neq 0,\ r + s \neq n + 1,\ 1 \leq r,\ s \leq n\}.$$

As $\mathcal{N}_{\alpha_r} = \Phi^+ - \{\alpha_{n-r+1}\}$ and $\mathcal{F}_{\alpha_r} = \{\alpha_r, \alpha_{n-r+1}\}$, we have $\{\alpha_r\}_1^\wedge = \{\alpha_r, \alpha_{n-r+1}\}$, $\{\alpha_r\}_2^\wedge = \{\alpha_r, \alpha_{n-r+1}\} = \{\alpha_r\}^\wedge$, for every $1 \leq r \leq n$.

The hypotheses of PROP. 10 are never satisfied. We know ([4] p. 703) that this domain has always reflections.

E) $$D = (\mathfrak{E}_{6(-14)},\ \mathfrak{so}(10) + \mathbf{R}).$$

This exceptional domain has Lie algebra $\mathfrak{g}^c = \mathfrak{E}_6$. All roots of $\mathfrak{E}_6$ are obtained by the Dynkin diagram



From [5] p. 494, we know that the simple root in the diagram belonging to Φ^+ is α_1 or α_5.

We have the sets of roots :

$$\begin{aligned}
\Phi^+ = \{&\alpha_1, & &\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha,\\
&\alpha_1 + \alpha_2, & &\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha,\\
&\alpha_1 + \alpha_2 + \alpha_3, & &\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha,\\
&\alpha_1 + \alpha_2 + {}_3 + \alpha, & &\alpha_1 + 2\alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha,\\
&\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4, & &\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha,\\
&\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha, & &\alpha_1 + 2\alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha,\\
&\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5, & &\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha,\\
&\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha, & &\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha\},
\end{aligned}$$

$$\Phi^- = \{\alpha / -\alpha \in \Phi^+\},$$

$$\begin{aligned}\Psi = \{&\pm\alpha,\ \pm\alpha_2,\ \pm\alpha_3,\ \pm\alpha_4,\ \pm\alpha_5,\ \pm(\alpha_2+\alpha_3),\ \pm(\alpha_3+\alpha_4),\\ &\pm(\alpha_4+\alpha_5),\ \pm(\alpha+\alpha_3),\ \pm(\alpha+\alpha_2+\alpha_3),\ \pm(\alpha_2+\alpha_3+\alpha_4),\\ &\pm(\alpha_3+\alpha_4+\alpha_5),\ \pm(\alpha+\alpha_3+\alpha_4),\ \pm(\alpha+\alpha_2+\alpha_3+\alpha_4),\\ &\pm(\alpha_2+\alpha_3+\alpha_4+\alpha_5),\ \pm(\alpha+\alpha_3+\alpha_4+\alpha_5),\ \pm(\alpha+\alpha_2+\alpha_3+\alpha_4+\alpha_5),\\ &\pm(\alpha+\alpha_2+2\alpha_3+\alpha_4),\ \pm(\alpha+\alpha_2+2\alpha_3+\alpha_4+\alpha_5)\},\end{aligned}$$

Now we can consider the sets

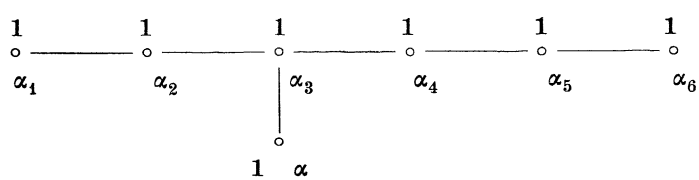
$$\begin{aligned}\mathcal{N}_{\alpha_1} = \{&\alpha_1,\ \alpha_1+\alpha_2,\ \alpha_1+\alpha_2+\alpha_3,\ \alpha_1+\alpha_2+\alpha_3+\alpha,\ \alpha_1+\alpha_2+\alpha_3+\alpha_4,\\ &\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha,\ \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5,\ \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha,\\ &\alpha_1+\alpha_2+2\alpha_3+\alpha_4+\alpha,\ \alpha_1+\alpha_2+2\alpha_3+\alpha_4+\alpha_5+\alpha\},\end{aligned}$$

$$\begin{aligned}\mathcal{F}_{\alpha_1} = \{&\alpha_1,\ \alpha_1+2\alpha_2+2\alpha_3+\alpha_4+\alpha,\ \alpha_1+2\alpha_2+2\alpha_3+\alpha_4+\alpha_5+\alpha,\\ &\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha,\ \alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha,\\ &\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+\alpha_5+\alpha,\ 2\alpha+\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+\alpha_5\}.\end{aligned}$$

It is easy to verify that $\overline{\mathcal{N}}_{\alpha_1} = \overline{\mathcal{F}}_{\alpha_1} = \Phi^+$. Then, as α_1 and $\alpha_1+\alpha_2$ are the only roots in the α_2-chain containing α_1 by PROP. 10 this domain has no reflections.

F) $$D = (\mathfrak{e}_{7(-25)},\ \mathfrak{e}_6 + \mathbf{R}).$$

All roots the Lie algebra $\mathfrak{g}^c = \mathfrak{e}_7$ are obtained from the Dynkin diagram



Only α_6 is the root in this diagram belonging to Φ^+. (see [5] p. 494).

