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TRIGONOMETRIC SUMS ASSOCIATED WITH PSEUDO-MEASURES

J. BENEDETTO

The purpose of this paper is to study the structure of pseudo-measures on closed sets $E \subseteq \Gamma \equiv \mathbb{R}/2\pi\mathbb{Z}$ of Lebesgue measure $m(E) = 0$. In particular, we find various conditions on E so that a given pseudo-measure T supported by E is actually a measure or, at least, the first derivative of a bounded function. Such questions are obviously related to the open problem of determining if, generally, a Helson is a set of spectral synthesis.

$A(\Gamma)$ is the space of absolutely convergent Fourier series $\varphi(\gamma) = \sum a_n e^{in\gamma}$ normed by $\|\varphi\|_A = \sum |a_n|$; and its dual is the space of pseudo-measures $A'(\Gamma)$.

$A'(E)$ (resp., $M(E)$) is the space of pseudo-measures (resp., measures) T supported by E such that the Fourier coefficient $\widehat{T}(0) = 0$. We let $D_b(E)$ be those first order distributions T supported by E such that T is the first derivative of an L^∞ function and $\widehat{T}(0) = 0$; the bounded pseudo-measures on E are $A'_b(E) \equiv D_b(E) \cap A'(E)$.

§ 1 is devoted to notation and the statement of some formulas for pseudo measures in terms of trigonometric sums; these results are basic to what follows. Next (§ 2), we characterize a useful subspace of $M(E)$ by a Stone-Weierstrass argument. In § 3, utilizing a summability technique, we prove that certain natural subspaces of $D_b(E)$ always contain non-pseudo-measures for infinite E . Then, with a metric hypothesis on E , we derive an estimate which is useful in characterizing those E for which $A'(E) = A'_b(E)$; metric conditions are generally not sufficient to establish the boundedness of $A'(E)$ -this is where we need arithmetic conditions. We give a functional analysis argument in § 4 to show the existence of a class of functions in $A(\Gamma)$ without any sort of local finite variation; and we use

this to examine subspaces of $A(\Gamma)$ on which measures with finite support approximate pseudo-measures. In § 5, after noting that unbounded pseudo-measures exist on countable sets with a single limit point, we give a technique which shows how arithmetic conditions on E lead to $A'(E) = A'_b(E)$. The results here are preliminary. Finally (§ 6), we illustrate that properties of Helson sets lead to a large class of topologies of summability type.

I would like to thank Mr. Gordon Woodward for his helpful advice.

1. Notation and Formulas for Pseudo-Measure.

We designate the complement of E by $CE \equiv \bigcup_0 I_j$ where, with $E \subseteq [0, 2\pi)$, $I_j \equiv (\lambda_j, \gamma_j) \subseteq [0, 2\pi)$ is an open interval of length ε_j ; since $m(E) = 0$, $\sum_0 \varepsilon_j = 2\pi$. For convenience, we set

$$c_{j,n}^{\pm} \equiv e^{\pm i\lambda_j n} - e^{\pm i\gamma_j n},$$

$$d_{j,n}^{\pm} \equiv c_{j+1,n}^{\pm} - c_{j,n}^{\pm};$$

generally, we drop the « + » in this notation.

Let $D_b(\Gamma)$ be the space of first order distributions T such that $T = f'$, $f \in L^\infty(\Gamma)$, and let $A'_b(\Gamma) = D_b(\Gamma) \cap A(\Gamma)$. Also define $D_1(E)$ to be those first order distributions T for which $T = f'$, $f \in L^1(\Gamma)$, $\text{supp } T \subseteq E$, and $\widehat{T}(0) = 0$. It is easy to see that $f = \sum_1 k_j \chi_{I_j}$ and, as such, we generally write $T \circ k_j$ for an element of $D_1(E)$. Besides the spaces indicated in the introduction we consider the following subspaces of $D_1(E)$:

$$D_\omega(E) \equiv \{T \circ k_j : f \in L^p(\Gamma) \text{ for all } p < \infty\},$$

$$A'_S(E) \equiv \{T \in A'(E) : \varphi = 0 \text{ on } E, \varphi \in A(\Gamma), \text{ implies } \langle T, \varphi \rangle = 0\},$$

$$M_d(E) \equiv \{T \in M(E) : T \text{ is discrete}\},$$

$$G(E) \equiv \{T \circ k_j \in D_b(E) : f(\gamma \pm) \text{ exists for all } \gamma \in \Gamma$$

and f has at most countably many jump discontinuities\}.

We multiply [2] $S \circ k_j$, $T \circ h_j \in D_\omega(E)$ by

$$ST = (\sum_1 k_j h_j \chi_{I_j})'.$$

Now, E is *Helson* if $A'_S(E) = M(E)$, *spectral synthesis* (S) if $A'(E) = A'_S(E)$, and *strong spectral resolution* if $A'(E) = M(E)$. E is a *Dirichlet*

set if

$$\lim_{n \rightarrow \infty} \sup_{\gamma \in E} |1 - e^{in\gamma}| = 0.$$

We set $A(E)$ to be the restrictions of $A(\Gamma)$ to E ; and $A_+(E)$ to be the restrictions of absolutely convergent Taylor series to E .

Next, let \mathcal{F} be the compact open sets of E so that $T \in D_1(E)$ is a finitely additive set function on $\mathcal{F}[1; 2]$; we norm such a T by

$$\|T\|_v \equiv \sup_{F \in \mathcal{F}} |T(F)|.$$

Also we write $I_j \leq I_k$ if, for $E \subseteq [0, 2\pi)$, $\lambda_j < \lambda_k$; $I_{j_1} \leq \dots \leq I_{j_m}$ is a partition P .

For detailed proofs of the following, as well as similar results, we refer to [3, § 2].

