ANNALI DELLA
SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

A. ANDREOTTI
F. NORGUET

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CYCLES OF ALGEBRAIC MANIFOLDS AND $\partial\bar{\partial}$-COHOMOLOGY

by A. ANDREOTTI and F. NORGUET *

Let X be an open set of an algebraic projective manifold Z and let $C_d^+(X)$ denote the space of d-dimensional positive cycles of Z which support in X. In [2] we have defined a map

$$\varrho_0 : H^d(X, \Omega^d) \longrightarrow \Gamma(\mathcal{C}_d^+(X), \mathcal{O})$$

and have remarked that in general ϱ_0 has a large kernel.

In this paper we study the map ϱ_0 in the case in which X is a Zariski open set in Z, $Y = Z - X$ is an algebraic submanifold of Z of codimension $d+1$ and X is d-pseudo convex. These conditions are verified if Y is a complete intersection on Z of $d+1$ divisors of holomorphic sections of a negative line bundle.

We are able to establish then that up to a finite dimensional vector space $\operatorname{Ker} \varrho_0 = \operatorname{Im} \{H^d(X, \Omega^{d-1}) \xrightarrow{d} H^d(X, \Omega^d)\}$.

This situation suggests by itself the introduction of the groups

$$V^{dd}(X) = \frac{\operatorname{Ker}(A^{dd}(X) \xrightarrow{\partial\bar{\partial}} A^{d+1\, d+1}(X)}{\partial A^{d-1\, d}(X) + \bar{\partial} A^{d\, d-1}(X)}$$

where $A^{rs}(X)$ is the space of (r, s) C^∞ forms on X. Indeed it turns out then that the corresponding map

$$\varrho_0 : V^{dd}(X) \longrightarrow \Gamma(\mathcal{C}_d^+(X), \mathcal{H})$$

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where $\mathcal{H}$ is the sheaf of germs of pluriharmonic functions, is, up to a finite dimensional space, injective.

The necessity to argue modulo finite dimensionality is unavoidable in this type of question as one can see in the examples. It is therefore convenient to make use of the terminology and properties of « classes » introduced by Serre in [6].

The possibility to pass from the Dolbeault cohomology to the $\partial\bar{\partial}$-cohomology is based on the study of Bigolin [3].

§ 1. On the Cohomology of Certain Domains.

0. *The class of finite dimensional vector spaces.* In the course of this research we will have to disregard finite dimensional vector spaces to enable us to clearly state some of our conclusions. It turns out, therefore, to be appropriate to introduce the *class Φ of all finite dimensional vector spaces over* $\mathbf{C}$ as it is explained in Serre [6].

We recall here for the convenience of the reader the main features of the class Φ. It satisfies the following conditions:

(*a*) every subspace and every quotient of an element of Φ is again in Φ;

(*b*) if $0 \to V' \to V \to V'' \to 0$ is an exact sequence of vector spaces over $\mathbf{C}$ and if V', V'' are in Φ, so is $V \in \Phi$;

(*c*) if V, V' are in Φ then $V \otimes_{\mathbf{C}} V' \in \Phi$.

Let E, F be any two vector spaces over $\mathbf{C}$ and let $f : E \to F$ be a linear map. We say that

f is Φ-injective if $\operatorname{Ker} f \in \Phi$

f is Φ surjective if $\operatorname{Coker} f \in \Phi$

f is Φ-isomorphism if it is Φ injective and Φ surjective.

Given two subspaces H, L of a vector space E, we will say that

H and L are Φ-equal if $H \cap L \subset H$ and $H \cap L \subset L$ are Φ-isomorphisms.

Given two vector spaces E, F over $\mathbf{C}$ a *Φ-homomorphism* from E to F is by definition a subspace $G \subset E \times F$ such that

i) $pr_E : G \to E$ is Φ-surjective,

ii) $(\{0\} \times F) \cap G \in \Phi$.

A Φ-homomorphism G from E to F is called

Φ-injective if $G \cap (E \times \{0\}) \in \Phi$

Φ-surjective if $pr_F: G \longrightarrow F$ is Φ surjective

Φ-isomorphism if it is Φ-injective and Φ-surjective.

We can then talk about Φ-exact sequences of vector spaces $\{E_i\}$ over $\mathbf{C}$ and Φ-homomorphisms G_i from E_i to E_{i+1}

$$\rightsquigarrow E_i \overset{G_i}{\rightsquigarrow} E_{i+1} \overset{G_{i+1}}{\rightsquigarrow} E_{i+2} \rightsquigarrow \cdots$$

if for each i, $pr_{E_{i+1}} G_i$ is Φ-equal to $(E_{i+1} \times \{0\}) \cap G_{i+1}$. The Φ-notions will be denoted with the usual symbols affected with the index Φ.

1. *Cohomology of certain elementary domains a)* Let $D = \{t \in \mathbf{C} \mid |t| < 1\}$ so that

$$D^q = \{(z_1, \ldots, z_q) \in \mathbf{C}^q \mid |t_i| < 1 \text{ for } 1 \leq i \leq q\}.$$

We set

$$V_q = (D^{q+1} - \{\mathbf{o}\}) \times D^{n-q-1} \qquad \text{for } 0 \leq q \leq n-1.$$

LEMMA 1.

$$H^r(V_q, Z) = \begin{cases} 0 & \text{for } r \neq 0, 2q+1 \\ \mathbf{Z} & \text{for } r = 0, 2q+1 \end{cases}.$$

PROOF. V_q is contractible on $D^{q+1} - \{0\}$ and this on the unit sphere S^{2q+1} of $\mathbf{C}^{q+1}$.

LEMMA 2.

a) $H^r(D^{q+1} - \{0\}, \mathcal{O}) = 0$ for $r \neq 0, q$.

b) $H^0(D^{q+1} - \{0\}, \mathcal{O}) \simeq H^0(D^{q+1}, \mathcal{O})$ if $q \geq 1$.

c) $H^q(D^{q+1} - \{0\}, \mathcal{O}) \simeq \left\{(z_1 \ldots z_{q+1})^{-1} \cdot \sum_{\alpha \in \mathbf{N}^{q+1}} C_\alpha z^{-\alpha} \;\middle|\; \lim \sqrt[|\alpha|]{|C_\alpha|} = 0\right\}.$

This lemma is due to J. Frenkel [4]. If $U_i = \{z \in D^{q+1} \mid z_i \neq 0\}$ then $\mathcal{U} = \{U_i\}_{1 \leq i \leq q+1}$ is a Stein covering of $D^{q+1} - \{0\}$, thus

$$H^q(D^{q+1} - \{0\}, \mathcal{O}) \simeq H^q(\mathcal{U}, \mathcal{O}).$$

Then the isomorphism of *c*) is given as follows. Taking Čech alternate cochains, one has [1]

$$Z^q(\mathcal{U}, \mathcal{O}) = \Gamma\left(\bigcap_{i=1}^{q+1} U_i, \mathcal{O}\right)$$

$$= B^q(\mathcal{U}, \mathcal{O}) \oplus \{(z_1 \dots z_{q+1})^{-1} \cdot \sum_{\alpha \in \mathbb{N}^{q+1}} C_\alpha z^{-\alpha} \mid \lim \sqrt[|\alpha|]{C_\alpha} = 0\}$$

and thus *c*) is obtained associating to each function on the right the cohomology class it represents in the Čech cohomology for the covering $\mathcal{U}$.

LEMMA 3.

a) $H^r(V, \Omega^p) = 0$ for $r \neq 0, q$,

b) $H^0(V, \Omega^p) = H^0(D^n, \Omega^p)$ if $q \geq 1$,

c) $H^q(V, \Omega^p) = \coprod_{s+r=p} H^q(D^{q+1} - \{0\}, \Omega^s) \widehat{\otimes} H^0(D^{n-q-1}, \Omega^r)$.

This is a consequence of Lemma 2 and the Künneth formula [5] [2].

b) Let us consider first the cohomology groups $D^{q+1} - \{0\} \subset \mathbb{C}^{q+1}$.

