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DISSIPATIVE SETS AND NONLINEAR PERTURBATED EQUATIONS IN BANACH SPACES

by VIOREL BARBU

ABSTRACT - Some existence results for abstract functional equations in Banach spaces are proved.

Introduction.

Let $X$ be a real Banach space $X^*$ its dual space, $(u,v)$ the pairing between $v$ in $X^*$ and $u$ in $X$. The duality mapping of $X$ in the subset $F$ of $X \times X$ defined by

$$F = \{ [x, x] ; x \in X, x^* \in X^* \text{ and } (x, x^*) = \| x \|^2 = \| x^* \|^2 \}$$

where $\| \|$ denotes the norm in $X$ (respectively $X^*$).

Let $A$ be a subset of $X \times X$. We define

$$Ax = y \in X; [x, y] \in A, D(A) = [x \in X; Ax = \emptyset], R(A) = \bigcup_{x \in D(A)} Ax,$$

and

$$A^{-1} = [y, x]; [x, y] \in A, \alpha A = \{ [x, \alpha y] ; [x, y] \in A \}$$

where $\alpha$ is real. If $B$ is a subset of $X \times X$ then,

$$A + B = [x, y + z]; [x, y] \in A \text{ and } [x, z] \in B.$$

A subset $A$ of $X \times X$ is called dissipative if for every $[x_i, y_i] \in A, i = 1, 2$ there exists $f \in F(x_1 - x_2)$ such that

$$(y_1 - y_2, f) \leq 0$$

or equivalently (see T. Kato [10], Lemma 1.1),

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(0.2) \[ \| x_i - x_2 \| \leq \| x_i - \lambda y_i - (x_0 - \lambda y_0) \| \]

for each \( \lambda > 0 \) and \( [x_i, y_i] \in A, i = 1, 2 \).

If \( A \) is dissipative one can define for \( \lambda > 0 \) a single valued operator \( A_\lambda = I^{-1} ((1 - \lambda A)^{-1} - 1) \) with \( D(A_\lambda) = R(1 - \lambda A) \). We notice some properties of \( A_\lambda \) which will be used frequently in this paper (for the proof see T. Kato [11]).

**Lemma 0.1.** Let \( A \) be dissipative, then

a) \( A_\lambda \) is dissipative and lipschitz with constant \( 2\lambda^{-1} \).

b) For \( x \in R(1 - \lambda A) \cap D(A), A_\lambda x \in A(1 - \lambda A)^{-1} x \) and \( \| A_\lambda x \| \leq \| Ax \| \)

We have denoted here, \( |Ax| = \inf \{ \| y \| : y \in Ax \} \).

A dissipative subset \( A \) of \( X \times X \) is called \( m \)-dissipative if \( R(1 - \lambda A) = X \) for every (or, equivalently, for some) \( \lambda > 0 \).

For other basic properties of dissipative sets and nonlinear semigroups of contractions we refer to Kömurs [12], Crundall and Pazy [6], T. Kato [11], F. Browder [2], Brezis and Pazy [4].

The purpose of this paper is to obtain existence results for perturbated nonlinear differential (respectively functional) equations on Banach spaces. Section 1 and 2 contain the main results. We start with an existence theorem for evolution equations, Theorem 1 which is the main tool used in proving principal perturbation results given in Section 2. Similar results were obtained previously by G. Da Prato (see [7]) in linear case. For related results see also [1], [2], [6], [9], [11].

In Sections 3 and 4 we apply these results in the study of certain nonlinear evolution equations.

§ 1. A class of nonlinear evolution equations.

Throughout this section we assume that \( X \) is a real Banach space and that the dual \( X^* \) of \( X \) is uniformly convex. In particular this implies that the duality mapping \( F \) of \( X \) is uniformly continuous on every bounded subset of \( X \) (see [10], Lemma 1.2).

Let \( C \) be a closed convex subset of \( X \).

In the present section we consider equations of evolution of the form

\[
\lambda u(t) + \frac{du(t)}{dt} \in A(t)u(t) + Bu(t) + \lambda f(t), \text{ a.e. on } (0, T)
\]
with the conditions

\( u(0) = x, u(t) \in C \) for \( 0 \leq t \leq T < \infty \),

on the space \( X \), where \( B \) is the infinitesimal generator of a strongly continuous semigroup of linear contractions on \( X \) and \( A(t) \) is a family of subsets of \( X \times X \) satisfying the following assumptions:

i) For every \( t \in [0, T] \), \( A(t) \) is a closed and dissipative subset of \( X \times X \). The domain \( D(A(t)) = D \) of \( A(t) \) is independent of \( t \).

ii) \( (1 - \lambda B)^{-1} C \subset C \) for every \( \lambda > 0 \).

iii) \( R(1 - \lambda A(t)) \) contains \( C \) and \( (1 - \lambda A(t))^{-1} C \subset C \) for every \( \lambda > 0 \) and for any \( t \in [0, T] \). Moreover,

\[
\| (1 - \lambda A(t))^{-1} x - (1 - \lambda A(s))^{-1} x \| \leq \lambda | t - s | \varphi (\| x \| + \| A_t(t) x \|)
\]

for each \( x \in C, t, s \in [0, T] \). Here \( \varphi : [0, \infty) \to [0, \infty) \) is an increasing continuous function such that \( \int_0^\infty \frac{dt}{\varphi(t)} = \infty \).

(iv) \( (1 - \lambda A(t))^{-1} (D(B) \cap C) \subset D(B) \cap C \) for \( \lambda > 0, t \in [0, T] \)

and

\[
\| B(1 - \lambda A(t))^{-1} x \| \leq \| Bx \| + \lambda \psi (\| x \| + \| A_t(t) x \|)
\]

for every \( x \in D(B) \cap C, \lambda > 0 \) and \( 0 \leq T \). Here \( \psi \) is an increasing continuous function from \([0, \infty)\) into itself such that \( \int_0^\infty \frac{dt}{\psi(t)} = \infty \).

Now we shall recall some definitions.

If \( X \) is a real Banach space with norm \( \| \cdot \|_X \) then \( L^p(0, T; X) \), \( 1 \leq p \leq \infty \), denotes the space of (classes of) measurable functions \( u : [0, T) \to X \) such that

\[
\| u \|_p^p = \int_0^T \| u(s) \|_X^p \, ds < \infty, \quad 1 \leq p < \infty
\]

and the usual modification in case \( p = \infty \).

If \( C \) is a closed subset of \( X \) we set

\( L^p(0, T; C) = \{ u \mid u \in L^p(0, T; X) \text{ and } u(t) \in C \text{ a.e. on } (0, T) \} \).

We denote also by \( W^{1,p}(0, T; X) \) the space of all absolutely continuous functions \( u : [0, T] \to X \) such that \( \frac{du(t)}{dt} \in L^p(0, T; X) \).
Finally, we set

\[ W_{0}^{1,p}(0, T; X) = \{ u : u \in W^{1,p}(0, T; X) \text{ and } u(0) = 0 \}. \]

**THEOREM 1.** Let \( C \) be a closed convex subset of \( X \) and let \( A(t) \) and \( B \) be closed dissipative subsets of \( X \times X \) satisfying Assumptions i) \( \cdots \) iv). Let \( f \in W^{1,1}(0, T; X) \cap L^{\infty}(0, T; D(B)) \) be such that \( f(t) \in C \) for \( 0 \leq t \leq T \).

Then for every \( x \in D \cap D(B) \cap C \) and for \( \lambda \geq 0 \), the initial value problem

\[
\begin{cases}
\lambda u(t) + \frac{du(t)}{dt} \in Bu(t) + A(t)u(t) + \lambda f(t), & 0 < t < T; \\
u(0) = x
\end{cases}
\]

(1.5)

has a unique solution \( u \in W^{1,\infty}(0, T; X) \cap L^{\infty}(0, T; D(B)) \) such that \( u(t) \in \epsilon C \) for all \( t \in [0, T] \).