PROPOSITION 1.1. For all $T \in A'(E)$ there is $f \in L^p(\Gamma)$, for each $p < \infty$, such that $f = \sum_1 k_j \chi_{I_j}$ a. e., $\sum e^{\delta |k_j|} \varepsilon_j < \infty$ for some $\delta > 0$, and

$$(1.1) \quad c_n \equiv \widehat{T}(n) = \frac{1}{2\pi} \sum_{j=1} k_j c_{j,n}^-.$$

PROPOSITION 1.2. For all $T \in A'(E)$, $T \infty k_j$ and $c_n \equiv \widehat{T}(n)$,

$$k_j = \frac{1}{\varepsilon_j} \sum'_n \frac{c_n}{n^2} c_{j,n}.$$

PROOF. We have

$$(1.2) \quad f(\gamma) \equiv \sum_1 k_m \chi_{I_m}(\gamma) = \sum'_n \frac{c_n}{in} e^{in\gamma} \quad \text{a. e.}$$

Since $f \in L^1(\Gamma)$ and since Fourier series can be integrated term by term, we integrate both sides of (1.2) over I_j . Thus

$$k_j \varepsilon_j = \sum'_n \frac{c_n}{in} \int_{\lambda_j}^{\gamma_j} e^{in\gamma} d\gamma = \sum'_n \frac{c_n}{n^2} c_{j,n}. \quad \text{q.e.d.}$$

PROPOSITION 1.3. If $T \infty k_j \in A'(E)$ and the partial sums $\sum_1^n k_j$ are bounded, then

$$(1.3) \quad 2\pi \widehat{T}(n) = - \sum_{j=1}^{\infty} \left(\sum_{p=1}^j k_p \right) d_{j,n}^-.$$

PROOF. By Abel's formula

$$\sum_{j=1}^J k_j c_{j,n}^- = \left(\sum_1^J k_j \right) c_{J+1,n}^- - \sum_{j=1}^J \left(\sum_{p=1}^j k_p \right) d_{j,n}^-.$$

Thus we have (1.3) by (1.1) and since $\lim_J c_{J+1,n}^- = 0$. q.e.d.

2. Structure of Measures.

Let us first recall [1,2] some characterization of $T \circ k_j \in M(E)$ in terms of k_j . Let $T \circ k_j \in D_1(E)$; $T \in M(E)$ if and only if any of the following equivalent conditions hold :

- i. $\| T \|_v < \infty$.
- ii. $\sup_p \sum_1^{m-1} |k_{n_{j+1}} - k_{n_j}| < \infty$.
- iii. There is M such that if $I_{n_1} \leq \dots \leq I_{n_{2m}}$ then

$$\left| \sum_{j=1}^m (k_{n_{2j-1}} - k_{n_{2j}}) \right| < M.$$

PROPOSITION 2.1. Let $\{k_j\}$ be bounded by a constant C and assume

$$\sum_1 |k_{j+1} - k_j| \equiv K < \infty.$$

Then $T \circ k_j \in M(E)$.

PROOF. Let

$$X \equiv \{ \varphi \in C^1(\Gamma) : \sum | \varphi(\lambda_j) |, \sum | \varphi(\gamma_j) | < \| \varphi \|_\infty \}.$$

X is a subalgebra of $C(\Gamma)$ which satisfies the conditions of the Stone-Weierstrass theorem.

Thus, $\bar{X} = C(\Gamma)$.

For each $\varphi \in X$, we have, for $a_j \equiv \varphi(\lambda_j) - \varphi(\gamma_j)$,

$$\left| \sum_{j=1}^J k_j a_j \right| = \left| \left(\sum_1^J a_j \right) k_{J+1} - \sum_{j=1}^J \left(\sum_{p=1}^j a_p \right) (k_{j+1} - k_j) \right| \leq$$

$$C \sum_{j=1}^J |a_j| + \sum_{j=1}^J \left(\sum_{p=1}^j |a_p| \right) |k_{j+1} - k_j| \leq 2(C + K) \| \varphi \|_\infty.$$

Since $\langle T, \varphi \rangle = \sum_1^\infty k_j a_j$, T is a continuous linear functional on a dense subspace of $C(\Gamma)$ and has support in E .

Consequently, $T \in M(E)$. q.e.d.

REMARK 1. Relative to our use of Abel's sum formula note that if $\{k_j\}$ is bounded then $\sum |k_{j+1} - k_j| < \infty$ if and only if $\sum k_j \cdot a_j$ converges for every convergent series $\sum a_j$.

2. PROP. 2.1 generalizes the easy fact that if $\sum |k_j| < \infty$ then $T \circ k_j \in M_d(E)$. A special case of PROP. 2.1 which is proved directly and simply by Schwarz's inequality, is that if $\sum_{j=1}^J |k_{j+1} - k_j| < \infty$ then $T \circ k_j \in M(E)$. It is also clear by Schwarz's inequality that if $T \circ k_j \in M(E)$ then

$$\|T\|_1 \leq \frac{1}{J} \sum_1^J |k_{j+1} - k_j|.$$

Finally note that if $\sum |k_{j+1} - k_j| < \infty$ then $\lim k_j$ exists.