LEMMA 4. *a*) *The cohomology group* $H^q(D^{q+1} - \{0\}, \mathcal{O})$ *is the Fréchet space generated by cohomology classes*

$$z_1^{-\alpha_1-1} \dots z_{q+1}^{-\alpha_{q+1}-1} \in \Gamma\left(\bigcap_{i=1}^{q+1} U_i, \mathcal{O}\right).$$

b) *By the Dolbeault isomorphism the cohomology class* $z_1^{-\alpha_1-1} \dots z_{q+1}^{-\alpha_q-1}$ *corresponds to the* $(0, q)$ $\bar{\partial}$*-closed form* (*up to a sign*)

$$q!\, \psi_{\alpha+1} = q!\, \frac{\sum (-1)^k \bar{z}_k^{\alpha_k+1}\, d\bar{z}_1^{\alpha_1+1} \wedge \dots \wedge \widehat{d\bar{z}_k^{\alpha_k+1}} \wedge \dots \wedge d\bar{z}_{q+1}^{\alpha_{q+1}+1}}{\left(\sum_{j=1}^{q+1} |z_j^{\alpha_j+1}|^2\right)^{q+1}}.$$

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(1) As usual $Z^k(\mathcal{U}, \mathcal{O})$ denote the k-cocyles and $B^k(\mathcal{U}, \mathcal{O})$ the k-coboundaries for the Čech complex of the covering $\mathcal{U}$.

(2) Note that by Lemma 2 the cohomology groups of $D^{q+1} - 0$ have all Hausdorff topology.

c) The Dolbeault representative so chosen is such that to the class $(z_1 \dots z_{q+1})^{-1} \Sigma C_\alpha z^\alpha$ *with* $\lim \sqrt[|\alpha|]{|C_\alpha|} = 0$, *corresponds the* $(0, q)$*-form expressed by the absolutely convergent series* $q! \, \Sigma C_\alpha \psi_{\alpha+1}$ *on* $\mathbf{C}^{q+1} - \{0\}$.

PROOF. *a*) is obvious from Lemma 2. To prove *b*), we proceed as follows. If we make the Dolbeault isomorphism explicit, we set (recall $\alpha + 1 = (\alpha_1 + 1, \dots, \alpha_{q+1} + 1)$)

$z^{-\alpha-1} = \Sigma (-1)^i \varphi_{1 \dots \hat{i} \dots q+1}$, where $\varphi_{1 \dots \hat{i} \dots q+1}$ are C^∞ functions on

$$U_1 \cap \dots \cap \widehat{U_i} \cap \dots \cap U_{q+1}$$

then

$$\Sigma (-1)^i \bar{\partial} \varphi_{1 \dots \hat{i} \dots q+1} = 0$$

and thus we can find $(0, 1)$ C^∞ forms $\varphi_{1 \dots \hat{i} \dots \hat{j} \dots q+1}$ on $U_1 \cap \dots \cap \widehat{U_i} \cap \dots \cap \widehat{U_j} \cap \dots \cap U_{q+1}$ such that

$$\bar{\partial} \varphi_{1 \dots \hat{i} \dots q+1} = \Sigma \pm \varphi_{1 \dots \hat{i} \dots \hat{j} \dots q+1}$$

then

$$\Sigma \pm \bar{\partial} \varphi_{1 \dots \hat{i} \dots \hat{j} \dots q+1} = 0 \quad \text{etc.}$$

At the end we get $q + 1$ $(0, q - 1)$ forms φ_j, C^∞ on U_j, with $\bar{\partial}\varphi_j = \bar{\partial}\varphi_i = \psi$ and ψ is the Dolbeault representative of $z^{-\alpha-1}$.

Now we remark that $z^{-\alpha-1}$ is characterized among the generators by the property

$$\int_{\substack{|z_1| = \varepsilon \\ \dots \\ |z_{q+1}| = \varepsilon}} z^{-\alpha-1} \Sigma C_\beta z^\beta dz_1 \dots dz_{q+1} = (2\pi i)^{q+1} C_\alpha$$

for any convergent power series $f = \Sigma C_\beta z^\beta$ in a neighborhood of the origin containing the polycylinder $P_\varepsilon = \{ |z_i| \leq \varepsilon \}$.

Now we have

$$\int_{\left\{\substack{|z_1| = \varepsilon \\ \dots \\ |z_{q+1}| = \varepsilon}\right.} z^{-\alpha-1} f dz_1 \dots dz_{q+1} = \Sigma \int_{\left\{\substack{|z_1| = \varepsilon \\ \dots \\ |z_{q+1}| = \varepsilon}\right.} (-1)^i \varphi_{1 \dots \hat{i} \dots q+1} f dz_1 \dots dz_{q+1}$$

$$= \Sigma (-1)^i \int_{\begin{cases} |z_1| = \varepsilon \\ \cdots \\ |z_i| \leq \varepsilon \\ \cdots \\ |z_{q+1}| = 1 \end{cases}} \bar{\partial} \varphi_{1 \ldots \hat{i} \ldots q+1} f dz_1 \ldots dz_{q+1}$$

$$= \ldots$$

$$= \pm \Sigma \int_{\substack{|z_i| = \varepsilon \\ |z_j| \leq \varepsilon \; j \neq i}} \psi f dz_1 \ldots dz_{q+1} = \pm \int_{\partial P_\varepsilon} \psi f dz_1 \ldots dz_{q+1}$$

$$= (2\pi i)^{q+1} C_\alpha .$$

It follows then that if B is a small ball with center at the origin and if ψ corresponds to $z^{-\alpha-1}$ we must have

$$\int_{\partial B} \psi f dz_1 \ldots dz_{q+1} = \pm (2\pi i)^{q+1} (\alpha !)^{-1} \frac{\partial^\alpha f}{\partial z^\alpha}(0).$$

Therefore according to [1] Proposition 1, we get that $z^{-\alpha-1}$ corresponds (up to sign possibly) to the Dolbeault class of $q ! \, \psi_{\alpha+1}$.

To prove c) we remark that if $\delta < 1$ and $\Sigma |z_i|^2 > \delta$ then for at least one i, $|z_i| > \delta/(q+1)$ thus $\Sigma |z_i|^{2\alpha_i+2} > (\delta/(q+1))^{2\alpha_i+2} > (\delta/(q+1))^{2|\alpha|+2}$. Then the coefficient of $d\bar{z}_1 \wedge \ldots \wedge \widehat{d\bar{z}_j} \ldots d\bar{z}_{q+1}$ in ψ_α is majorized by

$$|z_j| \cdot |z_1|^{\alpha_1} \ldots |z_{q+1}|^{\alpha_{q+1}} \cdot \left\{ \left(\frac{\delta}{q+1} \right)^{2|\alpha|+2} \right\}^{-q-1}$$

$$\leq |z_j| \cdot \left(\frac{\delta}{q+1} \right)^{-2q-2} \left(\frac{|z_1|}{\left(\frac{\delta}{q+1} \right)^{2q+2}} \right)^{\alpha_1} \cdots \left(\frac{|z_{q+1}|}{\left(\frac{\delta}{q+1} \right)^{2q+2}} \right)^{\alpha_{q+1}} .$$

Thus in the series $\Sigma C_\alpha \psi_\alpha$ the coefficient of $d\bar{z}_1 \wedge \ldots \wedge \widehat{d\bar{z}_j} \wedge \ldots \wedge d\bar{z}_{q+1}$ is majorized by the absolutely convergent series:

$$|z_j| \left(\frac{\delta}{q+1} \right)^{-2q-2} \Sigma |C_\alpha| w^\alpha \quad \text{where} \quad w_i = \frac{|z_i|}{\left(\frac{\delta}{q+1} \right)^{2q+2}} .$$

COROLLARY 1. *Every element of $H^q(D^{q+1} - \{0\}, \Omega^q)$ can be represented by a Čech cohomology class of the form*

$$\sum_{\alpha \in \mathbf{N}^{q+1}} \frac{1}{z^{\alpha+1}} \varphi_\alpha$$

where

$$\varphi_\alpha = \Sigma\, C_{\alpha i}\, dz_1 \wedge \ldots \wedge \widehat{dz_i} \wedge \ldots \wedge dz_{q+1} \qquad \lim_{|\alpha| \to \infty} \sqrt[|\alpha|]{|C_{\alpha i}|} = 0,$$

to which corresponds the Dolbeault representative

$$q\,!\; \Sigma\, \varphi_\alpha \wedge \psi_{\alpha+1}$$

expressed as an absolutely and uniformly convergent series on any compact set of $\mathbf{C}^{q+1} - \{0\}$.

PROOF. The proof follows from the fact that $\Omega^q = \underbrace{\mathcal{O} \oplus \ldots \oplus \mathcal{O}}_{q+1 \text{ times}}$.

c) Let us now consider $V = (D^{q+1} - \{0\}) \times D^{n-q-1}$ where in $D^{q+1} \subset \mathbf{C}^{q+1}$ we denote by $z_1, \ldots, z_{q+1}$ the coordinates while on $D^{n-q-1} \subset \mathbf{C}^{n-q-1}$ we will denote the coordinates by $w_1, \ldots, w_{n-q-1}$.