We preface the proof of Theorem 1 with the proof of some auxiliary lemmas.

**LEMMA 1.1.** Let \( Y \) be a real Banach space with uniformly convex adjoint space \( Y^* \). Let \( K \) be a closed convex subset of \( Y \) and let \( A \) and \( L \) be two closed dissipative sets of \( Y \times Y \). Suppose

a) \( A \) is continuous and bounded on every bounded subset of \( K = D(A) \). \( R(1 - \lambda A) \) contains \( K \) for every \( \lambda > 0 \).

b) \( K \subseteq \bigcap_{\lambda > 0} R(1 - \lambda L) \) and \( (1 - \lambda L)^{-1} K \subseteq K \) for every \( \lambda > 0 \).

Then for every \( \lambda > 0 \) and for any \( y \in K \), there exists a unique solution \( u \in D(L) \cap K \) of the equation

\[
\lambda u - Lu - Au \ni y.
\]

(1.6)

The proof is similar to that of Theorem 4.3 in [6] (see also the proof of Theorem 3 in § 2).

**LEMMA 1.2.** Let \( A \) and \( B \) satisfy Assumptions i), ii) and iii). Let \( f \in W^{1,1}(0, T; X) \) be such that \( f(t) \in C \) for all \( t \in [0, T] \).

Then for any \( \lambda > 0 \) and for any \( x \in D(B) \cap C \) there exists a unique \( u \in W^{1,\infty}(0, T; X) \in L^{\infty}(0, T; D(B)) \) such that \( u(0) = x, u(t) \in \epsilon C \) for all \( t \in [0, T] \) and

\[
\lambda u(t) + \frac{du(t)}{dt} = Bu(t) + A(t)u(t) + \lambda f(t); \text{ a. e. on } (0, T).
\]

(1.7)
PROOF. We may assume without loss of generality that \( x = 0 \).
This can be achieved by shifting \( C \). We fix \( p \in (1, \infty) \) and put \( K = L^p(0, T; C) \).

Let \( \tilde{A} \) denote the dissipative operator on \( Y = L^p(0, T; X) \) with domain \( K \) which is given by \( (\tilde{A} u)(t) = A_n(t) u(t) \) a.e. on \( (0, T) \) for \( u \in K \). Clearly \( \tilde{A} \) is well defined, continuous and bounded on every bounded subset of \( K \subset Y \) (see Lemma 0.1).

Let \( L \) be the linear operator defined in \( Y \) by

\[
D(L) = W^{1,p}_0(0, T; X) \cap L^\infty(0, T; D(B))
\]

and

\[
Lu = -\frac{du}{dt} + Bu \text{ for } u \in D(L).
\]

Here \( D(B) \) is considered as Banach space with norm defined by \( |x| = \|Bx\| + \|x\| \).

Since \( (1 - \lambda B)^{-1} \subset C \) for every \( \lambda > 0 \) it is easy to see that

\[
K \subset \bigcap_{\lambda > 0} R(1 - \lambda \tilde{L})
\]

and

\[
(1 - \lambda \tilde{L})^{-1} K \subset K \text{ for } \lambda > 0,
\]

where \( \tilde{L} \) is the closure of \( L \) in \( Y \times Y \).

We apply Lemma 1.1 to conclude that for every \( \lambda > 0 \) that there exists a unique solution \( u \in K \) of the equation

\[
\lambda u - \tilde{L} u = \tilde{A} u \neq \lambda f.
\]

By the definition of \( L \) there exists sequences \( \{u_k\} \subset K \cap D(L) \) and \( \{f_k\} \subset K \)

such that \( u_k \to u \) and

\[
(1.9) \quad \lambda u_k(t) + \frac{du_k(t)}{dt} - Bu_k(t) - A_n(t) u_k(t) = \lambda f_k(t) \to \lambda f(t)
\]

in \( L^p(0, T; Y) \) as \( k \to \infty \). Let \( k, j > 0 \). Since \( B \) and \( A_n \) are dissipative we obtain from (1.9) that

\[
\left( \frac{d}{dt} (u_k(t) - u_j(t)), F(u_k(t) - u_j(t)) \right) \leq -\lambda \|u_k(t) - u_j(t)\|^2 + \lambda \|f_k(t) - f_j(t)\| \||u_k(t) - u_j(t)| |
\]
for almost all \( t \in (0, T) \). By using the equality (see [10], Lemma 1.3)

\[
\left( \frac{d}{dt} (u_k(t) - u_j(t)), F(u_k(t) - u_j(t)) \right) = 2^{-1} \frac{d}{dt} \| u_k(t) - u_j(t) \|^2 \text{ a.e.}
\]

we obtain

\[
(1.10) \quad \| u_k(t) - u_j(t) \| \leq \exp(-\lambda t) \| u_k(0) - u_j(0) \| + \int_0^t \exp(-\lambda (t - s)) \| f_k(s) - f_j(s) \| ds.
\]

Since \( u_k(0) = u_j(0) = 0 \), we conclude that \( u_k(t) \) converges uniformly to \( u(t) \) on \([0, T]\). Let \( t, t + h \in (0, T) \) be such that \( \frac{d}{dt} (u_k(t + h) - u_k(t)) \) exists. Repeating the above argument we obtain

\[
(1.11) \quad \| u_k(t + h) - u_k(t) \| \leq \exp(-\lambda t) \| u_k(h) - u_k(0) \| + \int_0^t \exp(-\lambda (t - s)) (\| f_k(s + h) - f_k(s) \| +
\]

\[+ \lambda \int_0^t \exp(-\lambda (t - s)) \| f_k(s + h) - f_k(s) \| ds \]

and

\[
(1.12) \quad \| u_k(h) - u_k(0) \| \leq \int_0^h (\| A_n(s) \| \| f_k(s) \| ) ds,
\]

Passing to the limit \( k \to \infty \) in (1.11) and (1.12) we obtain

\[
(1.13) \quad \| u(t + h) - u(t) \| \leq \exp(-\lambda t) \int_0^h (\| A_n(s) \| \| f(s) \| ) ds + \lambda \int_0^t \exp(-\lambda (t - s)) (\| f(s + h) - f(s) \| + \lambda^{-1} (\| A_n(s + h) - A_n(s) \| u(s) \| ) ds.
\]

On the other hand by Assumption iii), \( \| A_n(s + h) - A_n(s) \| u(s) \| \leq h \| f(s) \| + \| A_n(s) u(s) \| \). Since \( \frac{df}{dt}(t) \in L^1(0, T; X) \) it follows from (1.13) that \( u \in W^{1, \infty}(0, T; X) \) (see Kömura [12], appendix).
Denote by $g(t)$ the function

$$g(t) = \lambda f(t) + A_n(t)u(t) - \lambda u(t).$$

Since $u_k(t)$ converges uniformly to $u(t)$, by (1.9) we have

$$u(t) = \int_0^t S(t-s)g(s)\,ds, \quad 0 \leq t \leq T,$$

where $S(t)$ denotes the semigroup generated by $B$ in $X$.

It is clear so that (1.14) implies that $u \in L^\infty(0, T; D(B))$ and

$$\frac{du(t)}{dt} = Bu(t) + A_n(t)u(t) + \lambda f(t) \text{ a.e. on } (0, T).$$

This proved Lemma 1.2. for $\lambda > 0$.

