ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

VIOREL BARBU

Dissipative sets and nonlinear perturbated equations in Banach spaces

Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série, tome 26, nº 2 (1972), p. 365-390

<http://www.numdam.org/item?id=ASNSP_1972_3_26_2_365_0>

DISSIPATIVE SETS AND NONLINEAR PERTURBATED EQUATIONS IN BANACH SPACES

by Viorel Barbu

Abstract - Some existence results for abstract functional equations in Banach spaces are proved.

Introduction.

Let X be a real Banach space X^* its dual space, (u, v) the pairing between v in X^* and u in X. The duality mapping of X in the subset F of $X \times X$ defined by

$$F = \{[x, x] ; x \in X, x^* \in X^* \text{ and } (x, x^*) = \| x \|^2 = \| x^* \|^2\} \tag{0.1}$$

where $\| \; \|$ denotes the norm in X (respectively X^*).

Let A be a subset of $X \times X$. We define

$$Ax = y \in X ; [x, y] \in A\}, D(A) = \{x \in X ; Ax \neq \emptyset\}, R(A) \bigcup_{x \in D(A)} Ax,$$

and

$$A^{-1} = \{[y, x] ; [x, y] \in A\}, \alpha A = \{[x, \alpha y] ; [x, y] \in A\}$$

where α is real. If B is a subset of $X \times X$ then,

$$A + B = \{[x, y + z] ; [x, y] \in A \text{ and } [x, z] \in B\}.$$

A subset A of $X \times X$ is called dissipative if for every $[x_i, y_i] \in A$, $i = 1, 2$ there exists $f \in F(x_1 - x_2)$ such that

$$(y_1 - y_2, f) \leq 0$$

or equivalently (see T. Kato [10], Lemma 1.1),

Pervenuto alla Redazione il 31 Marzo 1971.

$$\| x_1 - x_2 \| \leq \| x_1 - \lambda y_1 - (x_2 - \lambda y_2) \| \tag{0.2}$$

for each $\lambda > 0$ and $[x_i, y_i] \in A$, $i = 1,2$.

If A is dissipative one can define for $\lambda > 0$ a single valued operator $A_\lambda = \lambda^{-1} ((1 - \lambda A)^{-1} - 1)$ with $D(A_\lambda) = R(1 - \lambda A)$. We notice some properties of A_λ which will be used frequently in this paper (for the proof see T. Kato [11]).

LEMMA 0.1. Let A be dissipative, then

a) A_λ is dissipative and lipschitz with constant $2\lambda^{-1}$.

b) For $x \in R(1 - \lambda A) \cap D(A)$, $A_\lambda x \in A(1 - \lambda A)^{-1} x$ and $\| A_\lambda x \| \leq \leq | Ax |$.

We have denoted here, $| Ax | = \inf \{ \| y \| ; y \in Ax \}$.

A dissipative subset A of $X \times X$ is called m-dissipative if $R(1 - \lambda A) = X$ for every (or, equivalenty, for some) $\lambda > 0$.

For other basic properties of dissipative sets and nonlinear semigroups of contractions we reffer to Kömurs [12], Crandall and Pazy [6], T. Kato [11], F. Browder [2], Brezis and Pazy [4].

The purpose of this paper is to obtain existence results for perturbated nonlinear differential (respectively functional) equations on Banach spaces. Section 1 and 2 contain the main results. We start with an existence theorem for evolution equations, Theorem 1 which is the main tool used in proving principal perturbation results given in Section 2. Similar results were obtained proviously by G. Da Prato (see [7]) in linear case. For related results see also [1], [2], [6], [9], [11].

In Sections 3 and 4 we apply these results in the study of certain nonlinear evolution equations.

§ 1. A class of nonlinear evolution equations.

Throughout this section we assume that X is a real Banach space and that the dual X^* of X is uniformly convex. In particular this implies that the duality mapping F of X is uniformly continuous on every bounded sebset of X (see [10], Lemma 1.2).

Let C be a closed convex subset of X.

In lhe present section we consider equations of evolution of the form

$$\lambda u(t) + \frac{du(t)}{dt} \in A(t)\, u(t) + Bu(t) + \lambda f(t), \quad a.\,e. \text{ on } (0, T) \tag{1.1}$$

with the conditions

(1.2) $$u(0) = x,\ u(t) \in C \text{ for } 0 \leq t \leq T < \infty,$$

on the space X, where B is the infinitesimal generator of a strongly continuous semigroup of linear contractions on X and $A(t)$ is a family of subsets of $X \times X$ satisfying the following assumptions :

i) For every $t \in [0, T]$, $A(t)$ is a closed an dissipative subset of $X \times X$. The domain $D(A(t) = D$ of $A(t)$ is independent of t.

ii) $(1 - \lambda B)^{-1} C \subset C$ for every $\lambda > 0$.

iii) $R(1 - \lambda A(t))$ contains C and $(1 - \lambda A(t))^{-1} C \subset C$ for every $\lambda > 0$ and for any $t \in [0, T]$. Moreover,

(1.3) $$\|(1 - \lambda A(t))^{-1} x - (1 - \lambda A(s))^{-1} x\| \leq \lambda |t - s| \varphi(\|x\| + \|A_\lambda(t) x\|)$$

for each $x \in C$, $t, s \in [0, T]$. Here $\varphi : [0, \infty) \to [0, \infty)$ is an increasing continuous function such that $\int^{\infty} \frac{dt}{\varphi(t)} = \infty$.

(iv) $$(1 - \lambda A(t))^{-1} (D(B) \cap C) \subset D(B) \cap C \text{ for } \lambda > 0,\ t \in [0, T]$$

and

(1.4) $$\|B(1 - \lambda A(t))^{-1} x\| \leq \|Bx\| + \lambda \psi(\|x\| + \|A_\lambda x\|)$$

for every $x \in D(B) \cap C$, $\lambda > 0$ and $0 \leq T$. Here ψ is an increasing continuous function from $[0, \infty)$ into itself such that $\int^{\infty} \frac{dt}{\psi(t)} = \infty$.

Now we shall recall some definitions.

If X is a real Banach space with norm $\| \|_X$ then $L^p(0, T; X)$, $1 \leq p \leq \infty$, denotes the space of (classes of) measurable functions $u : [0, T) \to X$ such that

$$\|u\|_p^p = \int_0^T \|x(s)\|_X^p \, ds < \infty, \quad 1 \leq p < \infty$$

and the usual modification in case $p = \infty$.

If C is a closed subset of X we set

$$L^p(0, T; C) = \{u ; u \in L^p(0, T; X) \text{ and } u(t) \in C \text{ a. e. on } (0, T)\}.$$

We denote also by $W^{1,p}(0, T; X)$ the space of all absolutely continuous functions $u : [0, T] \to X$ such that $\frac{du(t)}{dt} \in L^p(0, T; X)$.

Finally, we set

$$W_0^{1,p}(0, T; X) = \{u : u \in W^{1,p}(0, T; X) \text{ and } u(0) = 0\}.$$

THEOREM 1. Let C be a closed convex subset of X and let $A(t)$ and B be closed dissipative subsets of $X \times X$ satisfying Assumptions i) $\sim$ iv). Let $f \in W^{1,1}(0, T; X) \cap L^\infty(0, T; D(B))$ be such that $f(t) \in C$ for $0 \leq t \leq T$.

Then for every $x \in D \cap D(B) \cap C$ and for $\lambda \geq 0$, the initial value problem

$$(1.5) \qquad \begin{cases} \lambda u(t) + \dfrac{du(t)}{dt} \in Bu(t) + A(t)u(t) + \lambda f(t), \ 0 < t < T, \\ u(0) = x \end{cases}$$

has a unique solution $u \in W^{1,\infty}(0, T; X) \cap L^\infty(0, T; D(B))$ such that $u(t) \in C$ for all $t \in [0, T]$.

We preface the proof of Theorem 1 with the proof of some auxiliary lemmas.