COROLLARY 2. *Every element of $H^q(V, \Omega^q)$ can be represented by a Čech cohomology class of the form*

$$\theta = \sum_{\alpha \in \mathbf{N}^{k+1}} \frac{1}{z^{\alpha+1}} \theta_\alpha$$

where

$$\theta_\alpha = \Sigma\, C_\alpha(w)_{i_1 \ldots i_p, j_1 \ldots j_{q-p}}\, dz^{i_1} \wedge \ldots \wedge dz^{i_p} \wedge dw^{j_1} \wedge \ldots \wedge dw^{j_{q-p}}$$

with $C_\alpha(w)$ holomorphic in D^{n-q-1} with $\lim \sqrt[|\alpha|]{|C_\alpha(w)|} = 0$, $\forall\, w \in D^{n-q-1}$. The form θ also corresponds to the Dolbeault representative:

$$\Theta = q\,! \sum_{\alpha \in \mathbf{N}^{q+1}} \theta_\alpha \wedge \psi_{\alpha+1}$$

expressed as an absolutely and uniformly convergent series on any compact subset of $(\mathbf{C}^{q+1} - 0) \times D^{n-q-1}$.

2. *$\partial\bar{\partial}$ groups and Dolbeault Groups.* Let X be a complex manifold, we will denote by $\mathcal{A}^{r,s}$ the sheaf of germs of C^∞ differential forms of type

r, s. We set $A^{r,s}(X) = \Gamma(X, \mathcal{A}^{r,s})$. We want to consider the following groups:

$$V^{q,q}(X) = \frac{\operatorname{Ker}\left(A^{q,q}(X) \xrightarrow{\partial\bar\partial} A^{q+1,q+1}(X)\right)}{\partial A^{q-1,q}(X) + \bar\partial A^{q,q-1}(X)}.$$

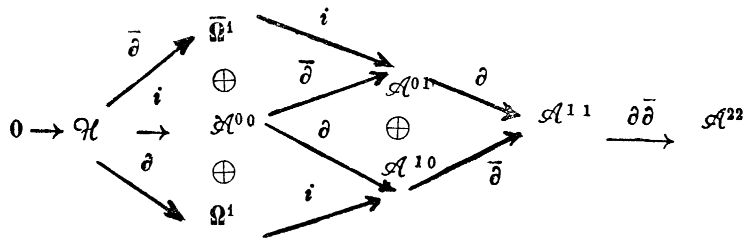
To relate them to the usual Dolbeault groups we will make use of the resolutions of the sheaf $\mathcal{H}$ of germs of pluriharmonic functions given in Bigolin's thesis [3].

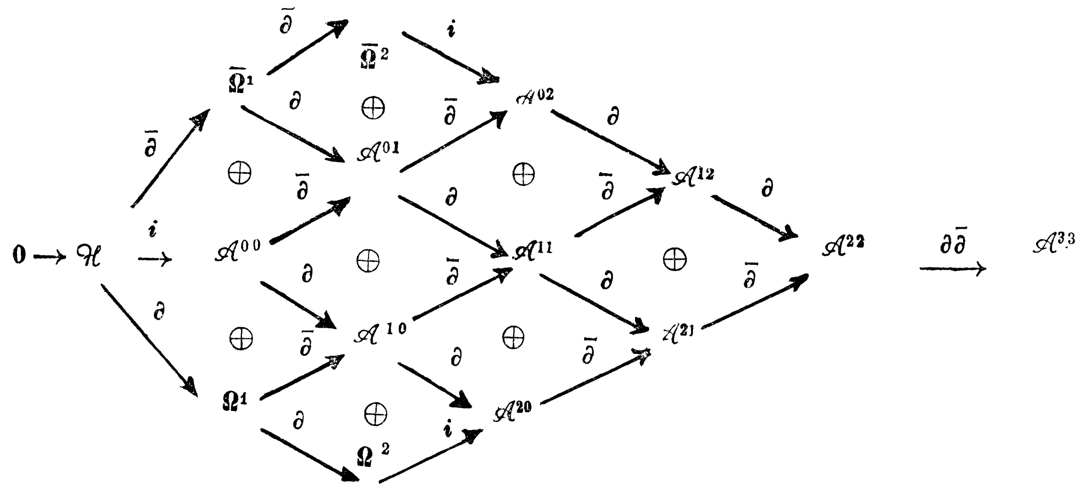
First of all we have the exact sequence

$$(1) \qquad 0 \to \mathbf{C} \xrightarrow{\alpha} \mathcal{O} \oplus \bar{\mathcal{O}} \xrightarrow{\beta} \mathcal{H} \to 0$$

where $\alpha(c) = c \oplus c$, $\beta(f \oplus g) = f - \bar g$ and where the bar denotes complex conjugate (so that, for instance, $\bar{\mathcal{O}}$ denotes the sheaf of germs of antiholomorphic functions).

Secondly we have the following resolutions

(11) 

(22) 

and in general a resolution of the form

$$\text{(qq)}\quad 0 \to \mathcal{H} \to \mathcal{A}^0 \xrightarrow{h_0} \mathcal{A}^1 \xrightarrow{h_1} \dots \to \mathcal{A}^{q-1} \xrightarrow{h_{q-1}} \mathcal{B}^q \xrightarrow{h_0} \dots \mathcal{B}^{q+1} \to$$

$$\dots \to \mathcal{B}^{2q} \to \mathcal{A}^{qq} \xrightarrow{\partial\bar{\partial}} \mathcal{A}^{q+1,q+1}$$

where

$$\left\{\begin{array}{l} \mathcal{B}^i = \bigoplus_{j=0}^{i} \mathcal{A}^{s,i-s} \\ \text{for } 0 \leq i \leq q \end{array}\right. ; \quad \left\{\begin{array}{l} \mathcal{B}^i \bigoplus_{s \geq i-q}^{q} \mathcal{A}^{s,i-s} \\ \text{for } i \geq q \end{array}\right. ; \quad \mathcal{A}^i = \bar{\Omega}^{i+1} \oplus \mathcal{B}^i \oplus \Omega^{i+1}$$

and where the maps are induced by $\partial, \bar{\partial}$ and injection according to the necessity of bigraduation.

PROPOSITION 1. *Let Φ be the Serre class of finite dimensional vector spaces (or the class of the 0 dimensional vector space). Let X be any complex manifold and let $q > 0$ be an integer.*

If

$$H^j(X, \Omega^k) \in \Phi \qquad \text{for } j + k \leq 2q, j > q$$

and

$$H^{2q}(X, \mathbf{C}) \in \Phi, H^{2q+1}(X, \mathbf{C}) \in \Phi,$$

then we have a Φ exact sequence

$$\text{(2)}\quad H^q(X, \Omega^{q-1}) \oplus H^q(X, \bar{\Omega}^{q-1}) \xrightarrow{\alpha} H^q(X,\Omega^q) \oplus H^q(X, \bar{\Omega}^q) \xrightarrow{\beta} V^{qq}(X) \to 0$$

where α is induced by the sheaf homomorphism given by exterior differentiation

$$d : \Omega^{q-1} \to \Omega^q, \; d : \bar{\Omega}^{q-1} \to \bar{\Omega}^q$$

and where

$$\beta(a \oplus b) = a - b.$$

Before proving Proposition 1 we state explicitly the following:

PROPOSITION. *If X is any complex manifold we have the exact cohomology sequence of* (1)

$$H^i(X, \mathbf{C}) \to H^i(X, \mathcal{O}) \oplus H^i(X, \bar{\mathcal{O}}) \xrightarrow{\beta} H^i(X, \mathcal{H}) \to H^{i+1}(X, \mathbf{C})$$

so that if $H^i(X, \mathbf{C}) \in \Phi$ *and* $H^{i+1}(X, \mathbf{C}) \in \Phi$ *we have the* Φ*-isomorphism*

$$H^i(X, \mathcal{O}) \oplus H^i(X, \bar{\mathcal{O}}) \underset{\Phi}{\xrightarrow{\overset{\beta}{\sim}}} H^i(X, \mathcal{H})$$

given by $\beta(a \oplus b) = a - b$.