EXAMPLE 2.1 a. We first show that there are $T \circ k_j \in M_d(E)$ such that $\sum |k_{j+1} - k_j|$ diverges. Take countable $E \subseteq [0, 2\pi]$ such that $\lambda_1 = \gamma_0$, $\lambda_{2j} = \gamma_{2j+2}$, $\gamma_{2j+1} = \lambda_{2j+3}$, $j = 0, 1, \dots$, and

$$\dots \leq I_{2j+2} \leq I_{2j} \leq \dots \leq I_2 \leq I_0 \leq I_1 \leq \dots \leq I_{2j+1} \leq \dots$$

Setting $k_j \equiv (-1)^j \frac{1}{j}$ we have $\sum |k_{j+1} - k_j| \equiv \sum \frac{2j+1}{j(j+1)} = \infty$ (as well as $\sum |k_j| = \infty$) and $f \equiv \sum k_j \chi_{I_j}$ a function of bounded variation. Thus $T \circ k_j \in M_d(E)$ and

$$T = \sum_0^\infty a_{\lambda_j} \delta_{\lambda_j}, \quad \sum |a_{\lambda_j}| < \infty,$$

where $a_{\lambda_0} = 2$, $a_{\lambda_{2j+1}} = 2/(2j+1)(2j+3)$, $j \geq 0$, and $a_{\lambda_{2j}} = -2/2j(2j-2)$, $j > 1$.

b. We must show that there are non-discrete measures $T \circ k_j \in M(E)$ such that $\sum |k_j - k_{j+1}| < \infty$. Let $E \subseteq [0, 2\pi]$ be perfect and set $k_j = \frac{1}{j}$; we show $\text{supp } T = E$ so that since $\sum \frac{1}{j(j+1)} < \infty$ we have $T \in M(E) - M_d(E)$. Since the accessible points are dense in E it is enough to prove that each λ_m (and γ_m) is in $\text{supp } T$. If $\lambda_m \notin \text{supp } T$ we find $\varphi \in C(\Gamma)$ such that $\varphi = 0$ on (a neighborhood of) $\text{supp } T$ and $\langle T, \varphi \rangle \neq 0$. To do this, first note that every subsequence of $\{k_j\}$ converges to 0 and so there is an open interval $V(\lambda_m)$ with center λ_m such that if $I_j \leq I_m$ and $I_j \cap V(\lambda_m) \neq \emptyset$

then $\frac{1}{j} < \frac{1}{m}$. Next take non-negative $\varphi \in C^1(\Gamma)$, $\text{supp } \varphi \subseteq V(\lambda_m)$, $\varphi(\lambda_m) = 1$ and φ symmetric about λ_m . Thus

$$|\langle T, \varphi \rangle| = \left| \frac{1}{m} - \alpha \right|, \quad \alpha < \frac{1}{m}.$$

c. We now observe that there are continuous measures $T \in M(E)$, such that $\sum |k_{j+1} - k_j|$ diverges. For example the continuous Cantor function on the Cantor set is of the form $f \equiv \sum k_j \chi_{I_j}$ on $\bigcup_0 I_j$ and there are an infinite number of pairs k_j, k_{j+1} such that $|k_j - k_{j+1}| \geq \frac{1}{2}$. More generally, if $E \subseteq [0, 2\pi)$ has more than one limit point and $f \equiv \sum k_j \chi_{I_j}$ is increasing, then there is $\varepsilon > 0$ such that $|k_j - k_{j+1}| > \varepsilon$ for infinitely many j ; thus there are no non trivial positive measures $T \in M(E)$ on such sets with the property $\sum |k_j - k_{j+1}| < \infty$.

3. Trigonometric Sums Associated with Accessible Points.

PROPOSITION 3.1 $\sup_n \left| \sum_{j=1}^{\infty} e_{j,n} \right| \leq 2.$

PROOF. Let $\varphi \in C^1(\Gamma)$, $|\varphi| \equiv 1$, and $T \equiv -\delta_{\lambda_0} + \delta_{\gamma_0}$. Then $g' = T$ distributionally, where $g = \chi_{\Gamma - \lambda_0}$. Now, let $f_T = \sum_1 \chi_{I_j}$ so that $f_T = g$ a. e.

By definition, $\langle T, \varphi \rangle = \varphi(\gamma_0) - \varphi(\lambda_0)$; and since $f_T' = T$,

$$\langle T, \varphi \rangle = -\langle f_T, \varphi' \rangle = \sum_1^{\infty} (\varphi(\lambda_j) - \varphi(\gamma_j)),$$

where the last equality follows by the Lebesgue dominated convergence theorem.

Consequently, for $\varphi(\gamma) = e^{in\gamma}$,

$$\left| \sum_1^{\infty} (e^{in\lambda_j} - e^{in\gamma_j}) \right| = |e^{in\lambda_0} - e^{in\gamma_0}| \leq 2. \quad \text{q.e.d.}$$

Obviously the bound of 2 in PROP. 3.1 can be refined depending on the arithmetic character of λ_0 and γ_0 . Note also that for each n , $\sum_j |e_{j,n}| < \infty$.

EXAMPLE 3.1 a. Let E be independent. Then $\{\varepsilon_j\} \subseteq [0, 2\pi)$ is independent. Thus, by Kronecker's theorem [7, pp. 176-177], if we take any k

there is N_k so that for real α satisfying $|1 - e^{i\alpha}| > 1$ there is $n \in [0, N_k]$ for which $|e^{in\epsilon_j} - e^{i\alpha}| < \frac{1}{2}$ if $j = 1, \dots, k$. For this n

$$|e^{in\epsilon_j} - 1| \geq \|e^{in\epsilon_j} - e^{i\alpha}\| - \|e^{i\alpha} - 1\| > \frac{1}{2}$$

if $j = 1, \dots, k$. Therefore

$$(3.1) \quad \sum_{j=1}^{\infty} |e^{in\epsilon_j} - 1| > k/2;$$

and for any k we can find n such that (3.1) holds; thus

$$\sup_n \sum_j |c_{j,n}| = \infty.$$

b. Now let E be a countable Helson set, or, more generally, a set of strong spectral resolution. Obviously such a set need not be independent; for example, let $E \equiv \left\{0, \frac{2\pi}{2^j} : j = 1, \dots\right\}$. Note that if

$$\sup_n \sum_{j=1}^{\infty} |c_{j,n}| < \infty$$

then $A'_b(E) = D_b(E)$; whereas, by the hypothesis on E , $A'_b(E) = M(E)$, a contradiction (for E infinite). Consequently, once again $\sup_n \sum_j |c_{j,n}| = \infty$.