LEMMA 1.1. Let Y be a real Banach space with uniformly convex adjoint space Y^*. Let K be a closed convex subset of Y and let A and L be two closed dissipative sets of $Y \times Y$.

Suppose

a) A is continuous and bounded on every bounded subset of $K = D(A)$. $R(1 - \lambda A)$ contains K for every $\lambda > 0$.

b) $K \subset \bigcap_{\lambda > 0} R(1 - \lambda L)$ and $(1 - \lambda L)^{-1} K \subset K$ for every $\lambda > 0$.

Then for every $\lambda > 0$ and for any $y \in K$, there exists a unique solution $u \in D(L) \cap K$ of the equation

$$(1.6) \qquad \lambda u - Lu - Au \ni \lambda y.$$

The proof is similar to that of Theorem 4.3 in [6] (see also the proof of Theorem 3 in § 2).

LEMMA 1.2. Let A and B satisfy Assumptions i), ii) and iii). Let $f \in W^{1,1}(0, T; X)$ be such that $f(t) \in C$ for all $t \in [0, T]$.

Then for any $\lambda > 0$ and for any $x \in D(B) \cap C$ there exists a unique $u \in W^{1,\infty}(0, T; X) \in L^\infty(0, T; D(B))$ such that $u(0) = x$, $u(t) \in C$ for all $t \in [0, T]$ and

$$(1.7) \qquad \lambda u(t) + \frac{du(t)}{dt} = Bu(t) + A_n(t)u(t) + \lambda f(t); \text{ a. e. on } (0, T).$$

PROOF. We may assume without loss of generality that $x = 0$. This can be achieved by shifting C. We fix $p \in (1, \infty)$ and put $K = L^p(0, T; C)$.

Let $\widetilde{A}$ denote the dissipative operator on $Y = L^p(0, T; X)$ with domain K which is given by $(\widetilde{A}u)(t) = A_n(t)\, u(t)$ a. e. on $(0, T)$ for $u \in K$. Clearly $\widetilde{A}$ is well defined, continuous and bounded on every bounded subset of $K \subset Y$ (see Lemma 0.1).

Let L be the linear operator defined in Y by

$$D(L) = W_0^{1,p}(0, T; X) \cap L^\infty(0, T; D(B))$$

and

$$Lu = -\frac{du}{dt} + Bu \text{ for } u \in D(L). \tag{1.8}$$

Here $D(B)$ is considered as Banach space with norm defined by $|x| = \| Bx \| + \| x \|$.

Since $(1 - \lambda B)^{-1}\, C \subset C$ for every $\lambda > 0$ it is easy to see that

$$K \subset \bigcap_{\lambda > 0} R(1 - \lambda \widetilde{L})$$

and

$$(1 - \lambda \widetilde{L})^{-1}\, K \subset K \text{ for } \lambda > 0,$$

where $\widetilde{L}$ is the closure of L in $Y \times Y$.

We apply Lemma 1.1 to conclude that for every $\lambda > 0$ that there exists a unique solution $u \in K$ of the equation

$$\lambda u - \widetilde{L}u - \widetilde{A}u \ni \lambda f.$$

By the definition of L there exists sequences $\{u_k\} \subset K \cap D(L)$ and $\{f_k\} \subset K$ such that $u_k \to u$ and

$$\lambda u_k(t) + \frac{du_k(t)}{dt} - Bu_k(t) - A_n(t)\, u_k(t) = \lambda f_k(t) \to \lambda f(t) \tag{1.9}$$

in $L^p(0, T; Y)$ as $k \to \infty$. Let $k, j > 0$. Since B and A_n are dissipative we obtain from (1.9) that

$$\left(\frac{d}{dt}(u_k(t) - u_j(t)), F(u_k(t) - u_j(t))\right) \leq -\lambda \| u_k(t) - u_j(t) \|^2 +$$

$$+ \lambda \| f_k(t) - f_j(t) \| \, \| u_k(t) - u_j(t) \|$$

for almost all $t \in (0, T)$. By using the equality (see [10], Lemma 1.3)

$$\left(\frac{d}{dt}(u_k(t) - u_j(t)), F(u_k(t) - u_j(t)) = 2^{-1}\frac{d}{dt}\| u_k(t) - u_j(t) \|^2 \text{ a. e.}\right.$$

we obtain

$$(1.10) \qquad \| u_k(t) - u_j(t) - u_j(t) \| \leq \exp(-\lambda t) \| u_k(0) - u_j(0) \| + \\ + \int_0^t \exp(-\lambda(t-s)) \| f_k(s) - f_s(s) \| \, ds.$$

Since $u_k(0) = u_j(0) = 0$, we conclude that $u_k(t)$ converges uniformly to $u(t)$ on $[0, T]$. Let $t, t+h \in (0, T)$ be such that $\frac{d}{dt}(u_k(t+h) - u_k(t))$ exists. Repeating the above argument we obtain

$$(1.11) \qquad \| u_k(t+h) - u_k(t) \| \leq \exp(-\lambda t) \| u_k(h) - u_k(0) \| + \\ + \lambda \int_0^t \exp(-\lambda(t-s)) (\| f_k(s+h) - f_k(s) \| + \\ + \lambda^{-1} \| A_n(s+h) - A_n(s)) u_k(s) \|) \, ds$$

and

$$(1.12) \qquad \| u_k(h) - u_k(0) \| \leq \int_0^h (\| A_n(s) 0 \| + \lambda \| f_k(s) \|) \, ds,$$

Passing to the limit $k \to \infty$ in (1.11) and (1.12) we obtain

$$(1.13) \qquad \| u(t+h) - u(t) \| \leq \exp(-\lambda t) \int_0^h (\| A_n(s) 0 \| + \lambda \| f(s) \|) \, ds + \\ + \lambda \int_0^t \exp(-\lambda(t-s)) (\| f(s+h) - f(s) \| + \lambda^{-1} \| (A_n(s+h) - \\ - A_n(s)) u(s) \|) \, ds.$$

On the other hand by Assumption iii), $\| A_n(s+h) - A_n(s)) u(s) \| \leq \leq h\varphi (\| u(s) \| + \| A_n(s) u(s) \|)$. Since $\frac{df(t)}{dt} \in L^1(0, T; X)$ it follows from (1.13) that $u \in W^{1,\infty}(0, T; X)$ (see Kömura [12], appendix).

Denote by $g(t)$ the function

$$g(t) = \lambda f(t) + A_n(t)\,u(t) - \lambda u(t).$$

Since $u_k(t)$ converges uniformly to $u(t)$, by (1.9) we have

$$u(t) = \int_0^t S(t-s)\,g(s)\,ds, \qquad 0 \le t \le T, \tag{1.14}$$

where $S(t)$ denotes the semigroup generated by B in X.

It is clear that $g \in W^{1,\infty}(0, T; X)$ so that (1.14) implies that $u \in L^\infty(0, T; D(B))$ and

$$\lambda u(t) + \frac{du(t)}{dt} = Bu(t) + A_n(t)\,u(t) + \lambda f(t) \text{ a. e. on } (0, T). \tag{1.15}$$

This proved Lemma 1.2. for $\lambda > 0$

Let $u_\lambda \in W_0^{1,\infty}(0, T; X) \cap L^\infty(0, T; D(B))$ be the solution of equation (1.15). Repeating the above argument it follows easily that $u_\lambda(t)$ is uniformly convergent on $[0, T]$ as $\lambda \to 0$ and that $\dfrac{du_\lambda(t)}{dt}$ is bounded uniformly on $(0, T)$. Thus passing to the limit $\lambda \to 0$ in (1.15) it follows Lemma 1.2 in the case $\lambda = 0$.

This completes the proof.