PROOF OF PROPOSITION 1. We use the resolution (qq). Set

$$\mathcal{Z}^j = \ker h_j$$

then we get the following exact sequence of sheaves:

$$(*) \qquad 0 \longrightarrow \mathcal{Z}^q \longrightarrow \mathcal{B}^q \xrightarrow{h_q} \mathcal{B}^{q+1} \dashrightarrow \ldots \longrightarrow \mathcal{B}^{2q} \longrightarrow \mathcal{A}^{qq} \xrightarrow{\partial\bar{\partial}} \mathcal{A}^{q+1,\, q+1}$$

and

$$(**) \qquad \begin{array}{l} 0 \longrightarrow \mathcal{Z}^{q-1} \longrightarrow \mathcal{A}^{q-1} \longrightarrow \mathcal{Z}^q \longrightarrow 0 \\ 0 \longrightarrow \mathcal{Z}^{q-2} \longrightarrow \mathcal{A}^{q-2} \longrightarrow \mathcal{Z}^{q-1} \longrightarrow 0 \\ \vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\quad \vdots \\ 0 \longrightarrow \mathcal{H} \longrightarrow \mathcal{A}^0 \longrightarrow \mathcal{Z}^1 \longrightarrow 0 \end{array}$$

From $(*)$ since the sheaves $\mathcal{B}^q$ are fine sheaves we get

$$(\alpha) \qquad V^{qq}(X) \simeq H^q(X, \mathcal{Z}^q).$$

From the first of $(**)$ we get, provided $q > 1$,

$$(\beta) \quad H^q(X, \mathcal{Z}^{q-1}) \longrightarrow H^q(X, \Omega^q) \oplus H^q(X, \bar{\Omega}^q) \longrightarrow H^q(X, \mathcal{Z}^q) \longrightarrow H^{q+1}(X, \mathcal{Z}^{q-1})$$

and from the other sequences $(**)$ we obtain the Φ homomorphisms

$$(\gamma) \quad H^{q+1}(X, \mathcal{Z}^{q-1}) \simeq_\Phi H^{q+2}(X, \mathcal{Z}^{q-2}) \simeq_\Phi \ldots \simeq_\Phi H^{2q}(X, \mathcal{H}) \simeq_\Phi H^{2q+1}(X, \mathbf{C}) \simeq_\Phi 0.$$

Moreover from the second of $(**)$ we get

$$(\delta) \qquad H^q(X, \Omega^{q-1}) \oplus H^q(X, \bar{\Omega}^{q-1}) \longrightarrow H^q(X, \mathcal{Z}^{q-1}) \longrightarrow H^{q+1}(X, \mathcal{Z}^{q-2})$$

and the Φ-isomorphisms

$$(\varepsilon) \quad H^{q+1}(X, \mathcal{Z}^{q-2}) \simeq_\Phi H^{q+2}(X, \mathcal{Z}^{q-3}) \simeq_\Phi \ldots \simeq_\Phi H^{2q-1}(X, \mathcal{H}) \simeq_\Phi H^{2q}(X, \mathbf{C}) \simeq_\Phi 0$$

provided $q > 2$.

From $(\alpha), (\beta), (\gamma), (\varepsilon)$ we deduce then the Φ-exact sequence

$$H^q(X, \Omega^{q-1}) \oplus H^q(X, \bar{\Omega}^{q-1}) \to H^q(X, \Omega^q) \oplus H^q(X, \bar{\Omega}^q) \to V^{qq}(X) \to 0$$

provided $q > 2$.

If $q = 2$ then (δ) is replaced by

$$H^q(X, \Omega^1) \oplus H^2(X, \bar{\Omega}^1) \to H^2(X, \mathscr{Z}^1) \to H^3(X, \mathscr{H})$$

and $H^3(X, \mathscr{H}) \simeq_\Phi H^4(X, \mathbf{C}) \simeq_\Phi 0$ so the conclusion holds also for $q = 2$. If $q = 1$ then (β) is replaced by

$$H^1(X, \mathscr{H}) \to H^1(X, \Omega^1) \oplus H^1(X, \bar{\Omega}^1) \to H^1(X, \mathscr{Z}^1) \to H^2(X, \mathscr{H})$$

while we have

$$H^2(X, \mathscr{H}) \simeq_\Phi H^2(X, \mathbf{C}) \simeq_\Phi 0$$

and

$$H^1(X, \mathscr{H}) \simeq_\Phi H^1(X, \mathcal{O}) \oplus H^1(X, \bar{\mathcal{O}}).$$

Thus the Φ-exactness of the sequence (2) is proved in any case.

Now we remark that if $q > 2$ the Φ homomorphism α is actually given by the composition of the two homomorphisms

$$\lambda\colon\ H^q(X, \Omega^{q-1}) \oplus H^q(X, \bar{\Omega}^{q-1}) \to H^q(X, \mathscr{Z}^{q-1})$$

$$\mu\colon\ H^q(X, \mathscr{Z}^{q-1}) \to H^q(X, \Omega^q) \oplus H^q(X, \bar{\Omega}^q)$$

which are induced by the sheaf homomorphism

$$\mathscr{A}^{q-2} \xrightarrow{h_{q-2}} \mathscr{A}^{q-1}$$

when we take cohomology groups. By the nature of h_{q-2} this reduces to the direct sum of the map

$$d\colon\ H^q(X, \Omega^{q-1}) \to H^q(X, \Omega^q)$$

$$d\colon\ H^q(X, \bar{\Omega}^{q-1}) \to H^q(X, \bar{\Omega}^q).$$

For $q = 2$, $q = 1$ a direct argument leads to the same conclusion.

An analogous argument shows that β is the natural map induced by the natural homomorphisms

$$H^q(X, \Omega^q) \longrightarrow V^{qq}(X)$$

$$H^q(X, \overline{\Omega}^q) \longrightarrow V^{qq}(X)$$

which associate to the class in $H^q(X, \Omega^q)$ $(H^q(X, \overline{\Omega}^q))$ represented by a (q, q) form φ $\overline{\partial}$-closed (∂-closed) the class represented by the same form in $V^{qq}(X)$.

COROLLARY. *Let X be a q-pseudoconvex manifold then*

$$V^{qq}(X) \simeq_{\Phi} \operatorname{Coker}\{H^q(X, \Omega^{q-1}) \xrightarrow{d} H^q(X, \Omega^q)\} \oplus$$

$$\oplus \operatorname{Coker}\{H^q(X, \overline{\Omega}^{q-1}) \xrightarrow{d} H^q(X, \overline{\Omega}^q)\}$$

if $H^{2q}(X, \mathbf{C})$ and $H^{2q+1}(X, \mathbf{C})$ are finite dimensional. In particular this Φ isomorphism is an isomorphism if X is a q-complete manifold and $H^{2q}(X, \mathbf{C}) = H^{2q+1}(X, \mathbf{C}) = 0$.

For q-complete manifolds, however, the homological condition $H^{2q}(X, \mathbf{C}) = 0$ may not be satisfied.

3. *The $\partial\overline{\partial}$-Groups for the Special Domains.* Let $V_q = (D^{q+1} - \{0\}) \times D^{n-q-1}$ be the special domain considered in n. 1. By Lemma 3 *a*) and Lemma 1 we see that V satisfies the conditions of Proposition 1 and actually, except for the group $H^{2q+1}(V, \mathbf{C}) = \mathbf{C}$ all other groups that are required to be in Φ are actually zero.

For the particular domains V_q we have the following more precise

PROPOSITION 2. *For the special domains V_q we have an exact sequence:*

$$(3) \quad H^q(X, \Omega^{q-1}) \oplus H^q(X, \overline{\Omega}^{q-1}) \xrightarrow{\alpha} H^q(X, \Omega^q) \oplus$$

$$\oplus H^q(X, \overline{\Omega}^q) \xrightarrow{\beta} V^{qq}(V) \xrightarrow{\lambda} \mathbf{C} \to 0$$

where α and β are as in Proposition 1 and where λ is the homomorphism

$$V^{qq}(V) \to H^{2q+1}(V, \mathbf{C}) = \mathbf{C}$$

which associates to the cohomology class $\{\varphi^{qq}\} \in V^{qq}(V)$ the class of the closed *form*

$$\lambda(\varphi) = \partial\varphi - \bar\partial\varphi.$$

Moreover the generator of $H^{2q+1}(V, \mathbf{C})$ can be written as $\lambda(\psi)$, $\{\psi\} \in V^{qq}(V)$ where

$$\psi = \frac{q!}{2(2i\pi)^{q+1}} \log \sum_1^{q+1} z_i \bar z_i \frac{\left| \sum_1^{q+1} (-1)^{i-1} z_i\, dz_1 \wedge \ldots \wedge \widehat{dz_i} \wedge \ldots \wedge dz_{q+1} \right|^2}{\left(\sum_1^{q+1} z_i \bar z_i \right)^{q+1}} .$$

PROOF. The exactness of (3) follows by the argument of Proposition 1, taking into account Lemma 1 and Lemma 3 *a*).

We now prove directly the fact that for $\lambda = \partial - \bar\partial$ the sequence is exact at $V^{qq}(V)$.

