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Trigonometric sums associated with pseudo-measures


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The purpose of this paper is to study the structure of pseudo-measures on closed sets $E \subseteq \mathbb{R} = R/2\pi \mathbb{Z}$ of Lebesgue measure $m(E) = 0$. In particular, we find various conditions on $E$ so that a given pseudo-measure $T$ supported by $E$ is actually a measure or, at least, the first derivative of a bounded function. Such questions are obviously related to the open problem of determining if, generally, a Helson is a set of spectral synthesis.

$A(\mathbb{R})$ is the space of absolutely convergent Fourier series $\varphi(y) = \sum a_n e^{iny}$ normed by $\|\varphi\|_A = \sum |a_n|$; and its dual is the space of pseudo-measures $A'(\mathbb{R})$.

$A'(E)$ (resp., $M(E)$) is the space of pseudo-measures (resp., measures) $T$ supported by $E$ such that the Fourier coefficient $\hat{T}(0) = 0$. We let $D_0(E)$ be those first order distributions $T$ supported by $E$ such that $T$ is the first derivative of an $L^\infty$ function and $\hat{T}(0) = 0$; the bounded pseudo-measures on $E$ are $A'_b(E) = D_0(E) \cap A'(E)$.

§ 1 is devoted to notation and the statement of some formulas for pseudo-measures in terms of trigonometric sums; these results are basic to what follows. Next (§ 2), we characterize a useful subspace of $M(E)$ by a Stone-Weierstrass argument. In § 3, utilizing a summability technique, we prove that certain natural subspaces of $D_0(E)$ always contain non-pseudo-measures for infinite $E$. Then, with a metric hypothesis on $E$, we derive an estimate which is useful in characterizing those $E$ for which $A'(E) = A'_b(E)$; metric conditions are generally not sufficient to establish the boundedness of $A'(E)$-this is where we need arithmetic conditions. We give a functional analysis argument in § 4 to show the existence of a class of functions in $A(\mathbb{R})$ without any sort of local finite variation; and we use
this to examine subspaces of $A(\Gamma)$ on which measures with finite support approximate pseudo-measures. In § 5, after noting that unbounded pseudo-measures exist on countable sets with a single limit point, we give a technique which shows how arithmetic conditions on $E$ lead to $A'(E) = A'_E(E)$. The results here are preliminary. Finally (§ 6), we illustrate that properties of Helson sets lead to a large class of topologies of summability type.

I would like to thank Mr. Gordon Woodward for his helpful advice.

1. Notation and Formulas for Pseudo-Measure.

We designate the complement of $E$ by $CE = \bigcup I_j$ where, with $E \subseteq [0, 2\pi)$,
$I_j = (\lambda_j, \gamma_j) \subseteq [0, 2\pi)$ is an open interval of length $\varepsilon_j$; since $m(E) = 0$,
$\sum \varepsilon_j = 2\pi$. For convenience, we set

$$e_{j, n}^\pm = e^{\pm i\lambda_j n} - e^{\pm i\gamma_j n},$$

$$d_{j, n}^\pm = e^{\pm i\alpha_j n} - e^{\pm i\beta_j n};$$

generally, we drop the $\pm$ in this notation.

Let $D_b(\Gamma)$ be the space of first order distributions $T$ such that $T = f'$,
$f \in L^\infty(\Gamma)$, and let $A'_b(\Gamma) = D_b(\Gamma) \cap A(\Gamma)$. Also define $D_1(E)$ to be those
first order distributions $T$ for which $T = f'$, $f \in L^1(\Gamma)$, supp $T \subseteq E$, and
$\hat{T}(0) = 0$. It is easy to see that $f = \sum k_j \chi_{E_j}$ and, as such, we generally
write $T \circ k_j$ for an element of $D_1(E)$. Besides the spaces indicated in the
introduction we consider the following subspaces of $D_1(E)$:

$$D_{oo}(E) = \{T \circ k_j : f \in L_p(\Gamma) \text{ for all } p < \infty\},$$

$$A'_b(E) = \{T \in A'(E) : \varphi = 0 \text{ on } E, \varphi \in A(\Gamma), \text{ implies } \langle T, \varphi \rangle = 0\},$$

$$M_d(E) = \{T \in M(E) : T \text{ is discrete}\},$$

$$G(E) = \{T \circ k_j \in D_b(E) : f(\gamma \pm) \text{ exists for all } \gamma \in \Gamma\},$$

and $f$ has at most countably many jump discontinuities.

We multiply $[2] S \circ k_j, T \circ h_j \in D_{oo}(E)$ by

$$ST = (\sum k_j h_j \chi_{E_j})'.$$

Now, $E$ is Helson if $A'_b(E) = M(E)$, spectral synthesis (S) if $A'(E) = A'_b(E)$, and strong spectral resolution if $A'(E) = M(E)$. $E$ is a Dirichlet
We set $A(E)$ to be the restrictions of $A(I)$ to $E$; and $A_+(E)$ to be the restrictions of absolutely convergent Taylor series to $E$.

Next, let $\mathcal{F}$ be the compact open sets of $E$ so that $T \in D_1(E)$ is a finitely additive set function on $\mathcal{F}[1; 2]$; we norm such a $T$ by

$$\|T\|_p = \sup_{F \in \mathcal{F}} |T(F)| .$$

Also we write $I_j \leq I_k$ if, for $E \subseteq [0, 2\pi)$, $I_j < \gamma_k$; $I_j \leq \ldots \leq I_{m_n}$ is a partition $P$.

For detailed proofs of the following, as well as similar results, we refer to [3, § 2].