We have the sets of roots :

$$\Phi^+ = \{\alpha_6,\ \alpha_5+\alpha_6,\ \alpha_4+\alpha_5+\alpha_6,\ \alpha+\alpha_3+\alpha_4+\alpha_5+\alpha_6,$$

$$\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6,\ \alpha+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6,\ \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6,$$

$$\alpha+\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6,\ \alpha+\alpha_2+2\alpha_3+\alpha_4+\alpha_5+\alpha_6,$$

$$\alpha+\alpha_1+\alpha_2+2\alpha_3+\alpha_4+\alpha_5+\alpha_6,\ \alpha+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6,$$

$$\alpha+\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6,\ \alpha+\alpha_1+2\alpha_2+2\alpha_3+\alpha_4+\alpha_5+\alpha_6,$$

$$\alpha+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6,\ \alpha+\alpha_1+\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6,$$

$$\alpha+\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+\alpha_5+\alpha_6,\ \alpha+\alpha_1+2\alpha_2+2\alpha_3+2\alpha_4+2\alpha_5+\alpha_6,$$

$$\alpha+\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+\alpha_5+\alpha_6,\ \alpha+\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+2\alpha_5+\alpha_6,$$

$$2\alpha+\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+\alpha_5+\alpha_6,\ 2\alpha+\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+2\alpha_5+\alpha_6,$$

$$\alpha+\alpha_1+2\alpha_2+3\alpha_3+3\alpha_4+2\alpha_5+\alpha_6,\ 2\alpha+\alpha_1+2\alpha_2+3\alpha_3+3\alpha_4+2\alpha_5+\alpha_6,$$

$$2\alpha+\alpha_1+2\alpha_2+4\alpha_3+3\alpha_4+2\alpha_5+\alpha_6,\ 2\alpha+\alpha_1+3\alpha_2+4\alpha_3+3\alpha_4+2\alpha_5+\alpha_6,$$

$$2\alpha+2\alpha_1+3\alpha_2+4\alpha_3+3\alpha_4+2\alpha_5+\alpha_6\},$$

$$\Phi^- = \{\alpha / -\alpha \in \Phi^+\},$$

$\Psi = \{$set of roots of $\mathfrak{E}_6\}$ (see the other exceptional domain).

Now it is easy to see that $\overline{\mathcal{N}}_{\alpha_6} = \overline{\mathcal{F}}_{\alpha_6} = \Phi^+$ and then $\{\alpha_6\}^\wedge = \Phi^+$. As α_6 and $\alpha_5+\alpha_6$ are the only roots of the α_5-chain containing α_6, by PROP. 10 this domain has no reflections.

Summing up we may state the

THEOR. 11. Let D be a symmetric bounded domain isomorphic to the cartesian product of irreducible bounded domains

$$D_1 \times D_2 \times \dots \times D_r.$$

D has reflections if, and only if, at least one of D_i is one of the following domains :

$$SU(1, n)/S(U(1) \times S(n)),\ n \geq 1,$$

$$SU(2, 2)/S(U(2) \times U(2)),$$

$$Sp\,(2, \mathbf{R})/U\,(2),$$

$$S0_0\,(n, 2)/S0\,(n) \times S0\,(2), \quad n \geq 5.$$

We point out that the results of [4] were only used to show that irreducible domain which do not satisfy PROP. 10 *do* have reflections.

However, for sake of completness, in next section we shall re obtain Gottschling's results.

§ 4. We prove uow that the bounded symmetric domains

$$SU\,(1, n)/S\,(U\,(1) \times U\,(n)), \quad n \geq 1\,;$$

$$SU\,(2, 2)/S\,(U\,(2) \times U\,(2))\,;$$

$$Sp\,(2, \mathbf{R})/U\,(2)\,;$$

$$S0_0\,(n, 2)/S0\,(n) \times S0\,(2), \quad n \geq 5\,;$$

have reflections. To do so, we construct explicitely, for each of them, a Lie algebra automorphism that satisfies i) ÷ iii) PROP. 4, and then, considering the realizations of these domains as subsets of certains sets of matrices, we give au explicit descripion of a reflection Σ at a fixed point o. Furthermore we show that the set of reflections at o is given by

$$\{h\,\Sigma\,h^{-1}/h \in K\}.$$

To prove these last statements we need a new

LEMMA 12. Let $(\beta_1, \alpha_1), (\beta_2, \alpha_2), \ldots, (\beta_m, \alpha_m)$ a finite sequence of elements in $\Phi^+ \times \Psi$ such that

i) only β_i and $\beta_i + \alpha_i$ are the roots of the α_i cain containing β_i.

ii) $\beta_i \pm \alpha_{i+r} \notin \Phi^+$ for every $0 < r \leq m - i$.

If $\omega : \mathfrak{l}^+ \longrightarrow \mathbf{C}$ is a linear map, there exists an element $h \in K$ such that

$$\omega\,[Ad\,(h)\,X_{\beta_i}] = 0 \text{ for every } 1 \leq i \leq m.$$

Proof. It $m = 1$, the lemma is reduced to lemma 9. Let us suppose that the lemma is true up to r, where $1 \leq r < m$, let $h' \in K$ be an element such that

$$\omega\,[Ad\,(h')\,X_{\beta_i}] = 0 \text{ for every } 1 \leq i \leq r.$$

If ω' is the linear map defined by $\omega'(X) = \omega[Ad(h')X]$, then, by lemma 9, there exists an element $h'' \in K$ of the form

$$h'' = \exp \varrho (e^{i\theta} X_{\alpha_{r+1}} - e^{-i\theta} X_{-\alpha_{r+1}})$$

such that

$$\omega' [Ad(h'') X_{\beta_{r+1}}] = 0.$$

Siuce $[X_{\pm\alpha_{r+1}}, X_{\beta_i}] = 0$ by ii), we have $Ad(h'')X_{\beta_i} = X_{\beta_i}$ for every $1 \leq i \leq r$. It follows that

$$\omega [Ad(h' h'') X_{\beta_i}] = 0 \text{ for every } 1 \leq i \leq r+1.$$

Our lemma follows by induction.

