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Classe di Scienze

D. E. EDMUNDS
L. A. PELETIER

Quasilinear parabolic equations

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QUASILINEAR PARABOLIC EQUATIONS

D. E. EDMUNDS and L. A. PELETIER

1. Introduction.

During the last decade an intensive study has been made of the existence in the large of solutions of the first initial-boundary value problem, or the Cauchy-Dirichlet problem as we prefer to call it, for quasilinear parabolic equations. This problem has, in fact, been reduced to that of obtaining *a priori* bounds for eventual solutions of the problem and for the first derivatives of such solutions. In the present paper we shall investigate the circumstances under which these a priori bounds may be obtained, and shall give conditions under which the Cauchy-Dirichlet problem has a solution: non-existence theorems for this problem will also be given.

The equations we shall study are of the form

$$\mathcal{A}_{ij}\left(x, t, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right)\frac{\partial^2 u}{\partial x_i\, \partial x_j} - \mathcal{B}\left(x, t, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right) - \frac{\partial u}{\partial t} = 0,$$

and for convenience we shall write this more concisely as

$$\mathcal{L} u \equiv \mathcal{A}(x, t, u, u_x)\, u_{xx} - \mathcal{B}(x, t, u, u_x) - u_t = 0, \tag{1}$$

where

$$\mathcal{A} = (\mathcal{A}_{ij}),\ \mathcal{A}_{ij} = \mathcal{A}_{ji},\ u_x = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right),\ u_{xx} = \left(\frac{\partial^2 u}{\partial x_i\, \partial x_j}\right).$$

The functions $\mathcal{A}(x, t, u, p)$ and $\mathcal{B}(x, t, u, p)$ which define the structure of equation (1) are assumed to be continuously differentiable for all u in R (the reals), all p in R^n, and all (x, t) in the closure of a space-time cylinder

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$Q = \Omega \times (0, T)$, where Ω is a bounded domain in R^n with C_4 boundary $\partial\Omega$. It is supposed that (1) is *parabolic*: this means that $\xi \mathscr{A} \xi \equiv \xi_i \mathscr{A}_{ij} \xi_j > 0$ for all non-zero ξ in R^n.

By the Cauchy-Dirichlet problem for equation (1) in Q we shall mean the question of determining a function u such that $\mathscr{L} u = 0$ in Q, with u assuming prescribed values on the base and the sides of Q. The existing literature on problems of this nature is extensive, but up to very recent times has been principally concerned either with equations possessing a rather simple structure or with *uniformly parabolic* equations, which are those with the property that there exist positive constants ν and μ such that for all u in R, all p and ξ in R^n, and all (x, t) in the closure of Q,

$$\nu \, | \xi |^2 \leq \xi \mathscr{A} (x, t, u, p) \, \xi \leq \mu \, | \xi |^2.$$

Particularly noteworthy in this connection are the contributions of Oleinik and Kruzhkov [6], Ladyzhenskaya and Ural'tseva [4], Ladyzhenskaya, Ural'tseva and Solonnikov [5], and Trudinger [8]. Amongst these authors there are differences in the kind of solution which is sought: for example, Trudinger is not concerned with solutions which are smooth right up to the boundary. Here, however, we shall deal only with solutions which do possess this degree of smoothness, so that the prescribed data has to satisfy certain compatibility conditions on the boundary of the base of Q.

We show that there is a class of (not necessarily uniformly parabolic) equations, the regularly parabolic equations which, so far as the Cauchy-Dirichlet problem is concerned behave nicely in that existence may under appropriate circumstances be obtained in arbitrary domains Q without the need for special restrictions on the curvatures of $\partial\Omega$. Some idea of the disasters which may occur if one wanders outside this class is given by providing examples of equations for which prescribed smooth data can be constructed such that the resulting Cauchy-Dirichlet problem has no solution. Our work may be thought of as a beginning of the extension to parabolic equations of the celebrated results of Serrin [7] concerning the Dirichlet problem for quasilinear elliptic equations, which provide necessary conditions and sufficient conditions for this problem, in a given domain and with arbitrarily given smooth data, to be soluble. The methods adopted are entirely natural analogues of those used by Serrin in the elliptic case, though as might be expected there are difficulties not present in that situation. We do not, however, deal with the analogue of that part of Serrin's work in which the curvatures of $\partial\Omega$ play a crucial role, and postpone our discussion of the subtle and complicated arguments that are then necessary to a future paper.

Our programme is as follows. In § 2 we give the fundamental existence theorem, which by means of a basic theorem due to Ladyzhenskaya and Ural'tseva reduces the Cauchy-Dirichlet problem to that of establishing appropriate a priori bounds. This theorem necessitates a more complex proof than does the comparable elliptic theorem, the extra complication being occasioned by the consistency condition. The next four sections relate to conditions under which the a priori bounds may be obtained, while §§ 7 and 8 give selections of existence and non-existence theorems. The paper contains proofs of sharper versions of results announced in [2]. The authors are grateful to Professors F. E. Browder and G. Stampacchia for helpful discussions about certain points dealt with in § 2.

2. Reduction of the problem to that of obtaining a priori bounds.

Let Ω be a bounded domain in R^n, with boundary $\partial\Omega$ and closure $\bar{\Omega}$, and denote by Q the cylinder $\Omega \times (0, T)$, where T is some fixed positive real number: points of $\bar{Q}$ will be written as (x, t), where $x = (x_1, \dots, x_n) \in \bar{\Omega}$ and $t \in [0, T]$. We shall let Γ represent the *parabolic boundary* of Q, that is, the set $(\bar{\Omega} \times \{0\}) \cup (\partial\Omega \times (0, T))$, and shall write $Q_T = \Omega \times \{T\}$. The set of real-valued functions u that are continuous on Q together with their derivatives $u_t, u_{x_i}, u_{x_i x_j}$ will be denoted by $C_{2,1}(Q)$; $C_{2,1}(\bar{Q})$ is defined in the obvious way. We shall need to use spaces of Hölder — continuous functions: a function u defined on a closed subset Q_1 of $\bar{Q}$ is called Hölder — continuous on Q_1 with exponent $\alpha \, (0 < \alpha \leq 1)$ and coefficient K if for all (x, t), (x_1, t_1) in Q_1,

$$| u(x, t) - u(x_1, t_1) | \leq K [|x - x_1|^2 + |t - t_1|]^{\alpha/2}.$$

By $C_\alpha(\bar{Q})$ we shall mean the linear space of those functions which satisfy a Hölder condition with exponent α in $\bar{Q}$. Endowed with the norm

$$\| u \|_{\alpha, Q} = \sup_Q | u(x, t) | + \sup \{ | u(x, t) - u(x_1, t_1) | [| x - x_1 |^2 + | t - t_1 |]^{-\alpha/2} : (x, t), (x_1, t_1) \in \bar{Q}, (x, t) \neq (x_1, t_1) \}$$

it becomes a Banach space. Next, $C_{2+\alpha}(\bar{Q})$ will represent the Banach space of those functions u in $C_\alpha(\bar{Q})$ such that u_t, u_{x_i} and $u_{x_i x_j}$ all belong to $C_\alpha(\bar{Q})$: the norm on this space is

$$\| u \|_{2+\alpha, Q} = \| u \|_{\alpha, Q} + \sum_i \| u_{x_i} \|_{\alpha, Q} + \sum_{ij} \| u_{x_i x_j} \|_{\alpha, Q} + \| u_t \|_{\alpha, Q} .$$

Similarly, $C_{1+\alpha}(\bar{Q})$ will stand for the Banach space of those members u of $C_\alpha(\bar{Q})$ such that $u_{x_i} \in C_\alpha(\bar{Q})$, with norm

$$\| u \|_{1+\alpha, Q} = \| u \|_{\alpha, Q} + \sum_i \| u_{x_i} \|_{\alpha, Q} .$$

Lastly, a function ψ defined on Γ will be said to be of class $C_{2+\alpha}$ if it is the restriction to Γ of a function, also denoted by ψ, in $C_{2+\alpha}(\bar{Q})$.