**Proposition 1.1.** For all $T \in A'(E)$ there is $f \in L^p(I)$, for each $p < \infty$, such that $f = \sum \limits_{j=1}^{n} k_j \mathcal{F}_j \mathcal{F}_j$, a.e., $\sum \mathcal{F}_j \mathcal{F}_j$ $\mathcal{F}_j < \infty$ for some $\delta > 0$, and

$$c_n = \hat{T}(n) = \frac{1}{2\pi} \sum \limits_{j=1}^{n} k_j \mathcal{F}_j \mathcal{F}_j .$$

**Proposition 1.2.** For all $T \in A'(E)$, $T \mathcal{F}_j$ and $c_n = \hat{T}(n)$,

$$k_j = \frac{1}{\varepsilon_j} \sum \limits_{n}^{j} \frac{c_n}{n^{\delta}} \mathcal{F}_j \mathcal{F}_j .$$

**Proof.** We have

$$f(\gamma) = \sum \limits_{m}^{j} \mathcal{F}_m \mathcal{F}_m(\gamma) = \sum \mathcal{F}_j \mathcal{F}_j \mathcal{F}_j(\gamma) .$$

Since $f \in L^1(I)$ and since Fourier series can be integrated term by term, we integrate both sides of (1.2) over $I_j$. Thus

$$\mathcal{F}_j \mathcal{F}_j \mathcal{F}_j = \sum \mathcal{F}_j \mathcal{F}_j \mathcal{F}_j \mathcal{F}_j \mathcal{F}_j .$$

**Proposition 1.3.** If $T \mathcal{F}_j$ $A'(E)$ and the partial sums $\sum \mathcal{F}_j \mathcal{F}_j$ are bounded, then

$$2\pi \hat{T}(n) = - \sum \mathcal{F}_j \mathcal{F}_j \mathcal{F}_j \mathcal{F}_j \mathcal{F}_j .$$
PROOF. By Abel's formula
\[
\sum_{j=1}^J k_j \ell_j - n = \left( \sum_{j=1}^J k_j \ell_j - n \right) - \sum_{j=1}^J \left( \sum_{p=1}^j k_p \right) d_j - n.
\]
Thus we have (1.3) by (1.1) and since \( \lim_{J} \ell_{J+1} = 0 \).

q.e.d.

2. Structure of Measures.

Let us first recall [1,2] some characterization of \( T \circ k_j \in M(E) \) in terms of \( k_j \). Let \( T \circ k_j \in D_1(E) \); \( T \in M(E) \) if and only if any of the following equivalent conditions hold:

i. \( \| T \|_1 < \infty \).

ii. \( \sup_{j=1}^{m-1} \sum_{p=1}^j |k_{n_j+1} - k_{n_j}| < \infty \).

iii. There is \( M \) such that if \( I_{n_1} \leq \ldots \leq I_{n_m} \) then

\[
\sum_{j=1}^m (k_{n_{j+1}} - k_{n_j}) < M.
\]

PROPOSITION 2.1. Let \( \{k_j\} \) be bounded by a constant \( C \) and assume

\[
\sum_{j=1}^J |k_{j+1} - k_j| = K < \infty.
\]

Then \( T \circ k_j \in M(E) \).

PROOF. Let \( X = \{ \varphi \in C^1(\Gamma) : \sum \varphi(\lambda_j), \sum \varphi(\gamma_j) \leq \| \varphi \|_\infty \} \).

\( X \) is a subalgebra of \( C(\Gamma) \) which satisfies the conditions of the Stone-Weierstrass theorem.

Thus, \( X = \mathbb{C}(\Gamma) \).

For each \( \varphi \in X \), we have for \( a_j = \varphi(\lambda_j) - \varphi(\gamma_j) \),

\[
\left| \sum_{j=1}^J k_j a_j \right| = \left| \left( \sum_{j=1}^J a_j \right) k_{j+1} - \sum_{j=1}^J \left( \sum_{p=1}^j a_p \right) (k_{j+1} - k_j) \right| \leq
C \sum_{j=1}^J |a_j| + \sum_{j=1}^J \left( \sum_{p=1}^j |a_p| \right) |k_{j+1} - k_j| \leq 2(C+K)\| \varphi \|_\infty.
\]
Since \( \langle T, \varphi \rangle = \sum_{j=1}^{\infty} k_j a_j \), \( T \) is a continuous linear functional on a dense subspace of \( C(T) \) and has support in \( E \).

Consequently, \( T \in M(E) \). q.e.d.

**Remark 1.** Relative to our use of Abel's sum formula note that if \( \{k_j\} \) is bounded then \( \Sigma |k_{j+1} - k_j| < \infty \) if and only if \( \Sigma k_j a_j \) converges for every convergent series \( \Sigma a_j \).

2. **Prop. 2.1** generalizes the easy fact that if \( \Sigma |k_j| < \infty \) then \( T \in M_d(E) \). A special case of **Prop. 2.1** which is proved directly and simply by Schwarz's inequality, is that if \( \Sigma j |k_{j+1} - k_j| < \infty \) then \( T \in M(E) \).

It is also clear by Schwarz's inequality that if \( T \in M(E) \) then

\[
\|T\|_1 \leq \frac{1}{j} \Sigma_{j=1}^{\infty} |k_{j+1} - k_j|.
\]

Finally note that if \( \Sigma |k_{j+1} - k_j| < \infty \) then \( \lim_{j \to \infty} k_j \) exists.