$\mathbf{A}_1$) $\qquad SU(1, n)/S(U(1) \times U(n)). \quad n \geq 1.$

We have the sets of roots:

$$\Phi^+ = \{\alpha_1 - \alpha_i / 1 < i \leq n+1\} \text{ and } \Psi = \{\alpha_r - \alpha_s / 1 < r, s \leq n+1, r \neq s\}.$$

LEMMA 13. Any automorphism σ' of $\mathfrak{g}^c$ satisfying i) $\div$ iii) PROP. 4 is in the set $\{Ad(h)\, \sigma\, Ad(h^{-1}) / h \in K\}$, where σ is an automorphism satisfying i) $\div$ iii) PROP. 4; furthermore

iv) $\qquad \sigma(X_{\alpha_1 - \alpha_2}) = \lambda X_{\alpha_1 - \alpha_2}, \ \sigma(X_{\alpha_1 - \alpha_i}) = X_{\alpha_1 - \alpha_i}$

for every $2 < i \leq n+1$.

Proof. If $n = 1$, clearly $\sigma = \sigma'$. If $n > 1$, let us consider the finite sequence $(\alpha_1 - \alpha_3, \alpha_3 - \alpha_2), \ldots, (\alpha_1 - \alpha_n, \alpha_n - \alpha_2)$. It satisfies hypothesis of lemma 12. Let $h \in K$ be an element such that

$$\omega [Ad(h) X_{\alpha_1 - \alpha_i}] = 0 \text{ for every } 3 \leq i \leq n+1,$$

(where $\omega : \mathfrak{l}^+ \to \mathbf{C}$ is any linear map such that $\operatorname{Ker} \omega = \{X \in \mathfrak{l}^+ / \sigma'(X) = X\}$). Clearly, $\sigma = Ad(h^{-1})\, \sigma'\, Ad(h)$ satisfies i) $\div$ iii) PROP. 4 and iv). QED.

As $\mathfrak{g}^{\alpha_r - \alpha_s} = \mathbf{C} E^r_s$, now we are led to look for an automorphism $\sigma : \mathfrak{sl}(n+1, \mathbf{C}) \to \mathfrak{sl}(n+1, \mathbf{C})$ that satisfies conditions:

$$\sigma(E^1_2) = \lambda E^1_2, \ \sigma(E^1_i) = E^1_i, \ \sigma(E^2_1) = \bar{\lambda} E^2_1, \ \sigma(E^i_1) = E^i_1,$$

for every $3 \leq i \leq n+1$, with $\lambda^n = 1$ and $\lambda \neq 1$, and such that

$$\sigma(\mathfrak{g}) = \mathfrak{g}, \ \sigma(\mathfrak{k}^c) = \mathfrak{k}^c.$$

Put

$$T = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Then we have

$$\sigma(A) = T^{-1} A T = T^{-1} \begin{pmatrix} 0 & a_2^1 & \cdots & a_{n+1}^1 \\ a_1^2 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ a_1^{n+1} & 0 & \cdots & 0 \end{pmatrix} T = \begin{pmatrix} 0 & \lambda a_2^1 & \cdots & a_{n+1}^1 \\ 1/\lambda a_1^2 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ a_1^{n+1} & 0 & \cdots & 0 \end{pmatrix}$$

the automorphism σ satisfies all above conditions. Since $\exp(T^{-1} A T) = T^{-1} \exp A \, T$, σ induces on $SU(1, n)$ the automorphism $S : g \longrightarrow T^{-1} g \, T$.

Let us consider now an explicit realization of $D = SU(m, n)/S(U(m) \times U(n))$. D is the set of $m \times n$ complex matrices Z such that $I - {}^t Z \bar{Z}$ is definite positive, and $SU(m, n)$ acts on it by the map

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} Z = (AZ + B)(CZ + D)^{-1}.$$

(See [6]). Since $SU(1, n)$ acts transitively on $SU(1, n)/S(U(1) \times U(u))$ the commutative diagram

$$\begin{array}{ccc} SU(1, n) \ni \begin{pmatrix} A & B \\ C & D \end{pmatrix} & \xrightarrow{S} & T^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} T. \\ \Big\downarrow p\pi & & \Big\downarrow p\pi \\ \mathbf{C}^n \supset D \ni BD^{-1} & \xrightarrow{\Sigma} & B D^{-1} \begin{pmatrix} \lambda & 0 \\ 0 & 1 \end{pmatrix} \end{array}$$

defines the reflection at 0

$$\Sigma : (z_1, z_2, \ldots, z_n) \longrightarrow (\lambda z_1, z_2, \ldots, z_n).$$