Proof of theorem 1. Let $f \in W^{1,2}(0, T; X) \cap L^\infty(0, T; D(B))$ be such that $f(t) \in C$ for all $t \in [0, T]$ and let x be an arbitrary element of $D \cap D(B) \cap C$. By Lemma 1.2 there exists a unique $u_n \in W^{1,\infty}(0, T: X) \cap L^\infty(0, T; D(B)$ such that $u_n(0) = x$, $u_n(t) \in C$ on $[0, T]$

$$\lambda u_n(t) + \frac{du_n(t)}{dt} = Bu_n(t) + A_n(t)\,u_n(t) + \lambda f(t) \text{ a. e. on } (0, T) \tag{1.16}$$

Obviously,

$$u_n(t) = \exp(-(n+\lambda)t)\,S(t)\,x + n\int_0^t \exp(-(n+\lambda)(t-s))\,S(t-s)\cdot$$

$$\cdot(1 - n^{-1}A(s))^{-1}\,u_n(s)\,ds + \lambda\int_0^t \exp(-(n+\lambda)(t-s))\,f(s)\,ds,$$

$$0 \le t \le T$$

where $S(t)$ is the semigroup generated by B. By Assumption iv) it follows that

$$(1.17) \qquad \| Bu_n(t) \| \leq \exp(-(n+\lambda)t) \| Bx \| + \| \lambda (n+\lambda)^{-1} \| Bf \|_\infty + \int_0^t \exp(-(n+\lambda)(t-s))(n \| Bu_n(s) \| + \psi(\| u_n(s) \| + \| A_n(s) u_n(s) \|)) \, ds.$$

Since $A_n(t)$ and B are dissipative, from (1.16) we obtain that

$$\frac{d}{dt} \| u_n(t) - x \| \leq -\lambda \| u_n(t) - x \| + \| A_n(t) x \| + \| Bx \| + \lambda \| f(t) \| \quad \text{a. e. on } (0, T);$$

therefore

$$(1.18) \qquad \| u_n(t) - x \| \leq \int_0^t \exp(-\lambda(t-s))(\| A_n(s) x \| + \| Bx \| + \lambda \| f(x) \|) \, ds, \quad 0 \leq t \leq T.$$

By using the same argument as in the proof of Lemma 1.2, we obtain

$$(1.19) \qquad \| u_n(t+h) - u_n(t) \| \leq \exp(-\lambda t) \| u_n(h) - x \| + \int_0^t \exp(-\lambda(t-s))(\lambda \| f(s+h) - f(s) \| + h\varphi(\| u_n(s) \| + \| A_n(s) u_n(s) \|)) \, ds$$

for all $t, t+h \in [0, T]$. On the other hand (1.18) implies that

$$\limsup_{t \to 0} t^{-1} \| u_n(t) - x \| \leq \| A_n(0) x \| + \lambda \| f(0) \| .$$

Using this estimate together with (1.19) we see that

$$(1.20) \qquad \left\| \frac{du_n(t)}{dt} \right\| \leq M \exp(-\lambda t) + \int_0^t \exp(-\lambda(t-s)) \left(\left\| \frac{df(s)}{ds} \right\| + \varphi(\| u_n(s) \| + \| A_n(s) u_n(s) \|) \right) ds$$

for almost all $t \in (0, T)$, where M is a positive constant independent of n.

Since $u_n(t)$ are uniformly bounded on $[0, T]$, from (1.17) and (1.20) we obtain

$$\left\|\frac{du_n(t)}{dt}\right\| + \| Bu_n(t) \| \leq \left(M + \left\| \frac{df}{dt} \right\|_1\right) \exp(-\lambda t) + \tag{1.21}$$

$$+ f_n(t) + y_n(t), \text{ a. e}$$

where

$$f_n(t) = \exp(-(n+\lambda)t) \| Bx \| + \lambda (n+\lambda)^{-1} \| Bf \|_\infty$$

while

$$y_n(t) = \int_0^t \exp(-\lambda(t-s))\, \varphi\left(k_0 + \| Bu_n(s) \| + \left\| \frac{du_n(s)}{ds} \right\|\right) ds +$$

$$+ \int_0^t \exp(-(n+\lambda)(t-s)) \left(n \| Bu_n(s) \| + \psi\left(k_0 + \| Bu_n(s) \| + \left\| \frac{du_n(s)}{ds} \right\|\right)\right) ds,$$

where k_0 is a constant independent of n.

By a simple computation it follows that

$$\frac{dy_n(t)}{dt} \leq -\lambda y_n(t) + \varphi(k_1 + y_n(t)) + \psi(k_1 + y_n(t)) + \tag{1.22}$$

$$+ n f_n(t), \quad 0 < t \leq T,$$

where k_1 is a suitable constant independent of n. Since $n f_n(t)$ is bounded we conclude from (1.22) dnd (1.21) that

$$\left\| \frac{du_n(t)}{dt} \right\| + \| Bu_n(t) \| \leq M_T < \infty \text{ for } 0 < t < T. \tag{1.23}$$

Thus by using the fact that the duality mapping F is uniformly continuous on every bounded subset of X it follows by a standard argument (see [10], Lemma 4.3) that $u_n(t)$ converges uniformly on $[0, T]$ as $n \to \infty$. Let $u(t) = \lim_{n \to \infty} u_n(t)$.

Clearly $u(t)$ is absolutely continuous on $[0, T]$. Since the space X is reflexive this implies that (see [12], Appendix) $\frac{du(t)}{dt}$ exists a. e. on $(0, T)$. Moreover the inequality (1.23) implies obviously that $u \in W^{1,\infty}(0, T; X) \cap L^\infty(0, T; D(B))$.

We shall prove that is the solution of initial value problem (1.6). For this latter purpose, choose $t_0 \in (0, T)$ such that $u(t)$ is differentiable at $t = t_0$. Let $[\tilde{x}, \tilde{y}]$ be an arbitrary element of $A(t_0)$ such that $\tilde{x} - \alpha\tilde{y} \in C$

for some positive α. This implies that $\widetilde{x}_n = \widetilde{x} - n^{-1}\widetilde{y}$ lies in C for some sufficiently large n. Since $\widetilde{y} = \widetilde{x} - A_n(t_0)\widetilde{x}_n$, we see from (1.19) that

$$2^{-1}\frac{d}{dt}\|u_n(t) - \widetilde{x}_n\|^2 \leq (Bu_n(t) + \widetilde{y} - \lambda u(t) + \lambda f(t), F(u_n(t) - \widetilde{x}_n)) +$$

$$+ \|(A_n(t) - A_n(t_0))\widetilde{x}_n\| \, \|u_n(t) - \widetilde{x}_n\|, \text{ a. e. on } (0, T).$$

Integrating this ineqnality over (t_0, t) and using Assumption iii) we obtain

$$(1.24) \qquad \|u_n(t) - \widetilde{x}_n\|^2 - \|u_n(t_0) - \widetilde{x}_n\|^2 \leq 2\int_{t_0}^{t}(Bu_n(s) + \widetilde{y} -$$

$$- \lambda u(s) + \lambda f(s), F(u_n(s) - \widetilde{x}_n))\,ds + M_0\,|t - t_0|^2\,\varphi(\|\widetilde{x}_n\| +$$

$$+ \|\widetilde{y} - \widetilde{x}\|)\,;\; 0 < t \leq T,$$

where M_0 is independent of n.

Now $Bu_n(s) \rightarrow Bu(s)$, $u_n(s) \rightarrow u(s)$ and $\widetilde{x}_n \rightarrow \widetilde{x}$ as $n \rightarrow \infty$. We pass to limit as $n \rightarrow \infty$ in (1.24) to obtain

$$\|u(t) - \widetilde{x}\|^2 - \|u(t_0) - \widetilde{x}\|^2 \leq 2\int_{t}^{t}(Bu(s) + \widetilde{y} - \lambda u(s) + \lambda f(s), F(u(s) -$$

$$- \widetilde{x}))\,ds + M_0\,|t - t_0|^2\,\varphi(\|\widetilde{x}\| + \|\widetilde{y} - \widetilde{x}\|), \quad 0 \leq t \leq T,$$

so that

$$(u(t) - u(t_0), F(u(t) - \widetilde{x})) \leq 2\int_{t_0}^{t}(Bu(s) + \widetilde{y} - \lambda u(s) + \lambda f(s),$$

$$F(u(s) - \widetilde{x}))\,ds + M_0\,|t - t_0|^2\,\varphi(\|\widetilde{x}\| + \|\widetilde{y} - \widetilde{x}\|), \quad 0 \leq t \leq T.$$

Since the function $t \rightarrow Bu(t)$ is weakly continuous, we obtain

$$(1.25) \qquad \left(\frac{du(t_0)}{dt} - Bu(t_0) - \widetilde{y} + \lambda u(t_0) - \lambda f(t_0), F(u(t_0) - \widetilde{x})\right) \leq 0.$$

Let $\{\varepsilon_n\}$ be a sequence of nonnegative numbers such that $\lim\limits_{n\to\infty} \varepsilon_n = 0$.