**Example 2.1 a.** We first show that there are \( T \in M_d(E) \) such that \( \Sigma |k_{j+1} - k_j| \) diverges. Take countable \( E \subseteq [0, 2\pi] \) such that \( \lambda_1 = \gamma_0 \), \( \lambda_{2j} = 2 \gamma_{2j+1}, \gamma_{2j+1} = \lambda_{2j+2}, j = 0, 1, \ldots \), and

\[
... \leq I_{2j+2} \leq I_{2j} \leq ... I_2 \leq I_0 \leq I_1 \leq ... \leq I_{2j+1} \leq ...
\]

Setting \( k_j = \frac{(-1)^j}{j} \) we have \( \Sigma |k_{j+1} - k_j| = \Sigma \frac{2j+1}{j(j+1)} = \infty \) (as well as \( \Sigma |k_j| = \infty \)) and \( f = \Sigma k_j \mathcal{X}_E \) a function of bounded variation. Thus \( T \in M_d(E) \) and

\[
T = \sum_{j=0}^{\infty} a_{k_j} \lambda_j, \quad \Sigma |a_{k_j}| < \infty,
\]

where \( a_{k_j} = 2, a_{2j+1} = \frac{2(2j+1)(2j+3)}{j} \), \( j \geq 0 \), and \( a_{2j} = -\frac{2(2j)(2j-2)}{j}, j > 1 \).

b. We must show that there are non-discrete measures \( T \in M(E) \) such that \( \Sigma |k_j - k_{j+1}| < \infty \). Let \( E \subseteq [0, 2\pi] \) be perfect and set \( k_j = \frac{1}{j} \); we show \( \text{supp} \ T = E \) so that since \( \Sigma \frac{1}{j(j+1)} < \infty \) we have \( T \in M(E) \). Since the accessible points are dense in \( E \) it is enough to prove that each \( \lambda_m \) (and \( \gamma_m \)) is in \( \text{supp} \ T \). If \( \lambda_m \notin \text{supp} \ T \) we find \( \varphi \in O(T) \) such that \( \varphi = 0 \) on (a neighborhood of) \( \text{supp} \ T \) and \( \langle T, \varphi \rangle \neq 0 \). To do this, first note that every subsequence of \( \{k_j\} \) converges to 0 and so there is an open interval \( V(\lambda_m) \) with center \( \lambda_m \) such that if \( I_j \subseteq I_m \) and \( I_j \cap V(\lambda_m) \neq \emptyset \)
then \( \frac{1}{j} < \frac{1}{m} \). Next take non-negative \( \varphi \in C^1(\Gamma) \), \( \text{supp } \varphi \subseteq V(\lambda_m) \), \( \varphi(\lambda_m) = 1 \) and \( \varphi \) symmetric about \( \lambda_m \). Thus

\[
| \langle T, \varphi \rangle | = \left| \frac{1}{m} - \alpha \right|, \quad \alpha < \frac{1}{m}.
\]

c. We now observe that there are continuous measures \( T \propto k_j \in M(E) \), such that \( \sum |k_{j+1} - k_j| \) diverges. For example the continuous Cantor function on the Cantor set is of the form \( f = \sum k_j \mathbb{I}_{I_j} \) on \( \bigcup I_j \) and there are

an infinite number of pairs \( k_j, k_{j+1} \) such that \( |k_j - k_{j+1}| \geq \frac{1}{2} \). More generally, if \( E \subseteq [0, 2\pi) \) has more than one limit point and \( f = \sum k_j \mathbb{I}_{I_j} \) is increasing, then there is \( \varepsilon > 0 \) such that \( |k_j - k_{j+1}| > \varepsilon \) for infinitely many \( j \); thus there are no non-trivial positive measures \( T \propto k_j \) on such sets with the property \( \sum |k_j - k_{j+1}| < \infty \).

3. Trigonometric Sums Associated with Accessible Points.

Proposition 3.1 \( \sup_n \left| \sum_{j=1}^{\infty} c_j, n \right| \leq 2. \)

Proof. Let \( \varphi \in C^1(\Gamma), |\varphi| = 1, \) and \( T = -\delta_0 + \delta_n \). Then \( g' = T \) distributionally, where \( g = \mathbb{I}_{(-\delta_n, \delta_n)} \). Now, let \( f_T = \sum \mathbb{I}_{I_j} \) so that \( f_T = g \) a.e.

By definition, \( \langle T, \varphi \rangle = \varphi(\gamma_0) - \varphi(\lambda_0) \); and since \( f_T = T \),

\[
\langle T, \varphi \rangle = -\langle f_T, \varphi' \rangle = \sum_{j=1}^{\infty} (\varphi(\lambda_j) - \varphi(\gamma_j)),
\]

where the last equality follows by the Lebesgue dominated convergence theorem.

Consequently, for \( \varphi(\gamma) = e^{in\gamma} \),

\[
\left| \sum_{j=1}^{\infty} (e^{in\gamma_j} - e^{in\delta}) \right| = |e^{in\delta} - e^{in\delta}| \leq 2. \quad \text{q.e.d.}
\]

Obviously the bound of 2 in Prop. 3.1 can be refined depending on the arithmetic character of \( \lambda_0 \) and \( \gamma_0 \). Note also that for each \( n, \sum_j |c_j, n| < \infty \).

Example 3.1 a. Let \( \mathcal{E} \) be independent. Then \( |\varepsilon_j| \subseteq [0, 2\pi) \) is independent. Thus, by Kronecker's theorem [7, pp. 176-177], if we take any \( k \)
there is \( N_k \) so that for real \( \alpha \) satisfying \( |1 - e^{i\alpha}| > 1 \) there is \( n \in [0, N_k] \) for which \( |e^{i\alpha} - e^{i\alpha}| < \frac{1}{2} \) if \( j = 1, \ldots, k \). For this \( n \)

\[
|e^{i\alpha} - 1| \geq \|e^{i\alpha} - e^{i\alpha} - e^{i\alpha} - 1\| > \frac{1}{2}
\]

if \( j = 1, \ldots, k \). Therefore

\[
(3.1) \quad \sum_{j=1}^{\infty} |e^{i\alpha j} - 1| > \frac{k}{2};
\]

and for any \( k \) we can find \( n \) such that (3.1) holds; thus

\[
\sup_n \sum_j |c_{j,n}| = \infty.
\]

b. Now let \( E \) be a countable Helson set, or, more generally, a set of strong spectral resolution. Obviously such a set need not be independent; for example, let \( E = \{0, \frac{2\pi}{2^j} : j = 1, \ldots\} \). Note that if

\[
\sup_n \sum_{j=1}^{\infty} |c_{j,n}| < \infty
\]

then \( A_1(E) = D_1(E) \); whereas, by the hypothesis on \( E \), \( A_1(E) = M(E) \), a contradiction (for \( E \) infinite). Consequently, once again \( \sup_n \sum_j |c_{j,n}| = \infty \).