The set of reflections of D is

$$\{g \, \Sigma \, g^{-1} / g \in G\}.$$

$\mathbf{A}_2$) $$D = SU(2, 2)/S(U(2) \times U(2)).$$

We have the sets of roots

$$\Phi^+ = \{\alpha_1 - \alpha_3,\ \alpha_1 - \alpha_4,\ \alpha_2 - \alpha_3,\ \alpha_2 - \alpha_4\},$$

$$\Psi = \{\pm(\alpha_1 - \alpha_2),\ \pm(\alpha_3 - \alpha_4)\}.$$

LEMMA 14. Any automorphism σ' of $\mathfrak{g}^c$ satisfying i) $\div$ iii) PROP. 4 is an element of the set $\{Ad(h^{-1})\,\sigma''\,Ad(h)/h \in K\}$, where σ'' is an automorphism satisfyng also

iv) $$\sigma''(X_{\alpha_1-\alpha_3}) = X_{\alpha_1-\alpha_3},\ \sigma''(X_{\alpha_2-\alpha_4}) = X_{\alpha_2-\alpha_4}$$

$$\sigma''(X_{\alpha_2-\alpha_3}) = a_1^1\, X_{\alpha_2-\alpha_3} + a_2^1\, X_{\alpha_1-\alpha_4}$$

$$\sigma''(X_{\alpha_1-\alpha_4}) = a_1^2\, X_{\alpha_2-\alpha_3} + a_2^2\, X_{\alpha_1-\alpha_4}.$$

Proof. We consider the couple $(\alpha_1 - \alpha_3,\ \alpha_3 - \alpha_4)$ and a linear form $\omega : \mathfrak{l}^+ \longrightarrow \mathbf{C}$ satisfying $Ker\ \omega = \{X \in \mathfrak{l}^+ : \sigma' X = X\}$. Then by lemma 9, we find a $h \in K$ such thai the automorphism

$$\sigma'' = Ad(h^{-1})\ \sigma'\ Ad(h)$$

satisfies i) $\div$ iii) PROP. 4 and

$$\sigma''(X_{\alpha_1-\alpha_3}) = X_{\alpha_1-\alpha_3}.$$

By lemma 5 we have (let us use the same notation)

$$\varrho_{\alpha_1-\alpha_3} = 0$$

and

$$\sigma''(X_{\alpha_2-\alpha_4}) = c\,\varrho_{\alpha_2-\alpha_4}(\bar{\varrho}_{\alpha_1-\alpha_4}\, X_{\alpha_1-\alpha_4} + \bar{\varrho}_{\alpha_2-\alpha_3}\, X_{\alpha_2-\alpha_3} + \bar{\varrho}_{\alpha_2-\alpha_4}\, X_{\alpha_2-\alpha_4}) + X_{\alpha_2-\alpha_4}.$$

Since $[X_{\alpha_2-\alpha_4},\ X_{\alpha_3-\alpha_1}] = 0$ it follows

$$\varrho_{\alpha_2-\alpha_4}(\bar{\varrho}_{\alpha_1-\alpha_4} N_{\alpha_1-\alpha_4,\,\alpha_3-\alpha_1}\, X_{\alpha_3-\alpha_4} + \bar{\varrho}_{\alpha_2-\alpha_3} N_{\alpha_2-\alpha_3,\,\alpha_3-\alpha_1}\, X_{\alpha_2-\alpha_1}) = 0$$

Since $X_{\alpha_3-\alpha_4}$ and $X_{\alpha_2-\alpha_1}$ are linearly independent and $N_{\alpha_1-\alpha_4,\,\alpha_3-\alpha_1}$, $N_{\alpha_2-\alpha_3,\,\alpha_2-\alpha_1} \neq 0$ we have two cases

a) $$\varrho_{\alpha_2-\alpha_4} = 0$$

b) $$\varrho_{\alpha_1-\alpha_4} = \varrho_{\alpha_2-\alpha_3} = 0.$$

Case *b*) is impossible because it yields

$$\sigma''(X_{\alpha_2-\alpha_3}) = X_{\alpha_2-\alpha_3}, \quad \sigma''(X_{\alpha_1-\alpha_4}) = X_{\alpha_1-\alpha_4}.$$

Ence

$$\sigma''[X_{\alpha_2-\alpha_3}[X_{\alpha_3-\alpha_1}, X_{\alpha_1-\alpha_4}]] = [X_{\alpha_2-\alpha_3}[X_{\alpha_3-\alpha_1}, X_{\alpha_1-\alpha_4}]]$$

and

$$N_{\alpha_3-\alpha_1,\alpha_1-\alpha_4} N_{\alpha_3-\alpha_4,\alpha_2-\alpha_3} \sigma''(X_{\alpha_2-\alpha_4}) = N_{\alpha_3-\alpha_1,\alpha_1-\alpha_4} N_{\alpha_3-\alpha_4,\alpha_2-\alpha_3} X_{\alpha_2-\alpha_4},$$

ence σ'' does not satisfy iii) PROP. 4.

By lemma 5, since

$$\varrho_{\alpha_1-\alpha_3} = \varrho_{\alpha_2-\alpha_4} = 0$$

it follows that σ'' satisfies also iv). QED.