Let $\{\eta^{qq}\} \in V^{qq}(V)$ and assume

$$\partial\eta - \bar\partial\eta = (\partial + \bar\partial)(\alpha^{0,2q} + \alpha^{1,2q-1} + \ldots + \alpha^{q-1,q+1} + \alpha^{q,q} + \alpha^{q+1,q-1} + \ldots + \alpha^{2q,0}).$$

From this we get

$$\bar\partial\alpha^{0,2q} = 0$$

$$\partial\alpha^{0,2q} + \bar\partial\alpha^{1,2q-1} = 0$$

$$\partial\alpha^{1,2q-1} + \bar\partial\alpha^{2,2q-2} = 0$$

$$\ldots \qquad \ldots$$

$$\partial\alpha^{q-2,q+2} + \bar\partial\alpha^{q-1,q+1} = 0$$

$$\partial\alpha^{q-1,q+1} + \bar\partial\alpha^{q,q} = -\bar\partial\eta$$

$$\partial\alpha^{q,q} + \bar\partial\alpha^{q+1,q-1} = \partial\eta$$

$$\ldots \qquad \ldots$$

$$\partial\alpha^{2q,0} = 0.$$

From the first row we get $\alpha^{0,2q} = \bar\partial\beta^{0,2q-1}$, thus from the second row we obtain

$$\bar\partial(-\partial\beta^{0,2q-1} + \alpha^{1,2q-1}) = 0$$

hence

$$\alpha^{1,2q-1} = \partial\beta^{0,2q-1} + \bar\partial\beta^{1,2q-2}.$$

Again

$$\bar{\partial}(-\partial\beta^{1,2q-2}+\alpha^{2,2q-2})=0$$

hence

$$\alpha^{2,2q-2}=\partial\beta^{1,2q-2}+\bar{\partial}\beta^{2,2q-3}.\ \ldots$$

Finally we obtain

$$\alpha^{q-1,q+1}=\partial\beta^{q-2,q+1}+\bar{\partial}\beta^{q-1,q}$$

and

$$-\bar{\partial}\eta=\bar{\partial}\alpha^{qq}+\partial\bar{\partial}\,\beta^{q-1,q}.$$

Similarly

$$\partial\eta=\partial\alpha^{qq}+\bar{\partial}\partial\,\beta^{q,q-1}.$$

Set

$$A=\eta+\alpha^{qq}-\partial\,\beta^{q-1,q}$$

$$B=\eta-\alpha^{qq}+\bar{\partial}\,\beta^{q,q-1}$$

so that $\bar{\partial}A=0$ and $\partial B=0$. Hence

$$\eta=A-\alpha^{qq}+\partial\beta^{q-1,q}$$

$$\eta=B+\alpha^{qq}+\bar{\partial}\,\beta^{q,q-1}.$$

From this it follows that

$$\eta^{qq}=\frac{1}{2}(A+\bar{\partial}\,\beta^{q,q-1})+\frac{1}{2}(B+\partial\,\beta^{q-1,q})$$

which shows that $\lambda\{\eta\}=0\Longrightarrow\{\eta\}\in\operatorname{Im}\beta$.

Conversely let $\{\eta\}\in\operatorname{Im}\beta$ then

$$\eta^{qq}=A^{qq}_{\bar{\partial}}+B^{qq}_{\partial}$$

where $\bar{\partial}A=0$ and $\partial B=0$ then

$$\begin{aligned}\lambda\eta=\partial\eta-\bar{\partial}\eta&=\partial A_{\bar{\partial}}-\bar{\partial}B_{\partial}\\&=(\partial+\bar{\partial})\,A+(\partial+\bar{\partial})(-B)\\&=d\,(A-B)\end{aligned}$$

i. e.,

$$\lambda\circ\beta=0.$$

Set now $m=2(q+1)$ and consider in $\mathbf{R}^m=\mathbf{C}^{q+1}$ the form

$$\sigma=\left(\sum_{1}^{q+1}z_i\bar{z}_i\right)^{-m}\Sigma\,(-1)^{j-1}x_j\,dx_1\ldots\widehat{dx_j}\ldots dx_m$$

where $z_j=x_j+ix_{(m/2)+j}$ are the coordinates.

We have $d\sigma = 0$ and $\sigma = \partial\psi - \bar{\partial}\psi$ as one verifies. This shows that $\partial\bar{\partial}\psi = 0$ thus $\{\psi\} \in V^{qq}(V)$. Moreover $\lambda\psi = \sigma$ and on the unit sphere S of $\mathbf{R}^m$ $\sigma|_S$ reduces to the volume element thus $\lambda\{\psi\} \neq 0$.

§ 2. Study of Certain Integrals (continuation of § 3 of [1]).

4. *Preliminaries.* In sections 4 and 5, we sum up some results on Leray's residue theory.

a) Let X be a complex manifold and let S be a complex hypersurface in X, nonsingular.

The inclusion $X - S \subset X$ gives rise to an exact cohomology sequence (coefficients in any abelian group G)

$$\ldots \rightarrow H^i(X \bmod X - S, G) \rightarrow H^i(X, G) \rightarrow$$
$$\rightarrow H^i(X - S, G) \xrightarrow{\delta} H^{i+1}(X \bmod X - S, G) \rightarrow \ldots.$$

Since S is nonsingular, a tubular neighborhood of S in X is isomorphic to a neighborhood of the 0-section of the normal bundle of S in X. This bundle being holomorphic is naturally oriented. We then have a Thom isomorphism

$$t : H^i(X \bmod X - S, G) \xrightarrow{\sim} H^{i-2}(S, G)$$

the map $\mathrm{Res} = t \circ \delta$:

$$\mathrm{Res} : H^i(X - S, G) \rightarrow H^{i-1}(S, G)$$

is called the *residue map.*

Dually we have in homology a homology sequence

$$\ldots \rightarrow H_i(X - S, G) \rightarrow H_i(X, G) \rightarrow H_i(X \bmod X - S, G) \xrightarrow{\delta} H_{i-1}(X - S, G) \rightarrow \ldots$$

and a Thom isomorphism

$$H_i(X \bmod X - S, G) \xleftarrow{\sim} H_{i-2}(S, G)$$

thus by composition a dual map to the residue

$$\varrho : H_i(S, G) \rightarrow H_{i+1}(X - S, G).$$

The duality between r and ϱ is given by the formula (when G is a ring):

$$\langle \operatorname{Res}(\alpha), \beta \rangle = \langle \alpha, \varrho(\beta) \rangle$$

$\forall\, \alpha \in H^i(X - S, G)$ and $\beta \in H_{i+1}(S, G)$ and where $\langle\, , \rangle$ denote the pairing between homology and cohomology.

b) If $G = \mathbf{C}$, cohomology classes can be represented via the de Rham theorem by classes of d-closed forms φ.

The following particular situation is of practical importance:

(i) S is given by a global holomorphic equation $s = 0, (ds)_{x \in S} \neq 0$.

(ii) the form $\varphi, d\varphi = 0$, on $X - S$ is such that $\chi = s\varphi$ extends C^∞ to the whole X.

Then

THEOREM. *There exist forms ψ and θ, C^∞ on X such that*

$$\varphi = \frac{ds}{s} \wedge \psi + \theta$$

and

$$\psi\,|_S \in \operatorname{Res}\{\varphi\}.$$

PROOF (α).

$$d\chi = ds \wedge \varphi \text{ since } d\varphi = 0 \text{ on } X - S$$

$$ds \wedge d\chi = 0 \text{ then } d\chi = ds \wedge \theta, \theta\ C^\infty \text{ on } X$$

$$ds \wedge (s\varphi - s\theta) = 0 \text{ thus } s\varphi - s\theta = ds \wedge \psi, \psi\ C^\infty \text{ on } X.$$

(β) Apply the formula of Cauchy to a small simplex σ on S. Then $\varrho(\sigma) = \{(x, s) \mid |s| = \varepsilon, x \in \sigma\}$ if x, s are local coordinates in a neighborhood of σ and

$$\int_{\varrho(\sigma)} \varphi = 2\pi i \int_\sigma \psi.$$

In general S will not be given by a global equation but we can select a system of coordinate patches $\mathcal{U} = \{U_i\}$ and local equations

$$s_i = 0,\ (ds_i)_{x \in U_i \cap S} \neq 0.$$

Then if $s_i\varphi$ extends C^∞ on the whole of U_i, we can write

$$\varphi\,|_{U_i} = \frac{ds_i}{s_i} \wedge \psi_i + \theta_i$$

with ψ_i and θ_i, C^∞ on U_i. We get

$$\psi_i|_S = \psi_j|_S = \psi \in \operatorname{Res}\{\varphi\}.$$

5. *a*). Let S_1, S_2 be two complex submanifolds of X of codimension 1 such that at each $x \in S_1 \cap S_2$, S_1 and S_2 intersect transversally. We then have the following inclusions

$$X - S_1 - S_2 \subset X - S_1 \quad \text{or} \quad X - S_1 - S_2 \subset X - S_2$$

$$S_2 - S_1 \cap S_2 \subset S_2, \quad S_1 - S_1 \cap S_2 \subset S_1 .$$

These give rise to iterated residues according to the diagram

$$\begin{array}{ccccc} & & H^i(X - S_1 - S_2, G) & & \\ & \swarrow^{\text{Res}} & & \searrow^{\text{Res}} & \\ H^{i-1}(S_2 - S_1 \cap S_2, G) & & \downarrow \text{Res}^2 & & H^{i-1}(S_1 - S_1 \cap S_2, G) \\ & \searrow_{\text{Res}} & & \swarrow_{\text{Res}} & \\ & & H^{i-2}(S_1 \cap S_2, G) & & \end{array}$$

The diagram is anticommutative so that Res^2 without specifications carries an ambiguity of sign.