Define

$$A(\varepsilon_n) = \varepsilon_n^{-1}(S(\varepsilon_n)\, u(t_0 - \varepsilon_n) - u(t_0)) - Bu(t_0) + \frac{du(t_0)}{dt}.$$

We notice that Assumption ii) implies that $S(t)\, C \subset C$ for all $t \geq 0$. Thus for every n there exists $[x_n, y_n] \in A(t_0)$ such that

$$S(\varepsilon_n)\, u(t_0 - \varepsilon_n) = x_n - \varepsilon_n y_n - \lambda \varepsilon_n (f(t_0) - x_n).$$

Consequently,

$$A(\varepsilon_n) = \varepsilon_n^{-1}(x_n - u(t_0)) - y_n - \lambda f(t_0) + \lambda x_n - Bu(t_0) + \frac{du(t_0)}{dt}. \tag{1.26}$$

Now, we use (1.25) where $[\tilde{x}, \tilde{y}] = [x_n, y_n]$ to obtain that

$$(\lambda + \varepsilon_n^{-1})\, \| u(t_0) - x_n \|^2 \leq (A(\varepsilon_n), F(u(t_0) - x_n)).$$

It is clear that $\lim_{n \to \infty} A(\varepsilon_n) = 0$. So that letting $n \to \infty$, we see that

$$\lim_{n \to \infty} \varepsilon_n^{-1}(u(t_0) - x_n) = 0.$$

This last observation together (1.26) imply that $x_n \to u(t_0)$ and $y_n \to \frac{du(t_0)}{dt} - Bu(t_0) + \lambda u(t_0) - \lambda f(t_0)$ as $n \to \infty$. Since $A(t_0)$ is closed we conclude that

$$\lambda u(t_0) + \frac{du(t_0)}{dt} \in Bu(t_0) + A(t_0)\, u(t_0) + \lambda f(t_0).$$

The uniqueness of of solution u follows immediately from the dissipativeness property of B and $A(t)$.

This completes the proof.

§ 2. Some perturbation results.

As in preceding section X is a real Banach space with uniformly convex adjoint and C is a closed convex subset of X.

We consider the functional equation in X of the form

$$\lambda u - Au - Bu \ni \lambda f, \quad f \in X, \; u \in C, \tag{2.1}$$

where A and B are dissipative subsets of $X \times X$, which satisfy the following conditions :

j) A is closed dissipative subset of $X \times X$. $R(1 - \lambda A)$ contains C for $\lambda > 0$ and

$$(1 - \lambda A)^{-1} C \subset C \text{ for every } \lambda > 0. \tag{2.2}$$

jj) B is a densely defined, linear and m-dissipative operator in X. $(1 - \lambda B)^{-1} C \subset C$ for every $\lambda > 0$.

jjj) $(1 - \lambda A)^{-1} (D(B) \cap C) \subset D(B)$ for every $\lambda > 0$ and

$$\| B (1 - \lambda A)^{-1} x \| \leq \| Bx \| + \lambda \psi (\| x \| + \| A_\lambda x \|) \tag{2.3}$$

holds for every $x \in D(B) \cap C$ and for each $\lambda > 0$.

Here $\psi : [0, \infty) \to [0, \infty)$ is an increasing function such that $\int^{\infty} \frac{dt}{\psi(t)} = \infty$.

The main result of this section may be stated as follows :

THEOREM 2. Let A and B be dissipative subsets of $X \times X$ satisfying conditions j), jj) and jjj).

Then

$$C \subset (1 - \lambda \overline{A + B}) (D(\overline{A + B}) \cap C) \quad \text{for all } \lambda > 0 \tag{2.4}$$

and

$$(1 - \lambda \overline{A + B})^{-1} y = \lim_{n \to \infty} (1 - \lambda (A_n + B))^{-1} y \tag{2.5}$$

for every $y \in C$ and $\lambda > 0$. (Here $\overline{A + B}$ denotes the closure of $A + B$ in $X \times X$).

A stronger version of Theorem 2 is

THEOREM 5. Let A and B be dissidative and closed subsets of X X satisfying assumptions j), jj) with the inequality (2.3) replaced by

$$\| B (1 - \lambda A)^{-1} x \| \leq \| Bx \| + M\lambda (\| x \| + \| A_\lambda x \|), \text{ for } x \in D(B) \cap C, \tag{2.6}$$

where M is a nonnegative constant independent of λ.

Then

$(1 - \lambda (A + B)) (D(A) \cap D(B) \cap C) \supset C$ for some sufficienty large λ.

COROLLARY 2.1. Let A and B satisfy hypotheses of Theorem 2. Suppose in addition that X is uniformly convex and that the following condition holds jv) $|(A+B)x| \longrightarrow \infty$ as $\|x\| \longrightarrow \infty, x \in D(A) \cap D(B) \cap C$.

Then

$$0 \in \overline{A+B}(D(\overline{A+B}) \cap C).$$

PROOF OF THEOREM 2. Let y be an arbitrary element of $D(B) \cap C$ and let $\lambda > 0$. We fix $x \in D(A) \cap D(B) \cap C$ and denote by $u(t)$ the solution of problem (1.5) where $A(t) \equiv A$ and $f(t) \equiv y$. It is clear that $u(t)$ can be extended as solution of the eqnation (1.5) on $(0, \infty)$. From the proof of Theorem 1 (see (1.18) and (1.19)) we obtain

$$\|u(t) - x\| \leq \lambda^{-1}(1 - \exp(-\lambda t))(|Ax))(|Ax| + \|Bx\| + \lambda \|y\|), \quad 0 < t < \infty$$

therefore

$$\|u(t+h) - u(t)\| \leq \lambda^{-1} \exp(-\lambda t)(1 - \exp(-\lambda h))(|Ax| + \|Bx\| + \lambda \|y\|). \tag{2.7}$$

This estimate implies immediately that $u(t)$ converges as $t \longrightarrow \infty$ and

$$\limsup_{t \to \infty} \frac{du(t)}{dt} = 0. \tag{2.8}$$

Let $u = \lim\limits_{t \to \infty} u(t)$. Letting $t \longrightarrow \infty$ in (1.5) we see that

$$(\lambda - \overline{A+B})\, u \ni \lambda y.$$

Note that $(1 - \lambda\overline{A+B})^{-1}$ is well defined and nonexpansive on $D(B)$ C in consquence of the fact that $\overline{A+B}$ is dissipative. On the other hand condition j) implies that $D(B) \cap C$ is a dense subset of C. Hence $R(1 - \lambda\overline{A+B})$ contains C for every $\lambda > 0$ which proves (2.4).

By using a standard fixed point technique it follows easily that for any $n, (1 - \lambda(A_n + B))^{-1}$ is well defined and nonexpansive on C. It suffices to prove (2.5) for every $y \in D(B) \cap C$.

Let $u_u(t) \in C$ be the solution of equation

$$\lambda\, u_n(t) + \frac{du_n(t)}{dt} = A_n\, u_n(t) + Bu_n(t) + \lambda y, \quad 0 < t < \infty,$$

with initial condition $u_n(0) = x \in D(A) \cap D(B) \cap C$.