The phenomena of EXAMPLE 3.1 is general; in fact, the following lemma is straightforward.

LEMMA 3.1. Let E be infinite. Then, for any infinite sequence $\{p_j\}$ of natural numbers,

$$\sup_n \sum_{j=1}^{\infty} \|c_{p_{j+1},n} - c_{p_j,n}\| = \infty.$$

In the following theorem, part *a* is, of course, proved independent of LEMMA 3.1.

THEOREM 3.1 *a*.

$$(3.3) \quad \sup_n \sum_j |d_{j,n}|$$

diverges if and only if there is $T \infty k_j \in D_1(E) - A'(E)$ such that $\sum k_j$ converges.

b. There is $T \in k_j \in D_1(E) - A'(E)$ such that $\sum k_j$ converges for every infinite E .

PROOF. *b* is immediate from *a* and LEMMA 3.1.

a. Assume there is such a T and let (3.3) be finite. Then from PROP. 1.3 we have

$$2\pi |\widehat{T}(n)| \leq \sum_{j=1}^{\infty} \left(\sum_{p=1}^j k_p \right) |d_{j,n}^-|;$$

so that with our hypothesis on (3.3) we get the desired contradiction since we've proved $T \in A'(E)$.

For the converse assume without loss of generality that

$$(3.4) \quad \sup_{n \geq 0} \sum_j |d_{j,n}| = \infty.$$

We shall choose a sequence of j 's and n 's inductively such that for a given j_r we'll choose j_{r+1} and n_r .

Beginning with j_1 assume we have j_1, \dots, j_r and n_1, \dots, n_{r-1} . Take $n_r > n_{r-1}$ such that

$$\sum_{j=1}^{\infty} |d_{j,n_r}| > 8rj_r + r^2 + 2r + 2,$$

by (3.4).

Note that

$$\sum_{j=1}^{j_r} |d_{j,n_r}| < 4j_r + 1.$$

Now for our n_r take $j_{r+1} > j_r$ such that

$$\sum_{j=j_{r+1}+1}^{\infty} |d_{j,n_r}| < 1.$$

Combining these three inequalities gives

$$(3.5) \quad \sum_{j=j_r+1}^{j_{r+1}} |d_{j,n_r}| = \sum_{j=1}^{\infty} - \sum_{j=1}^{j_r} - \left(\sum_{j=1}^{\infty} - \sum_{j=1}^{j_{r+1}} \right) >$$

$$8rj_r + r^2 + 2r + 2 - 4j_r - 1 - > 4j_r + r^2 + 2r$$

since $2r - 1 > r$.

Next we define $T \in k_j$.

Let $s_j \equiv \sum_1^j k_m$ and let $s_j = 0$ for $j \leq j_1$; thus define

$$k_1 = \dots = k_{j_1} = 0.$$

For $j_r < j \leq j_{r+1}$ take

$$s_j = \frac{1}{r} \frac{\overline{d_{j,n}}}{|d_{j,n_r}|},$$

noting that $s_j \rightarrow 0$.

In this manner we define all k_j . For example, let

$$k_{j_1+1} = \frac{\overline{d_{j_1+1,n_1}}}{|d_{j_1+1,n_1}|};$$

and since $s_{j_1+2n_1} = \overline{d_{j_1+2,n_1}} / |d_{j_1+2,n_1}|$ we set

$$k_{j_1+2} = \operatorname{sgn} \overline{d_{j_1+2,n_1}} - \sum_{j=1}^{j_1+2} k_j.$$

Now, from PROP. 1.3,

$$2\pi |\widehat{T}(-n_r)| = \left| \sum_{j=1}^{\infty} s_j d_{j,n_r} \right|,$$

and so

$$2\pi |\widehat{T}(-n_r)| \geq \left| \left| \sum_{j=j_r+1}^{j_r+1} \right| - \left| \sum_{j=1}^{j_r} \right| + \sum_{j=j_r+1}^{\infty} s_j d_{j,n_r} \right| \geq$$

$$\frac{1}{r} \sum_{j=j_r+1}^{j_r+1} |d_{j,n_r}| - \left| \sum_{j=1}^{j_r} s_j d_{j,n_r} \right| - \left| \sum_{j=j_r+1}^{\infty} s_j d_{j,n_r} \right|.$$

Notice that for any domain D of summation

$$\left| \sum_D s_j d_{j,n_r} \right| \leq \sum_D |d_{j,n_r}|.$$

Consequently from (3.5)

$$2\pi |\widehat{T}(-n_r)| > 4j_r + r + 2 - (4j_r + 1) - 1 = r,$$

and hence $T \notin A'(E)$.

q.e.d.

Independent of THEOREM 3.1, it is trivial to see that if there is $T \in D_b(E) - A'_b(E)$ then $\sup_n \sum_j |c_{j,n}| = \infty$; and a proof similar to that of the second part of THEOREM 3.1 *a* shows that the converse is also true.

Such T exist since every infinite E has a countably infinite Helson subset; in this regard, we further refer to [8].