Let $u_1 \in W^{1,\infty}_0(0, T; X) \cap L^\infty(0, T; D(B))$ be the solution of equation (1.15). Repeating the above argument it follows easily that $u_1(t)$ is uniformly convergent on $[0, T]$ as $\lambda \to 0$ and that $\frac{du_1(t)}{dt}$ is bounded uniformly on $(0, T)$. Thus passing to the limit $\lambda \to 0$ in (1.15) it follows Lemma 1.2 in the case $\lambda = 0$.

This completes the proof.

**Proof of Theorem 1.** Let $f \in W^{1,2}(0, T; X) \cap L^\infty(0, T; D(B))$ be such that $f(t) \in C$ for all $t \in [0, T]$ and let $x$ be an arbitrary element of $D \cap D(B) \cap C$. By Lemma 1.2 there exists a unique $u_n \in W^{1,\infty}(0, T; X) \cap L^\infty(0, T; D(B))$ such that $u_n(t) = x$, $u_n(t) \in C$ on $[0, T]$.

$$\lambda u_n(t) + \frac{du_n(t)}{dt} = Bu_n(t) + A_n(t)u_n(t) + \lambda f(t) \text{ a.e. on } (0, T)$$

Obviously,

$$u_n(t) = \exp\left(-(n+\lambda)t\right)S(t)x + \int_0^t \exp\left(-(n+\lambda)(t-s)\right)S(t-s)\cdot$$

$$(1 - n^{-1}A(s))^{-1}u_n(s)\,ds + \lambda \int_0^t \exp\left(-(n+\lambda)(t-s)\right)f(s)\,ds, \quad 0 \leq t \leq T$$

$$0 \leq t \leq T$$
where $S(t)$ is the semigroup generated by $B$. By Assumption iv) it follows that

\begin{equation}
\|Bu_n(t)\| \leq \exp\left(-(n + \lambda) t\right) \|Bx\| + \|\lambda(n + \lambda)^{-1}Bf\|_\infty + \\
+ \int_0^t \exp\left(-(n + \lambda)(t - s)\right)(n \|Bu_n(s)\| + \varphi(\|u_n(s)\|) + \\
+ \|A_n(s)u_n(s)\|)\,ds.
\end{equation}

Since $A_n(t)$ and $B$ are dissipative, from (1.16) we obtain that

\[\frac{d}{dt}\|u_n(t) - x\| \leq -\lambda \|u_n(t) - x\| + \|A_n(t)x\| + \|Bx\| + \lambda \|f(t)\|\]

therefore

\begin{equation}
\|u_n(t) - x\| \leq \int_0^t \exp\left(-\lambda(t - s)\right)(\|A_n(s)x\| + \|Bx\| + \\
+ \lambda \|f(x)\|)\,ds, \ 0 \leq t \leq T.
\end{equation}

By using the same argument as in the proof of Lemma 1.2, we obtain

\begin{equation}
\|u_n(t + h) - u_n(t)\| \leq \exp\left(-\lambda t\right) \|u_n(h) - x\| + \\
+ \int_0^t \exp\left(-\lambda(t - s)\right)(\|f(s + h) - f(s)\| + \lambda \|u_n(s)\| + \\
+ \|A_n(s)u_n(s)\|)\,ds
\end{equation}

for all $t, t + h \in [0, T]$. On the other hand (1.18) implies that

\[\lim_{t \to 0} t^{-1} \|u_n(t) - x\| \leq \|A_n(0)x\| + \lambda \|f(0)\|.
\]

Using this estimate together with (1.19) we see that

\begin{equation}
\frac{\|du_n(t)\|}{dt} \leq M \exp\left(-\lambda t\right) + \int_0^t \exp\left(-\lambda(t - s)\right)\left(\|\frac{df(s)}{ds}\| + \\
+ \varphi(\|u_n(s)\| + \|A_n(s)u_n(s)\|)\right)\,ds
\end{equation}

for almost all $t \in (0, T)$, where $M$ is a positive constant independent of $n$. 

Since \( u_n(t) \) are uniformly bounded on \([0, T]\), from (1.17) and (1.20) we obtain
\[
\left( \frac{du_n(t)}{dt} \right) + \| B u_n(t) \| \leq \left( M + \left\| \frac{df}{dt} \right\|_1 \right) \exp (-\lambda t) + f_n(t) + y_n(t), \text{ a.e.}
\]
where
\[
f_n(t) = \exp (- (n + \lambda) t) \| B x \| + \lambda (n + \lambda)^{-1} \| B f \|_{\infty}
\]
while
\[
y_n(t) = \int_0^t \exp (- \lambda (t - s)) \phi \left( k_0 + \| B u_n(s) \| + \left\| \frac{du_n(s)}{ds} \right\| \right) ds + \int_0^t \exp (- (n + \lambda) (t - s)) \left( \phi (k_1 + \| B u_n(s) \| + \| B u_n(s) \| + \left\| \frac{du_n(s)}{ds} \right\| \right) ds,
\]
where \( k_0 \) is a constant independent of \( n \).

By a simple computation it follows that
\[
\frac{dy_n(t)}{dt} \leq - \lambda y_n(t) + \phi (k_1 + y_n(t)) + \psi (k_0 + \| B u_n(t) \| + \left\| \frac{du_n(t)}{ds} \right\|) + n f_n(t), \quad 0 < t \leq T,
\]
where \( k_1 \) is a suitable constant independent of \( n \). Since \( n f_n(t) \) is bounded we conclude from (1.22) and (1.21) that
\[
\| \frac{du_n(t)}{dt} \| + \| B u_n(t) \| \leq M_T < \infty \quad \text{for} \ 0 < t < T.
\]

Thus by using the fact that the duality mapping \( F \) is uniformly continuous on every bounded subset of \( X \), it follows by a standard argument (see [10], Lemma 4.3) that \( u_n(t) \) converges uniformly on \([0, T]\) as \( n \to \infty \). Let \( u(t) = \lim_{n \to \infty} u_n(t) \).

Clearly \( u(t) \) is absolutely continuous on \([0, T]\). Since the space \( X \) is reflexive this implies that (see [12], Appendix) \( \frac{du(t)}{dt} \) exists a.e. on \((0, T)\). Moreover the inequality (1.23) implies obviously that \( u \in W^{1,\infty}(0, T; X) \cap L^\infty(0, T; D(B)) \).

We shall prove that is the solution of initial value problem (1.6). For this latter purpose, choose \( t_0 \in (0, T) \) such that \( u(t) \) is differentiable at \( t = t_0 \). Let \( \tilde{x}, \tilde{y} \) be an arbitrary element of \( A(t_0) \) such that \( \tilde{x} - z \tilde{y} \in C \).
for some positive \( \alpha \). This implies that \( \tilde{x}_n = x - n^{-1} \tilde{y} \) lies in \( C \) for some sufficiently large \( n \). Since \( \tilde{y} = x - A_n(t_0) \tilde{x}_n \), we see from (1.19) that

\[
2^{-1} \frac{d}{dt} \| u_n(t) - \tilde{x}_n \|^2 \leq (Bu_n(t) + \tilde{y} - \lambda u(t) + \lambda f(t), F(u_n(t) - \tilde{x}_n)) + \\
+ \| (A_n(t) - A_n(t_0)) \tilde{x}_n \| \| u_n(t) - \tilde{x}_n \|, \text{ a.e. on } (0, T).
\]

Integrating this inequality over \((t_0, t)\) and using Assumption iii) we obtain

\[
\| u_n(t) - \tilde{x}_n \|^2 - \| u_n(t_0) - \tilde{x}_n \|^2 \leq 2 \int_{t_0}^{t} (Bu_n(s) + \tilde{y} - \\
- \lambda u(s) + \lambda f(s), F(u_n(s) - \tilde{x}_n)) \, ds + M_0 \| t - t_0 \|^2 \varphi (\| \tilde{y} \|) + \\
+ \| \tilde{y} - \tilde{x} \|, \quad 0 < t \leq T,
\]

where \( M_0 \) is independent of \( n \).