The Cauchy-Dirichlet problem for equation (1) in Q consists in determining a function u in $C_{2,1}(Q) \cap C(Q \cup \Gamma)$ such that

$$\text{(2)} \qquad \begin{cases} \mathcal{L}u = 0 \text{ in } Q, \\ u = \psi \text{ (a given continuous function) on } \Gamma. \end{cases}$$

We remark that if the solution u is in the class $C_{2,1}(\bar{Q})$ then plainly the condition $\mathcal{L}\psi = 0$ must be satisfied on $\partial\Omega \times \{0\}$: this necessary condition will be referred to subsequently as the *consistency condition.*

We can now show how the question of the existence of solutions of the Cauchy-Dirichlet problem which are smooth up to Γ may be reduced to that of obtaining suitable a priori bounds. The fundamental tool is a theorem due to Ladyzhenskaya and Ural'tseva [5, p. 533] which provides bounds for the Hölder norms of the solution and its space gradient given bounds for the solution and its first derivatives. More precisely, let $\partial\Omega \in C_3$, and suppose $v \in C_{2,1}(\bar{Q})$ is such that $\mathcal{L}v = 0$ in Q and $v = \psi$ on Γ, where ψ is a given function of class C_{2+1} which satisfies the consistency condition. Suppose that

$$\sup_Q (|v| + |v_x|) + \sup_{\Omega\times\{0\}} |v_t| \leq M,$$

and also that ν is a positive constant such that for all ξ in R^n and all (x, t) in $\bar{Q}$, $\xi \mathcal{A}(x, t, v, v_x) \xi \geq \nu |\xi|^2$. Under these assumptions, the theorem of Ladyzhenskaya and Ural'tseva implies that there are positive constants N and γ $(\gamma < 1)$ such that $\| v \|_{1+\gamma, Q} \leq N$. These constants depend only on ν, M, bounds for $\mathcal{A}, \mathcal{B}$ and their derivatives, and on $\partial\Omega$ and the data ψ: they are independent of the particular function v.

The reduction mentioned above is a consequence of the following theorem:

Theorem 1. Let $\partial\Omega$ be of class C_3, and suppose the given data ψ is of class C_{2+1} and satisfies the consistency condition $\mathcal{L}\psi = 0$ on $\partial\Omega \times \{0\}$. Let τ be any real number in $[0, 1]$. Suppose there is a number M, independent of τ, such that if $v \in C_{2,1}(\bar{Q})$ satisfies

$$(3)\quad \begin{cases} \tau \mathcal{A}(x, t, v, v_x) v_{xx} + (1-\tau)\Delta v - \tau \mathcal{B}(x, t, v, v_x) + \\ \qquad (1-\tau)(\psi_t - \Delta\psi) - v_t = 0 \text{ in } Q, \\ \qquad v = \psi \text{ on } \Gamma \end{cases}$$

then

$$(4)\quad \sup_Q (|v| + |v_x|) + \sup_{\Omega \times \{0\}} |v_t| \leq M.$$

Then there is a solution $u \in C_{2,1}(\bar{Q})$ of the Cauchy-Dirichlet problem (2).

Proof. Suppose $v \in C_{2,1}(\bar{Q})$ satisfies (3) for some τ in $[0, 1]$. Then by (4) and the Ladyzhenskaya and Ural'tseva theorem there are positive constants N and γ, independent of τ, such that $\| v \|_{1+\gamma, Q} \leq N$.

Define X to be the closed subset of $C_{1+\gamma}(\bar{Q})$ consisting of those functions which coincide with ψ on $\bar{\Omega} \times \{0\}$. Let $w \in X$, $\tau \in [0, 1]$, and consider the linear problem

$$(5)\quad \begin{cases} \tau \mathcal{A}(x,t,w,w_x) W_{xx} + (1-\tau)\Delta W - \tau \mathcal{B}(x,t,w,w_x) + (1-\tau)(\psi_t - \Delta\psi) - W_t = 0 \text{ in } Q, \\ \qquad W = \psi \text{ on } \Gamma. \end{cases}$$

By linear theory (note that the consistency condition is satisfied) there is exactly one solution $W \in C_{2+\gamma}(\bar{Q})$ of this problem, and evidently $W \in X$. Hence there is a well defined map $T : X \times [0, 1]$ $(\subset C_{1+\gamma}(\bar{Q}) \times [0, 1]) \to X$ given by $T(w, \tau) = W$. We assert that T is completely continuous, that is, continuous, and compact. To see that T is compact let $K \subset X \times [0, 1]$ be bounded. By the linear a priori estimates the elements W of $T(K)$ have uniformly bounded second order x derivatives, so that for each fixed t the W_x satisfy a Hölder condition with exponent 1. Similarly, since the W_t are uniformly bounded the elements W satisfy for each fixed x a Hölder condition with exponent 1. Thus by a lemma of Ladyzhenskaya and Ural'tseva [4, p. 276] each W_x satisfies a Hölder condition in t with exponent $\frac{1}{2}$, and so $W \in C_{1+1}(\bar{Q})$ and there is a constant C such that for all W in $T(K)$, $\| W \|_{1+1, Q} \leq C$. The compactness of T is now an immediate consequence of the Arzela-Ascoli theorem. That T is continuous follows from an elementary *reductio ad absurdum* argument: we omit the details.

Finally, define $X_1 = \{w - \psi : w \in X\}$ and $T_1 : X_1 \times [0, 1] \to X_1$ by $T_1(w - \psi, \tau) = T(w, \tau) - \psi$. Evidently X_1 is a closed linear subspace of

$C_{1+\gamma}(\bar{Q})$, and hence is a Banach space when given the induced topology, and from the properties derived above for T it is plain that T_1 is completely continuous. Let B denote the closed ball in X_1 with centre 0 and radius $N + \| \psi \|_{1+\gamma, Q} + 1$. It is easy to see that $T_1(w - \psi, 0) = 0$ for all $w \in X$, and moreover $T_1(w - \psi, \tau) \neq w - \psi$ on the boundary of B, for all $\tau \in [0, 1]$. Hence by the Leray-Schauder theorem (see [1], p. 106), $T_1(\cdot, 1)$ has a fixed point in B, and so there exists u in X such that $T(u, 1) = u$. Such a function u is a solution of the Cauchy-Dirichet problem (2): since moreover u evidently belongs to $C_{2,1}(\bar{Q})$ the proof is complete.

To apply Theorem 1 it is necessary to derive a priori bounds related to the solutions of a whole family of Cauchy-Dirichlet problems, namely those given by (3), with τ varying from 0 to 1. While we can derive such bounds in a variety of situations, and indeed shall do precisely this in the succeeding sections, it turns out that for some purposes there is an advantage in having a different homotopy family of equations to handle. For this reason we give the following reduction theorem, even though the class of data ψ to which it applies is rather severely limited.

THEOREM 2. Let $\partial\Omega$ be of class C_3, and suppose the given data ψ is of class C_{2+1} and satisfies $\psi = 0$, $\psi_x = 0$ and $\mathcal{A}(x, 0, 0, 0)\, \psi_{xx} - \mathcal{B}(x, 0, 0, 0) - \psi_t = 0$ on $\partial\Omega \times \{0\}$. Let τ be any real number in $[0, 1]$. Suppose there is a number M, independent of τ, such that if $v \in C_{2,1}(\bar{Q})$ satisfies

$$
(6) \qquad \begin{cases} \mathcal{A}(x, t, v, v_x)\, v_{xx} - \tau \mathcal{B}(x, t, v, v_x) - v_t = 0 \text{ in } Q \\ v = \tau\psi \text{ on } \Gamma \end{cases}
$$

then

$$
\sup_{Q} (|v| + |v_x|) + \sup_{\Omega \times \{0\}} |v_t| \leq M.
$$

Then there is a solution $u \in C_{2,1}(\bar{Q})$ of the Cauchy-Dirichlet problem (2).

We omit the proof because of its general similarity to that of Theorem 1, and merely remark that the conditions imposed on ψ in the theorem are designed to ensure that problem (6) has data which satisfies the consistency condition.

The question of the existence of solutions of the Cauchy-Dirichlet problem can therefore be regarded as settled if we can obtain a priori bounds of the kind used in Theorems 1 or 2. These bounds may be obtained for equations having a suitable form by carrying out the following four steps:

(A) Estimate $\max_{\bar{Q}} |u|$.

(B) Given that (A) has been achieved, estimate $|u_x|$ at $S = \partial\Omega \times [0, T]$.

(C) Given (A) and (B), estimate $|u_x|$ in the entire domain.

(D) Estimate $|u_t|$ on $\partial\Omega \times \{0\}$.

It should be emphasised that these steps have to be carried out for a whole family of problems (either (3) or (6)), and that the estimates obtained must be independent of the homotopy parameter τ. Despite this it will suffice to discuss the various steps in terms of the original Cauchy-Dirichlet problem, provided care is taken to describe the dependence of the estimates on the structure of the equation and the prescribed data. The next four sections are devoted to the a priori estimates required by steps (A)-(D).

Before passing to these estimates, however, it will be convenient to record here certain conventions and notations. We shall write

$$c_0 = \sup_Q |\psi|, \; c_1 = \sup_Q |\psi_x|, \; c_2 = \sup_Q \|\psi_{xx}\|, \; c_3 = \sup_Q |\psi_t|,$$

where the norm $\|\psi_{xx}\|$ is defined to be the square root of the sum of the squares of the entries in the matrix ψ_{xx}.