In the same way one defines Res^k, $\forall k \in \mathbf{N}$ provided one has k hypersurfaces $S_1, \ldots, S_k$ nonsingular and in general position. In particular let us consider the case

$$X = \mathbf{C}^n, \quad S_i = \{z \in \mathbf{C}^n \mid z_i = 0\}$$

so that

$$X - S_1 - \ldots - S_n = \{z \in \mathbf{C}^n \mid z_1 \ldots z_n \neq 0\}, \cap S_i = \{0\}$$

$$\operatorname{Res}^n : H^n(X - \cup S_i, \mathbf{C}) \to H^0(\{0\}, \mathbf{C}) = \mathbf{C}.$$

For instance the forms

$$\varphi_\alpha = z^{-\alpha}\, dz_1 \ldots dz_n \quad \text{for } \alpha \in \mathbf{Z}^n$$

have

$$\operatorname{Res}_0^n(\varphi_\alpha) = \begin{cases} (2\pi i)^n & \text{if } \alpha = (1, \ldots, 1) \\ 0 & \text{if } \alpha \neq (1, \ldots, 1). \end{cases}$$

Indeed if some $\alpha_i \neq 1$ then $\varphi_\alpha = dg$ with some g holomorphic on $X - \cup S_i$ if $\alpha_1 = \ldots = \alpha_n = 1$ then we repeatedly apply Leray's theorem n times.

In particular one obtains the following

PROPOSITION 3. *Let U be a neighbourhood of the origin in $\mathbf{C}^n$ and let*

$$Y = \{z \in U \mid z_1 \neq 0, \ldots, z_n \neq 0\}.$$

Let

$$\theta = \Big(\sum_{\alpha \in \mathbf{N}^n} C_\alpha z^{-\alpha-1} \Big) dz_1 \wedge \ldots \wedge dz_n \qquad (C_\alpha \in \mathbf{C})$$

be a holomorphic n-form in Y. Then $\theta = 0$ is equivalent to the set of conditions

$$\mathrm{Res}_0^n (z^\alpha \theta) = 0 \qquad \forall\, \alpha \in \mathbf{N}^n.$$

6. *A residue formula in $\mathbf{C}^n$.* Let B be a domain of holomorphy containing the origin in $\mathbf{C}^n$. We will assume $n \geq 2$.

LEMMA 5. We have, if $n \geq 2$,

$$H^r(B - \{0\}, \mathcal{O}) = 0 \qquad \text{if} \quad r \neq 0,\ n-1$$

$$H^0(B - \{0\}, \mathcal{O}) \simeq H^0(B, \mathcal{O})$$

$$H^{n-1}(B - \{0\}, \mathcal{O}) \simeq H^{n-1}(\mathbf{C}^n - \{0\}, \mathcal{O}).$$

PROOF. It follows from the commutative diagram

$$\begin{array}{ccc} H^r(\mathbf{C}^n - \{0\}, \mathcal{O}) & \xrightarrow{\ \delta\ } & H_0^{r+1}(\mathbf{C}^n, \mathcal{O}) \\ \downarrow r_1 & & \downarrow r_2 \\ H^r(B^n - \{0\}, \mathcal{O}) & \xrightarrow{\ \delta\ } & H_0^{r+1}(B, \mathcal{O}) \end{array}$$

where if $r \geq 1$, δ is isomorphism and r_2 is an isomorphism by excision.

COROLLARY.

$$H^{n-1}(\mathbf{C}^n - \{0\}, \Omega^p) \xrightarrow{\sim} H^{n-1}(B - \{0\}, \Omega^p)$$

$$H^r(B - \{0\}, \Omega^p) = 0 \qquad \text{if} \quad r \neq 0,\ n-1, \qquad \forall\, p \quad \text{if} \quad n \geq 2.$$

Let us consider now a cohomology class (for $n \geq 2$)

$$\xi \in H^{n-1}(B_n - \{0\}, \Omega^{n-1})$$

this will be represented by a Čech cocycle:

$$\check{\varphi} = \sum_{\alpha \in \mathbf{N}^n} \frac{1}{z^{\alpha+1}} \varphi_\alpha \tag{1}$$

where

$$\varphi_\alpha = \Sigma\, C_{\alpha i}\, dz_1 \wedge \ldots \wedge \widehat{dz_i} \wedge \ldots \wedge dz_n \tag{2}$$

$$\lim_{|\alpha| \to \infty} \sqrt[|\alpha|]{|C_{\alpha i}|} = 0, \qquad C_{\alpha i} \in \mathbf{C}. \tag{3}$$

To this class is associated the Dolbeault representative

$$\varphi = (n-1)! \sum_{\alpha \in \mathbf{N}^n} \varphi_\alpha \wedge \psi_{\alpha+1}\,. \tag{4}$$

In what follows the condition for the coefficient $C_{\alpha i}$ to be constant will not be used.

We will therefore suppose only that the forms

$$\varphi_\alpha = \Sigma\, C_{\alpha i}\, dz_1 \wedge \ldots \wedge \widehat{dz_i} \wedge \ldots \wedge dz_n \tag{2'}$$

have holomorphic coefficients in B subjected to the conditions

$$\lim_{|\alpha| \to \infty} \sqrt[|\alpha|]{|C_{\alpha i}(z)|} = 0 \tag{3'}$$

for every point $z \in B$.

Then to the form $\check{\varphi}$ will again correspond the Dolbeault representative

$$\varphi = (n-1)!\, \Sigma\, \varphi_\alpha \wedge \psi_{\alpha+1}\,. \tag{4'}$$

Indeed as in Lemma 4 it follows that the class $(n-1)!\, z^\beta\, \psi_{\alpha+1}$ corresponds to the Čech class of $z^{-(\alpha+1)+\beta}$.

Let $f \in \mathcal{H}(B) - \mathcal{H}_0(B)$ be a holomorphic function on B which does not vanish at the origin and let F denote the divisor of f in B.

From [1] Proposition 2, we derive the following residue formula:

PROPOSITION 4. *Let ϱ be a C^∞ function with compact support in B and equal to 1 near the origin. Then*

$$\int_F \varrho\varphi = \frac{1}{2i\pi}\int_B \frac{df}{f}\wedge\bar\partial\varrho\wedge\varphi - (2i\pi)^n \operatorname{Res}_0^n\left(\frac{df}{f}\wedge\check{\varphi}\right).$$

PROOF. For $\varphi = (n-1)!\ \varphi_\alpha \wedge \psi_{\alpha+1}$ from the Proposition 2 of [1] we get:

$$\int_F \varrho\varphi_\alpha\wedge\psi_{\alpha+1} = \frac{1}{2i\pi}\int_B \frac{df}{f}\wedge\bar\partial\varrho\wedge\varphi_\alpha\wedge\psi_{\alpha+1} - (2i\pi)^n \operatorname{Res}_0^n\left(\frac{1}{z^{\alpha+1}}\frac{df}{f}\wedge\varphi_\alpha\right).$$

Now the series $\Sigma\,\varphi_\alpha\wedge\psi_{\alpha+1}$ is a series of differential forms whose coefficients converge absolutely and uniformly on any compact set $K\subset B-\{0\}$ (see Lemma 4 and the convergence condition for the coefficients of φ_α).

Thus we can sum over α under the integral sign. This shows that the series

$$(2\pi i)^n\,\Sigma\operatorname{Res}_0^n\left(\frac{1}{z^{\alpha+1}}\frac{df}{f}\wedge\varphi_\alpha\right)$$

is an absolutely convergent series converging to $(2\pi i)^n\operatorname{Res}_0^n\left(\frac{df}{f}\wedge\check{\varphi}\right)$.