Now \( Bu_n(s) \rightarrow Bu(s), u_n(s) \rightarrow u(s) \) and \( \tilde{x}_n \rightarrow \tilde{x} \) as \( n \rightarrow \infty \). We pass to limit as \( n \rightarrow \infty \) in (1.24) to obtain

\[
\| u(t) - \tilde{x} \|^2 - \| u(t_0) - \tilde{x} \|^2 \leq 2 \int_{t_0}^{t} (Bu(s) + \tilde{y} - \lambda u(s) + \lambda f(s), F(u(s) - \\
- \tilde{x})) \, ds + M_0 \| t - t_0 \|^2 \varphi (\| \tilde{x} \| + \| \tilde{y} - \tilde{x} \|), \quad 0 \leq t \leq T,
\]

so that

\[
(u(t) - u(t_0), F(u(t) - \tilde{x})) \leq 2 \int_{t_0}^{t} (Bu(s) + \tilde{y} - \lambda u(s) + \lambda f(s), \\
F(u(s) - \tilde{x})) \, ds + M_0 \| t - t_0 \|^2 \varphi (\| \tilde{x} \| + \| \tilde{y} - \tilde{x} \|), \quad 0 \leq t \leq T.
\]

Since the function \( t \rightarrow Bu(t) \) is weakly continuous, we obtain

\[
(1.25) \quad \left( \frac{du(t_0)}{dt} - Bu(t_0) - \tilde{y} + \lambda u(t_0) - \lambda f(t_0), F(u(t_0) - \tilde{x}) \right) \leq 0.
\]

Let \( \{ \varepsilon_n \} \) be a sequence of nonnegative numbers such that \( \lim_{n \rightarrow \infty} \varepsilon_n = 0 \).
Define

\[ A(\varepsilon_n) = \varepsilon_n^{-1} \left( S(\varepsilon_n) u(t_0 - \varepsilon_n) - u(t_0) \right) - Bu(t_0) + \frac{du(t_0)}{dt}. \]

We notice that Assumption ii) implies that \( S(t) C \subseteq C \) for all \( t \geq 0 \). Thus for every \( n \) there exists \( A(t_0) \) such that

\[ S(\varepsilon_n) u(t_0 - \varepsilon_n) = x_n - \varepsilon_n y_n - \lambda \varepsilon_n (f(t_0) - x_n). \]

Consequently,

\[ (1.26) \quad A(\varepsilon_n) = \varepsilon_n^{-1} (x_n - u(t_0)) - y_n - \lambda f(t_0) + \lambda x_n - Bu(t_0) + \frac{du(t_0)}{dt}. \]

Now, we use (1.25) where \( \tilde{x}, \tilde{y} = [x_n, y_n] \) to obtain that

\[ (\lambda + \varepsilon_n^{-1}) \| u(t_0) - x_n \| ^2 \leq (A(\varepsilon_n), F(u(t_0) - x_n)). \]

It is clear that \( \lim_{n \to \infty} A(\varepsilon_n) = 0 \). So that letting \( n \to \infty \), we see that

\[ \lim_{n \to \infty} \varepsilon_n^{-1} (u(t_0) - x_n) = 0. \]

This last observation together (1.26) imply that \( x_n \to u(t_0) \) and \( y_n \to \frac{du(t_0)}{dt} \) as \( n \to \infty \). Since \( A(t_0) \) is closed we conclude that

\[ \lambda u(t_0) + \frac{du(t_0)}{dt} \in Bu(t_0) + A(t_0) u(t_0) + \lambda f(t_0). \]

The uniqueness of solution \( u \) follows immediately from the dissipativeness property of \( B \) and \( A(t) \).

This completes the proof.

\section{Some perturbation results.}

As in preceding section \( X \) is a real Banach space with uniformly convex adjoint and \( C \) is a closed convex subset of \( X \).

We consider the functional equation in \( X \) of the form

\[ \lambda u - Au - Bu \geq \lambda f, \quad f \in X, \ u \in C, \]

where \( A \) and \( B \) are dissipative subsets of \( X \times X \),
which satisfy the following conditions:

j) A is closed dissipative subset of $X \times X$. $R(1 - \lambda A)$ contains $C$ for $\lambda > 0$ and

$$ (1 - \lambda A)^{-1} C \subseteq C \text{ for every } \lambda > 0. $$

ji) $B$ is a densely defined, linear and $m$-dissipative operator in $X$.

$$ (1 - \lambda B)^{-1} C \subseteq C \text{ for every } \lambda > 0. $$

jj) $(1 - \lambda A)^{-1} (D(B) \cap C) \subseteq D(B)$ for every $\lambda > 0$ and

$$ \| B(1 - \lambda A)^{-1} x \| \leq \| Bx \| + \lambda \psi(\| x \| + \| A_1 x \|) $$

holds for every $x \in D(B) \cap C$ and for each $\lambda > 0$.

Here $\psi : [0, \infty) \rightarrow [0, \infty)$ is an increasing function such that $\int_0^\infty \frac{dt}{\psi(t)} = \infty$.

The main result of this section may be stated as follows:

**Theorem 2.** Let $A$ and $B$ be dissipative subsets of $X \times X$ satisfying conditions j), ji) and jj).

Then

$$ C \subseteq (1 - \frac{\lambda A + B}{\lambda A + B}) (D(A + B) \cap C) \text{ for all } \lambda > 0 $$

and

$$ (1 - \frac{\lambda A + B}{\lambda A + B})^{-1} y = \lim_{n \to \infty} (1 - \frac{\lambda}{\lambda A + B})^{-1} y $$

for every $y \in C$ and $\lambda > 0$. (Here $\overline{A + B}$ denotes the closure of $A + B$ in $X \times X$).

A stronger version of Theorem 2 is

**Theorem 5.** Let $A$ and $B$ be dissipative and closed subsets of $X \times X$ satisfying assumptions j), ji) with the inequality (2.3) replaced by

$$ \| B(1 - \lambda A)^{-1} x \| \leq \| Bx \| + M\lambda (\| x \| + \| A_1 x \|), \text{ for } x \in D(B) \cap C, $$

where $M$ is a nonnegative constant independent of $\lambda$.

Then

$$ (1 - \frac{\lambda (A + B)}{\lambda A + B}) (D(A) \cap D(B) \cap C) \supseteq C \text{ for some sufficiently large } \lambda. $$

**Corollary 2.1.** Let $A$ and $B$ satisfy hypotheses of Theorem 2. Suppose in addition that $X$ is uniformly convex and that the following condition holds
Then
\[ 0 \in A + B(D(A) \cap D(B) \cap C). \]

**Proof of Theorem 2.** Let \( y \) be an arbitrary element of \( D(B) \cap C \) and let \( \lambda > 0 \). We fix \( x \in D(A) \cap D(B) \cap C \) and denote by \( u(t) \) the solution of problem (1.5) where \( A(t) \equiv A \) and \( f(t) \equiv y \). It is clear that \( u(t) \) can be extended as solution of the equation (1.5) on \((0, \infty)\). From the proof of Theorem 1 (see (1.18) and (1.19)) we obtain

\[ \| u(t) - x \| \leq \lambda^{-1} (1 - \exp(-\lambda t))(\| Ax \| + \| Bx \| + \lambda \| y \|), \quad 0 < t < \infty \]

therefore

\[ \| u(t + h) - u(t) \| \leq \lambda^{-2} \exp(-\lambda t)(1 - \exp(-\lambda h))(\| Ax \| + \| Bx \| + \lambda \| y \|). \]

This estimate implies immediately that \( u(t) \) converges as \( t \to \infty \) and

\[ \lim_{t \to \infty} \sup_{t} \frac{du(t)}{dt} = 0. \]

Let \( u = \lim_{t \to \infty} u(t) \). Letting \( t \to \infty \) in (1.5) we see that

\[ (\lambda - A + B)u \equiv 0. \]

Note that \((1 - \lambda A + B)^{-1}\) is well defined and nonexpansive on \( D(B) \cap C \) in consequence of the fact that \( A + B \) is dissipative. On the other hand condition \( j) \) implies that \( D(B) \cap C \) is a dense subset of \( C \). Hence \( R(1 - 1A + B) \) contains \( C \) for every \( \lambda > 0 \) which proves (2.4).