The phenomena of Example 3.1 is general; in fact, the following lemma is straightforward.

**Lemma 3.1.** Let \( E \) be infinite. Then, for any infinite sequence \( \{p_j\} \) of natural numbers,

\[
\sup_n \sum_{j=1}^{\infty} \|c_{p_j+1,n} - c_{p_j,n}\| = \infty.
\]

In the following theorem, part a is, of course, proved independent of Lemma 3.1.

**Theorem 3.1 a.**

\[
(3.3) \quad \sup_n \sum_j \|d_{j,n}\|
\]

diverges if and only if there is \( T \in k_j \in D_1(E) - A'(E) \) such that \( \sum k_j \) converges.
b. There is $T \in A(E) = A'(E)$ such that $\sum k_j$ converges for every infinite $E$.

PROOF. b is immediate from a and Lemma 3.1.

a. Assume there is such a $T$ and let $(3.3)$ be finite. Then from Prop. 1.3 we have

$$2\pi \left| \widehat{T}(n) \right| \leq \sum_{j=1}^{\infty} \left( \sum_{p=1}^{j} k_p \right) \left| \hat{d}_{j,n} \right| ;$$

so that with our hypothesis on $(3.3)$ we get the desired contradiction since we've proved $T \in A'(E)$.

For the converse assume without loss of generality that

$$(3.4) \quad \sup_{n \geq 0} \sum_{j} \left| \hat{d}_{j,n} \right| = \infty.$$ 

We shall choose a sequence of $j$'s and $n$'s inductively such that for a given $j_r$ we'll choose $j_{r+1}$ and $n_r$.

Beginning with $j_1$ assume we have $j_1, \ldots, j_r$ and $n_1, \ldots, n_{r-1}$. Take $n_r > n_{r-1}$ such that

$$\sum_{j=1}^{\infty} \left| \hat{d}_{j,n_r} \right| > 8rj_r + r^2 + 2r + 2,$$

by (3.4).

Note that

$$\sum_{j=1}^{j_r} \left| \hat{d}_{j,n_r} \right| < 4j_r + 1.$$ 

Now for our $n_r$ take $j_{r+1} > j_r$ such that

$$\sum_{j=r+1}^{\infty} \left| \hat{d}_{j,n_r} \right| < 1.$$ 

Combining these three inequalities gives

$$(3.5) \quad \sum_{j=j_r+1}^{j_{r+1}} \left| \hat{d}_{j,n_r} \right| = \sum_{j=1}^{\infty} \sum_{j=1}^{j_r} \left| \hat{d}_{j,n_r} \right| - \sum_{j=1}^{\infty} \sum_{j=1}^{j_r} \left| \hat{d}_{j,n_r} \right| > 8rj_r + r^2 + 2r + 2 - 4j_r - 1 > 4j_r + r^2 + 2r$$

since $2r - 1 > r$.

Next we define $T \in k_j$.

Let $j = j_k$ and let $j = 0$ for $j \leq j_1$; thus define

$$k_1 = \ldots = k_{j_1} = 0.$$
For $j_r < j \leq j_{r+1}$ take

$$s_j = \frac{1}{r} \frac{d_{j, n}}{|d_{j, n_r}|},$$

noting that $s_j \to 0$.

In this manner we define all $k_j$. For example, let

$$k_{j_{r+1}} = \frac{d_{j_{r+1}, n_r}}{|d_{j_{r+1}, n_r}|};$$

and since $s_{j_{r+1}+2n} = \frac{d_{j_{r+1}+2, n_r}}{|d_{j_{r+1}+2, n_r}|}$ we set

$$k_{j_{r+2}} = \text{sgn} \frac{d_{j_{r+2}, n_r}}{|d_{j_{r+2}, n_r}|} - \sum_{j=1}^{j_{r+2}} k_j.$$

Now, from Prop. 1.3,

$$2\pi |\hat{T}(-n_r)| = \left| \sum_{j=1}^{\infty} s_j d_{j, n_r} \right|,$$

and so

$$2\pi |\hat{T}(-n_r)| \geq \left| \sum_{j=j_r+1}^{j_r+1} s_j d_{j, n_r} \right| - \sum_{j=j_r+1}^{\infty} s_j d_{j, n_r} \right| \geq$$

$$\frac{1}{r} \sum_{j=j_r+1}^{j_r+1} |d_{j, n_r}| - \left| \sum_{j=j_r+1}^{\infty} s_j d_{j, n_r} \right| \geq \frac{1}{r} \sum_{j=j_r+1}^{j_r+1} |d_{j, n_r}| - \left| \sum_{j=j_r+1}^{\infty} s_j d_{j, n_r} \right|.$$

Notice that for any domain $D$ of summation

$$|\sum_D s_j d_{j, n_r}| \leq \sum_D |d_{j, n_r}|.$$

Consequently from (3.5)

$$2\pi |\hat{T}(-n_r)| > 4j_r + r + 2 - (4j_r + 1) - 1 = r,$$

and hence $T \notin A'(E)$.

Independent of Theorem 3.1, it is trivial to see that if there is $T \in D_b(E) - A'_b(E)$ then $\sup_n \sum_j |c_{j, n}| = \infty$; and a proof similar to that of the second part of Theorem 3.1 a shows that the converse is also true.

Such $T$ exist since every infinite $E$ has a countably infinite Helson subset; in this regard, we further refer to [8].