Lat us consider the element

$$H = \begin{pmatrix} i/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i/2 \end{pmatrix}$$

for any $\beta \in \Phi+$ we have

$$Ad(\exp \varrho H) X_\beta = e^{\varrho\beta(H)} X_\beta;$$

in particular

$$Ad(\exp \varrho H) X_{\alpha_1-\alpha_4} = X_{\alpha_1-\alpha_4}$$

$$Ad(\exp \varrho H) X_{\alpha_2-\alpha_3} = e^{i\varrho} X_{\alpha_2-\alpha_3}.$$

As $X_{\alpha_2-\alpha_3} = c_1 E_3^2$ and $X_{\alpha_1-\alpha_4} = c_2 E_4^1$, it is now clear that we can find a ϱ such that

$$\sigma = Ad(\exp(-\varrho H) h^{-1}) \sigma' Ad(h \exp \varrho H)$$

satisfies i) ÷ iii) PROP. 4 and

iv′) $\sigma(E_3^1) = E_3^1, \ \sigma(E_1^3) = E_1^3, \ \sigma(E_4^2) = E_4^2, \ \sigma(E_2^4) = E_2^4$

$$\sigma(E_3^2) = b_1^1 E_3^2 + b_2^1 E_4^1 \qquad \sigma(E_2^3) = \bar{b}_1^1 E_2^3 + \bar{b}_2^1 E_1^4$$

$$\sigma(E_4^1) = b_1^2 E_3^2 + b_2^2 E_4^1 \qquad \sigma(E_1^4) = \bar{b}_1^2 E_2^3 = \bar{b}_2^2 E_1^4$$

$$b_2^1 = \bar{b}_2^1 \geq 0$$

Conditions i) ÷ iii) PROP. 4 and iv′) determine uniquely the automorphism of $\mathfrak{G}^c$

$$\sigma : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \longmapsto \begin{pmatrix} -{}^tD & {}^tB \\ {}^tC & -{}^tA \end{pmatrix}.$$

This shows that D has reflections.

It is easy to verify that $\sigma/_{\mathfrak{G}}$ can be expressed by

$$\sigma/_{\mathfrak{G}} : \begin{pmatrix} A & B \\ {}^t\bar{B} & D \end{pmatrix} \longmapsto \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \cdot \begin{pmatrix} \bar{A} & \bar{B} \\ {}^tB & \bar{D} \end{pmatrix} \cdot \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

and then its extension to $SU(2, 2)$ by

$$S : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \longmapsto \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{pmatrix} \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Now let us consider the realization of the above domain.

Let $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in SU(2, 2)$. Then ${}^tA\bar{A} - {}^tC\bar{C} = I$, ${}^tD\bar{D} - {}^tB\bar{B} = I$, ${}^tA\bar{B} = {}^tC\bar{D}$. So the commutative diagram

$$\begin{array}{ccc} \begin{pmatrix} A & B \\ C & D \end{pmatrix} & \xmapsto{S} & \begin{pmatrix} \bar{D} & \bar{C} \\ \bar{B} & \bar{A} \end{pmatrix} \\ {\scriptstyle p\pi}\downarrow & & {\scriptstyle p\pi}\downarrow \\ BD^{-1} & \xrightarrow{\Sigma} & \bar{C}\bar{A}^{-1} = {}^t(BD^{-1}) \end{array}$$

defines a reflection at o

$$\Sigma : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

The set of reflections of this domain is

$$\{g\,\Sigma\,g^{-1} / g \in SU(2, 2)\}.$$

$\mathbf{B}_1$) $$D = Sp(2, \mathbb{R})/U(2).$$

We have the sets of roots

$$\Phi^+ = \{2\alpha_1,\ \alpha_1 + \alpha_2,\ 2\alpha_2\} \text{ and } \Psi = \{\pm(\alpha_1 - \alpha_2)\}.$$

LEMMA 15. Any automorphism σ' of $\mathfrak{g}^c$ satisfying i) $\div$ iii) PROP. 4 is an element of the set $\{Ad(h)\,\sigma\, Ad(h^{-1})/h \in K\}$, where σ is an automorphism satisfying beside

iv) $\sigma(X_{2\alpha_1}) = X_{2\alpha_1}$, $\sigma(X_{2\alpha_2}) = X_{2\alpha_2}$, $\sigma(X_{\alpha_1+\alpha_2})\, X_{\alpha_1+\alpha_2}$.

Proof. As $\{2\alpha_1\}^\wedge = \{2\alpha_2\}^\wedge = \{2\alpha_1, 2\alpha_2\}$, lemma 9 implies that, if σ' does not satisfy iv), then we have

$$\sigma'(X_{2\alpha_1}) \neq X_{2\alpha_1} \text{ and } \sigma'(X_{2\alpha_2}) \neq X_{2\alpha_2}.$$

Since we have

$$N^2_{\alpha_1-\alpha_2,\,\alpha_1+\alpha_2} = N^2_{\alpha_2-\alpha_1,\,\alpha_1+\alpha_2} = (\alpha_1 - \alpha_2)(H_{\alpha_1-\alpha_2}),$$

by well known properties of coefficient $N_{\beta,\gamma}$, it follows that

$$N_{\alpha_1-\alpha_2,\,\alpha_1+\alpha_2} = N_{\alpha_1+\alpha_2,\,-2\alpha_1} = N_{\alpha_2-\alpha_1,\,2\alpha_1} = \varepsilon_1 \sqrt{(\alpha_1 - \alpha_2)(H_{\alpha_1-\alpha_2})}$$