COROLLARY 1. *Under the same assumptions setting $D^\alpha = \frac{1}{\alpha!}\frac{\partial^{|\alpha|}}{\partial z^\alpha}$ we have*

$$\int_F \varrho\varphi = \frac{1}{2i\pi}\int_B \frac{df}{f}\wedge\bar\partial\varrho\wedge\varphi + (2i\pi)^n \sum_{\alpha\in\mathbf{N}^n}(D^\alpha\log f)(0)\operatorname{Res}_0^n(z^\alpha\, d\check{\varphi})$$

where the series is absolutely convergent.

PROOF. In a small neighborhood W of 0, selecting a determination for $\log f$ in W,

$$\frac{df}{f}\wedge\check{\varphi} = d(\log f\check{\varphi}) - \log f\, d\check{\varphi}.$$

Now $\operatorname{Res}_0^n d(\log f\cdot\varphi) = 0$ because the cohomology class of $d(\log f\varphi)$ in $H^n(W', \mathbf{C})$ (where $W' = W - \{z_1 \dots z_n = 0\}$) is the zero class. Therefore

$$\operatorname{Res}_0^n\left(\frac{df}{f}\wedge\check{\varphi}\right) = -\operatorname{Res}_0^n(\log f\, d\check{\varphi})$$

$$= - \operatorname{Res}_0^n \Big(\sum_{\alpha \in \mathbf{N}^n} D^\alpha \log f(0)\, z^\alpha\, d\check{\varphi} \Big)$$

$$= - \sum_{\alpha \in \mathbf{N}^n} D^\alpha \log f(0) \operatorname{Res}_0^n (z^\alpha\, d\check{\varphi}).$$

To prove the absolute convergence of this last series we proceed as follows.

First we remark that we have

$$\check{\varphi} = \Sigma \frac{1}{z^{\beta+1}} \varphi_\beta$$

with

$$\varphi_\beta = \Sigma\, C_{\beta i}(z)\, dz_1 \wedge \dots \wedge \widehat{dz_i} \wedge \dots \wedge dz_n$$

$$\lim_{|\beta| \to \infty} \sqrt[|\beta|]{\| C_{\beta_i}(z) \|_K} = 0$$

where K is compact in B and $\| \ \|_K = \sup_K | \ |$. Therefore we can write

$$d\check{\varphi} = \left(\Sigma \frac{1}{z^{\beta+2}} \gamma_\beta(z) \right) dz_1 \dots dz_n$$

while $\beta + 2 = (\beta_1 + 2, \dots, \beta_n + 2)$ and where

$$(*) \qquad \lim_{|\beta| \to \infty} \sqrt[|\beta|]{\| \gamma_\beta(z) \|_K} = 0.$$

Let

$$\gamma_\beta(z) = \Sigma\, \gamma_{\beta\lambda}(0)\, z^\lambda$$

$$\operatorname{Res}_0^n (z^\alpha\, d\check{\varphi}) = (2\pi i)^n \sum_\beta \gamma_{\beta,\, \beta-\alpha+1}(0).$$

Now for any r sufficiently small, $r \leq \delta$, taking $K = \{ |z_i| \leq \delta,\ 1 \leq i \leq n \}$ we have:

$$| \gamma_{\beta,\, \beta-\alpha+1}(0) | \leq \frac{c}{r^{|\beta-\alpha+1|}} \| \gamma_\beta \|_K .$$

Therefore

$$(**) \qquad | \operatorname{Res}_0^n (z^\alpha\, d\check{\varphi}) | \leq (2\pi)^n \sum_\beta \frac{1}{r^{|\beta-\alpha+1|}} \| \gamma_\beta \|_K$$

$$\leq (2\pi)^n\, C(\varepsilon)\, r^{|\alpha-1|} \frac{1}{1 - \varepsilon/r}$$

$$\leq C\, r^{|\alpha-1|}$$

the second inequality being obtained from $(*)$ in the form $\| \gamma_\beta(z) \|_K \leq C(\varepsilon) \varepsilon^{|\beta|}$ for any $\varepsilon > 0$.

From the other end $\log f$ is holomorphic in a neighborhood of the origin, say $\{ |z_i| < \varrho$ for $1 \leq i \leq n\}$. Thus

$$(***) \qquad | D^\alpha \log (0) | \leq \frac{C}{\varrho^{|\alpha|}} .$$

From $(**)$ and $(***)$ we get the absolute convergence of the series we consider taking $r < \varrho$.

Let now G denote a holomorphic function in B, vanishing at the origin of order ≥ 2.

Let $\widehat{\Sigma}_G$ denote the set of holomorphic functions in B of the form

$$f = t + a_1 z_1 + \dots + a_n z_n + b\, G(z_1, \dots, z_n).$$

Let Σ_G denote the set of divisors of the functions in $\widehat{\Sigma}_G$. We have

COROLLARY 2. *a) For* $f \in \widehat{\Sigma}_G$:

$$\mathrm{Res}_0^n \left(\frac{df}{f} \wedge \check{\varphi} \right) = E \left(\frac{a_1}{t}, \dots, \frac{a_n}{t}, \frac{b}{t} \right)$$

where E is an entire function in $\mathbf{C}^{n+1}$.

b) We have an expansion in power series of the form:

$$\mathrm{Res}_0^n \left(\frac{df}{f} \wedge \check{\varphi} \right) = \sum_{k \in \mathbf{N}^*} (-1)^{k-1} \frac{(k-1)!}{t^k} \sum_{\substack{\gamma \in \mathbf{N}^n \\ |\gamma| = k}} \frac{a^\gamma}{\gamma!} \mathrm{Res}_0^n (z^\gamma\, \check{d\varphi})$$

$$+ \frac{b}{t} H \left(\frac{a_1}{t}, \dots, \frac{a_n}{t}, \frac{b}{t} \right)$$

where H is an entire function in $\mathbf{C}^{n+1}$.

PROOF. From Corollary 1 it follows that $\mathrm{Res}_0^n \left(\frac{df}{f} \wedge \check{\varphi} \right)$ depends holomorphically on the parameters $a_1/t, \dots, a_n/t, b/t$. Since it is defined for all values of these parameters, it follows that its expression is an entire function of these variables.

To prove contention $b)$ it is enough to prove the statement for $b = 0$. In this case by direct calculation, we get, setting $k = |\alpha|$,

$$D^\alpha \log (t + a_1 z_1 + \dots + a_n z_n)(0) = \frac{1}{\alpha!} (-1)^{k-1} (k-1)! \frac{1}{t^k} a^\alpha .$$

From Corollary 1 we thus get the conclusion.

COROLLARY 3. *If there exists a constant $C > 0$ such that for any $f \in \widehat{\Sigma}_G$ we have*

$$\left| \operatorname{Res}_0^n \left(\frac{df}{f} \wedge \check{\varphi} \right) \right| < C$$

then we must have identically

$$\operatorname{Res}_0^n \left(\frac{df}{f} \wedge \check{\varphi} \right) \equiv 0$$

and in particular

$$\operatorname{Res}_0^n (z^\gamma \, d\check{\varphi}) = 0 \qquad \forall \gamma \in \mathbf{N}^n.$$

PROOF. By Liouville's theorem E must be a constant. Since $E(0, \dots, 0) = 0$ then $E \equiv 0$. In particular due to the explicit expression of E we must have

$$\operatorname{Res}_0^n (z^\gamma \, d\check{\varphi}) = 0 \qquad \forall \gamma \in \mathbf{N}^n - \{0\}$$

but also for $\gamma = 0$ since $\operatorname{Res}_0^n (d\check{\varphi}) = 0$.

REMARK. *Suppose that*

$$\check{\varphi} = \sum_{\alpha \in \mathbf{N}^n} \frac{1}{z^{\alpha+1}} \sum_i C_{\alpha i} \, dz_1 \wedge \dots \wedge \widehat{dz_i} \wedge \dots \wedge dz_n, \quad C_{\alpha i} \in \mathbf{C}$$

is in the «canonical form». Then the conditions

$$\operatorname{Res}_0^n (z^\gamma \, d\check{\varphi}) = 0 \qquad \forall \gamma \in \mathbf{N}^n$$

are equivalent to

$$d\check{\varphi} = 0.$$

Indeed

$$d\check{\varphi} = \left\{ - \sum_{\alpha \in \mathbf{N}^n} \frac{1}{z^{\alpha+1}} \left(\Sigma (-1)^i \alpha_i \, C_{\alpha_1, \dots, \alpha_i - 1, \dots, \alpha_n, i} \right) \right\} dz_1 \wedge \dots \wedge dz_n .$$

Thus the conclusion by virtue of Proposition 3.

In general the relevance of these residue conditions are explained by the following.