Certain invariant functions prove to be of great importance, just as in the elliptic situation: these are

$$\mathcal{C}(x, t, u, p) = \mathcal{B}(x, t, u, p)/|p| \quad (p \neq 0),$$

$$\mathcal{E}(x, t, u, p) = p\,\mathcal{A}(x, t, u, p)\,p,$$

$$\mathcal{F}(x, t, u, p, p_0) = (p - p_0)\,\mathcal{A}(x, t, u, p)\,(p - p_0),$$

$$a = \text{trace}\,\mathcal{A}.$$

A final point is that when dealing with various functions of the variables $x, t, u,$ and p the range of any one of the variables is, if not specifically delineated, to be taken to be $\bar{\Omega}$, $[0, T]$, R and R^n respectively.

3. Estimates for $|u|$ in Q.

Here we give a number of maximum principles, all closely related to those presented by Serrin [7] in his analysis of the corresponding elliptic situation.

THEOREM 3. Let $\omega \in C_{2,1}(Q \cup Q_T) \cap C(\bar{Q})$ be such that $\mathcal{L}(\omega + b) \leq 0$ in $Q \cup Q_T$ for all positive constants b. Suppose also that $u \in C_{2,1}(Q \cup Q_T) \cap C(\bar{Q})$ is a solution of equation (1) in Q such that $u \leq \omega$ on Γ. Then $u \leq \omega$ on $\bar{Q}$.

PROOF. Put $v = u - \omega$, $K = \sup_{Q \cup Q_T} v$, and suppose $K > 0$. Then there exists a point $P = (x_P, t_P)$ in $Q \cup Q_T$ at which v takes the value K, and evidently we may suppose that there is a neighbourhood N of P such that in N, $v > 0$, and in $N \cap \{t \leq t_P\}$, $v \not\equiv K$. We know that

$$\mathcal{A}(x, t, \omega + b, \omega_x)\, \omega_{xx} - \mathcal{B}(x, t, \omega + b, \omega_x) - \omega_t \leq 0$$

for all (x, t) in N, and for all $b > 0$; hence this inequality will also hold for all positive functions b. Thus if we take $b = v$ we obtain

$$\mathcal{A}(x, t, u, \omega_x)\, \omega_{xx} - \mathcal{B}(x, t, u, \omega_x) - \omega_t \leq 0 \quad \text{in } N. \tag{7}$$

Since u is a solution of (1) we also have

$$\mathcal{A}(x, t, u, u_x)\, u_{xx} - \mathcal{B}(x, t, u, u_x) - u_t = 0. \tag{8}$$

Subtraction of (7) from (8) and application of the mean value theorem now shows that

$$\mathcal{A}(x, t, u, u_x)\, v_{xx} - v_t + (\text{linear function of } v_x) \geq 0 \quad \text{in } N.$$

However, Nirenberg's maximum principle implies that $v \equiv K$ in $N \cap \{t \leq t_P\}$, which is a contradiction. Hence $K \leq 0$, which proves the theorem.

The proof is identical wlth that given by Serrin [7] for the corresponding result in the elliptic situation, save that we have to use the Nirenberg maximum principle rather than the Hopf one. We now give a number of applications of this result which correspond to ones given by Serrin : because the proofs are similar in every respect to those of Serrin we shall give only the barest indication (if any at all) of them.

THEOREM 4. Suppose that $u\,\mathcal{B}(x, t, u, 0) \geq 0$ for $|u| > M$, and let $u \in C_{2,1}(Q \cup Q_T) \cap C(\bar{Q})$ be a solution of (1) in Q such that $|u| \leq m$ on Γ. Then $|u| \leq \max(m, M)$ in Q.

PROOF. Take $\omega = \max(m, M)$ and apply Theorem 3.

THEOREM 5. Let Ω be contained in a ball (in R^n) of radius R, and assume that

$$\operatorname{sign} u \cdot \mathcal{C}(x, t, u, p) \geq -\frac{1}{R} \operatorname{trace} \mathcal{A}(x, t, u, p)$$

for $|u| \geq M$ and $|p| \geq l > 0$. Let $u \in C_{2,1}(Q \cup Q_T) \cap C(\bar{Q})$ be a solution of (1) in Q such that $|u| \leq m$ on Γ. Then $|u| \leq \max(m, M) + 2l R$ in Q.

PROOF. Apply Theorem 3 with $\omega(x, t) = \max(m, M) + l R (e - e^{r/R})$ $(0 \leq r \leq R)$, where r denotes the distance of (x, t) to (x_0, t) and x_0 corresponds to the centre of the ball containing Ω.

THEOREM 6. Suppose that for some fixed direction ν and for all $\varrho < 0$,

$$\frac{|\mathcal{B}(x, t, u, p)|}{\mathcal{E}(x, t, u, p)} \leq \varphi(\varrho), p = \varrho\nu,$$

where φ is a positive continuous function such that $\int_0^\infty \{\varrho^2 \varphi(\varrho)\}^{-1} d\varrho = \infty$.

Let $u \in C_{2,1}(Q \cup Q_T) \cup C(\bar{Q})$ be a solution of (1) such that $|u| \leq m$ on Γ. Then $|u| \leq K$ in Q, where K depends only on m, φ and the diameter of Ω.

THEOREM 7. Suppose $\mathcal{A}$ is independent of u and $\partial\mathcal{B}/\partial u \geq 0$. Let $u, \omega \in C_{2,1}(Q \cup Q_T) \cup C(\bar{\Omega})$ be such that $\mathcal{L}(\omega) \leq 0$ and $\mathcal{L}(u) = 0$ in Q, and suppose $u \leq \omega$ on Γ. Then $u \leq \omega$ in $\bar{\Omega}$.

PROOF. Evidently $\mathcal{L}(\omega + b) \leq \mathcal{L}(\omega) \leq 0$ for all positive constants b.

4. Estimates for $|u_x|$ at S.

Step (B) of our programme is achieved by the method of *global barriers*. Let $d(x)$ denote the distance from points (x, t) in $\bar{Q}$ to $S = \partial\Omega \times [0, T]$: it follows from the work of Serrin [7, p. 420] that d is twice continuously differentiable for $0 \leq d < d_0$, where d_0 depends on $\partial\Omega$. Let N be the neighbourhood of S defined by the inequalities $0 \leq d < d_1 \leq d_0$. A function v in $C_{2,1}(N)$ is called a *global barrier function* if (i) $\mathcal{L}(v + b) \leq 0$ in N for all positive constants b, and (ii) $v(x, t)$ can be written as $\psi(x, t) + h(d)$, where h is non-negative and continuous in the closure of N and is such that $h(0) = 0$, $h(d_1) = M$.

Given the existence of an appropriate global barrier function it is easy to see that step (B) can be carried out: more precisely, we have the following

LEMMA 1. Let $u \in C_{2,1}(\bar{Q})$ be a solution of the Cauchy-Dirichlet problem (2), and suppose $u \leq m$. Suppose also that there is a global barrier function corresponding to $M = m + c_0$. Then $\frac{\partial u}{\partial n} \leq C$ on S for every spacelike direction $\underset{\sim}{n}$ into Q, where C is a constant which depends on the global barrier function.

PROOF. Let $v = \psi + h$ be the global barrier function. Then $u = v$ on S, $u \leq v$ on $\bar{N} \cap (\bar{\Omega} \times \{0\})$, and when $d = d_1$, $u \leq m \leq m + \psi + c_0 = \psi + M = v$. It follows from Theorem 3 that $u \leq v$ in N, so that $\frac{\partial u}{\partial n} \leq \frac{\partial v}{\partial n} \leq \max_S |v_x| = C$ on S, for every direction $\underset{\sim}{n}$ into Q.

It follows that step (B) is reduced to the problem of constructing a suitable global barrier function. As it turns out there is a class of equations, which by analogy with Serrin's terminology in the elliptic case we shall call the *regularly parabolic* equations, for which global barrier functions can be constructed for any cylinder Q with $\partial\Omega$ of class C_3.