PROPOSITION 3.2. If $\sum_1 \varepsilon_j e^{-\varepsilon_j} \log \left(\frac{1}{\varepsilon_j} \right) < \infty$ then

$$(3.6) \quad \sum_j \sum'_n \frac{|c_{j,n}|}{n^2} < \infty.$$

PROOF. Because $|c_{j,n}| = 2 \left| \sin \frac{n\varepsilon_j}{2} \right|$ and by the Fourier series expansion of $|\sin x|$ it is sufficient to show

$$\sum_{j=1} \sum_{n=1} \frac{1}{n^2} \left(\sum_{m=1} \frac{\sin^2 \frac{mn\varepsilon_j}{2}}{4m^2 - 1} \right) < \infty.$$

Further, by an elementary calculation with residues,

$$\int_1^\infty \frac{\sin^2 \frac{xn\varepsilon_j}{2}}{4x^2 - 1} dx \leq \frac{1}{2} \int_0^\infty \frac{dx}{x^2 + 1} - \frac{1}{2} \int_0^\infty \frac{\cos xn\varepsilon_j}{x^2 + 1} dx = \frac{\pi}{4} (1 - e^{-n\varepsilon_j}),$$

for $n \geq 1$ and $j \geq 1$.

Thus, since we can estimate $\sin^2 \left(\frac{mn\varepsilon_j}{2} \right) / (4m^2 - 1)$ in terms of

$$\int_{m-1}^m \frac{\sin^2 \frac{xn\varepsilon_j}{2}}{4x^2 - 1} dx$$

for $m \geq 2$, it is sufficient to prove

$$(3.7) \quad \sum_{j=1} \sum_{n=1} \frac{1 - e^{-n\varepsilon_j}}{n^2} < \infty.$$

Noting that $\varepsilon_j \in (0, 2\pi)$ we have by the mean-value theorem that $|e^{-\varepsilon_j} - 1| \leq \varepsilon_j$ so that (3.7) reduces to showing

$$\sum_{j=1} \sum_{n=2} \frac{1 - e^{-n\varepsilon_j}}{n^2} < \infty.$$

Letting $f(x) = (1 - e^{-x\varepsilon_j})/x^2$ on $[1, \infty)$ we see that $f' < 0$ so that f is

decreasing, and, hence, by the integral test we need only prove

$$(3.8) \quad \sum_{j=1}^{\infty} \int_1^{\infty} \frac{1 - e^{-x\varepsilon_j}}{x^2} dx < \infty.$$

We have

$$\int_1^{\infty} \frac{1 - e^{-x\varepsilon}}{x^2} dx = \varepsilon_j \int_{\varepsilon_j}^{\infty} \frac{1 - e^{-u}}{u^2} du = (1 - e^{-\varepsilon_j}) - \varepsilon_j e^{-\varepsilon_j} \log \varepsilon_j + \varepsilon_j \int_{\varepsilon_j}^{\infty} (\log u) e^{-u} du.$$

As is well known

$$\int_0^{\infty} (\log u) e^{-u} du = L,$$

Euler's constant, and so by hypothesis and the fact that

$$\sum (1 - e^{-\varepsilon_j}) < \infty,$$

we have (3.8).

q.e.d.

Note that generally, by PROP. 1.1, if $T \infty k_j \in A'(E)$ then $k_j = 0 \left(\log \frac{1}{\varepsilon_j} \right)$, $j \rightarrow \infty$, whereas for E satisfying the hypothesis of PROP. 3.2, $k_j = O \left(e^{-\varepsilon_j} \log \frac{1}{\varepsilon_j} \right)$, $j \rightarrow \infty$.

In [2] it is made clear that closure of the multiplication operation of (bounded) pseudo-measures is important on Helson sets. For example, when $A'(E)$ is a Banach algebra for this multiplication not only does $A'(E) \subseteq \subseteq G(E)$, as we showed in [2], but, by the open mapping theorem, $A'(E) \neq \neq G(E)$ — for if there was equality we'd have $\overline{M(E)} = A'(E)$ since $\overline{M(E)} = = G(E)$, a contradiction since $M(E) \neq G(E)$ and $M(E)$ is closed in $A'(E)$.

4. Subspaces of Bounded Variation in $A(\Gamma)$.

PROPOSITION 4.1. Given any infinite E . There is $\varphi \in A(\Gamma)$ such that

$$(4.1) \quad \sum_{j=1}^{\infty} |\varphi(\lambda_j) - \varphi(\gamma_j)|$$

diverges.

PROOF. Assume (4.1) is finite for all $\varphi \in A(\Gamma)$.

Take any $T \in \mathcal{D}_b(E)$ and define measures μ_J (on $A(\Gamma)$) by

$$(4.2) \quad \langle \mu_J, \varphi \rangle = \sum_1^J k_j(\varphi(\lambda_j) - \varphi(\gamma_j)).$$

Since (4.1) is finite we have that given $\varphi \in A(\Gamma)$ there is $K_\varphi > 0$ such that for all J , $|\langle \mu_J, \varphi \rangle| \leq K_\varphi$.

By (4.2) we consider $\varphi \in A(E)$ and so by the uniform boundedness principle $\{\mu_J\}$ is bounded in $A'_s(E)$. Hence, by Alaoglu, the fact that $\mu_J \rightarrow T$ on $C^1(\Gamma)$, and T is arbitrary in $D_b(E)$, we have $D_b(E) = A'_b(E)$.

This contradicts THEOREM 3.1.

q.e.d.

REMARK a. PROP. 4.1 tells us something more than the well known fact that there are functions of infinite variation in $A(\Gamma)$; it tells us that locally — that is, on any given infinite set of points — there are elements of $A(\Gamma)$ with infinite variation.

b. PROP. 4.1 has some interest from the point of view of Helson sets. More precisely, if E were Helson and (4.1) were finite for all $\varphi \in A(\Gamma)$ then the argument of PROP. 4.1 is used to show $A'_b(E) = M(E)$; in fact, for $T \in A'_b(E)$ a weak * convergent subnet of $\{\mu_J\}$ converges to an element of $A'_s(E)$, and hence to a measure (for Helson sets). Thus there is some relation between the structure of $A'_b(E)$ and the variation of $A(\Gamma)$ on the accessible points of E . Of course, if an even stronger variation criterion held on $A(\Gamma)$, we could get conditions that $A'(E) = M(E)$.