$$N_{\alpha_2-\alpha_1,\,\alpha_2+\alpha_1} = N_{\alpha_1+\alpha_2,\,-2\alpha_2} = N_{\alpha_1-\alpha_2,\,2\alpha_2} = \varepsilon_2 \sqrt{(\alpha_1 - \alpha_2)(H_{\alpha_1-\alpha_2})},$$

where

$$\varepsilon_1^2 = \varepsilon_2^2 = 1.$$

From

$$[X_{\alpha_1+\alpha_2}, X_{-2\alpha_2}] = N_{\alpha_1+\alpha_2,\,-2\alpha_2}\, X_{\alpha_1-\alpha_2}$$

and

$$[X_{2\alpha_1}, X_{-\alpha_1-\alpha_2}] = N_{2\alpha_1,\,-\alpha_1-\alpha_2}\, X_{\alpha_1-\alpha_2}$$

we have

$$\varepsilon_2 [X_{\alpha_1+\alpha_2}, X_{-2\alpha_2}] - \varepsilon_1 [X_{2\alpha_1}, X_{-\alpha_1-\alpha_2}] = 0.$$

Since $\mathfrak{g}^{\alpha_1-\alpha_2}$, $\mathfrak{g}^{\alpha_2-\alpha_1}$ and $\mathfrak{c}^c$ are three linearly independent subalgebras of $\mathfrak{g}^C$, the coefficients of $X_{\alpha_1-\alpha_2}$ and $X_{\alpha_2-\alpha_1}$ in the first members of

$$\sigma'[X_{2\alpha_1}, X_{2\alpha_2}] = 0$$

$$\sigma'(\varepsilon_2 [X_{\alpha_1+\alpha_2}, X_{-2\alpha_2}] - \varepsilon_1 [X_{2\alpha_1}, X_{-\alpha_1-\alpha_2}]) = 0$$

must be zero. Therefore, by lemma 5, we have

$$\varrho_{2\alpha_1}\bar{\varrho}_{2\alpha_2}(\varepsilon_1 \varrho_{2\alpha_1}\bar{\varrho}_{\alpha_1+\alpha_2} + \varepsilon_2 \bar{\varrho}_{2\alpha_2}\varrho_{\alpha_1+\alpha_2}) = 0$$

$$c\,\bar{c}(\bar{\varrho}^2_{2\alpha_2}\varrho^2_{\alpha_1+\alpha_2} - \varrho^2_{2\alpha_1}\bar{\varrho}^2_{\alpha_1+\alpha_2}) + \varepsilon_1\varepsilon_2\bar{\varrho}_{2\alpha_2}\varrho_{2\alpha_1}(c - \bar{c}) = 0$$

$$c\bar{c}\,\varrho_{2\alpha_1}\bar{\varrho}_{2\alpha_2}(\varepsilon_1\bar{\varrho}_{2\alpha_1}\varrho_{\alpha_1+\alpha_2} + \varepsilon_2\varrho_{2\alpha_2}\bar{\varrho}_{\alpha_1+\alpha_2}) + \bar{(c}\,\varepsilon_1\bar{\varrho}_{2\alpha_2}\varrho_{\alpha_1+\alpha_2} + c\,\varepsilon_2\varrho_{2\alpha_1}\bar{\varrho}_{\alpha_1+\alpha_2}) = 0.$$

These relations are satisfied if and only if

$$|\varrho_{2\alpha_1}| = |\varrho_{2\alpha_2}|$$

and

$$\arg \varrho_{2\alpha_1} + \arg \varrho_{2\alpha_2} - 2 \arg \varrho_{\alpha_1+\alpha_2} + \frac{1-\varepsilon_1 \varepsilon_2}{2}\pi = 2n\pi.$$

Let h be

$$h = \exp t\,(e^{i\theta} X_{\alpha_2-\alpha_1} - e^{-i\theta} X_{\alpha_1-\alpha_2}) \in K,$$

where

$$\theta = \frac{1}{2}\left(\arg \varrho_{2\alpha_1} - \arg \varrho_{\alpha_1+\alpha_2} + \frac{3-\varepsilon_1}{2}\pi\right)$$

and

$$t = \frac{1}{\sqrt{2(\alpha_1-\alpha_2)(H_{\alpha_1-\alpha_2})}} \arcsin \frac{\sqrt{2}\,|\varrho_{2\alpha_1}|}{\sqrt{|\varrho_{\alpha_1+\alpha_2}|^2 + 2|\varrho_{2\alpha_1}|^2}}$$

$$0 \leq t\sqrt{2(\alpha_1-\alpha_2)(H_{\alpha_1-\alpha_2})} \leq \pi/2.$$

We have

$$Ad(h)\,X_2 = \frac{1}{2}(1 + \cos t\sqrt{2(\alpha_1-\alpha_2)(H_{\alpha_1-\alpha_2})})\,X_{2\alpha_1} +$$

$$+ \frac{\varepsilon_1 e^{i\theta}}{2} \sin t\sqrt{2(\alpha_1-\alpha_2)(H_{\alpha_1-\alpha_2})})\,X_{\alpha_1+\alpha_2} +$$

$$+ \frac{\varepsilon_1 \varepsilon_2 e^{i\theta}}{2}(-1 + \cos t\sqrt{2(\alpha_1-\alpha_2)(H_{\alpha_1-\alpha_2})})\,X_{2\alpha_2}.$$

Let us define the linear map $\omega : \mathfrak{l}^- \to \mathbf{C}$ by

$$(u_1 X_{2\alpha_1} + u_2 X_{\alpha_1+\alpha_2} + u_3 X_{2\alpha_2}) = u_1 \varrho_{2\alpha_1} + u_2 \varrho_{\alpha_1+\alpha_2} + u_3 \varrho_{2\alpha_2},$$

we have $\operatorname{Ker} \omega = \{X \in \mathfrak{l}^+ / \sigma'(X) = X\}$.