We shall say that equation (1) is *m-regularly parabolic* if

$$\frac{1 + \operatorname{trace} \mathcal{A} + |\mathcal{C}|}{\mathcal{E}} \leq \Phi(|p|) \quad \text{for } |u| \leq m, p \neq 0,$$

where $\Phi : R_+ \to R_+$ is a decreasing continuous function such that

$$\int_1^\infty \{\varrho^2 \, \Phi(\varrho)\}^{-1} \, d\varrho = \infty. \tag{9}$$

If $\Phi(\varrho)$ can be taken of the form $C\varrho^{-1} \log \varrho$ for large ϱ equation (1) will be called regularly parabolic *in the strong sense.*

Various examples will serve to illustrate this definition. Thus it is easy to verify that uniformly parabolic equations are m-regularly parabolic in the strong sense if

$$|\mathcal{B}| \leq \gamma(m)\,|p|^2 \log(1 + |p|) \text{ for } |p| \geq 1 \text{ and } |u| \leq m.$$

The class of regularly parabolic equations includes various non-uniformly parabolic equations: as an example may be cited the equation (in two space variables)

$$(1 + u_x^2)\, u_{xx} + 2u_x\, u_y\, u_{xy} + (1 + u_y^2)\, u_{yy} - u_t = 0.$$

Moreover, all equations with genre $g \leq 1$ and trace $\mathcal{A}$ bounded away from zero are regularly parabolic in the strong sense provided

$$|\mathcal{B}|/\mathcal{E} \leq \gamma(m) \log(1 + |p|) \text{ when } |u| \leq m, |p| \geq 1.$$

We recall that equation (1) is said to have a well-defined *genre* if there are positive constants μ_1, μ_2 and a real number l such that $\mu_1 |p|^l \leq \mathcal{E}/\text{trace}\ \mathcal{A} \leq \mu_2 |p|^l$ for $|p| \geq 1$: the genre g is defined to be $2 - l$, which is necessarily non negative, and zero for uniformly parabolic equations. If $g > 1$ the equation is not regularly parabolic.

Before we can exhibit the construction of the global barrier functions we need two preliminary lemmas. Let N be the strip defined by $0 < d < d_1 \leq d_0$, and write $v(x, t) = \psi(x, t) + h(d)$, where $h \in C_2(0, d_1) \cap C[0, d_1]$ and $h' > 0$.

LEMMA 2. For (x, t) in N and for all positive constants b we have

$$\mathcal{L}(v + b) = \mathcal{F} h''/h'^2 - \mathcal{J} h' + \mathcal{A}\psi_{xx} - \mathcal{B} - \psi_t .$$

Here $\mathcal{A} = \mathcal{A}(x, t, v + b, p)$, $\mathcal{B} = \mathcal{B}(x, t, v + b, p)$, $p = p_0 + \nu h'$, $p_0 = \psi_x$, and ν is the inner unit normal at the unique point $y(x)$ on $\partial\Omega$ nearest to x. Also

$$\mathcal{F} = (p - p_0)\mathcal{A}(p - p_0), \quad \mathcal{J} = \sum_{i=1}^{n-1} \frac{\lambda_i \mathcal{A} \lambda_i}{1 - k_i d} k_i,$$

where $k_1, \ldots, k_{n-1}$ and $\lambda_1, \ldots, \lambda_{n-1}$ are respectively the principal curvatures and principal directions of $\partial\Omega$ at $y(x)$.

PROOF. Exactly as in lemma 4.1 of Serrin [7].

LEMMA 3. Suppose that $|p_0| \leq c$. Then

$$\frac{1}{2}\mathcal{E} - 2c^2 \text{ trace } \mathcal{A} \leq \mathcal{F} \leq 2\mathcal{E} + 2c^2 \text{ trace } \mathcal{A}.$$

PROOF. An elementary modification of that of lemma 7.2 of Serrin [7].

THEOREM 8. Let $u \in C_{2,1}(\bar{Q})$ be a solution of the Cauchy-Dirichlet problem (2), and suppose $|u| \leq m$ in Q. Then if (1) is m-regularly parabolic we have $|u_x| \leq C$ on S, where C depends only on c_0, c_1, c_2, c_3, m, K and Φ.

PROOF. Initially we suppose that the inequality

$$\text{(10)} \qquad \frac{1 + \text{trace}\,\mathcal{A} + |\mathcal{C}|}{\mathcal{E}} \leq \Phi(|p|) \qquad (p \neq 0)$$

holds for all u, not merely for $|u| \leq m$. For $0 \leq d \leq d_1 < d_0$ put

$$v(x, t) = \psi(x, t) + h(d),$$

where h is twice continuously differentiable and $h(0) = 0$, $h(d_1) = M$, $h'(d) \geq \alpha$,

α being a positive constant to be specified later. By lemma 3 we have for all $b > 0$,

$$\mathcal{L}(v + b) = \mathcal{F}h''/h'^2 - \mathcal{G}h' + \mathcal{A}\psi_{xx} - \mathcal{B} - \psi_t,$$

and since

$$\mathcal{G} = \sum_{i=1}^{n-1} \frac{\lambda_i \mathcal{A} \lambda_i}{1 - k_i d} k_i \geq \sum_{i=1}^{n-1} \lambda_i \mathcal{A} \lambda_i k_i \geq -K \sum_{i=1}^{n-1} \lambda_i \mathcal{A} \lambda_i \geq -K \operatorname{trace} \mathcal{A}$$

we obtain

$$\mathcal{L}(v + b) \leq \mathcal{F}h''/h'^2 + K \operatorname{trace} \mathcal{A} h' + \mathcal{A}\psi_{xx} - \mathcal{B} - \psi_t. \tag{11}$$

Next, we notice that

$$\mathcal{E}/\operatorname{trace} \mathcal{A} \geq \mathcal{E}/(1 + \operatorname{trace} \mathcal{A} + |\mathcal{C}|) \geq 1/\Phi \to \infty \text{ as } |p| \to \infty,$$

the last statement holding because the integral in (9) diverges. It follows that there exists a positive constant α_1 such that $\mathcal{E}/\operatorname{trace} \mathcal{A} \geq 8 c_1^2$ for $|p| \geq \alpha_1$.

Now choose $\alpha = \max(c_1 + \alpha_1, MK, M/d_0, 1)$. Since $p = p_0 + \nu h'$ and $|p_0| \leq c_1$ we have

$$|p| = |p_0 + \nu h'| \leq |p_0| + h' \leq c_1 + h' \leq 2h',$$

and

$$|p| \geq h' - |p_0| \geq h' - c_1 \geq \alpha_1.$$

Thus

$$|\mathcal{B}| = |p|\,|\mathcal{C}| \leq 2h'|\mathcal{C}|, \text{ and by lemma 3,}$$

$$\mathcal{F} \geq \frac{1}{2}\mathcal{E} - 2c_1^2 \operatorname{trace} \mathcal{A} \geq \frac{1}{4}\mathcal{E}.$$

Moreover,

$$|\mathcal{A}\psi_{xx}| \leq \|\psi_{xx}\| \operatorname{trace} \mathcal{A} \leq c_2 \operatorname{trace} \mathcal{A}.$$

If we use these inequalities in (11) we obtain

$$\begin{aligned}
\mathcal{L}(v + b) &\leq \mathcal{F}h''/h'^2 + h' K \operatorname{trace} \mathcal{A} + c_2 \operatorname{trace} \mathcal{A} + 2h'|\mathcal{C}| + c_3 \\
&\leq \mathcal{F}h' \left\{ h''/h'^3 + \frac{4}{\mathcal{E}} \left(K \operatorname{trace} \mathcal{A} + \frac{c_2 \operatorname{trace} \mathcal{A} + c_3}{\alpha} + 2|\mathcal{C}| \right) \right\} \\
&\leq \mathcal{F}h' \{ h''/h'^3 + c\Phi(|p|) \} \\
&\leq \mathcal{F}h' \{ h''/h'^3 + c\,\Phi(h' - c_1) \}
\end{aligned}$$

where $c = 4 \max\left(K + \dfrac{c_2}{\alpha}, \dfrac{c_3}{\alpha}, 2\right)$.

It remains to choose h in such a manner that $h''/h'^3 + c\Phi(h' - c_1) = 0$, which will ensure that $\mathcal{L}(v + b) \leq 0$. To do this we check, exactly as Serrin [7, p. 433] that the following parametric definition of h does what is required:

$$ch = \int_{\varrho}^{\beta} \frac{ds}{s^2\, \Phi(s - c_1)}, \qquad cd = \int_{\varrho}^{\beta} \frac{ds}{s^3\, \Phi(s - c_1)}, \qquad \text{for } \alpha \leq \varrho \leq \beta,$$

where β is defined by the relation $cM = \int_{\alpha}^{\beta} \frac{ds}{s^2\, \Phi(s - c_1)}$. Lemma 1 may now be invoked to show that $\frac{\partial u}{\partial n} \leq C$ on S. Since we may replace u by $-u$ in the equation without affecting the construction we conclude that $\frac{\partial u}{\partial n} \geq -C$ on S, which completes the proof of the theorem, subject to the assumption that (10) holds for all u. To remove this assumption we construct new functions $\widehat{\mathcal{A}}$ and $\widehat{\mathcal{B}}$ according to the prescription

$$\widehat{\mathcal{A}}(x, t, u, p) = \begin{cases} \mathcal{A}(x, t, -m, p) & \text{if } \quad u < -m, \\ \mathcal{A}(x, t, u, p) & \text{if } \quad |u| \leq m, \\ \mathcal{A}(x, t, m, p) & \text{if } \quad u > m, \end{cases}$$

with $\widehat{\mathcal{B}}$ similarly defined. Evidently u is a solution of the equation $\widehat{\mathcal{A}} u_{xx} - \widehat{\mathcal{B}} - u_t = 0$, and since $\frac{1 + \operatorname{trace} \widehat{\mathcal{A}} + |\widehat{\mathcal{C}}|}{\mathcal{E}} \leq \Phi(|p|)\ (p \neq 0)$, the preceding argument may be repeated. This concludes the proof of Theorem 8.