By using a standard fixed point technique it follows easily that for any \( u, (1 - \lambda (A + B))^{-1}\) is well defined and nonexpansive on \( C \). It suffices to prove (2.5) for every \( y \in D(B) \cap C \).

Let \( u_n(t) \in C \) be the solution of equation

\[ \lambda u_n(t) + \frac{du_n(t)}{dt} = A_n u_n(t) + Bu_n(t) + \lambda y, \quad 0 < t < \infty, \]

with initial condition \( u_n(0) = x \in D(A) \cap D(B) \cap C \).

From the proof of Theorem 1 (see (1.18) and (1.19)) we deduce that \( u_n = \lim_{t \to \infty} u_n(t) \) exists uniformly with respect to \( t \). Moreover since \( A_n \) are
continuous and $B$ is closed it follows that $u_n \in D(B) \cap C$ and

$$
\lambda u_n - A_n u_n - B u_n = \lambda y, \quad n = 1, 2, ...
$$

We know by the proof of Theorem 1 that $u_n(t)$ converges uniformly on every bounded interval of $[0, \infty)$ to the solution $u(t)$ of problem (1.5). On the other hand according to first part of the proof we have

$$
(1 - \lambda A + B)^{-1} y = \lim_{t \to \infty} u(t).
$$

Thus by a simple computation it follows that

$$
\lim_{n \to \infty} u_n = (1 - \lambda^{-1} A + B)^{-1} y
$$

which concludes the proof.

**Proof of Theorem 3.** Consider the equation

(2.9)  

$$
\lambda u_n - B u_n - A_n u_n = \lambda y, \quad n = 1, 2, ...
$$

which is equivalent to

(2.10)  

$$
 u_n = \lambda (\lambda + n - B)^{-1} y + n (\lambda + n - B)^{-1} (1 - n^{-1} A)^{-1} u_n.
$$

By using the contraction fixed point theorem it follows easily that for every $y \in C$ and any fixed $\lambda > 0$ this equation has a unique solution $u_n \in D(A) \cap D(B) \cap C$. Let $x$ be fixed in $D(A) \cap D(B) \cap C$. Multiplying (2.9) by $F(u_n - x)$ yields

(2.11)  

$$
\lambda \| u_n - x \| \leq \lambda \| y \| + \| B x \| + \| A x \| + \lambda \| x \|
$$

since $A_n$ and $B$ are dissipative.

Suppose now that $y \in D(B) \cap C$. Then from (2.5) and (2.10) we obtain

$$
\| B u_n \| \leq \lambda (\lambda + \lambda)^{-1} \| B y \| + n (\lambda + \lambda)^{-1} \| B u_n \| + M(n + \lambda)^{-1} (\| u_n \| + \| A_n u_n \|).
$$

Consequently

$$
\| B u_n \| \leq \lambda (\lambda - M)^{-1} \| B y \| + M (\lambda + 1) (\lambda - M)^{-1} \| u_n \| + M \lambda (\lambda - M)^{-1} \| y \|
$$

if $M < \lambda$. This estimate together (2.11) show that $\| B u_n \|$ and $\| A_n u_n \|$ are bounded as $n \to \infty$ if $\lambda$ is sufficiently large. We fix $\lambda > M$.

Thus following a standard method (see [1], [6]), we see that $[u_n]$ converges as $n \to \infty$. Let $u = \lim_{n \to \infty} u_n$. Letting $n \to \infty$ in (2.9) we obtain
where $\tilde{A}$ is the smallest demiclosed extension of $A$.

Using the fact that duality mapping $F$ is continuous we see easily that $\tilde{A}$ is dissipative in $X \times X$.

Let $|\epsilon_n|$ be a sequence of nonnegative numbers such that $\lim_{n \to \infty} \epsilon_n = 0$. We set $A_n = \epsilon_n^{-1} (S(\epsilon_n) u - u) - Bu$,

where $S$ is the semigroup generated by $B$. Since $S(\epsilon_n) C \subseteq C$, in view of assumption j), for every $n$ there exists $[x_n, y_n] \in A$ such that $B(\epsilon_n) u = x_n - \epsilon_n y_n - \lambda \epsilon_n (y + x_n)$.

Consequently

\begin{equation}
A_n = \epsilon_n^{-1} (x_n - u) - Bu - y_n - \lambda (y + x_n).
\end{equation}

Multiplying (2.13) by $F(x_n - u)$ and using (2.12) we obtain

$$(\lambda + \epsilon_n^{-1}) \| x_n - u \| \leq \| A_n \|$$

Since $A_n \to 0$ it follows from (2.13) that $y_n \to \lambda u - Bu - \lambda y$.

Hence

$$\lambda u - Au - Bu \geq \lambda y$$

Since $A$ is closed. This completes the proof.

**Proof of Corollary 2.1.** In view of Theorem 2, for every $y \in C$ and for each $\lambda > 0$ the equation

\begin{equation}
\lambda u_\lambda - \overline{A + B} u_\lambda \geq \lambda y
\end{equation}

has a unique solution $u_\lambda \in D(\overline{A + B}) \cap C$. Let $x$ be arbitrary but fixed in $D(\overline{A + B})$. We multiply (2.14) by $F(u_\lambda - x)$. We obtain

$$\lambda \| u_\lambda - x \| \leq \| A + Bx \| + \| y \| + \lambda \| x \|$$

since $A + B$ is dissipative. Using this estimate together jv) and (2.14) we see that $|u_\lambda|$ is bounded as $\lambda \to 0$. Without loss of generality we may as-
sume that $u_k \to u$ as $\lambda \to 0$. Let $h > 0$ we have

$$\| (1 - hA + B)^{-1} u_k - u_k \| \leq h \| A + B \| u_k \|, \text{ for } \lambda > 0.$$ 

From (2.14) it follows that $\lim_{\lambda \to 0} (1 - hA + B)^{-1} u_k = u$. Since $X$ is uniformly convex and $(1 - hA + B)^{-1}$ is nonexpansive on $C$ we conclude that $(1 - hA + B) u \ni u$ (see [2], Theorem 8.2).

Hence

$$0 \in A + Bu$$

which concludes the proof.

A slightly modified version of Theorem 2 is useful in some applications.

**Corollary 2.2.** Let $A$ and $B$ satisfy hypotheses of Theorem 2 with Assumption \( \| u \| \) replaced by

$$D(A) \cap D(B) \neq \emptyset \text{ and } (Bu, F(A_n u) \geq 0$$

for every $u \in D(B) \cap C$ and $n = 1, 2, ...$.