From the proof of Theorem 1 (see (1.18) and (1.19)) we deduce that $u_n = \lim\limits_{t \to \infty} u_n(t)$ exists uniformly with respect to t. Moreover since A_n are continuous and B is closed it follows that $u_n \in D(B) \cap 0$ and

$$\lambda u_n - A_n u_n - Bu_n = \lambda y, \quad n = 1,2, ...$$

We know by the proof of Theorem 1 that $u_n(t)$ converges uniformly on every bounded interval of $[0, \infty)$ to the solution $u(t)$ of problem (1.5). On the other hand according to first part of the proof we have

$$(1 - \lambda \overline{A + B})^{-1} y = \lim_{t \to \infty} u(t).$$

Thus by a simple computation it follows that $\lim_{n \to \infty} u_n = (1 - \lambda^{-1} \overline{A + B})^{-1} y$ which concludes the proof.

PROOF OF THEOREM 3. Consider the equation

$$\lambda u_n - Bu_n - A_n u_n = \lambda y, \quad n = 1,2, ... \tag{2.9}$$

which is equivalent to

$$u_n = \lambda(\lambda + n - B)^{-1} y + n(\lambda + n - B)^{-1}(1 - n^{-1} A)^{-1} u_n . \tag{2.10}$$

By using the contraction fixed point theorem it follows easily that for every $y \in C$ and any fixed $\lambda > 0$ this equation has a unique solution $u_n \in$ $\in D(B) \cap C$. Let x be fixed in $D(A) \cap D(B) \cap C$. Multiplying (2.9) by $F(u_n - x)$ yelds

$$\lambda \| u_n - x \| \leq \lambda \| y \| + \| Bx \| + | Ax | + \lambda \| x \| \tag{2.11}$$

since A_n and B are dissipative.

Suppose now that $y \in D(B) \cap C$. Then from (2.5) and (2.10) we obtain

$$\| Bu_n \| \leq \lambda (n + \lambda)^{-1} \| By \| + n(n + \lambda)^{-1} \| Bu_n \| + M(n + \lambda)^{-1}(\| u_n \| + \| A_n u_n \|).$$

Consequently

$$\| Bu_n \| \leq \lambda(\lambda - M)^{-1} \| By \| + M(\lambda + 1)(\lambda - M)^{-1} \| u_n \| + M\lambda(\lambda - M)^{-1} \| y \|$$

if $M < \lambda$. This estimate together (2.11) show that $\| Bu_n \|$ and $\| A_n u_n \|$ are bounded as $n \to \infty$ if λ is sufficiently large. We fix $\lambda > M$.

Thus following a standard method (see [1], [6]), we see that $\{u_n\}$ converges as $n \to \infty$. Let $u = \lim_{n \to \infty} u_n$. Letting $n \to \infty$ in (2.9) we obtain

$$(2.12) \qquad \lambda u - Bu - \widetilde{A} u \ni \lambda f,$$

where $\widetilde{A}$ is the smallest demiclosed extension of A.

Using the fact that duality mapping F is continuous we see easily that $\widetilde{A}$ is dissipative in $X \times X$.

Let $\{\varepsilon_n\}$ be a sequence of nonnegative numbers such that $\lim_{n \to \infty} \varepsilon_n = 0$. We set

$$A_n = \varepsilon_n^{-1} (S(\varepsilon_n) u - u) - Bu,$$

where S is the semigroup generated by B. Since $S(\varepsilon_n) C \subset C$, in view of assumption j), for every n there exists $[x_n, y_n] \in A$ such that

$$B(\varepsilon_n) u = x_n - \varepsilon_n y_n - \lambda \varepsilon_n (y + x_n).$$

Consequently

$$(2.13) \qquad A_n = \varepsilon_n^{-1} (x_n - u) - Bu - y_n - \lambda (y + x_n).$$

Multiplying (2.13) by $F(x_n - u)$ and using (2.12) we obtain

$$(\lambda + \varepsilon_n^{-1}) \| x_n - u \| \leq \| A_n \|$$

Since $A_n \longrightarrow 0$ it follows from (2.13) that $y_n \longrightarrow \lambda u - Bu - \lambda y$.
Hence

$$\lambda u - Au - Bu \ni \lambda y$$

Since A is closed. This completes the proof.

PROOF OF COROLLARY 2.1. In view of Theorem 2, for every $y \in C$ and for each $\lambda > 0$ the equation

$$(2\text{-}14) \qquad \lambda u_\lambda - \overline{A + B}\, u_\lambda \ni \lambda y$$

has a unique solution $u_\lambda \in D(\overline{A + B}) \cap C$. Let x be arbitrary but fixed in $D(\overline{A + B})$. We multiply (2.14) by $F(u_\lambda - x)$. We obtain

$$\lambda \| u_\lambda - x \| \leq | \overline{A + B x} | + \| y \| + \lambda \| x \|$$

since $\overline{A + B}$ is dissipative. Using this estimate together jv) and (2.14) we see that $\{u_\lambda\}$ is bounded as $\lambda \longrightarrow 0$. Without loss of generality we may assume that $u_\lambda \rightharpoonup u$ as $\lambda \to 0$. Let $h > 0$ we have

$$\| (1 - h\,\overline{A + B})^{-1} u_\lambda - u_\lambda \| \leq h \, | \overline{A + B}\, u_\lambda | , \text{ for } \lambda > 0.$$

From (2.14) it follows that $\lim\limits_{\lambda \to 0} (1 - h\overline{A + B})^{-1} u_\lambda = u$. Since X is uniformly convex and $(1 - h\overline{A + B})^{-1}$ is nonexpansive on C we conclude that $(1 - h\overline{A + B})\, u \ni u$ (see [2], Theorem 8.2).

Hence

$$0 \in \overline{A + B u}$$

which concludes the proof.

A slightly madified version of Theorem 2 is useful in some applications.

COROLLARY 2.2. Let A and B satisfy hypotheses of Theorem 2 with Assumption jjj) replaced by

$$D(A) \cap D(B) \neq \varnothing \text{ and } (Bu, F(A_n u) \geq 0 \tag{2.15}$$

for every $u \in D(B) \cap C$ and $n = 1,2, \ldots$

Then

$$(1 - \lambda(A + B))(D(A) \cap D(B) \cap C) \supset C \text{ for every } \lambda > 0. \tag{2.16}$$

PROOF. Let $y \in C$, $\lambda > 0$ and let $u_n \in D(B) \cap D$ be the solution of the equation

$$\lambda u_n - Bu_n - A_n u_n = \lambda y.$$

Condition (2.15) implies that $\| Bu_n \|$ and $\| A_n u_n \|$ are bounded as $n \to \infty$. From this the proof proceeds exactly as the proof of Theorem 3.

REMARK 2.1. By the proof we see easily that if A and B satisfy to assumption of Theorem 2 with (2.6) replaced by the following stronger assumption

$$\| B(1 - \lambda A)^{-1} x \| \leq \| Bx \|, \text{ for } x \in D(B) \cap C \text{ and } \lambda \supset 0, \tag{2.17}$$

then $(1 - \lambda(A + B))(D(A) \cap D(B) \cap C) \supset C$ for all $\lambda > 0$.

§ 3. Periodic problems.

We consider in this section evolution equations of the form

$$\lambda u(t) + \frac{du(t)}{dt} \in A(t)\, u(t) + \lambda f(t), \text{ a. e. on } (0, T) \tag{3.1}$$

with the conditions

$$(3.2) \qquad u(0) = u(T)\,;\; u(t) \in C \text{ for all } t \in [0, T]$$

on a real Banach space X, where C is a closed convex subset of X and $A(t)$ is a family of dissipative subsets of $X \times X$, which satisfies the following condition:

CONDITION P. For every $t \in [0, T]$, $A(t)$ is a closed and dissipative subset of $X \times X$. The domain D of $A(t)$ is independent of t and for every $\lambda > 0$ and $t \in [0, T]$, $R(1 - \lambda A(t))$ contains C. In addition,

a) There exists a constant $c > 0$ such that for all $x \in C$ and $s, t \in \in [0, T]$ and $s, t \in [0, T]$ and $\lambda > 0$,

$$(3.4) \quad \| (1 - \lambda A(t))^{-1} x - (1 - \lambda A(s))^{-1} x \| \leq c\lambda \, | t - s | \, (\| x \| + A_\lambda (t) x \|).$$

b) $(1 - \lambda A(0))^{-1} x = (1 - \lambda A(T))^{-1} x$ for $x \in C, \lambda > 0$.