5. Interior estimates for $|u_x|$.

For these estimates the cases $n = 2$ and $n > 2$ seem to require separate treatment. Since, however, the details of the arguments are often quite similar to those of the corresponding elliptic theorems in [7] we shall in these cases present only the barest outline of the proofs, leaving the rest of the proof to the reader as a useful exercise.

We begin with a discussion of the case $n = 2$. For $p \neq 0$ set $\mathcal{E}^* = \mathcal{E}/|p|^2$, $\mathcal{D} = \frac{\mathcal{C} + u_t |p|^{-1}}{a - \mathcal{E}^*}$, $\mathcal{D}_x = (\partial \mathcal{D}/\partial x_1, \ldots, \partial \mathcal{D}/\partial x_n)$, $\mathcal{D}_u = \partial \mathcal{D}/\partial u$, $\dot{\mathcal{D}} = \sigma \mathcal{D}_x + |p| \mathcal{D}_u$, $\sigma = p/|p|$.

THEOREM 9. Suppose $\dot{\mathcal{D}} + \mathcal{D}^2 \geq 0$ for $|p| \geq C'$. Let $u \in C_{2,1}(\bar{\Omega})$ be a solution of (1) in Ω such that $u_x \in C_{2,1}(\Omega)$ and $|u_x| \leq C$ on Γ. Then $|u_x| \leq \leq \max(C, C')$ in Q.

PROOF. Put $w = |u_x|^2$, and let Q' be the open subset of Q on which $w \neq 0$. It can now be shown, by following closely the procedure used by Serrin in the proof of his Theorem 12.1, that in Q' the function w satisfies an equation of the form

$$\mathcal{A}\, w_{xx} - \mathcal{M}(x, t)\, w_x - 2\,(a - \mathcal{E}^*)\,(\dot{\mathcal{D}} + \mathcal{D}^2)\, w - w_t = 0.$$

Here the arguments of $\mathcal{A}$, $\mathcal{D}$ and $\mathcal{E}^*$ are x, t, u, u_x and u_t (for $\mathcal{D}$), while $\mathcal{M}$ is a continuous function on Q'. An application of Nirenberg's maximum principle now finishes the proof.

Theorem 9 has the effect of reducing step (C) to the problem of finding conditions on the structure of equation (1) under which it can be shown that $\dot{\mathcal{D}} + \mathcal{D}^2 \geq 0$, and it turns out that such conditions exist in situations compatible with (1) being regularly parabolic.

THEOREM 10. Suppose that $n = 2$ and that $\mathcal{E}^* \leq (1 - \mu)$ trace $\mathcal{A}$ for some positive constant μ. Suppose also that $\mathcal{B}, \mathcal{B}_x, \mathcal{E}_x, |p|^2 a = 0(\mathcal{E})$ and $\mathcal{B}_u, \mathcal{B}_p, \mathcal{E}_u, \mathcal{E}_p, |p|^2 a_u, |p|^2 a_p = 0(\mathcal{E}/|p|)$ as $|p| \to \infty$, uniformly on compact sets. Let $u \in C_{2,1}(\bar{\Omega})$ be a solution of (1) with $u_x \in C_{2,1}(Q)$ and satisfying $|u| \leq m$ in Q and $|u_x| \leq C$ on Γ. Then $|u_x| \leq M$ in Q, where M depends only on μ, m, C and bounds for the order terms above.

PROOF. Since $|u| \leq m$ there is no loss of generality in assuming that the order terms arising are bounded independently of u. We introduce a new dependent variable $\bar{u}$ by means of the transformation $u = \varphi(\bar{u})$, where $\varphi'(\bar{u}) > 0$. It is easy to see that $\bar{u}$ satisfies the equation

$$\bar{\mathcal{A}}\, \bar{u}_{xx} - \bar{\mathcal{B}} - \bar{u}_t = 0 \tag{12}$$

where $\bar{\mathcal{A}} = \mathcal{A}$, $\bar{\mathcal{B}} = (\mathcal{B} + \omega\,\mathcal{E})/\varphi'$, $\omega = -\varphi''/\varphi'^2$, and the arguments of $\mathcal{A}, \mathcal{B}$ and $\mathcal{E}$ are x, t, u and $p = \varphi'(\bar{u})\,\bar{p}$. Also,

$$\frac{\bar{\mathcal{B}} + \bar{u}_t}{|\bar{p}|} = \mathcal{C} + \frac{\omega}{|p|}\mathcal{E} + \frac{\bar{u}_t}{|\bar{p}|},\ \bar{\mathcal{E}}^* = \mathcal{E}^*, \text{ trace } \bar{\mathcal{A}} = \text{trace } \mathcal{A}.$$

It follows that if we write $\delta = \dfrac{\bar{p}}{|\bar{p}|}\dfrac{\partial}{\partial x} + |\bar{p}|\dfrac{\partial}{\partial \bar{u}} = \dfrac{p}{|p|}\dfrac{\partial}{\partial x} + |p|\left(\dfrac{\partial}{\partial u} - \omega p \dfrac{\partial}{\partial p}\right)$

we obtain

$$\dot{\bar{\mathcal{D}}} + \bar{\mathcal{D}}^2 = \delta\left(\frac{\mathcal{C} + \omega\mathcal{E}/|p| + \bar{u}_t/|\bar{p}|}{a - \mathcal{E}^*}\right) + \left(\frac{\mathcal{C} + \omega\mathcal{E}/|p| + \bar{u}_t/|\bar{p}|}{a - \mathcal{E}^*}\right)^2.$$

Hence $\dot{\bar{\mathcal{D}}} + \bar{\mathcal{D}}^2$ will be non-negative provided that

$$(13)\qquad I \equiv (a - \mathcal{E}^*)\,\delta(\mathcal{C} + \omega\mathcal{E}/|p| + \bar{u}_t/|\bar{p}|) - (\mathcal{C} + \omega\mathcal{E}/|p| + \bar{u}_t/|\bar{p}|)\,\delta(a - \mathcal{E}^*) + (\mathcal{C} + \omega\mathcal{E}/|p| + \bar{u}_t/|\bar{p}|)^2 \geq 0.$$

For simplicity set $I_1 = \mathcal{C} + \omega\mathcal{E}/|p| + \bar{u}_t/|\bar{p}|$. Then

$$(14)\qquad I = (a - \mathcal{E}^*)\,\delta(\mathcal{C} + \omega\mathcal{E}/|p|) - I_1\,\delta(a - \mathcal{E}^*) + I_1^2 \geq (a - \mathcal{E}^*)\,\delta(\mathcal{C} + \omega\mathcal{E}/|p|) - \frac{1}{4}\{\delta(a - \mathcal{E}^*)\}^2.$$

Now assume $\omega > 0$. Then

$$\delta\mathcal{C} \geq -C_1(|p|^{-1} + \omega)\,\mathcal{E};\ \delta\mathcal{E} \leq C_1(1 + \omega|p|)\,\mathcal{E}$$

and, as $\delta\omega = |p|\,(\omega'/\varphi')$:

$$\delta(\omega\mathcal{E}/|p|) \geq \left\{\frac{\omega'}{\varphi'} + \omega^2 - C_1(1 + \omega)(|p|^{-1} + \omega)\right\}$$

for $|p| \geq \alpha_1$ say. For convenience C_1 will denote any constant, unless an especial one is needed. Hence for $|p| \geq \alpha_1$,

$$\delta(\mathcal{C} + \omega\mathcal{E}/|p|) \geq \left\{\frac{\omega'}{\varphi'} + \omega^2 - C_1(\omega^2 + \omega + |p|^{-1})\right\}\mathcal{E}.$$

Moreover

$$\delta\mathcal{E}^* \leq C_1(|p|^{-1} + \omega)\frac{\mathcal{E}}{|p|};\quad \delta a \leq C_1(|p|^{-1} + \omega)\frac{\mathcal{E}}{|p|}$$

for $|p| \geq \alpha_2$ say, so that

$$|\delta(a - \mathcal{E}^*)|^2 \leq C_1(\omega^2 + \omega + |p|^{-1})\frac{\mathcal{E}^2}{|p|^2} \leq C_1(1 - \mu)\,a\mathcal{E}(\omega^2 + \omega + |p|^{-1}).$$