Then

$$\lambda(A + B) (D(A) \cap D(B) \cap C) \supset C \text{ for every } \lambda > 0.$$ 

**Proof.** Let $y \in C$, $\lambda > 0$ and let $u_n \in D(B) \cap D$ be the solution of the equation

$$\lambda u_n - Bu_n = A_n u_n = \lambda y.$$ 

Condition (2.15) implies that $\| Bu_n \|$ and $\| A_n u_n \|$ are bounded as $n \to \infty$. From this the proof proceeds exactly as the proof of Theorem 3.

**Remark 2.1.** By the proof we see easily that if $A$ and $B$ satisfy to assumption of Theorem 2 with (2.6) replaced by the following stronger assumption

$$\| B(1 - \lambda A)^{-1} x \| \leq \| Bx \|, \text{ for } x \in D(B) \cap C \text{ and } \lambda \geq 0,$$

then $(1 - \lambda(A + B)) (D(A) \cap D(B) \cap C) \supset C$ for all $\lambda > 0$.

§ 3. Periodic problems.

We consider in this section evolution equations of the form

$$\lambda u(t) + \frac{du(t)}{dt} \in A(t) u(t) + \lambda f(t), \text{ a. e. on } (0, T)$$
with the conditions
\[(3.2) \quad u(0) = u(T); \quad u(t) \in C \text{ for all } t \in [0, T]\]
on a real Banach space \(X\), where \(C\) is a closed convex subset of \(X\) and \(A(t)\) is a family of dissipative subsets of \(X \times X\), which satisfies the following condition:

\textbf{Condition P.} For every \(t \in [0, T]\), \(A(t)\) is a closed and dissipative subset of \(X \times X\). The domain \(D\) of \(A(t)\) is independent of \(t\) and for every \(\lambda > 0\) and \(t \in [0, T]\), \(R(1 - \lambda A(t))\) contains \(C\). In addition,

\textit{a)} There exists a constant \(c > 0\) such that for all \(x \in C\) and \(s, t \leq T\) and \(s, t \in [0, T]\) and \(\lambda > 0\),

\[(3.4) \quad \| (1 - \lambda A(t))^{-1} x - (1 - \lambda A(s))^{-1} x \| \leq c \lambda | t - s | (\| x \| + \| A(t) x \|).
\]

\textit{b)} \((1 - \lambda A(0))^{-1} x = (1 - \lambda A(T))^{-1} x\) for \(x \in C, \lambda > 0\).

We introduce the notation
\[(3.3) \quad W_{\lambda}^{1, p}(0, T; X) = \{ u \in W^{1, p}(0, T; X) \text{ and } u(0) = u(T), 1 \leq p \leq \infty \}\]

\textbf{Definition 3.1 (see [3]).} Let \(1 \leq p < \infty\). The function \(u \in L^p(0, T; X)\) is said to be generalized solution of problem (3.1), (3.2) if exist sequences \([u_n] \subset W_{\lambda}^{1, p}(0, T; X)\) and \([y_n] \subset L^p(0, T; X)\) such that the following conditions hold:

\textit{a)} \(u_n(t) \in D(A(t)) \cap C\) and \(y_n(t) \in A(t) u_n(t)\) a. e. on \((0, T)\);

\textit{b)} \(u_n \rightarrow u\) in \(L^p(0, T; X)\) as \(n \rightarrow \infty\):

\textit{c)} \(\lambda u_n \frac{du_n}{dt} - y_n \rightarrow \lambda f\) in \(L^p(0, T; X)\) as \(n \rightarrow \infty\).

\textbf{Theorem 4.} Let \(X\) be a real Banach space with uniformly convex adjoint space and let \(C\) be a closed convex subset of \(X\). Let \(A(t)\) be a family of dissipative subsets of \(X \times X\) satisfying Condition P. Then for every \(f \in L^p(0, T; C), 1 < p < \infty,\) and \(\lambda > 0\) the problem (3.1), (3.2) has a unique generalized solution \(u\) in \(L^p(0, T; X)\). Moreover \(u\) is continuous on \([0, T]\) and \(u(0) = u(T)\).

If \(f \in W_{\lambda}^{1, p}(0, T; X)\) and \(\lambda\) is sufficiently large then \(u \in W_{\lambda}^{1, p}(0, T; X)\) and it is strong solution of equation (3.1).
PROOF. Let $p \in (1, \infty)$ be arbitrary. We introduce the following subset of $L^p(0, T; X) \times L^p(0, T; X)$

\[
A = \{[u, v] \mid u, v \in L^p(0, T; X) \text{ and } v(t) \in A(t)u(t) \text{ a.e. on } (0, T)\}
\]

(3.5) Clearly $A$ is dissipative and closed. Moreover, Condition $P$ implies that $L^p(0, T; C) \ni R(1 - \lambda A)$ for all $\lambda > 0$ and

\[
(1 - \lambda A)^{-1} L^p(0, T; C) \subset L^p(0, T; C) \text{ for } \lambda > 0.
\]

(3.6) In order to verify (3.6) it suffices to show that the function $t \mapsto (1 - \lambda A(t))^{-1}f(t)$ is strongly measurable for every $\lambda > 0$ and $f \in L^p(0, T; C)$.

For this latter purpose we approximate $f(t)$ by $f_\varepsilon(t) = \int f(x) \chi_\varepsilon(t - s) \, ds$ where $\chi(t)$ is a real valued function of class $C^1$ with $\int \chi(t) \, dt = 1$, supp $\chi \subset (0, 1)$ and $\chi_\varepsilon(t) = \varepsilon^{-1} \chi(t/\varepsilon)$. If $f$ is suitable defined outside the interval $(0, T)$ then $f_\varepsilon(t) \in C$ on $(0, T)$. Then $u_\varepsilon(t) = (1 - \lambda A(t))^{-1}f_\varepsilon(t)$ are well defined, continuous functions on $[0, T]$, and $u_\varepsilon(t) \to u(t)$ a.e. on $(0, T)$ as $\varepsilon \to 0$. This proves (3.6).

Let $D(B) = W^{1,p}_n(0, T; X)$ and let $Bu = -\frac{du}{dt}$ for $u \in D(B)$. It is known (see [7]), that $B$ generates on $L^p(0, T; X)$ a strongly continuous semigroup of linear contractions. From Condition $P$ it follows easily that hypotheses of Theorem of § 2 are satisfied for every $\lambda > 0$ and $\lambda > 0$.

We have

\[
\lambda u - A + Bu \geq \lambda f
\]

(3.7) has a unique solution $u \in L^p(0, T; C)$. Clearly $u$ is a generalized for problem (3.1), (3.2) in the sense of Definition 3.1.

Now we shall prove that $u \in C(0, T; X)$ and that $u(0) = u(T)$. Let $[u_n] \subset W^{1,p}_n(0, T; X)$ and $[y_n] \subset L^p(0, T; X)$ be chosen as in Definition (3.1).

We have

\[
\frac{d}{dt} \| u_n(t) - u_m(t) \| \leq -\lambda \| u_n(t) - u_m(t) \| + \| f_n(t) - f_m(t) \|
\]

a.e. on $(0, T)$ since $A(t)$ are dissipative. Here $f_n \to f$ in $L^p(0, T; X)$ as $n \to \infty$. 

V. BARRU: Dissipative Sets
Consequently

\[(3.8) \quad \| u_n(t) - u_m(t) \| \leq \exp(-\lambda t) \| u_n(0) - u_m(0) \| + \lambda \int_0^t \exp(-\lambda (t-s)) \| f_n(s) - f_m(s) \| \, ds, \]

Hence

\[(3.9) \quad \| u_n(0) - u_m(0) \| \leq \lambda (1 - \exp(-\lambda T))^{-1} \int_0^T \exp(-\lambda (T-s)) \| f_n(s) - f_m(s) \| \, ds \]

since \( u_n(0) = u_n(T) \) for all \( n \). This inequality together (3.8) imply that \( u_n(t) \) converges uniformly on \([0, T]\) to \( u(t) \). Hence \( u(t) \) is continuous on \([0, T]\), \( u(t) \in C \) for every \( t \in [0, T] \) and \( u(0) = u(T) \).