We introduce the notation

$$(3.3) \qquad W_\pi^{1,p}(0, T; X) = \{ u \in W^{1,p}(0, T; X \text{ and } u(0) = u(T) \},\ 1 \leq p \leq \infty \}$$

DEFINITION 3.1 (see [3]). Let $1 \leq p \leq \infty$. The function $u \in L^p(0, T; X)$ is said to be generalized solution of problem (3.1), (3.2) if exist sequences $\{u_n\} \subset W_\pi^{1,p}(0, T; X)$ and $\{y_p\} \subset L^p(0, T; X)$ such that the following conditions hold:

a) $u_n(t) \in D(A(t)) \cap C$ and $y_n(t) \in A(t)\, u_n(t)$ a. e. on $(0, T)$;

b) $u_n \longrightarrow u$ in $L^p(0, T; X)$ as $n \longrightarrow \infty$:

c) $\lambda u_n + \dfrac{du_n}{dt} - y_u \longrightarrow \lambda f$ in $L^p(0, T; X)$ as $n \longrightarrow \infty$.

THEOREM 4. Let X be a real Banach space with uniformly convex adjoint space and let C be a closed convex subset of X. Let $A(t)$ be a family of dissipative subsets of $X \times X$ satisfying Condition P. Then for every $f \in L^p(0, T; C)$, $1 < p < \infty$, and $\lambda > 0$ the problem (3.1), (3.2) has a unique generalized solution u in $L^p(0, T; X)$. Moreover u is continuous on $[0, T]$ and $u(0) = u(T)$.

If $f \in W_\pi^{1,p}(0, T; X)$ and λ is sufficiently large then $u \in W_\pi^{1,p}(0, T; X)$ and it is strong solution of equation (3.1).

Proof. Let $p \in (1, \infty)$ be arbitrary. We introduce the following subset of $L^p(0, T; X) \times L^p(0, T; X)$

$$(3.5) \qquad \mathcal{A} = \{[u, v]\,;\ u, v \in L^p(0, T; X) \text{ and } v(t) \in A(t)\, u(t) \text{ a. e. on } (0, T)\}$$

Clearly $\mathcal{A}$ is dissipative and closed. Moreover, Condition P implies that $L^p(0, T; C) \supset R(1 - \lambda \mathcal{A})$ for all $\lambda > 0$ and

$$(3.6) \qquad (1 - \lambda \mathcal{A})^{-1} L^p(0, T; C) \subset L^p(0, T; C) \text{ for } \lambda > 0.$$

In order to verify (3.6) it suffices to show that the function $t \to (1 - \lambda A(t))^{-1} f(t)$ is strongly measurable for every $\lambda > 0$ and $f \in L^p(0, T; C)$. For this latter purpose we approximate $f(t)$ by $f_\varepsilon(t) = \int f(x)\, \chi_\varepsilon(t - s)\, ds$ where $\chi(t)$ is a real valued function of class C^1 with $\int \chi(t)\, dt = 1$, supp $\chi \subset (0, 1)$ and $\chi_\varepsilon(t) = \varepsilon^{-1} \chi(t/\varepsilon)$. If f is suitable defined outside the interval $(0, T)$ then $f_\varepsilon(t) \in C$ on $(0, T)$. Then $u_\varepsilon(t) = (1 - \lambda A(t))^{-1} f_\varepsilon(t)$ are well defined, continuous functions on $[0, T]$, and $u_\varepsilon(t) \to u(t)$ a. e. on $(0, T)$ as $\varepsilon \to 0$. This proves (3.6).

Let $D(\mathcal{B}) = W_\pi^{1,p}(0, T; X)$ and let $\mathcal{B}u = -\dfrac{du}{dt}$ for $u \in D(\mathcal{B})$. It is known (see [7]), that $\mathcal{B}$ generates on $L^p(0, T; X)$ a strongly continuous semigroup of linear contractions. From Condition P it follows easily that hypotheses of Theorem of § 2 are satisfied with $\mathcal{X} = L^p(0, T; X)$, $\mathcal{C} = L^p(0, T; C)$ and $\mathcal{A}, \mathcal{B}$ defined above. Applying Theorem 3 (or Theorem 2) we obtain that for every $f \in L^p(0, T; C)$ and $\lambda > 0$ the equation

$$(3.7) \qquad \lambda u - \overline{\mathcal{A} + \mathcal{B}}\, u \ni \lambda f$$

has a unique solution $u \in L^p(0, T; C)$. Clearly u is a generalized for problem (3.1), (3.2) in the sense of Definition 3.1.

Now we shall prove that $u \in C(0, T; X)$ ond that $u(0) = u(T)$. Let $\{u_n\} \subset W^{1,p}(0, T; X)$ and $\{y_n\} \subset L^p(0, T; X)$ be choosen as in Definition (3.1).

We have

$$\frac{d}{dt} \| u_n(t) - u_m(t) \| \leq -\lambda \| u_n(t) - u_m(t) \| + \lambda \| f_n(t) - f_m(t) \|$$

a. e. on $(0, T)$

since $A(t)$ are dissipative Here $f_n \to f$ in $L^p(0, T; X)$ as $n \to \infty$.

Consequently

$$(3.8) \qquad \| u_n(t) - u_m(t) \| \leq \exp(-\lambda t) \| u_n(0) - u_m(0) \| + \\ + \lambda \int_0^t \exp(-\lambda(t-s)) \| f_n(s) - f_m(s) \| \, ds,$$

Hence

$$(3.9) \qquad \| u_n(0) - u_m(0) \| \leq \lambda (1 - \exp(-\lambda T))^{-1} \int_0^T \exp(-\lambda (T-s)) \| f_n(s) - \\ - f_m(s) \| \, ds$$

since $u_n(0) = u_n(T)$ for all n. This Inequality together (3.8) imply that $u_n(t)$ converges uniformly on $[0, T]$ to $u(t)$. Hence $u(t)$ is continuous on $[0, T]$, $u(t) \in C$ for every $t \in [0, T]$ and $u(0) = u(T)$.

Second part of Theorem 4 is a direct cousequence of Theorem 3.

REMARK 3.1. Theorem 4 may be proved under more general assumptions, by a slight modification of the argument for Theorem 1.

Nevertheless we have prefered to prove it in this form for illustrating one of possible applications of the perturbation results established before.

§. 4. Second order abstract differential equations.

Let V and H be a pais of Hilbert spaces snch that $V \subset H \subset V^*$ with esch inclusion mapping continuous and dense. Let L be a continuous self-adjoint linear operator from V into its adjoint space V^* such that $(Lv, v) \geq \geq \gamma |v|^2$ for $v \in V$. Here γ is a positive constant and $| \; |$ denotes the norm in V.

We are now going to consider evolution equation of the form

$$(4.1) \qquad \frac{d^2 u}{dt^2} + L(u(t)) \in A(t) \left(\frac{du}{dt} \right) + f(t), \text{ a. e. on } (0, T)$$

with the initial conditions

$$(4.2) \qquad u(0) = u_0 \, , \frac{d}{dt} u(0) = u_1$$

on H, where $A(t)$ is a family of m-dissipative subsets of $H \times H$ satisfying the following conditions :

I. The domain $D(A(t)) = D$ of $A(t)$ is independent of t and

$$\| (1 - \lambda A(t))^{-1} x - (1 - \lambda A(s))^{-1} x \| \leq \lambda \, | t - s | \, \varphi (\| x \| + \| A_\lambda (t) x \|) \tag{4.3}$$

for every $x \in H, \lambda > 0$ and $t, s \in [0, T]$

II. $(1 - \lambda A(t))^{-1} V \subset V$ for every $\lambda > 0$ and

$$(L (1 - \lambda A(t))^{-1} x, (1 - \lambda A(t))^{-1} x)^{1/2} \leq (Lx, x)^{1/2} + \\ + \lambda \psi (\| x \| + \| A_\lambda (t) x \|) \tag{4.4}$$

for every $x \in V, \lambda > 0$ and $t \in [0, T]$.