PROPOSITION 3.2. If \( \sum \varepsilon_j e^{-\varepsilon_j} \log \left( \frac{1}{\varepsilon_j} \right) < \infty \) then

\[
\sum_j \frac{\varepsilon_j}{n^2} < \infty.
\]

\((3.6)\)

PROOF. Because \( |\varepsilon_{j, n}| = \frac{1}{2} |\sin \frac{n\varepsilon_j}{2}| \) and by the Fourier series expansion of \(|\sin x|\) it is sufficient to show

\[
\sum_j \frac{1}{n^2} \left( \sum_{m=1}^n \frac{\sin^2 \frac{mn \varepsilon_j}{2}}{4m^2 - 1} \right) < \infty.
\]

Further, by an elementary calculation with residues,

\[
\int \frac{\sin^2 \frac{x \varepsilon_j}{2}}{4x^2 - 1} \, dx \leq \frac{1}{2} \int_0^\infty \frac{dx}{x^2 + 1} - \frac{1}{2} \int_0^\infty \frac{dx}{x^2 + 1} \cos \frac{x \varepsilon_j}{2} \, dx = \frac{\pi}{4} \left( 1 - e^{-\varepsilon_j} \right),
\]

for \( n \geq 1 \) and \( j \geq 1 \).

Thus, since we can estimate \( \sin^2 \left( \frac{mn \varepsilon_j}{2} \right) / (4m^2 - 1) \) in terms of

\[
\int_{n=1}^m \frac{\sin^2 \frac{x \varepsilon_j}{2}}{4x^2 - 1} \, dx
\]

for \( m \geq 2 \), it is sufficient to prove

\[
\sum_j \frac{1 - e^{-n \varepsilon_j}}{n^2} < \infty.
\]

\((3.7)\)

Noting that \( \varepsilon_j \in (0, 2\pi) \) we have by the mean-value theorem that \( |e^{-\varepsilon_j} - 1| \leq \varepsilon_j \) so that \((3.7)\) reduces to showing

\[
\sum_{j=1}^\infty \frac{1 - e^{-n \varepsilon_j}}{n^2} < \infty.
\]

Letting \( f(x) = (1 - e^{-x \varepsilon_j})/x^2 \) on \([1, \infty)\) we see that \( f' < 0 \) so that \( f \) is
decreasing, and, hence, by the integral test we need only prove

\[
\sum_{j=1}^{\infty} \int_{1}^{\infty} \frac{1 - e^{-xj}}{x^2} \, dx < \infty.
\]

We have

\[
\int_{1}^{\infty} \frac{1 - e^{-x_{j}x}}{x_{j}^2} \, dx_{j} = \epsilon_{j} \int_{\epsilon_{j}}^{\infty} \frac{1 - e^{-u}}{u^2} \, du = (1 - e^{-\epsilon_{j}}) - \epsilon_{j} e^{-\epsilon_{j}} \log \epsilon_{j} + \epsilon_{j} \int_{\epsilon_{j}}^{\infty} (\log u) e^{-u} \, du.
\]

As is well known

\[
\int_{0}^{\infty} (\log u) e^{-u} \, du = I,
\]

Euler's constant, and so by hypothesis and the fact that

\[
\sum_{j=1}^{\infty} (1 - e^{-\epsilon_{j}}) < \infty,
\]

we have (3.8).

Note that generally, by Prop. 1.1, if \( T \supset k_{j} A'(E) \) then \( k_{j} = 0 \left( \log \frac{1}{\epsilon_{j}} \right) \), \( j \to \infty \), whereas for \( E \) satisfying the hypothesis of Prop. 3.2, \( k_{j} = O \left( e^{-\epsilon_{j}} \log \frac{1}{\epsilon_{j}} \right) \), \( j \to \infty \).

In [2] it is made clear that closure of the multiplication operation of (bounded) pseudo-measures is important on Helson sets. For example, when \( A'(E) \) is a Banach algebra for this multiplication not only does \( A'(E) \subseteq \subseteq G(E) \), as we showed in [2], but, by the open mapping theorem, \( A'(E) \supseteq \supseteq G(E) \) — for if there was equality we'd have \( M(E) = A'(E) \) since \( M(E) = G(E) \), a contradiction since \( M(E) \supseteq G(E) \) and \( M(E) \) is closed in \( A'(E) \).

4. Subspaces of Bounded Variation in \( \Lambda(\Gamma) \).

Proposition 4.1. Given any infinite \( E \). There is \( \varphi \in \Lambda(\Gamma) \) such that

\[
\sum_{j=1}^{\infty} | \varphi(\lambda_{j}) - \varphi(\gamma_{j}) | \text{ diverges.}
\]
PROOF. Assume (4.1) is finite for all \( \varphi \in A (\Gamma) \).

Take any \( T \in D_b (E) \) and define measures \( \mu_j \) (on \( A (\Gamma) \)) by

\[
\langle \mu_j, \varphi \rangle = \sum_{i=1}^{J} k_i (\varphi (\lambda_i) - \varphi (\gamma_i)).
\]

Since (4.1) is finite we have that given \( \varphi \in A (\Gamma) \) there is \( K_\varphi > 0 \) such that for all \( J, | \langle \mu_j, \varphi \rangle | \leq K_\varphi \).

By (4.2) we consider \( \varphi \in A (E) \) and so by the uniform boundedness principle \( \{ \mu_j \} \) is bounded in \( A_1^E (E) \). Hence, by Alaoglu, the fact that \( \mu_j \to T \) on \( C^1 (\Gamma) \), and \( T \) is arbitrary in \( D_b (E) \), we have \( D_b (E) = A_1^E (E) \).

This contradicts THEOREM 3.1. q.e.d.