We finally obtain

$$(15)\qquad I \geq (a - \mathcal{E}^*)\,\mathcal{E}\left\{\frac{\omega'}{\varphi'} + \omega^2 - C_1(\omega^2 + \omega + |p|^{-1})\right\} - C_1(1 - \mu)\,a\mathcal{E}(\omega^2 + \omega + |p|^{-1}),$$

so that if we can choose φ in such a way that $\omega > 0$ and

$$\frac{\omega'}{\varphi'} + \omega^2 - C_2 (\omega^2 + \omega + |p|^{-1}) \geq 0 \qquad (C_2 = C_1/\mu)$$

for $|p|$ sufficiently large, we have shown, remembering $a - \mathcal{E}^* \geq \mu a$, that $I \geq 0$. To effect such a choice we define φ by means of the inverse relation

$$\bar{u} = \int_{-2m-1}^{\varphi-m-1} \frac{dt}{(1 - e^{2C_2 t})^{1/2 C_2}} .$$

Use of this relation in (15) gives

$$I \geq C_2 \mu \, \mathcal{E} \, a (\omega^2 + \omega - |p|^{-1}) \geq C_2 \, \mu \, \mathcal{E} \, a \, (e^{2C_2(\varphi-m-1)} - |p|^{-1}).$$

Thus $I \geq 0$ provided $|p| \geq \max(\alpha_1, \alpha_2, 1, e^{2C_2(2m+1)}) = C'$ say.

We have therefore shown that $\dot{\bar{\mathcal{D}}} + \bar{\mathcal{D}}^2 \geq 0$ provided $|p| \geq C'$; that is, provided $|\bar{p}| \geq C'/\min \varphi'$. By Theorem 9, $|\bar{u}_x| \leq \max(C, C')/\min \varphi'$, and so

$$|u_x| \leq \frac{\max \varphi'}{\min \varphi'} \max(C, C') < 2 \max(\alpha_1, \alpha_2, e^{4C_2(1+m)}, C) = M.$$

The proof is complete.

For $n > 2$ further restrictions on the equation appear to be necessary, and it proves possible to deal with equations for which $\mathcal{A}$ can be written as

$$(16) \qquad \mathcal{A}(x, t, u, p) = \mathcal{G}(x, t, u, p) \, \mathcal{A}'(p) + \mathcal{G}_1(x, t, u, p) \, pp$$

where $\mathcal{G} (> 0)$ and $\mathcal{G}_1$ are real-valued functions, and $\mathcal{A}'$ is a positive definite matrix with unit trace. Given that (16) holds we write $\mathcal{S} = (\mathcal{B} + u_t)/(|p| \mathcal{G})$ $(p \neq 0)$, and define $\dot{\mathcal{S}}$ just as $\dot{\mathcal{D}}$ was defined earlier. The argument from this point on is close to that given for $n = 2$, with $\mathcal{D}$ replaced by $\mathcal{S}$: that is, it can be shown that step (C) may be carried out if $\dot{\mathcal{S}} + \mathcal{S}^2 \geq 0$ for large enough $|p|$, and conditions under which this inequality holds are given in terms of the structure of the equation. The precise results are as follows:

THEOREM 11. Suppose that (16) holds and that $\dot{\mathcal{S}} + \mathcal{S}^2 \geq 0$ for $|p| \geq C'$. Let $u \in C_{2,1}(\bar{Q})$ be such that $u_x \in C_{2,1}(Q)$ and $|u_x| \leq C$ on Γ. Then $|u_x| \leq \max(C, C')$ in Q.

PROOF. As in the proof of Theorem 9 we put $w = |u_x|^2$ and let Q' be the open subset of Q on which $w \neq 0$. A routine adaptation of the proof of Theorem 13.1 of Serrin [7] enables us to show that in Q',

$$\mathcal{A}\, w_{xx} - 2\,\mathcal{G}(\dot{\mathcal{S}} + \mathcal{S}^2)\, w - \mathcal{N}(x,t)\, w_x - w_t \geq 0.$$

Here the arguments of $\mathcal{A}$, $\mathcal{G}$ and $\mathcal{S}$ are x, t, u, u_x and $\mathcal{N}$ is a continuous vector-valued function on Q'. The rest of the proof follows immediately from Nirenberg's maximum principle.

The analogue of Theorem 10 is obtained under the assumption that (16) holds in a sharper form. More precisely, we require that $\mathcal{A}$ be of the form

$$\mathcal{A}(x,t,u,p) = \mathcal{G}(x,t,u,p)\,\mathcal{A}'(\sigma) + \mathcal{G}_1(x,t,u,p)\,pp, \tag{17}$$

where $\sigma = p/|p|$, $\mathcal{G}(>0)$ and $\mathcal{G}_1$ are real-valued functions, and $\mathcal{A}'$ is a positive definite matrix with unit trace.

THEOREM 12. Suppose that (17) holds. Assume that $n > 2$ but otherwise let the hypothesis of Theorem 10 be satisfied. Then the conclusion of that theorem also holds.

PROOF. As in the proof of Theorem 10 we introduce a transformation $u = \varphi(\bar{u})$, so that $\bar{u}$ satisfies the equation

$$\bar{\mathcal{A}}\,\bar{u}_{xx} - \bar{\mathcal{B}} - \bar{u}_t = 0, \tag{18}$$

where

$$\bar{\mathcal{A}} = \mathcal{A} = \bar{\mathcal{G}}(x,t,\bar{u},\bar{p})\,\mathcal{A}'(\bar{\sigma}) + \bar{\mathcal{G}}_1(x,t,\bar{u},\bar{p})\,\varphi'^2\,\bar{p}\,\bar{p},$$

and

$$\bar{\mathcal{G}}(x,t,\bar{u},\bar{p}) = \mathcal{G}(x,t,\varphi(\bar{u}),\varphi'\bar{p}),\quad \bar{\mathcal{G}}_1(x,t,\bar{u},\bar{p}) = \mathcal{G}_1(x,t,\varphi(\bar{u}),\varphi'\bar{p}),$$

$$\bar{\mathcal{B}} = (\mathcal{B} + \omega\,\mathcal{E})/\varphi'.$$

Theorem 11 may be applied to equation (18) since $\bar{\mathcal{A}}$ has the required form, and so we have to express $\bar{\mathcal{S}} = \dfrac{1}{\mathcal{G}}\left(\mathcal{C} + \dfrac{\omega}{|p|}\mathcal{E} + \dfrac{\bar{u}_t}{|\bar{p}|}\right)$ in as convenient a way as possible. If we take the trace of (17) we obtain

$$\operatorname{trace}\mathcal{A} = \mathcal{G} + |p|^2\,\mathcal{G}_1\,.$$

Moreover,

$$\mathcal{E}^* = \sigma\,\mathcal{A}\,\sigma = \mathcal{G}(\sigma\,\mathcal{A}'\,\sigma) + \mathcal{G}_1\,|p|^2.$$

Thus if we eliminate $\mathcal{G}_1$ from these equations we find that

$$\mathcal{G} = \frac{a - \mathcal{E}^*}{1 - \mathcal{E}'^*}, \quad \text{where } \mathcal{E}'^* = \sigma \mathcal{A}' \sigma.$$

It follows that $\bar{\mathcal{S}} = (1 - \mathcal{E}'^*) \bar{\mathcal{D}}$, and so

$$\dot{\bar{\mathcal{S}}} + \bar{\mathcal{S}}^2 = (1 - \mathcal{E}'^*) \dot{\bar{\mathcal{D}}} + (1 - \mathcal{E}'^*)^2 \bar{\mathcal{D}}^2.$$

Since $\mathcal{E}'^*$ is bounded away from 1 it is easy to modify the proof of Theorem 10 to cope with this new situation : we omit the entirely routine details.

6. Estimates for $|u_t|$ on $\Omega \times \{0\}$.

This estimate is by far the easiest to obtain.

THEOREM 13. Let $u \in C_{2,1}(\bar{Q})$ be a solution of (2), and suppose $|u| \leq m$. Then $|u_t| \leq C$ on $\Omega_0 = \Omega \times \{0\}$, where C depends only on ψ and m.