Let $A_1(\Gamma)$ be the elements φ of $A(\Gamma)$ for which there is $\{\varphi_n\} \subseteq C^1(\Gamma)$ such that $\|\varphi_n - \varphi\|_A \rightarrow 0$ and

$$(4.3) \quad \sup_n \int |\varphi'_n| < \infty.$$

$A_{1+}(\Gamma)$ is the subspace of $A_1(\Gamma)$ in which the condition (4.3) is replaced by

$$(4.4) \quad \sup_n \int |\varphi'_n|^p < \infty, \text{ some } 1 < p < \infty.$$

The vector space is normed by

$$\|\varphi\| \equiv \|\varphi\|_A + K_\varphi,$$

where

$$K_\varphi \equiv \inf \left\{ \sup_n \int |\varphi'_n| : \{\varphi_n\} \subseteq C^1(\Gamma), \|\varphi_n - \varphi\|_A \rightarrow 0, \text{ and (4.3)} \right\}.$$

Because of PROP. 1.1 we define, for each $T \in A'(E)$, the sequence of measures with finite support

$$(4.5) \quad \mu_J \equiv \sum_1^J k_j (\delta_{\lambda_j} - \delta_{\gamma_j}).$$

As might be expected, generally, μ_J does not converge to T in the weak * topology. We do have

PROPOSITION 4.2. For all $T \in A'(E)$ and for all $\varphi \in A_1(\Gamma)$,

$$\lim_J \langle \mu_J - T, \varphi \rangle = 0.$$

PROOF. Let $\{\varphi_n\} \subseteq C^1(\Gamma)$ correspond to φ , and note that

$$\langle \mu_J, \varphi_n \rangle = - \sum_1^J k_j \int_{\lambda_j}^{\gamma_j} \varphi_n'.$$

Further, $\lim_J \langle \mu_J - T, \varphi_n \rangle = 0$ since $\varphi_n \in C^1(\Gamma)$, and

$$\lim_n \langle \mu_J, \varphi - \varphi_n \rangle = 0 \text{ since } \mu_J \in A'(E).$$

Letting K be a bound for $\int |\varphi_n'|$, we have

$$|\langle T - \mu_J, \varphi_n \rangle| \leq K \sum_{j=1}^{\infty} |k_j| \varepsilon_j,$$

and so $\lim_J \langle T - \mu_J, \varphi_n \rangle = 0$ uniformly in n by PROP. 1.1. Consequently we apply the Moore-Smith theorem and have

$$\langle T, \varphi \rangle = \lim_n \langle T, \varphi_n \rangle = \lim_n \lim_J \langle \mu_J, \varphi_n \rangle = \lim_J \lim_n \langle \mu_J, \varphi_n \rangle = \lim_J \langle \mu_J, \varphi \rangle,$$

since $\|\varphi - \varphi_n\|_A \rightarrow 0$. q.e.d.

COROLLARY 4.2.1 $A_1(\Gamma) \neq A(\Gamma)$.

PROOF. If $A_1(\Gamma) = A(\Gamma)$ then every E (of measure 0) is \mathcal{S} , a contradiction. (Note that the triadic Cantor set has non- \mathcal{S} subsets). q.e.d.

REMARK. Note that if, in the definition of $A_1(\Gamma)$, we demanded that $\varphi_n \equiv \varphi * \varrho_n$, ϱ_n some mollifier — that is, $\varrho_n \geq 0$, $\int \varrho_n = 1$, $\varrho_n(0) \rightarrow \infty$, then it is trivial to show $A_1(\Gamma) \neq A(\Gamma)$ by the fundamental theorem of calculus.

There are several other natural subspace of $A(\Gamma)$ with bounded variation properties, with the corresponding questions of topologies, duals, category, and inter-relation, that seem interesting to investigate.

5. Bounded Pseudo-Measures.

We begin by showing that even on countable E there is no reason to expect $A'(E) = A'_b(E)$ unless E has some additional, generally arithmetic, properties.

EXAMPLE 5.1. To define E we adopt a construction of G. Salmons [8]; E will be a subset of $\left\{0, \frac{1}{n} : n = 1, \dots\right\} \subseteq [0, 2\pi)$. We then construct an unbounded pseudo-measure on E . Let $F_n \subseteq [0, 2\pi)$ be a finite arithmetic progression with $2M_n + 1$ terms such that if $\gamma \in F_{n+1}$ then $\gamma < \lambda$ for each $\lambda \in F_n$; inductively we choose $M_n > M_{n-1}$ so that

$$\sum_{j=1}^{M_n} \frac{1}{j} \geq n^3,$$

and let $E = \overline{\bigcup F_n}$. On F_n we define a measure μ_n which has mass 0 at the « center » of F_n and mass $1/j$ ($-1/j$) at the j -th point (of F_n) to the right (to the left) of the center. A standard calculation shows that $\|\mu_n\|_{A'} \leq 2(\pi + 1)$. Next, we calculate h_n so that $h'_n = \mu_n$ and note that $|h_n| = \sum_1^{M_n} \frac{1}{j}$ on the two intervals contiguous to the center of F_n . Hence, setting

$$\nu_k = \sum_{n=1}^k \frac{1}{n^2} \mu_n \quad \text{and} \quad f_k \equiv \sum_{n=1}^k \frac{1}{n^2} h_n,$$

we have $\|\nu_k\|_{A'} \leq 2(\pi + 1) \sum_1^k \frac{1}{n^2}$ and $|f_k| = |h_k|/k^2 \geq k$ (on the two intervals contiguous to the center of F_k).

Consequently, a subset of $\{\nu_p\}$ converges to $T \in A'(E) - M(E)$ in the weak * topology, $f_p \rightarrow f$ pointwise a. e., $f' = T$, and f is unbounded.