REMARK a. PROP. 4.1 tells us something more than the well known fact that there are functions of infinite variation in \( A (\Gamma) \); it tells us that locally — that is, on any given infinite set of points — there are elements of \( A (\Gamma) \) with infinite variation.

b. PROP. 4.1 has some interest from the point of view of Helson sets. More precisely, if \( E \) were Helson and (4.1) were finite for all \( \varphi \in A (\Gamma) \) then the argument of PROP. 4.1 is used to show \( A_1^E (E) = M (E) \); in fact, for \( T \in A_1^E (E) \) a weak \( * \) convergent subnet of \( \{ \mu_j \} \) converges to an element of \( A_1^E (E) \), and hence to a measure (for Helson sets). Thus there is some relation between the structure of \( A_1^E (E) \) and the variation of \( A (\Gamma) \) on the accessible points of \( E \). Of course, if an even stronger variation criterion held on \( A (\Gamma) \), we could get conditions that \( A' (E) = M (E) \).

Let \( A_1 (\Gamma) \) be the elements \( \varphi \) of \( A (\Gamma) \) for which there is \( \{ \varphi_n \} \subseteq C^1 (\Gamma) \) such that \( \| \varphi_n - \varphi \|_A \to 0 \) and

\[
\sup_n \int | \varphi_n | < \infty.
\]

\( A_{1+} (\Gamma) \) is the subspace of \( A_1 (\Gamma) \) in which the condition (4.3) is replaced by

\[
\sup_n \int | \varphi_n |^p < \infty, \text{ some } 1 < p < \infty.
\]

The vector space is normed by

\[
\| \varphi \| = \| \varphi \|_A + K_\varphi,
\]

where

\[
K_\varphi = \inf \left\{ \sup_n \int | \varphi_n | : \{ \varphi_n \} \subseteq C^1 (\Gamma), \| \varphi_n - \varphi \|_A \to 0, \text{ and } (4.3) \right\}.
\]
Because of Prop. 1.1 we define, for each $T \otimes k_j \in A'(E)$, the sequence of measures with finite support

$$\mu_j = \sum_{i}^J k_j(\delta_j - \delta_t).$$

As might be expected, generally, $\mu_j$ does not converge to $T$ in the weak * topology. We do have

**Proposition 4.2.** For all $T \otimes k_j \in A'(E)$ and for all $\varphi \in A_1(\Gamma)$,

$$\lim_j \langle \mu_j - T, \varphi \rangle = 0.$$

**Proof.** Let $\{\varphi_n\} \subseteq C^1(\Gamma)$ correspond to $\varphi$, and note that

$$\langle \mu_j, \varphi_n \rangle = -\sum_{j}^J k_j \int_{\Gamma} \varphi'_n.$$

Further, $\lim_j \langle \mu_j - T, \varphi_n \rangle = 0$ since $\varphi_n \in C^1(\Gamma)$, and

$$\lim_n \langle \mu_j, \varphi - \varphi_n \rangle = 0$$

since $\mu_j \in A'(E)$.

Letting $K$ be a bound for $\int |\varphi'_n|$, we have

$$|\langle T - \mu_j, \varphi_n \rangle| \leq K \sum_{j+1}^{\infty} |k_j| \epsilon_j,$$

and so $\lim_j \langle T - \mu_j, \varphi_n \rangle = 0$ uniformly in $n$ by Prop. 1.1. Consequently we apply the Moore-Smith theorem and have

$$\langle T, \varphi \rangle = \lim_n \langle T, \varphi_n \rangle = \lim_n \langle \mu_j, \varphi_n \rangle = \lim_{j} \langle \mu_j, \varphi_n \rangle = \lim_{j} \langle \mu_j, \varphi \rangle,$$

since $\| \varphi - \varphi_n \|_A \to 0.$

**Corollary 4.2.1** $A_1(\Gamma) \cong A(\Gamma)$.

**Proof.** If $A_1(\Gamma) = A(\Gamma)$ then every $E$ (of measure 0) is $S$, a contradiction. (Note that the triadic Cantor set has non-$S$ subsets). q.e.d.
Remark. Note that if, in the definition of $A_1(I)$, we demanded that $\varphi_n = \varphi \ast \varphi_n$, $\varphi_n$ some mollifier — that is, $\varphi_n \geq 0$, $\int \varphi_n = 1$, $\varphi_n(0) \to \infty$, then it is trivial to show $A_1(I) = A(I)$ by the fundamental theorem of calculus.

There are several other natural subspaces of $A(I)$ with bounded variation properties, with the corresponding questions of topologies, duals, category, and inter-relation, that seem interesting to investigate.


We begin by showing that even on countable $E$ there is no reason to expect $A'(E) = A_1(E)$ unless $E$ has some additional, generally arithmetic, properties.

Example 5.1. To define $E$ we adopt a construction of G. Salmons [8]; $E$ will be a subset of $\{0, \frac{1}{n} : n = 1, \ldots \} \subseteq [0, 2\pi)$. We then construct an unbounded pseudo-measure on $E$. Let $F_n \subseteq [0, 2\pi)$ be a finite arithmetic progression with $2M_n + 1$ terms such that if $\gamma \in F_{n+1}$ then $\gamma < \lambda$ for each $\lambda \in F_n$; inductively we choose $M_n > M_{n-1}$ so that

$$M_n \geq n^2,$$

and let $E = \overline{F_n}$. On $F_n$ we define a measure $\mu_n$ which has mass 0 at the «center» of $F_n$ and mass $1/j(-1/j)$ at the $j$-th point (of $F_n$) to the right (to the left) of the center. A standard calculation shows that $\| \mu_n \|_{A'} \leq 2(\pi + 1)$. Next, we calculate $h_n$ so that $h_n = \mu_n$ and note that $|h_n| = \sum_{j=1}^{M_n} \frac{1}{j}$ on the two intervals contiguous to the center of $F_n$. Hence, setting

$$v_k = \sum_{n=1}^{k} \frac{1}{n^2} \mu_n \quad \text{and} \quad f_k = \sum_{n=1}^{k} \frac{1}{n^2} h_n,$$

we have $\|v_k\|_{A'} \leq 2(\pi + 1) \sum_{i=1}^{k} \frac{1}{n^2}$ and $|f_k| = |h_k| \geq k$ (on the two intervals contiguous to the center of $F_n$).