Now it is only a matter of trivial computation to verify that

$$\omega\,[Ad(h)\,X_{2\alpha_1}] = 0.$$

It follows that

$$\sigma = Ad(h^{-1})\;\sigma'\,Ad(h)$$

satisfies i) ÷ iii) PROP. 4 and the first two relations of iv), ence, it satisfies also the third of them by iii) PROP. 4. QED.

Since we have the following root spaces

$$\mathfrak{g}^{\alpha_r+\alpha_s} = \mathbf{C}\begin{pmatrix} E_s^r + E_r^s & -i(E_s^r + E_r^s) \\ -i(E_s^r + E_r^s) & -(E_s^r + E_r^s) \end{pmatrix}$$

$$\mathfrak{g}^{-\alpha_r-\alpha_s} = \mathbf{C}\begin{pmatrix} E_s^r + E_r^s & i(E_s^r + E_r^s) \\ i(E_s^r + E_r^s) & -(E_s^r + E_r^s) \end{pmatrix}$$

our automorphism must satisfy i) $\div$ iii) PROP. 4 and iv′)

$$\mathfrak{l}^+ \ni \begin{pmatrix} a & b & -ia & -ib \\ b & c & -ib & -ic \\ -ia & -ib & -a & -b \\ -ib & -ic & -b & -c \end{pmatrix} \xrightarrow{\sigma} \begin{pmatrix} a & \lambda b & ia & -\lambda ib \\ \lambda b & c & -\lambda ib & -ic \\ -ia & -\lambda ib & -a & -\lambda b \\ -\lambda ib & -ic & -\lambda b & -c \end{pmatrix}$$

$$\mathfrak{l}^- \ni \begin{pmatrix} a & b & ia & ib \\ b & c & ib & ic \\ ia & ib & -a & -b \\ ib & ic & -b & -c \end{pmatrix} \xrightarrow{\sigma} \begin{pmatrix} a & \bar{\lambda} b & ia & \bar{\lambda} ib \\ \bar{\lambda} b & c & \bar{\lambda} ib & ic \\ ia & \bar{\lambda} ib & -a & -\bar{\lambda} b \\ \bar{\lambda} ib & ic & -\bar{\lambda} b & -c \end{pmatrix}$$

where $\lambda \neq 1$ and $\lambda^n = 1$.

These conditions determine uniquely the automorphism given by

$$\sigma : \begin{pmatrix} a & b & c & d \\ e & f & d & h \\ i & l & -a & -e \\ l & m & -b & -f \end{pmatrix} \longmapsto \begin{pmatrix} a & b & c & -d \\ -e & f & -d & h \\ i & -l & -a & e \\ -l & m & b & -f \end{pmatrix}.$$

Consider now

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is easely cheaked that $\sigma(X) = T^{-1} X T'$ for every $X \in \mathfrak{g}^c$. It follows that the extension of σ to $Sp(n.\mathbf{R})$ is given by $\varrho : g \longmapsto T^{-1}\, g\, T$.

The domains of type $Sp(n, \mathbf{R})/U(n)$ can be realized as subsets of matrix spaces considering all the complex $n \times n$ symmetric matrices such that

$I - {}^tZ\bar{Z}$ is definite positive. $Sp(n, \mathbf{R})$ acts on D by the mapping

$$\begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} \cdot Z = (AZ + B)(\bar{B}Z + \bar{A})^{-1}.$$

The commutative diagramm

$$\begin{array}{ccc} \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} & \overset{S}{\longmapsto} & T^{-1} \begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix} T \\ p\pi \downarrow & & \downarrow p\pi \\ B\bar{A}^{-1} & \overset{\Sigma}{\longmapsto} & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} B\bar{A}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{array}$$

shows that all reflections are obtained by

$$\Sigma : \begin{pmatrix} a & b \\ b & c \end{pmatrix} \longrightarrow \begin{pmatrix} a & -b \\ -b & c \end{pmatrix}.$$

$\mathbf{D}_1$) $\qquad S_0(n, 2)/S0(n) \times S0(2).$

LEMMA 16. If there exists any antomorphism σ' of $\mathfrak{g}^c$ that satisfies i) ÷ iii) PROP. 4, then it is in the set $\{Ad(h)\,\sigma\,Ad(h^{-1})/h \in K\}$, where σ is an automorphism such that

iv) $\quad \sigma(X_{\alpha_r}) = X_{\alpha_r}$ for every $\quad 2 \leq r \leq n-1$

$$\sigma(X_{\alpha_1}) = a_1^1 X_{\alpha_1} + a_2^1 X_{\alpha_n}$$

$$\sigma(X_{\alpha_n}) = a_1^2 X_{\alpha_1} + a_2^2 X_{\alpha_n}$$

To prove this statement we need only to apply lemma 12 to the finite sequence $(\alpha_{n-1}, \alpha_1 - \alpha_{n-1}), \ldots (\alpha_2, \alpha_1 - \alpha_2)$ for a suitable linear map and remember lemma 5. QED.