Second part of Theorem 4 is a direct consequence of Theorem 3.

**Remark 3.1.** Theorem 4 may be proved under more general assumptions, by a slight modification of the argument for Theorem 1.

Nevertheless we have preferred to prove it in this form for illustrating one of possible applications of the perturbation results established before.

\[ \text{§ 4. Second order abstract differential equations.} \]

Let \( V \) and \( H \) be a pairs of Hilbert spaces such that \( V \subset H \subset V^* \) with each inclusion mapping continuous and dense. Let \( L \) be a continuous self-adjoint linear operator from \( V \) into its adjoint space \( V^* \) such that \( (Lv, v) \geq \gamma \| v \|^2 \) for \( v \in V \). Here \( \gamma \) is a positive constant and \( \| \cdot \| \) denotes the norm in \( V \).

We are now going to consider evolution equation of the form

\[ (4.1) \quad \frac{d^2 u}{dt^2} + L(u(t)) \in A(t) \left( \frac{du}{dt} \right) + f(t), \text{ a.e. on } (0, T) \]

with the initial conditions

\[ (4.2) \quad u(0) = u_0, \frac{d}{dt} u(0) = u_1 \]

on \( H \), where \( A(t) \) is a family of \( m \)-dissipative subsets of \( H \times H \) satisfying the following conditions:
1. The domain $D(A(t)) = D$ of $A(t)$ is independent of $t$ and

$$
\| (1 - \lambda A(t))^{-1} x - (1 - \lambda A(s))^{-1} x \| \leq \lambda | t - s | \varphi(\| x \| + \| A(A(t)x) \|)
$$

for every $x \in H$, $\lambda > 0$ and $t, s \in [0, T]$. 

II. $(1 - \lambda A(t))^{-1} V \subset V$ for every $\lambda > 0$ and

$$
(L(1 - \lambda A(t))^{-1} x, (1 - \lambda A(t))^{-1} x)^{1/2} \leq (Lx, x)^{1/2} + \lambda \psi(\| x \| + \| A(A(t)x) \|)
$$

for every $x \in V$, $\lambda > 0$ and $t \in [0, T]$.

Here $\varphi$ and $\psi$ are non-decreasing functions from $[0, \infty)$ into itself.

Let us denote by $L_{H}$ the restriction of $L$ to $H$ i.e. $D(L_{H}) = \{ u \in V, Lu \in H \}$, $L_{H} u = Lu$ for $u \in D(L_{H})$. It is known that $L_{H}$ is m-dissipative in $H \times H$.

Let $Y$ denote the space $D(L_{H})$ normed by

$$
\| u \|_{Y} = \| Lu \| + \| u \|; u \in D(L_{H}).
$$

**Theorem 4.** Suppose that Conditions I, II are satisfied. Then $f \in W^{1, 1}(0, T; H) \cap L^{\infty}(0, T; V)$. Then for every $u_{0} \in Y$ and $u_{1} \in D \cap V$ the problem (4.1), (4.2) has a unique solution $u \in C(0, T; H) \cap L^{\infty}(0, T; Y)$ with

$$
\frac{du}{dt} \in C(0, T; H) \cap L^{\infty}(0, T; V) \text{ and } \frac{d^2 u}{dt^2} \in L^{\infty}(0, T; H).
$$

**Proof.** Let $\mathcal{V}$ denote the direct sum of $V$ and $H$

$$
\mathcal{V} = V \oplus H
$$

with the scalar product defined by

$$
\langle U, V \rangle = (Lu_{1}, v_{1}) + (u_{2}, v_{2})
$$

where $W = \{ u_{1}, u_{2} \}$ are generic elements of $\mathcal{V}$.

Thus the problem (4.1), (4.2) is equivalent to

$$
\frac{d}{dt} U(t) = \mathcal{A} U(t) + \mathcal{A}(t) V(t) + F(t), \ a.e.
$$

and

$$
U(0) = U_{0}.
$$
where \( U(t) = \left\{ u(t), \frac{du(t)}{dt} \right\}, F(t) = [0, f(t)], W_0 = [u_0, u_1] \) and \( \mathcal{B}, \mathcal{A}(t) \) are dissipative subsets of \( \mathcal{H} \times \mathcal{H} \) defined by

\[
\mathcal{B} = \begin{pmatrix} 0 & 1 \\ -L & 0 \end{pmatrix}, \quad D(\mathcal{B}) = Y \oplus V
\]

respectively

\[
\mathcal{A}(t) = \begin{pmatrix} 0 & 0 \\ 0 & A(t) \end{pmatrix}, \quad D(\mathcal{A}(t)) = V \oplus D.
\]

We shall verify the hypotheses of Theorem 1 where \( X = C = \mathcal{H}, B = \mathcal{B} \) and \( A(t) = \mathcal{A}(t) \).

We have

\[
(1 - \lambda \mathcal{A}(t))^{-1} F = [f_1, (1 - \lambda A(t))^{-1} f_2], \quad F = [f_1, f_2]
\]

Now Assumptions i), ii), iii) of Theorem are simple consequences of Conditions I. and II. Let us verify iv). Indeed if \( F \in D(\mathcal{B}) = Y \oplus V, F = [f_1, f_2] \) then

\[
| \mathcal{B}(1 - \lambda \mathcal{A}(t))^{-1} F |_{\mathcal{H}}^2 = (L(1 - \lambda(t))^{-1} f_2, (1 - \lambda A(t))^{-1} f_2) + \| Lf_1 \|^2
\]

Using (4.4) we obtain

\[
| \mathcal{B}(1 - \lambda \mathcal{A}(t))^{-1} F |_{\mathcal{H}} \leq | \mathcal{B} F |_{\mathcal{H}} + \lambda \psi(| F |_{\mathcal{H}}) + | \mathcal{A}(t) F |_{\mathcal{H}}
\]

which proves iv).

Thus according to Theorem 1, the initial value problem (4.6), (4.7) has a unique solution \( U \in W^{1, \infty}(0, T; \mathcal{H} \cap L^\infty(0, T; Y \oplus V)) \).

This concludes the proof.

**Example 4.1.** Let \( \Omega \) be an open bounded subset in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \) and let \( L \) be a differential operator of second order

\[
Eu = - \sum_{i, j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij}(x) \frac{\partial u}{\partial x_j} \right),
\]

where \( a_{ij} \) are real functions of class \( C^1 \) on \( \Omega \). In addition suppose that

\[
a_{ij}(x) = a_{ij}(x) \text{ for } i, j = 1, 2, \ldots, n
\]
and

\[
\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \geq \gamma |\xi|^2, \quad x \in \Omega, \xi \in \mathbb{R}^n
\]

where \( \gamma \) is a positive constant independent of \( x \).

Let \( L \) denote the self-adjoint operator from \( H_0^1(\Omega) \) into \( (H_0^1(\Omega))^* = H^{-1}(\Omega) \) which is given by \( Lu = Eu \) for \( u \in D(L) \). The restriction of \( L \) to \( L^2(\Omega) \) has the domain \( H^2(\Omega) \cap H_0^1(\Omega) \) and generates a continuous semigroup of linear contractions on \( L^2(\Omega) \). Here \( H_0^1(\Omega) \) and \( H^2(\Omega) \) are usual Sobolev spaces.