Consequently, a subset of $|v_p|$ converges to $T \in A'(E) - M(E)$ in the weak * topology, $f_p \to f$ pointwise a.e., $f' = T$, and $f$ is unbounded.
PROPOSITION 5.1. $A'(E) = A'_b(E)$ if and only if

$$A'(E) \times D_4(E) \rightarrow D_4(E)$$

(5.1)

$$S \circ h_j, T \circ h_j \rightarrow ST \circ h_j h_j$$

is a well-defined multiplication.

PROOF. If $A'(E) = A'_b(E)$, $S \circ h_j, T \circ h_j \in A'_b(E)$, and $T \circ h_j \in D_4(E)$, then

$$\sum h_j k_j \mathcal{F}_j(\gamma) \mid d \gamma \leq K \int \mathcal{F}(h_j \mathcal{F}_j) d \gamma < \infty.$$ 

Conversely, if $A'(E) \neq A'_b(E)$ let $T \circ h_j \in A'_b(E)$ where $\lim_j |h_j| = \infty$.

Without loss of generality take $|k_{n_j}| \geq j$ and define $g = \sum h_j \mathcal{F}_j$ such that $h_{n_j} = 1/(j^2 \epsilon_{n_j})$ and $h_m = 0$ if $m \neq n_j$.

Then

$$\int |g| = \int \sum \mid h_j \mathcal{F}_j(\gamma) \mid d \gamma = \sum \frac{1}{j^2 \epsilon_{n_j}} \int \mathcal{F}_{n_j} = \sum \frac{1}{j^2} = \infty.$$ 

On the other hand

$$\int \sum \mid h_j h_j \mathcal{F}_j \mid \geq \left( \frac{1}{j \epsilon_{n_j}} \mathcal{F}_{n_j}(\gamma) \right) d \gamma \geq \sum \frac{1}{j},$$

a contradiction.

q.e.d.

Obviously, Prop. 5.2 is just a usual duality property between $\ell^\infty$ and $L^1$, and has nothing to do with $A'_b(E)$ per se.

REMARK. Note that $A'(E) = A'_b(E)$ if $\Sigma_{n} |c_{j,n}/n^2 = 0 (\epsilon), j \rightarrow \infty$, from Prop. 1.2; and that the metric condition of Prop. 3.2 is much weaker than this.

In [4, Theorem 19], Hardy and Littlewood prove that if $\varphi \in \mathcal{H}(E)$ [5, pp. 70-71] has the Fourier series $\Sigma_{n=0}^{\infty} a_n e^{i\varphi}$ then

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n} \leq \pi \| \varphi \|_1.$$
They show by counter-example that if $\varphi(\gamma) = \sum_{n = N}^{\infty} a_n e^{in\gamma}$ then (5.2) is not necessarily true. We shall give another type of counter example as well as showing

**Proposition 5.2.** For all $T \in A'(I')$ there is $S \in D_b(I')$ such that $\hat{S}(n) = \hat{T}(n)$ for $n \geq 1$.

**Proof.** Given $T$. A direct application of (5.2) says that if $\varphi(\gamma) = \sum_{0}^{N} a_n e^{in\gamma}$ then

$$\sum_{n=1}^{N} \frac{|a_n|}{n} \leq \pi \int_{0}^{2\pi} \left| \sum_{0}^{N} a_n e^{in\gamma} \right| d\gamma.$$  

Now, if $f' = T$ we have $\hat{f}(n) = 0 \left( \frac{1}{|n|} \right), |n| \to \infty$; and hence there is a constant $K_T$ such that for all trigonometric polynomials of the form $\varphi(\gamma) = \sum_{1}^{N} a_n e^{in\gamma}$

$$\left\| f_{\varphi} \right\| = |\left\langle f, \varphi \right\rangle| \leq K_T \| \varphi \|_1.$$  

Consequently, by the Hahn-Banach theorem there is $g \in L^\infty$ such that $\left\langle f - g, \varphi \right\rangle = 0$ for all $\varphi(\gamma) = \sum_{1}^{N} a_n e^{in\gamma}$.

In particular, $\hat{f}(n) = \hat{g}(n)$ for all $n \geq 0$. q.e.d.

Because of Prop. 5.2 we say that $E$ has **bounded halves** if for all $T \in A'(E)$ there is $S \in D_b(E)$ such that $\hat{T}(n) = \hat{S}(n)$ for $n \geq 1$. The question is, of course, to determine for given $E \subset I'$ the type of subset $X \subset Z$ such that for all $T \in A'(E)$ there is $S \in D_b(E)$ for which $\hat{T} = \hat{S}$ on $X$. Obviously the problem is meaningful in a much more general context.