PROOF. Set $v(x, t) = \psi(x, t) + \alpha t$, where α is a positive constant to be chosen later. Define $\widehat{\mathcal{A}}$, $\widehat{\mathcal{B}}$ as in § 4 : evidently u is a solution of $\widehat{\mathcal{L}} u = 0$ in Q, $u = \psi$ on Γ. For all positive constants b,

$$\widehat{\mathcal{L}}(v + b) = \widehat{\mathcal{A}}(x, t, v + b, \psi_x) \psi_{xx} - \widehat{\mathcal{B}}(x, t, v + b, \psi_x) - \psi_t - \alpha \leq C_1 - \alpha$$

in Q, where C_1 is independent of α and b, and depends only on ψ and m (and Q, of course). Choose $\alpha = C_1$, so that $\widehat{\mathcal{L}}(v + b) \leq 0$ in Q for all $b > 0$. Moreover, $u \leq v$ on Γ. It follows from Theorem 3 (note that this is applicable to $\widehat{\mathcal{L}}$) that $u \leq v$ in $\bar{Q}$. Hence $u_t \leq v_t \leq c_3 + \alpha$ on Ω_0. If we replace u by $-u$ in the equation and thus obtain a lower bound for u_t, the proof becomes complete.

7. Existence theorems.

Now that steps (A)-(D) have been carried out, under certain conditions, we can give various theorems asserting the existence of a solution of the Cauchy-Dirichlet problem.

THEOREM 14. Let $\Omega \subset R^n$ be a bounded domain with boundary of class C_4. Assume either $n = 2$ or that (17) holds. Suppose there is a positive constant μ such that

$$(1 + a)\,\mu\,|p| \leq \mathcal{E} \leq (1 - \mu)\,|p|^2\, a$$

for all large enough $|p|$, and that as $|p| \to \infty$,

$$|p|^2\, a_x\,,\ \mathcal{B},\ \mathcal{B}_x\,,\ \mathcal{E}_x = 0\,(\mathcal{E}) \text{ and } \mathcal{B}_u\,,\ \mathcal{B}_p\,,\ \mathcal{E}_u\,,\ \mathcal{E}_p\,,\ |p|^2 a_u\,,\ |p|^2\, a_p = 0\,(\mathcal{E}/|p|\,).$$

Lastly suppose that $\mathcal{B}$ satisfies the hypotheses of Theorem 4. Then the Cauchy-Dirichlet problem (2) is soluble for arbitrarily given data ψ such that $\psi \in C_{2+1}(\overline{Q})$ and $\psi_x \in C_{2+1}(Q)$.

PROOF. The main object is to arrange matters so that the basic theorem 1 may applied. Thus let $v \in C_{2,1}(\overline{Q})$ be such that for some τ in $[0, 1]$,

$$\tau\,\mathcal{A}(x, t, v, v_x)\, v_{xx} + (1 - \tau)\,\Delta v - \tau\,\mathcal{B}(x, t, v, v_x) + \\ + (1 - \tau)(\psi_t - \Delta\psi) - v_t = 0 \text{ in } Q \tag{19}$$

and $v = \psi$ on Γ. Clearly $|v| \leq c_0$ on Γ: to obtain a global bound independent of τ for v we have to obtain a maximum principle for solutions of (19). This follows just as Theorem 4 follows from Theorem 3, save that we take $\omega = e^{\lambda t} \max(m, M)$ for an appropriately large and positive λ.

To obtain bounds for $|v_x|$ we first note that equation (19) is regularly parabolic: for all $\tau \in [0, 1]$,

$$\frac{1 + tr\,\mathcal{A}_\tau + |\mathcal{C}_\tau|}{\mathcal{E}_\tau} \leq \frac{1 + \tau\, tr\,\mathcal{A} + (1 - \tau)\, n + \tau\,|\mathcal{C}| + (1 - \tau)\, C}{\tau\,\mathcal{E} + (1 - \tau)\,|p|^2}$$

$$\leq \frac{1 + tr\,\mathcal{A} + |\mathcal{C}|}{\mathcal{E}} + \frac{C}{|p|^2}$$

$$\leq \frac{C}{|p|} + \frac{C}{|p|^2}$$

for large enough $|p|$, and $\int_1^\infty \frac{d\varrho}{\varrho + 1} = \infty$. It follows that $|v_x| \leq m_1$ on S, where m_1 is independent of τ.

We must now estimate $|v_x|$ throughout Q. Since v belongs to $C_{2,1}(\overline{Q})$, it follows from Lemma 2 on p. 276 of Ladyzhenskaya and Ural'tseva [4] that v_x satisfies a Hölder condition in t with exponent $\frac{1}{2}$. Hence the coefficients in (19), regarded as functions of (x, t), are in $C_\alpha(\overline{Q})$ for some α, $0 < \alpha \leq 1$, so that by linear theory ([3], p. 65), $v \in C_{2+\alpha}(\overline{Q})$. This now implies that the cofficients of (19) have x-derivatives in $C_\alpha(\overline{Q})$, and a second application of linear theory ([3], p. 72) enables us to conclude that $v_x \in C_{2+\alpha}(Q)$. From Theorem 10 it follows that $|v_x| \leq m_2$ in Q, where m_2 is independent of τ.

Conditions are thus right for Theorem 1 to be applied, and this completes the proof.

REMARKS. 1) The condition $(1 + a)\,\mu\,|p| \leq \mathcal{E}$ is made simply to ensure that equation (1) is regularly parabolic in the strong sense, and the theorem is true if this condition is replaced by that of requiring (1) to be regularly parabolic in the strong sense. It is not clear whether we may merely require (1) to be regularly parabolic, since (19) may then fail to be regularly parabolic.

2) A variant of Theorem 14 may be obtained by appealing to Theorem 6 rather than to Theorem 4: more precisely, we replace the condition on $\mathcal{B}$ required by Theorem 4 by those on $\mathcal{B}$ and $\mathcal{E}$ necessitated by Theorem 6, with $\varphi(\varrho) = C/\varrho$ for large ϱ. In the proof we would then have to show that the hypotheses of Theorem 6 were satisfied by the invariants $\mathcal{E}_\tau$ and $\mathcal{B}_\tau$ which correspond to equation (19). However, it is clear that

$$|\mathcal{B}_\tau|/\mathcal{E}_\tau \leq \frac{\tau\,|\mathcal{B}| + (1-\tau)\,C}{\tau\,\mathcal{E} + (1-\tau)\,\varrho^2} \leq \frac{|\mathcal{B}|}{\mathcal{E}} + \frac{C}{\varrho^2}$$

$$\leq C(\varrho + 1)/\varrho^2 \qquad (p = \varrho v),$$

and since $\int_0^\infty \frac{d\varrho}{\varrho + 1} = \infty$, Theorem 6 can indeed be applied to give $|v| \leq m$ in $\overline{Q}$, where m is a constant independent of τ.

The assumption that $\mathcal{B}$ and $\mathcal{E}$ satisfy the hypotheses of Theorem 6, with $\varphi(\varrho) = C/\varrho$ for large ϱ. is made so as to be able to obtain a maximum principle for solutions of (19): if one wishes to use Theorem 6 it is clear that the demand that $\varphi(\varrho)$ be eventually equal to C/ϱ may be replaced by the requirement that $\int_0^\infty \frac{d\varrho}{C + \varphi(\varrho)\,\varrho^2} = \infty$ for every positive constant C.

Remark 2 makes it plain that the use of the homotopy family of equations necessitated by appealing to Theorem 1 is a source of some difficulty in that it is sometimes necessary to impose extra conditions on the structure of equation (1) merely to ensure that equation (19) has an equally pleasant structure. This difficulty is, of course, removed if instead of using Theorem 1 we choose to employ Theorem 2, with its much simpler homotopy. We pay a penalty for doing this, however, as the data is then much more restricted than hitherto. The kind of theorem which is obtainable by adopting this procedure is the following:

THEOREM 15. Suppose the hypotheses of Theorem 14 hold, save for the final condition concerning $\mathcal{B}$. Let $\mathcal{B}$ and $\mathcal{E}$ satisfy the hypotheses of either Theorem 4 or Theorem 6. Assume further that the given data ψ is of class $C_{2+1}(\overline{Q})$ and satisfies $\psi = 0$, $\psi_x = 0$ and $\mathcal{A}(x, 0, 0, 0)\, \psi_{xx} - \mathcal{B}(x, 0, 0, 0) - \psi_t = 0$ on $\partial\Omega \times \{0\}$. Then the Cauchy-Dirichlet problem (2) has a solution.

8. Non-existence Theorems.

For certain classes of non-regularly parabolic equations it is possible to construct smooth data for which no solution of the Cauchy-Dirichlet problem is possible. We give one such class, analogous to the irregularly elliptic equations of Serrin.