PROPOSITION 5.1. $A'(E) = A'_b(E)$ if and only if

$$(5.1) \quad \begin{aligned} A'(E) \times D_1(E) &\rightarrow D_1(E) \\ (S \circ k_j, T \circ h_j) &\rightarrow ST \circ k_j h_j \end{aligned}$$

is a well-defined multiplication.

PROOF. If $A'(E) = A'_b(E)$, $S \circ k_j \in A'_b(E)$, and $T \circ h_j \in D_1(E)$, then $\sum h_j k_j \chi_{I_j} \in L^1(\Gamma)$ since

$$\int |\sum h_j k_j \chi_{I_j}(\gamma)| d\gamma \leq K \int (\sum |h_j| \chi_{I_j}(\gamma)) d\gamma < \infty.$$

Conversely if $A'(E) \neq A'_b(E)$ let $T \circ k_j \in A'(E)$ where $\lim_j |k_{n_j}| = \infty$.

Without loss of generality take $|k_{n_j}| \geq j$ and define $g \equiv \sum h_j \chi_{I_j}$ such that $h_{n_j} = 1/(j^2 \varepsilon_{n_j})$ and $h_m = 0$ if $m \neq n_j$.

Then

$$\int |g| = \int \sum |h_j| \chi_{I_j}(\gamma) d\gamma = \sum_j \frac{1}{j^2 \varepsilon_{n_j}} \int \chi_{I_{n_j}} = \sum \frac{1}{j^2} = \infty.$$

On the other hand

$$\int |\sum k_j h_j \chi_{I_j}| \geq \int \left(\frac{1}{j \varepsilon_{n_j}} \chi_{I_{n_j}}(\gamma) d\gamma \geq \sum \frac{1}{j}, \right.$$

a contradiction.

q.e.d.

Obviously, PROP. 5.2 is just a usual duality property between L^∞ and L^1 , and has nothing to do with $A'_b(E)$ per se.

REMARK. Note that $A'(E) = A'_b(E)$ if $\sum'_n |e_{j,n}|/n^2 = 0(\varepsilon_j)$, $j \rightarrow \infty$, from PROP. 1.2; and that the metric condition of PROP. 3.2 is much weaker than this.

In [4, THEOREM 19], Hardy and Littlewood prove that if $\varphi \in H^1$ [5, pp. 70-71] has the Fourier series $\sum_0^\infty a_n e^{in\gamma}$ then

$$(5.2) \quad \sum_{n=1}^\infty \frac{|a_n|}{n} \leq \pi \|\varphi\|_1.$$

They show by counter-example that if $\varphi(\gamma) = \sum_{|n| \leq N} a_n e^{in\gamma}$ then (5.2) is not necessarily true. We shall give another type of counter example as well as showing

PROPOSITION 5.2. For all $T \in A'(I')$ there is $S \in D_b(I')$ such that $\widehat{S}(n) \equiv \widehat{T}(n)$ for $n \geq 1$.

PROOF. Given T .

A direct application of (5.2) says that if $\varphi(\gamma) = \sum_0^N a_n e^{in\gamma}$ then

$$\sum_1^N \frac{|a_n|}{n} \leq \pi \int_0^{2\pi} \left| \sum_0^N a_n e^{in\gamma} \right| d\gamma.$$

Now, if $f' = T$ we have $\widehat{f}(n) = 0 \left(\frac{1}{|n|} \right)$, $|n| \rightarrow \infty$; and hence there is a constant K_T such that for all trigonometric polynomials of the form $\varphi(\gamma) \equiv \sum_1^N a_n e^{in\gamma}$

$$\left| \int f \bar{\varphi} \right| \equiv |\langle f, \varphi \rangle| \leq K_T \|\varphi\|_1.$$

Consequently, by the Hahn-Banach theorem there is $g \in L^\infty$ such that $\langle f - g, \varphi \rangle = 0$ for all $\varphi(\gamma) = \sum_1^N a_n e^{in\gamma}$.

In particular, $\widehat{f}(n) = \widehat{g}(n)$ for all $n > 0$. q.e.d.

Because of PROP. 5.2 we say that E has *bounded halves* if for all $T \in A'(E)$ there is $S \in D_b(E)$ such that $\widehat{T}(n) = \widehat{S}(n)$ for $n \geq 1$. The question is, of course, to determine for given $E \subseteq I'$ the type of subset $X \subseteq Z$ such that for all $T \in A'(E)$ there is $S \in D_b(E)$ for which $\widehat{T} = \widehat{S}$ on X . Obviously the problem is meaningful in a much more general context.

Now, assuming E has bounded halves we wish to find conditions so that $A'(E) = A'_b(E)$. Arithmetic properties definitely play a role here. In fact, using a (by now) standard approximation technique [6,10], we have

PROPOSITION 5.3. Let E be a Dirichlet set with bounded halves. Then $A'(E) = A'_b(E)$.

PROOF. Let $T \in A'(E)$ and $S \in D_b(E)$, $\widehat{S} = \widehat{T}$ for $n \geq 1$.

Observe that E Dirichlet is equivalent to

$$(5.3) \quad \lim_{n \rightarrow \infty} \sup_{\gamma \in E} |\sin n \gamma| = 0.$$

From (5.3) we know that for all $\varepsilon > 0$ there is a positive integer n_ε such that

$$(5.4) \quad \sup_{\gamma \in E} |\sin n_\varepsilon \gamma| < \frac{\varepsilon}{2}$$

and

$$\lim_{\varepsilon \rightarrow 0} n_\varepsilon = \infty.$$

Next we define the continuous ε — diminishing — M function M_ε in $[-\pi, \pi]$ to be 0 at 0 and outside $(-2\varepsilon, 2\varepsilon)$, ε at $\pm\varepsilon$, and linear otherwise.