Now, assuming $E$ has bounded halves we wish to find conditions so that $A'(E) = A'_b(E)$. Arithmetic properties definitely play a role here. In fact, using a (by now) standard approximation technique [6,10], we have

**Proposition 5.3.** Let $E$ be a Dirichlet set with bounded halves. Then $A'(E) = A'_b(E)$.

**Proof.** Let $T \in A'(E)$ and $S \in D_b(E), \hat{S} = \hat{T}$ for $n \geq 1$. 
Observe that \( E \) Dirichlet is equivalent to

\[
\lim_{n \to \infty} \sup_{\gamma \in E} |\sin n \gamma| = 0.
\]

From (5.3) we know that for all \( \varepsilon > 0 \) there is a positive integer \( n_\varepsilon \) such that

\[
\sup_{\gamma \in E} |\sin n_\varepsilon \gamma| < \frac{\varepsilon}{2}
\]

and

\[
\lim_{\varepsilon \to 0} n_\varepsilon = \infty.
\]

Next we define the continuous \( \varepsilon \)-diminishing \(-M\) function \( M_\varepsilon \) in \([-\pi, \pi)\) to be 0 at 0 and outside \((-2 \varepsilon, 2 \varepsilon)\), \( \varepsilon \) at \( \pm \varepsilon \), and linear otherwise. Then from (5.4) we have for \( S = g' \),

\[
(\hat{S} - \hat{T})(2n_\varepsilon) - (\hat{S} - \hat{T})(0) = -\frac{i}{\pi} \langle S - T, e^{-i n_\varepsilon \gamma} M_\varepsilon(\sin n \gamma) \rangle,
\]

since there is a neighborhood of \( E \) in which \( |\sin n \gamma| \leq \varepsilon \). A main feature of \( M_\varepsilon \) is that \( \|M_\varepsilon\|_A \to 0 \) and so, since \( (\hat{S} - \hat{T})(2n_\varepsilon) = 0, (\hat{S} - \hat{T})(0) = 0 \). A similar calculation shows \( (\hat{S} - \hat{T})(n) = 0 \) for all \( n \leq 0 \). Thus \( S = T \).

Example 5.2. If the analogue of (5.2) were true for \( \varphi(\gamma) = \sum_{|n| \leq N} a_n e^{in\gamma} \) then the proof of Prop. 5.2 shows that \( A'(T) \subseteq D_T(I) \) which contradicts Example 5.1.


Using Wik's theorem that \( A(E) = A_+ (E) \) characterizes Helson sets \([7]\) we have
PROPOSITION 6.1. Let \( E \) be Helson. For all \( m < 0 \) there is \( \sum_{n=0}^{\infty} |a_{n,m}| < \infty \) so that for each \( T \in A'(E) \) we have

\[
\widehat{T}(-m) = \lim_{J \to \infty} \sum_{n=0}^{\infty} a_{n,m} \widehat{\mu}_J(-n),
\]

where \( \{\mu_J\} \) is the sequence of measure corresponding to \( T \) (as in (4.5)).

PROOF. \( e^{in\gamma} = \sum_{n=0}^{\infty} a_{n,m} e^{in\gamma} = \varphi(\gamma) \) on \( E \), \( \sum_{n=0}^{\infty} |a_{n,m}| < \infty \) since \( E \) is Helson.

Thus, using the notation of (4.5) for \( T \in A'(E) \), we have

\[
2\pi \widehat{\mu}_J(-m) = \langle \mu_J, \varphi \rangle;
\]

and hence \( \lim_{J \to \infty} \langle \mu_J, \varphi \rangle \) exists. q.e.d.

Now, if \( \varphi(\gamma) = \sum_{n=0}^{\infty} a_n e^{in\gamma} \in A_+(E) \) we write

\[
\varphi_r(\gamma) = \sum_{n=0}^{\infty} a_n r^n e^{in\gamma}, r \in (0,1).
\]

Note that \( \varphi_r \in C^\infty(\Gamma) \), and hence for each \( r \in (0,1) \), \( T \in A'(E) \), and \( \varphi \in A_+(E) \) we have \( \lim_{J \to \infty} \langle \mu_J - T, \varphi_r \rangle = 0 \).

PROPOSITION 6.2 Let \( E \) be Helson. Assume \( T \in A'(E) \) has the property that for each \( \varphi \in A_+(E) \), there exists

\[
\lim_{J \to \infty} \langle \mu_J, \varphi \rangle, \quad \text{uniformly in } r \in \left[\frac{1}{2}, 1\right].
\]

Then \( T \in \mathcal{M}(E) \).

PROOF. (6.2) allows us to use Moore-Smith so that \( \langle \mu_J, \varphi \rangle \) converges for all \( \varphi \in A(\Gamma) \).

Thus by the uniform boundedness principle and the fact that \( E \) is Helson we have \( \|\mu_J\|_1 \) bounded. Consequently by Alaoglu and Prop. 1.1, \( T \in \mathcal{M}(E) \). q.e.d.

For example, if \( r = 1 - \frac{1}{n} \) then for \( \varphi \in \sum_{n=0}^{\infty} a_n e^{in\gamma} \in A_+(E) \) and \( T \in A'(E) \),

\[
\langle \mu_J, \varphi_r \rangle = 2\pi \sum_{j=1}^{\infty} \left( \sum_{p=1}^{j} a_p \widehat{\mu}_J(-p) \right) \left( 1 - \frac{1}{n} \right)^j \frac{1}{n},
\]

noting that \( 1 - \frac{1}{n} = \sum_{j=1}^{\infty} \frac{1}{n} \left( 1 - \frac{1}{n} \right)^j \).
Prompted by Prop. 6.2 consider diagonal sums
\[ \sum_{n=0}^{\infty} a_n \mu_j (-n) F(J)^n \]
where \( 0 < F(J) < 1 \) and \( F(J) \to 1 \).

Generally, in a dual system \((X, Y)\) of \( T_2 \) locally convex spaces we say that a directed system \([T_a] \subseteq X\) converges in the \( \sigma\sigma(X, Y)\) topology to \( T \in X\) if for all \( \varphi \in Y \) there is \([\varphi_a] \subseteq Y\) such that \( \varphi_a \) converges to \( \varphi \) and
\[ \lim_{a} \langle T_a - T, \varphi_a \rangle = 0. \]

Although significantly weaker than the weak * topology, it is not generally minimal [9, p. 191] and the intermediate topologies between \( \sigma(X, Y)\) and \( \sigma\sigma(X, Y)\) become interesting in light of Prop. 6.2, the lack of weak * convergence in § 4, and the convergence in Prop. 1.1 (in terms of (4.5)).

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