Equation (1) is said to be *irregularly parabolic* if

$$|\mathcal{C}|/a \to \infty \text{ as } |p| \to \infty, \text{ uniformly in } u, \tag{20}$$

and

$$\frac{|\mathcal{C}|}{\mathcal{E}} \geq \Psi(|p|) \text{ for } |u| \geq M,\ |p| \geq l, \tag{21}$$

where l, M are positive constants, and Ψ is a positive continuous function such that

$$\int^{\infty} \frac{d\varrho}{\varrho^2\, \Psi(\varrho)} < \infty.$$

For example, if $\mathcal{E} \geq \mu\, |p|\, a$ when $|p| \geq 1$, equation (1) is irregularly parabolic if

$$|\mathcal{B}|/\mathcal{E} \geq (\log |p|)^{1+\theta} \text{ for large } u \text{ and } |p|,$$

where θ is a positive constant. Hence a uniformly parabolic equation is irregularly parabolic if

$$|\mathcal{B}| \geq |p|^2 (\log |p|)^{1+\theta} \text{ for large } u \text{ and } |p|:$$

the equation $\Delta u - |u_x|^\beta - u_t = 0$ is thus irregularly parabolic if $\beta > 2$.

To obtain our non-existence theorem we need a variant of the maximum principle analogous to that introduced by Serrin for a similar purpose in the elliptic situation [7, p. 459]. Let $x \in \partial\Omega$, and suppose N is a neighbourhood of x in R^n: the set $\partial\Omega \cap N$ is called a boundary neighbourhood of x, and a subset of $\partial\Omega$ is called open if it contains a boundary neighbourhood of each of its points. In the following theorem $\partial\Omega_1$ will stand for an open subset of $\partial\Omega$, and we shall write $S_1 = \partial\Omega_1 \times (0, T)$.

THEOREM 16. Let $\omega \in C_{2,1}(Q \cup Q_T) \cap C(\bar{Q})$ be such that $\mathcal{L}(\omega + b) \leq 0$ in $Q \cup Q_T$ for all positive constants b, and let $\dfrac{\partial \omega}{\partial n} = -\infty$ at each point of S_1: here n denotes the normal into Q. Suppose $u \in C_{2,1}(Q \cup Q_T \cup S_1) \cap C(\bar{Q})$ is a solution of (1) in Q such that $u \leq \omega$ on $\Gamma - S_1$. Then $u \leq \omega$ in $\bar{Q}$.

PROOF. Suppose the theorem were false. Then $v = u - \omega$ must become positive on S_1, by Theorem 3. But on S_1, $\dfrac{\partial v}{\partial n} = \dfrac{\partial u}{\partial n} - \dfrac{\partial \omega}{\partial n} = \infty$, since $u \in C_{2,1}(Q \cup Q_T \cup S_1)$. This gives a contradiction, and establishes the theorem.

The promised non-existence theorem follows.

THEOREM 17. Let Ω be a bounded domain in R^n. Let equation (1) be irregularly parabolic, and suppose $\mathcal{B}(x, 0, 0, 0) = 0$ for $x \in \partial\Omega$. Then there exists C_∞ data, satisfying the consistency condition, such that the Cauchy-Dirichlet problem (2) has no solution.

PROOF. We shall assume, without loss of generality that (20) and (21) hold with the absolute value signs removed from $\mathcal{C}$. Let P be a point of $\partial\Omega$ at which there is an internally touching sphere $\Sigma \subset R^n$. Let the radius of Σ be 2β, let the diameter of Ω be δ, and denote distance from P by r. Put $\{P\} \times (0, T) = S_P$.

Consider the function ω defined by

$$\omega(x, t) = M + h(r), \quad \beta \leq r \leq \delta,$$

where $h \in C_2$ for $\beta < r \leq \delta$, $h'(\beta) = -\infty$, $h(\delta) = 0$, $h'(r) \leq -l$, and l, M are the quantities arising in (21). Then for any positive constant b,

$$\begin{aligned}\mathcal{L}(\omega + b) &= \mathcal{A}(x, t, \omega + b, \omega_x)\,\omega_{xx} - \mathcal{B}(x, t, \omega + b, \omega_x)\\ &\leq \mathcal{C}(x, t, \omega + b, \omega_x)\,h''/h'^2 - \mathcal{B}(x, t, \omega + b, \omega_x)\\ &\leq \mathcal{C}h'\,\{h''/h'^3 + \Psi(|h'|)\},\end{aligned}$$

by (21). As in Serrin [7], h may be chosen so that $\mathcal{L}(\omega + b) \leq 0$: we give the details briefly for convenience. Put $C = \max\left(1, (\delta - \beta)/\int_l^\infty \frac{d\varrho}{\varrho^3 \Psi(\varrho)}\right)$, and define a constant $\alpha \geq l$ by

$$\int_\alpha^\infty \frac{d\varrho}{\varrho^3 \Psi(\varrho)} = \delta - \beta \text{ if } C = 1, \alpha = l \text{ if } C > 1.$$

Now let h and r be parametrically related by

$$h = C\int_\alpha^\varrho \frac{d\varrho}{\varrho^3 \Psi(\varrho)}, \quad r = \beta + C\int_\varrho^\infty \frac{d\varrho}{\varrho^3 \Psi(\varrho)}, \quad \text{where } \alpha \leq \varrho \leq \infty.$$

It is easy to verify that $h' = -\varrho \leq -l, h'(\beta) = -\infty, h(\delta) = 0$, and

$$\frac{h''}{h'^3} + \Psi(|h'|) = \frac{C-1}{C}\Psi(|h'|) \geq 0.$$

Hence $\mathcal{L}(\omega + b) \leq 0$.

Next we apply Theorem 16 to the region $Q \cap \{(x, t) : r = d(x, P) > \beta, 0 < t < T\} = Q \cap C_\beta = Q_\beta$ say. Let $S_\beta = S \cap C_\beta$, and let $\sup_{\Gamma - S_\beta} u = m$. Then by Theorem 16,

$$u \leq \max(m, M) + h(r) \leq \max(m, M) + h(\beta)$$

$$\leq \max(m, M) + C\int_l^\infty \frac{d\varrho}{\varrho^2 \Psi(\varrho)} = m^*$$

in Q_β. To obtain an estimate for u at (P, t) we proceed as follows. Let $0 < \varepsilon < \beta$ and let s denote distance (in R^n) from the centre of Σ. Define a function ω_1 by

$$\omega_1(x, t) = m^* + h_1(s), \beta \leq s \leq 2\beta - \varepsilon$$

where $h_1 \in C_2$ for $\beta \leq s < 2\beta - \varepsilon$ and $h_1'(2\beta - \varepsilon) = \infty$, $h_1(\beta) = 0$, $h_1'(s) \geq l^* \geq 0$. For any positive constant b,

$$\mathcal{L}(\omega_1 + b) \leq \mathcal{E} h_1' \left\{ h_1''/h_1'^3 + \frac{a\beta^{-1} - \mathcal{C}}{\mathcal{E}} \right\}.$$

Since $\mathcal{C}/a \to \infty$ as $|p| \to \infty$, there exists a positive constant l_1 such that $\mathcal{C}/a \geq 2^{-1}$ for $|p| \geq l_1$. Using (21) we finally obtain

$$\mathcal{L}(\omega_1 + b) \leq \mathcal{E}h_1' \left\{h_1''/h_1'^3 - \frac{1}{2}\,\Psi(h_1')\right\}$$

if we take $l^* = \max(l, l_1)$. Just as in the first part of the proof we may choose h_1 in such a way that $\mathcal{L}(\omega_1 + b) \leq 0$. Now apply Theorem 16 to the domain $Q^\varepsilon = \{(x, t): r < \beta, 0 < t < T\} \cap \{(x, t): s < 2\beta - \varepsilon, 0 < t < T\}$. This gives

$$u \leq m^* + C^* \int_{l^*}^{\infty} \frac{d\varrho}{\varrho^2\,\Psi(\varrho)} \equiv m^* + C_1^* \text{ in } Q^\varepsilon,$$

where $C^* = \max\left(2, \beta / \int_{l^*}^{\infty} \frac{d\varrho}{\varrho^3\,\Psi(\varrho)}\right)$. If we let $\varepsilon \to 0$ and use the continuity of u we obtain the same inequality at (P, t) for $0 < t < T$.

To construct data ψ which gives rise to a contradiction we merely need to take a C_∞ function ψ such that $\psi = 0$ on $\Gamma \cap \{(x, t): x \in \overline{Q}, 0 \leq \leq t < \eta < T\}$, $\psi = 0$ on $S \cap \{(x, t): r \geq \beta, 0 < t < T\}$, $\psi(P, t) > m^* + C_1^*$ for some $t, \eta < t < T$.

Remark. The condition $\mathcal{B}(x, 0, 0, 0) = 0$ for $x \in \partial\Omega$ is imposed so that the C_∞ data ψ which is constructed above will satisfy the consistency condition. This condition can be relinquished if we are prepared to be satisfied with somewhat less smooth data.

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Mathematics Division,
University of Sussex.