Here φ and ψ are non-decreasing functions from $[0, \infty)$ into itself.

Let us denote by L_H the restriction of L to H i. e. $D(L_H) = \{u \in V, Lu \in H\}$, $L_H u = Lu$ for $u \in D(L_H)$. It is known that L_H is m-dissipative in $H \times H$.

Let Y denote the space $D(L_H)$ normed by

$$\| u \|_Y = \| Lu \| + \| u \| ; u \in D(L_H).$$

THEOREM 4. Suppose that Conditions I, II are satisfied. Let $f \in W^{1,1}(0, T; H) \cap L^\infty(0, T; V)$. Then for every $u_0 \in Y$ and $u_1 \in D \cap V$ the problem (4.1), (4.2) has a unique solution $u \in C(0, T; H) \cap L^\infty(0, T; Y)$ with $\dfrac{du}{dt} \in C(0, T; H) \cap L^\infty(0, T; V)$ and $\dfrac{d^2 u}{dt^2} \in L^\infty(0, T; H)$.

PROOF. Let $\mathcal{H}$ denote the direct sum of V and H

$$\mathcal{H} = V \oplus H$$

with the scalar product defined by

$$\langle U, V \rangle = (Lu_1, v_1) + (u_2, v_2) \tag{4.5}$$

where $W = \{u_1, u_2\}$ are generic elements of $\mathcal{H}$.

Thus the problem (4.1), (4.2) is equivalent to

$$\frac{d}{dt} U(t) = \mathcal{B} U(t) + \mathcal{A}(t) V(t) + F(t), \text{ a. e.} \tag{4.6}$$

and

$$U(0) = U_0, \tag{4.7}$$

where $U(t) = \left\{u(t), \frac{du(t)}{dt}\right\}$, $F(t) = \{0, f(t)\}$, $W_0 = \{u_0, u_1\}$ and $\mathcal{B}, \mathcal{A}(t)$ are dissipative subsets of $\mathcal{H} \times \mathcal{H}$ defined by

$$\mathcal{B} = \begin{pmatrix} 0 & 1 \\ -L & 0 \end{pmatrix} \quad D(\mathcal{B}) = Y \oplus V \tag{4.8}$$

respectively

$$\mathcal{A}(t) = \begin{pmatrix} 0 & 0 \\ 0 & A(t) \end{pmatrix}, \; D(\mathcal{A}(t)) = V \oplus D. \tag{4.9}$$

We shall verify the hypotheses of Theorem 1 where $X = C = \mathcal{H}$, $B = \mathcal{B}$ and $A(t) = \mathcal{A}(t)$.

We have

$$(1 - \lambda\mathcal{A}(t))^{-1} F = \{f_1, (1 - \lambda A(t))^{-1} f_2\}, \; F = \{f_1, f_2\}$$

Now Assumptions i), ii), iii) of Theorem are simple consequences of Conditions I. and II. Let us verify iv). Indeed if $F \in D(\mathcal{B}) = Y \oplus V$, $F = \{f_1, f_2\}$ then

$$|\mathcal{B}(1 - \lambda\mathcal{A}(t))^{-1} F|^2_{\mathcal{H}} = (L(1 - \lambda(t))^{-1} f_2, (1 - \lambda A(t))^{-1} f_2) + \|Lf_1\|^2$$

Using (4.4) we obtain

$$|\mathcal{B}(1 - \lambda\mathcal{A}(t))^{-1} F|_{\mathcal{H}} \leq |\mathcal{B}F|_{\mathcal{H}} + \lambda\psi(|F|_{\mathcal{H}} + |\mathcal{A}_\lambda(t) F|_{\mathcal{H}})$$

which proves iv).

Thus according to Theorem 1, the initial value problem (4.6), (4.7) has a unique solution $U \in W^{1,\infty}(0, T; \mathcal{H} \cap L^\infty(0, T; Y \oplus V)$.

This concludes the proof.

Example 4.1. Let Ω be an open bounded subset in R^n with smooth boundary $\partial\Omega$ and let L be a differential operator of second order

$$Eu = - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x) \frac{\partial u}{\partial x_j}\right), \tag{4.10}$$

where a_{ij} are real functions of class C^1 on Ω. In addition suppose that

$$a_{ij}(x) = a_{ji}(x) \text{ for } i, j = 1, 2, \dots, n \tag{4.11}$$

and

$$\sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i\,\xi_j \geq \gamma\,|\,\xi\,|^2,\ x \in \Omega,\ \xi \in R^n \tag{4.12}$$

where γ is a positive constant independent of x.

Let L denote the self adjoint operator from $H_0^1(\Omega)$ into $(H_0^1(\Omega))^* = H^{-1}(\Omega)$ which is given by $Lu = Eu$ for $u \in D(L)$. The restriction of L to $L^2(\Omega)$ has the domain $H^2(\Omega) \cap H_0^1(\Omega)$ and generates a continuous semigroup of linear contractions on $L^2(\Omega)$. Here $H_0^1(\Omega)$ and $H^2(\Omega)$ are usual Sobolev spaces.

Finally, let $A(t)$ be the family of m-dissipative subsets of $L^2(\Omega) \times \times L^2(\Omega)$ defined by

$$A(t) = \{[u, v]\,;\ u, v \in L^2(\Omega) \text{ and } v(x) \in \Gamma(t)(v(x)) \text{ a. e. in } \Omega\} \tag{4.13}$$

where $-\Gamma(t) \subset R \times R$ is a family of maximal monotone sets in $R \times R$ such that $D(\Gamma(t))$ is independent of t and contains 0. Moreover assume that

$$|\,(1 - \lambda\Gamma(t))^{-1}\,v - (1 - \lambda\Gamma(s))^{-1}\,v\,| \leq M\lambda\,|\,t - s\,|\,(\,|\,v\,| + |\,\Gamma_\lambda(t)\,v\,|\,) \tag{4.14}$$

for every $v \in R$, $t, s \in [0, T]$ and $\lambda > 0$. Here M is a nonnegative constant independent of λ, t and s.

Let us observe that hypotheses of Theorem 4 are satisfied with $H = = L^2(\Omega)$, $V = H_0^1(\Omega)$, H and $A(t)$ defined as above Indeed Condition I· follows from the corresponding properties oe $\Gamma(t)$ and II. is a consequence of the fact that $\left|\dfrac{\partial}{\partial u}(1 - \lambda\,\Gamma(t))^{-1}\,u\right| \leq 1$ for every $\lambda > 0$, and $u \in R$.