Finally, let \( A(t) \) be the family of \( m \)-dissipative subsets of \( L^2(\Omega) \times L^2(\Omega) \) defined by

\[
A(t) = \{ [u, v] ; u, v \in L^2(\Omega) \text{ and } v(x) \in \Gamma(t)(v(x)) \text{ a.e. in } \Omega \}
\]

where \( \Gamma(t) \subset \mathbb{R} \times \mathbb{R} \) is a family of maximal monotone sets in \( \mathbb{R} \times \mathbb{R} \) such that \( D(\Gamma(t)) \) is independent of \( t \) and contains 0. Moreover assume that

\[
| (1 - \lambda \Gamma(t))^{-1} v - (1 - \lambda \Gamma(s))^{-1} v | \leq M| t - s | \left( |v| + |\Gamma_2(t)v| \right)
\]

for every \( v \in \mathbb{R}, t, s \in [0, T] \) and \( \lambda > 0 \). Here \( M \) is a nonnegative constant independent of \( \lambda, t \) and \( s \).

Let us observe that hypotheses of Theorem 4 are satisfied with \( H = L^2(\Omega), V = H_0^1(\Omega), H \) and \( A(t) \) defined as above. Indeed Condition I. follows from the corresponding properties of \( \Gamma(t) \) and II. is a consequence of the fact that \( \frac{\partial}{\partial u} (1 - \lambda \Gamma(t))^{-1} u \leq 1 \) for every \( \lambda > 0 \), and \( u \in \mathbb{R} \).

Thus Theorem 4 yields the following Corollary:

**Corollary 4.1.** Let \( f, u_0, u_1 \) be given, satisfying

\[
f \in W^{1,1}(0, T ; L^2(\Omega)) \cap L^\infty(0, T ; H_0^1(\Omega))
\]

and

\[
u_0 \in H^2(\Omega) \cap H_0^1(\Omega), u_1 \in H_0^1(\Omega) \cap D(A(t)).
\]

Then the problem

\[
\frac{\partial^2 u}{\partial t^2} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial u}{\partial x_i} \right) \in A(t) \left( \frac{\partial u}{\partial t} \right) + f(t), \quad \text{in } \Omega \times (0, \infty)
\]

\[
|u(x, 0) = u_0(x), \frac{\partial u}{\partial t}(x, 0) = u_1(x), \text{ in } \Omega|
\]
has a unique solution $u \in C(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^2(\Omega))$ such that

\begin{equation}
\frac{\partial u}{\partial t} \in C(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^1(\Omega))
\end{equation}

and

\begin{equation}
\frac{\partial^2 u}{\partial t^2} \in L^\infty(0, T; L^2(\Omega)).
\end{equation}

Now we consider the differential equation of the form

\begin{equation}
\lambda u(t) - \frac{d^2 u}{dt^2} \in A(u(t)) + f(t), \quad 0 < t < T
\end{equation}

with Dirichlet conditions

\begin{equation}
 u(0) = u(T) = x
\end{equation}

on a Hilbert space $H$, where $A$ is dissipative subset of $H \times H$.

Let $1 \leq p \leq \infty$. Then $W^{2,p}(0, T; H)$ denote the space of vectorial distributions $u \in \mathcal{D}'(0, T; H)$ such that $\frac{d^k u}{dt^k} \in L^p(0, T; H)$ for $0 \leq k \leq 2$.

We recall that if $u \in W^{2,p}(0, T; H)$ then $\frac{du}{dt}$ coincides a.e. on $(0, T)$ with an absolutely continuous function.

**Theorem 5.** Let $A$ be a closed and dissipative subset of $H \times H$ and let $Q$ be a closed convex cone of $H$. Suppose that $\mathcal{R}(1 - \lambda A)$ contains $Q$ for every $\lambda > 0$ and

\begin{equation}
(1 - \lambda A)^{-1} Q \subset Q \text{ for } \lambda > 0.
\end{equation}

Let $x$ be in $D(A) \cap Q$ such that $Ax \cap Q \neq \emptyset$. Then for every $f \in L^p(0, T; Q)$, $1 < p < \infty$ and for each $\lambda > 0$ the problem (4.18), (4.19) has a unique solution $u \in W^{2,p}(0, T; H) \cap L^p(0, T; D(A))$ such that $u(t) \in Q$ for $0 \leq t \leq T$.

**Proof.** We may assume without loss of generality that $x = 0 \in D(A) \cap Q$ and $0 \in AO$. Let $B$ denote the operator on $L^p(0, T; H)$ with domain $D(B) = \{u; u \in H^{2,p}(0, T; H); u(0) = u(T) = 0\}$, which is given by $Bu = \frac{d^2 u}{dt^2}$ for $u \in D(B)$. 
It is known (see [7]) that $B$ is the infinitesimal generator of a continuous semigroup of linear contractions on $L^p(0, T; H)$ defined by

$$(4.21) \quad (S(t)u)(s) = \int_0^T K(\bar{C}, s, t) u(\bar{C}) \, d\bar{C}, \quad u \in C^0_0(0, T; H)$$

where $K(\bar{C}, s, t) = 2T/\pi \sum_{n=1}^{\infty} \exp(-n^2t) \sin \frac{n\pi \bar{C}}{T} \sin \frac{n\pi s}{T}$. Since $K(\bar{C}, s, t) \leq 0$ for $\bar{C} \in (0, \infty)$ and $t, s \in (0, T)$, from (4.21) it follows that $S(t)Q \subset Q$ for every $t \geq 0$. This implies that

$$(4.22) \quad (1 - \lambda B)^{-1} Q \subset Q \text{ for every } \lambda > 0.$$ 

We introduce the following operator

$$(4.23) \quad \widetilde{A} = \{[u, v]; u, v \in L^p(0, T; H) \text{ and } v(t) \in A(u(t)) \text{ a.e. on } (0, T)\}$$

Clearly $\widetilde{A}$ is dissipative and closed in $L^p(0, T; H) \times L^p(0, T; H)$. Moreover, assumption (4.20) implies immediately that

$$(4.24) \quad (1 - \lambda \widetilde{A})^{-1} L^p(0, T; Q) \subset L^p(0, T; Q) \text{ for all } \lambda > 0.$$ 

We now verify hypotheses of Corollary 2.2 where $X = L^p(0, T; H)$, $C = L^p(0, T; Q)$, $A = \widetilde{A}$ and $B$ is defined above. Obviously $j)$ and $jj)$ are implied by (4.22) and (4.24). It remains to prove (2.15).

Let $u$ be arbitrary in (2.15). Recalling that

$$F(u)(t) = u(t) \| u(t) \|^{p-2} \|u\|_{L^p(0, T; H)}$$

is the duality mapping of $X = L^p(0, T; H)$ we obtain

$$(4.25) \quad \langle Bu, F(A_n u) \rangle_X = -(p - 1) \| A_n u \|_{L^p(0, T; H)}^{p-2} \langle \frac{d}{dt} A_n u(t), \frac{d}{dt} A_n u(t) \rangle$$

Since

$$\left( \frac{du(t)}{dt}, \frac{d}{dt} A_n u(t) \right) = \lim_{h \to 0} \left( \frac{u(t+h) - u(t)}{h}, \frac{A_n u(t+h) - A_n u(t)}{h} \right) \text{ a.e. on } (0, T)$$

it follows from (4.25) that

$$(4.26) \quad \langle Bu, F(A_n u) \rangle_X \geq 0$$
since $A_n$ are dissipative in $H \times H$ for every $n$. By Corollary 2.2 we conclude that there exists a unique solution $u \in D(B) \cap D(A) \cap L^p(0, T; Q)$ of the equation

$$\lambda u - Bu - Av \geq f, \quad \lambda > 0, \ f \in L^p(0, T; Q).$$

This completes the proof of Theorem 5.

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