Then from (5.4) we have for $S = g'$,

$$(\widehat{S} - \widehat{T})(2n_\varepsilon) - (\widehat{S} - \widehat{T})(0) = -\frac{i}{\pi} \langle S - T, e^{-in_\varepsilon \gamma} M_\varepsilon(\sin n_\varepsilon \gamma) \rangle,$$

since there is a neighborhood of E in which $|\sin n_\varepsilon \gamma| \leq \varepsilon$. A main feature of M_ε is that $\|M_\varepsilon\|_A \rightarrow 0$ and so, since $(\widehat{S} - \widehat{T})(2n_\varepsilon) = 0$, $(\widehat{S} - \widehat{T})(0) = 0$. A similar calculation shows $(\widehat{S} - \widehat{T})(n) = 0$ for all $n < 0$. Thus $S = T$.

q.e.d.

Note that every Kronecker set is both Helson and Dirichlet, and that there are Dirichlet sets which aren't Helson and vice-versa. Further, Dirichlet sets are not only sets of uniqueness, but Kahane [6] has shown that if E is Dirichlet then for all $T \in A'(E)$

$$\overline{\lim}_{|n| \rightarrow \infty} |\widehat{T}(n)| = \|T\|_{A'}.$$

Observe that Kronecker sets E are S [10] so that, in particular, $A'(E) = A'_b(E)$ in this case.

EXAMPLE 5.2. If the analogue of (5.2) were true for $\varphi(\gamma) = \sum_{|n| \leq N} a_n e^{in\gamma}$ then the proof of PROP. 5.2 shows that $A'(I) \subseteq D_b(I)$ which contradicts **EXAMPLE 5.1.**

6. Helson Sets and Summability Topologies.

Using Wik's theorem that $A(E) = A_+(E)$ characterizes Helson sets [7] we have

PROPOSITION 6.1. Let E be Helson. For all $m < 0$ there is $\sum_{n=0}^{\infty} |a_{n,m}| < \infty$ so that for each $T \in A'(E)$ we have

$$(6.1) \quad \widehat{T}(-m) = \lim_J \sum_{n=0}^{\infty} a_{n,m} \widehat{\mu}_J(-n),$$

where $\{\mu_J\}$ is the sequence of measure corresponding to T (as in (4.5)).

PROOF. $e^{in\gamma} = \sum_{n=0}^{\infty} a_{n,m} e^{in\gamma} \equiv \varphi(\gamma)$ on E , $\sum_{n=0}^{\infty} |a_{n,m}| < \infty$ since E is Helson. Thus, using the notation of (4.5) for $T \in A'(E)$, we have

$$2\pi \widehat{\mu}_J(-m) = \langle \mu_J, \varphi \rangle;$$

and hence $\lim_J \langle \mu_J, \varphi \rangle$ exists.

q.e.d.

Now, if $\varphi(\gamma) \equiv \sum_0^{\infty} a_n e^{in\gamma} \in A_+(E)$ we write

$$\varphi_r(\gamma) \equiv \sum_{n=0}^{\infty} a_n r^n e^{in\gamma}, \quad r \in (0, 1).$$

Note that $\varphi_r \in C^\infty(\Gamma)$, and hence for each $r \in (0, 1)$, $T \in A'(E)$, and $\varphi \in A_+(E)$ we have $\lim_J \langle \mu_J - T, \varphi_r \rangle = 0$.

PROPOSITION 6.2 Let E be Helson. Assume $T \in A'(E)$ has the property that for each $\varphi \in A_+(E)$, there exists

$$(6.2) \quad \lim_{J \rightarrow \infty} \langle \mu_J, \varphi_r \rangle, \quad \text{uniformly in } r \in \left[\frac{1}{2}, 1 \right).$$

Then $T \in M(E)$.

PROOF. (6.2) allows us to use Moore-Smith so that $\langle \mu_J, \varphi \rangle$ converges for all $\varphi \in A(\Gamma)$.

Thus by the uniform boundedness principle and the fact that E is Helson we have $\{\|\mu_J\|_1\}$ bounded. Consequently by Alaoglu and PROP. 1.1, $T \in M(E)$.

q.e.d.

For example, if $r = 1 - \frac{1}{n}$ then for $\varphi \in \sum a_n e^{in\gamma} \in A_+(E)$ and $T \in A'(E)$,

$$\langle \mu_J, \varphi_r \rangle = 2\pi \sum_{j=1}^{\infty} \left(\sum_{p=1}^j a_p \widehat{\mu}_J(-p) \right) \left(1 - \frac{1}{n} \right)^j \frac{1}{n},$$

noting that $1 - \frac{1}{n} = \sum_{j=1}^{\infty} \frac{1}{n} \left(1 - \frac{1}{n} \right)^j$.

Prompted by PROP. 6.2 consider diagonal sums

$$\sum_{n=0}^{\infty} a_n \widehat{\mu}_J(-n) F(J)^n$$

where $0 < F(J) < 1$ and $F(J) \rightarrow 1$.

Generally, in a dual system (X, Y) of T_2 locally convex spaces we say that a directed system $\{T_\alpha\} \subseteq X$ converges in the $\sigma\sigma(X, Y)$ topology to $T \in X$ if for all $\varphi \in Y$ there is $\{\varphi_\alpha\} \subseteq Y$ such that φ_α converges to φ and

$$\lim_{\alpha} \langle T_\alpha - T, \varphi_\alpha \rangle = 0.$$

Although significantly weaker than the weak * topology, it is not generally minimal [9, p. 191] and the intermediate topologies between $\sigma(X, Y)$ and $\sigma\sigma(X, Y)$ become interesting in light of PROP. 6.2, the lack of weak * convergence in § 4, and the convergence in PROP. 1.1 (in terms of (4.5)).

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