Thus Theorem 4 yields the following Corollary:

Corollary 4.1. Let f, u_0, u_1 be given, satisfiying

$$f \in W^{1,1}(0, T\,;\, L^2(\Omega)) \cap L^\infty(0, T\,;\, H_0^1(\Omega))$$

and

$$u_0 \in H^2(\Omega) \cap H_0^1(\Omega),\ u_1 \in H_0^1(\Omega) \cap D(A(t)).$$

Then the problem

$$\frac{\partial^2 u}{\partial t^2} - \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\,\frac{\partial u}{\partial x_i}\right) \in A(t)\left(\frac{\partial u}{\partial t}\right) + f(t), \text{ in } \Omega \times (0, \infty) \tag{4.14}$$

$$u(x, 0) = u_0(x),\ \frac{\partial u}{\partial t}(x, 0) = u_1(x), \text{ in } \Omega,$$

has a unique solution $u \in C(0, T; L^2(\Omega)) \cap L^\infty(0, T; H^2(\Omega))$ such that

$$\frac{\partial u}{\partial t} \in C(0, T; L^2(\Omega)) \cap L^\infty(0, T; H_0^1(\Omega)) \tag{4.16}$$

and

$$\frac{\partial^2 u}{\partial t^2} \in L^\infty(0, T; L^2(\Omega)). \tag{4.17}$$

Now we consider the differential equation of the form

$$\lambda u(t) - \frac{d^2 u}{dt^2} \in A(u(t)) + f(t), \qquad 0 < t < T \tag{4.18}$$

whith Dirichlet conditions

$$u(0) = u(T) = x \tag{4.19}$$

on a Hilbert space H, where A is dissipative subset of $H \times H$

Let $1 \leq p \leq \infty$. Then $W^{2,p}(0, T; H)$ denote the space of vectorial distributions $u \in \mathcal{D}'(0, T; H)$ such that $\dfrac{d^k u}{dt^k} \in L^p(0, T; H)$ for $0 \leq k \leq 2$

We recall that if $u \in W^{2,p}(0, T; H)$ then $\dfrac{du}{dt}$ coincides a. e. on $(0, T)$ with an absolutely continuous function.

THEOREM 5. Let A be a closed and dissipative subset of $H \times H$ and let Q be a closed convex cone of H. Suppose that $R(1 - \lambda A)$ contains Q for every $\lambda > 0$ and

$$(1 - \lambda A)^{-1} Q \subset Q \text{ for } \lambda > 0. \tag{4.20}$$

Let x be in $D(A) \cap Q$ such that $Ax \cap Q \neq \emptyset$. Then for every $f \in L^p(0, T; Q)$, $1 < p < \infty$ and for each $\lambda > 0$ the problem (4.18), (4.19) has a unique solution $u \in W^{2,p}(0, T; H) \cap L^p(0, T; D(A))$ such that $u(t) \in Q$ for $0 \leq t \leq T$.

PROOF. We may assume without loss of generality that $x = 0 \in D(A) \cap Q$ and $0 \in AO$. Let B denote the operator on $L^p(0, T; H)$ with domain $D(B) = \{u ; u \in H^{2,p}(0, T; H); u(0) = u(T) = 0\}$, which is given by $Bu = \dfrac{d^2 u}{dt^2}$ for $u \in D(B)$.

It is known (see [7]) that B is the infinitesimal generator of a continuous semigroup of linear contractions on $L^p(0, T; H)$ defined by

$$(4.21) \qquad (S(t)\,u)(s) = \int_0^T K(\mathcal{T}, s, t)\, u(\mathcal{T})\, d\mathcal{T}, \quad u \in C_0^\infty(0, T; H)$$

where $K(\mathcal{T}, s, t) = 2T/\pi \sum_{n=1}^{\infty} \exp(-n^2 t) \sin \frac{n\pi\mathcal{T}}{T} \sin \frac{n\pi s}{T}$. Since $K(\mathcal{T}, s, t) \leq 0$ for $\mathcal{T} \in (0, \infty)$ and $t, s \in (0, T)$, from (4.21) it follows that $S(t)\, Q \subset Q$ for every $t \geq 0$. This implies that

$$(4.22) \qquad (1 - \lambda B)^{-1}\, Q \subset Q \text{ for every } \lambda > 0.$$

We introduce the following operator

$$(4.23) \quad \widetilde{A} = \{[u, v]\,;\, u, v \in L^p(0, T; H) \text{ and } v(t) \in A(u(t)) \text{ a. e. on } (0, T)\}$$

Clearly $\widetilde{A}$ is dissipative and closed in $L^p(0, T; H) \times L^p(0, T; H)$. Moreover, assumption (4.20) implies immediately that

$$(4.24) \qquad (1 - \lambda A)^{-1}\, L^p(0, T; Q) \subset L^p(0, T; Q) \text{ for all } \lambda > 0.$$

We now verify hypotheses of Corollary 2.2 where $X = L^p(0, T; H)$, $C = L^p(0, T; Q)$, $A = \widetilde{A}$ and B is defined above. Obviously j) and jj) are implied by (4.22) and (4.24). It remains to prove (2.15).

Let u be erbitrary in (2.15). Recalling that

$$F(u)(t) = u(t)\, \| u(t) \|^{p-2} / | u |^{p-2}_{L^p(0, T; H)}$$

is the duality mapping of $X = L^p(0, T; H)$ we obtain

$$(4.25) \qquad \langle Bu, F(A_n u) \rangle_X = -(p-1)\, | A_n u |^{2-p}_{L^p} \int_0^T \left(\frac{du(t)}{dt}, \frac{d}{dt} A_n u(t) \right) \| A_n u(t) \|^{p-2}\, dt.$$

Since $\left(\frac{du(t)}{dt}, \frac{d}{dt} A_n u(t) \right) = \lim_{h \to 0} \left(\frac{u(t+h) - u(t)}{h}, \frac{A_n u(t+h) - A_n u(t)}{h} \right)$ a.e. on $(0, T)$ it follows from (4.25) that

$$(4.26) \qquad \langle Bu, F(A_n u) \rangle_X \geq 0$$

since A_n are dissipative in $H \times H$ for every n. By Corollary 2.2 we conclude that there exists a uuique solution $u \in D(B) \cap D(A) \cap L^p(0, T; Q)$ of of the equation

$$\lambda u - Bu - Au \ni f, \lambda > 0, f \in L^p(0, T; Q).$$

This completes the proof of Theorem 5.

Acknowledgement

Part of this paper was written menwhile the author was visiting the Istituto per le Applicazioni del Calcolo of the C. N. R., Rome, Italy. The author wishes to thank G. Da Prato and G. Stampacchia for very useful discussions.

REFERENCES

[1] V. BARBU, *Weak solutions for nonlinear functional equations in Banach spaces,* Ann. Mat. Pura ed Appl., vol. LXXXVII, 1970, pp. 87-110.

[2] F. BROWDER, *Nonlinear operators and nonlinear equations of evolution in Banach spaces,* Proceedings of the Symposium on Nonlinear Functional Analysis A. M. S., 1968.

[3] H. BREZIS, *Semi-groupes nonlinéaires et applications.* Proceedings of the Symposium on Evolution Equations, Rome, 1970 (to appear).

[4] H. BREZIS and A. PAZY, *Semigroup of nonlinear contractions on convex sets.* J. Functional Analysis (to appear).

[5]. H. BREZIS, M. G. CRANDALL and A. PAZY, *Perturbations of Nonlinear Maximal Monotone Sets in Banach Space.* Comm. Pure Appl. Math., vol. XXIII, 1 (1970), pp. 123-144.

[6] M. G. CRANDALL and A. PAZY, *Semigroups of nonlinear contractions and dissipative sets.* J. Functional Analysis, 3 (1969), pp. 376-418.

[7] G. DA PRATO, *solutions for linear abstract differential equations in Banach Space.* Advances in Math., vol. 5 (1970), pp. 181-245.

[8] G. DA PRATO, *Somme d'applications non-linéaires dans des cônes et équations d'évolution dans des espaces d'opérateurs.* J. Math. Pures Appl., vol. 49 (1970), pp. 289-348.

[9] T. KATO, *Nonlinear semigroups and evolution equations.* J. Math. Soc. Japan, 19 (1967), pp. 508-520.

[10] T KATO *Accretive operators and nonlinear evolulion equations in Banach spaces.* Proceedings of the Symposium of Nonlinear Functional Analysis A. M. S. 1968.

[11] Y. KÖMURS, *Nonlinear semigroups in Hilbert spaces.* J. Math. Soc. Japan, 19 (1967), pp- 493-507.

[12] J. L. LIONS, *Quelques méthodes de résolution des problèmes aux limites non-linéaires.* Dunod-Gauthier Villars, 1969.

Faculty of Mathematics
University of Jassy, Romania