The root spaces are given by

$$\mathfrak{g}^{\alpha_p} = \begin{pmatrix} 0 & A_p & -iA_p \\ {}^tA_p & 0 & 0 \\ -i\,{}^tA_p & 0 & 0 \end{pmatrix}, \quad \mathfrak{g}^{-\alpha_p} = \begin{pmatrix} 0 & \bar{A}_p & i\bar{A}_p \\ {}^t\bar{A} & 0 & 0 \\ i\,{}^t\bar{A}_p & 0 & 0 \end{pmatrix}.$$

where A_p is a complex $n \times 1$ matrix whose elements are $a_p^r = \delta_p^r + i\delta_{n-p+1}^r$.

Thus we are led to look for an automorphism of $\mathfrak{g}^c$ that satisfy i) $\div$ iii) PROP. and

$$\text{iv}') \qquad \sigma\begin{pmatrix} 0 & A_p & -iA_p \\ {}^tA_p & 0 & 0 \\ -i\,{}^tA_p & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & A_p & -iA_p \\ {}^tA_p & 0 & 0 \\ -i\,{}^tA_p & 0 & 0 \end{pmatrix}$$

for every $2 \leq p \leq n-1$,

$$\sigma\begin{pmatrix} 0 & A_1 & -iA_1 \\ {}^tA_1 & 0 & 0 \\ -i\,{}^tA_1 & 0 & 0 \end{pmatrix} = b_1^1\begin{pmatrix} 0 & A_1 & -iA_1 \\ {}^tA_1 & 0 & 0 \\ -i\,{}^tA_1 & 0 & 0 \end{pmatrix} + b_2^1\begin{pmatrix} 0 & A_n & -iA_n \\ {}^tA_n & 0 & 0 \\ -i\,{}^tA_n & 0 & 0 \end{pmatrix}$$

$$\sigma\begin{pmatrix} 0 & A_n & -iA_n \\ {}^tA_n & 0 & 0 \\ -i\,{}^tA_n & 0 & 0 \end{pmatrix} = b_1^2\begin{pmatrix} 0 & A_1 & -iA_1 \\ {}^tA_1 & 0 & 0 \\ -i\,{}^tA_1 & 0 & 0 \end{pmatrix} + b_2^2\begin{pmatrix} 0 & A_n & -iA_n \\ {}^tA_n & 0 & 0 \\ -i\,{}^tA_n & 0 & 0 \end{pmatrix}$$

and the complex conjugate relations,

v) $\quad -ib_2^1 \in \mathbb{R}$ and $-ib_2^1 \geq 0$.

To prove v), procede as in A_2) considering the element

$$H = \begin{pmatrix} \frac{i}{2}(E_1^n - E_n^1) & 0 & 0 \\ 0 & 0 & \frac{i}{2} \\ 0 & -\frac{i}{2} & 0 \end{pmatrix} \in \mathfrak{c}^c$$

Conditions i) $\div$ iii) PROP. 4, iv′) and v) determine uniquely the automorphism σ of $\mathfrak{g}^c$ given by

$$\sigma\begin{pmatrix} A & C \\ {}^tC & B \end{pmatrix} = T\begin{pmatrix} A & C \\ {}^tC & B \end{pmatrix} T$$

where T the $(n+2) \times (n+2)$ matrix

$$T = \begin{pmatrix} -1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

Clearly σ can be extended to an automorphism of $S0_0(n, 2)$ wich is expressed by $S: g \longrightarrow TgT$.

We can consider D as the set of complex $n \times 1$ matrices Z satisfying the two conditions

$$2\,{}^tZ\overline{Z} < 1 + |\,{}^tZZ\,|^2\,;\ |\,{}^tZZ\,| < 1.$$

$S0_0(n, 2)$ acts on D by the relation (see [6])

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} Z = \left(AZ + \frac{1}{2} B \begin{pmatrix} {}^tZZ + 1 \\ i\,{}^tZZ - i \end{pmatrix}\right) \left((1\ \ i)\, CZ + \frac{1}{2}(1\ \ i)\, D \begin{pmatrix} {}^tZZ + 1 \\ i\,{}^tZZ - i \end{pmatrix}\right)^{-1}.$$

From the commutative diagram

$$\begin{array}{ccc} \begin{pmatrix} A & B \\ C & D \end{pmatrix} & \longmapsto & T \begin{pmatrix} A & B \\ C & D \end{pmatrix} T \\ \Big\downarrow p\pi & & \Big\downarrow p\pi \\ B \begin{pmatrix} & 1 \\ - & i \end{pmatrix} \left((1\ \ i)\, D \begin{pmatrix} & 1 \\ - & i \end{pmatrix}\right)^{-1} & \xmapsto{\Sigma} & T'\, B \begin{pmatrix} & 1 \\ - & i \end{pmatrix} \left((1\ \ i)\, D \begin{pmatrix} & 1 \\ - & i \end{pmatrix}\right)^{-1} \end{array}$$

(where T' is the $n \times n$ diagonal matrix obtained from T by suppressing the last two rows and columns) we find that the set of all reflections is $\{g\,\Sigma\,g^{-1}/g \in S0_0(n, 2)\}$ where Σ is the reflection

$$\Sigma(z^1,\ z^2,\ \dots\ z^n) = (-\,z^1\,,\ z^2,\ \dots\ z^n).$$

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