PROPOSITION 5. *Let B be a convex neighborhood of the origin $0 \in \mathbf{C}^n$, let $\xi \in H^{n-1}(B - \{0\}, \Omega^{n-1})$ be represented by the Čech cocycle:*

$$\check{\varphi} = \sum_{\alpha \in \mathbf{N}^n} \frac{1}{z^{\alpha+1}} \varphi_\alpha$$

with φ_α holomorphic $(n-1)$-forms in B.

Let $d: \Omega^{n-2} \longrightarrow \Omega^{n-1}$ be the sheaf homomorphism induced by the exterior differentiation. Then the necessary and sufficient condition for

$$\xi \in \operatorname{Im}\{H^{n-1}(B - \{0\}, \Omega^{n-2}) \xrightarrow{d} H^{n-1}(B - \{0\}, \Omega^{n-1})\}$$

is that

$$\operatorname{Res}_0^n (z^\gamma \, d\check{\varphi}) = 0 \qquad \forall \, \gamma \in \mathbf{N}^n .$$

PROOF. We can write

$$\check{\varphi} = \check{\varphi}_0 + \delta\mu$$

on the covering $\mathcal{U} = \{U_i\}$, $U_i = \{z \in B \mid z_i \neq 0\}$ of B, where $\check{\varphi}_0$ has the form (1) with φ_α's as in (2) subjected to the condition (3).

We have

$$\delta\mu = \Sigma (-1)^i \mu_i$$

where μ_i are holomorphic $(n-1)$ forms on the sets $\bigcap_{j \neq i} U_j$.

From n. 5 remark before Proposition 3, it follows that

$$\operatorname{Res}_0^n (z^\gamma \, d\delta\mu) = 0 \qquad \forall \, \gamma \in \mathbf{N}^n .$$

Thus the residue conditions reduce to

$$\operatorname{Res}_0^n (z^\gamma \, d\check{\varphi}_0) = 0 \qquad \forall \, \gamma \in \mathbf{N}^n$$

i. e., (by virtue of the previous remark) to

$$d\check{\varphi}_0 = 0.$$

Let us consider the exact sequence of sheaves:

$$0 \longrightarrow \mathcal{Z}^{r-1} \longrightarrow \Omega^{r-1} \xrightarrow{d} \mathcal{Z}^r \longrightarrow 0$$

where $\mathcal{Z}^j$ denotes the sheaf of germs of closed holomorphic forms of degree j.

Since $d\check{\varphi}_0 = 0$ it follows from that sequence, for $r = n$,

$$(*) \qquad \{\check{\varphi}\} \in \operatorname{Im}\{H^{n-1}(B - \{0\}, \mathcal{Z}^{n-1}) \to H^{n-1}(B - \{0\}, \Omega^{n-1})\}.$$

Using the same exact sequence and Lemma 5, we get for $r = n-1, n-2, \ldots$

$$H^n(B - \{0\}, \mathcal{Z}^{n-2}) \simeq H^{n+1}(B - \{0\}, \mathcal{Z}^{n-3}) \simeq H^{2n-2}(B - \{0\}, \mathbf{C}) = 0.$$

Therefore, the map

$$H^{n-1}(B - \{0\}, \Omega^{n-2}) \xrightarrow{\ d\ } H^{n-1}(B - \{0\}, \mathcal{Z}^{n-1})$$

is surjective. This with (*) gives the result.

COROLLARY. *For the Dolbeault representative φ given by (4′) the residue conditions*

$$\operatorname{Res}_0^n (z^\gamma\, d\check{\varphi}) = 0 \qquad \forall\, \gamma \in \mathbf{N}^n$$

are equivalent to the existence of C^∞ forms ψ of degree $(n-2, n-1)$ and χ of degree $(n-1, n-2)$ on $B - \{0\}$ such that

$$\varphi = \partial\psi + \bar{\partial}\chi \quad \textit{with} \quad \bar{\partial}\psi = 0.$$

7. *Application of the residue formula. a)* We maintain the notations of the previous section.

Let $B' \subset\subset B$ be a bounded subdomain of B containing the origin.

LEMMA 6. *There exists a constant $C > 0$ such that for every $F \subset \Sigma_G$ we have*

$$\operatorname{vol}(F \cap B') \leq C.$$

PROOF. Let ϱ be a C^∞ function on B compactly supported, $\varrho \geq 0$, and $\varrho \mid B' = 1$. Let ω^{n-1} be the $(n-1)$ power of the exterior form of the Kähler metric so that

$$\operatorname{vol}(F \cap B') = C \int_{F \cap B'} \omega^{n-1} \leq C \int_F \varrho\, \omega^{n-1}.$$

It is enough therefore to prove that these last integrals are uniformly bounded.

Let $u : \mathbf{C}^{n+1} \to \mathcal{H}(B) - \mathcal{H}_0(B)$ defined by

$$(\alpha_1, \ldots, \alpha_n, \beta) \to f = 1 + \alpha_1 z_1 + \ldots + \alpha_n z_n + \beta\, G(z_1, \ldots, z_n)$$

and let $v : \mathcal{H}(B) - \mathcal{H}_0(B) \to \mathbf{R}$ defined by

$$f \to \int_F \varrho \omega^{n-1} = \int_B \frac{df}{f} \bar{\partial} \varrho \, \omega^{n-1} .$$

Both u, v are continuous thus $v \circ u$ is a continuous function on $\mathbf{C}^{n+1}$.

Now we remark that if $z^0 \in F$ we have

$$1 = | \Sigma \alpha_i z_i^0 + \beta \, G(z^0) | \leq c_1 (\Sigma | \alpha_i |) \| z^0 \| + c_2 | \beta | \| z_0 \|^2$$

as it follows from an inequality of type

$$| G(z) | \leq c_2 \| z \|^2 \qquad \forall z \in \operatorname{supp} \varrho .$$

Thus

$$\operatorname{dist}(F, 0) \geq c_2 (c_1 \Sigma | \alpha_i | + c_3 | \beta |)^{-1}$$

$c_3 = \sup_{z \in \operatorname{supp} \varrho} c_2 \| z \|$. It follows then that $v \circ u$ is a function in $\mathbf{C}^{n+1}$ with compact support. Thus being continuous it has a maximum.

Let $\check{\varphi}, \varphi$ be as in the previous section (1), (2′), (3′), (4′).

Lemma 7. *If*

$$\sup_{F \in \Sigma_G} \left| \int_F \varrho \check{\varphi} \right| < \infty$$

then

$$\operatorname{Res}_0^n (z^\gamma \, d\check{\varphi}) = 0 \qquad \forall \gamma \in \mathbf{N}^n .$$

Proof. We apply the formula of Proposition 4. First we remark that:

$$\text{for } f \in \widehat{\Sigma}_G , \int \frac{df}{f} \bar{\partial} \varrho \wedge \varphi \text{ is uniformly bounded.}$$

Indeed as in the previous lemma, we get a continuous map

$$g : \mathbf{C}^{n+1} \to \mathbf{C}$$

defined by associating to $\alpha_1 , \dots , \alpha_n , \beta$ the function $f = 1 + \Sigma \alpha_i z_i + \beta G$ and to this the above integral.

By the same argument as in the lemma we see that g is a compactly supported function in $\mathbf{C}^{n+1}$ (1). Thus our contention follows.

---

(1) For large values of α and β, $\log f$ is holomorphic on $\operatorname{supp} \bar{\partial} \varrho \wedge \varphi$ thus

$$\int \frac{df}{f} \bar{\partial} \varrho \wedge \varphi = \int d (\log f \, \bar{\partial} \varrho \wedge \varphi) = 0$$

since $\log f \, \bar{\partial} \varrho \wedge \varphi$ is compactly supported.

From this remark and the assumption, it follows then that we are in the condition to apply Corollary 3 of Proposition 4.

b) We now shift notations. We identify the set B of *a*) and the previous section with $D^{q+1} \subset \mathbf{C}^{q+1}$. We consider the set

$$V = (D^{q+1} - \{0\}) \times D^{n-q-1} \subset \mathbf{C}^n = \mathbf{C}^{q+1} \times \mathbf{C}^{n-q-1}.$$

As usual $z_1, \dots, z_{q+1}$ will denote coordinates in $\mathbf{C}^{q+1}$ while $w_1, \dots, w_{n-q-1}$ will denote the coordinates on $\mathbf{C}^{n-q-1}$.

Let $\mathcal{U} = \{V_i\}$, $V_i = \{(z, w) \in V \mid z_i \neq 0\}$ will be a Stein covering of V. Any cohomology class

$$\xi \in H^q(V, \Omega^q)$$

will have a Čech representative on the covering $\mathcal{U}$ of the form

$$\check{\theta} = \sum_{\alpha \in \mathbf{N}^{q+1}} \frac{1}{z^{\alpha+1}} \theta_\alpha \tag{1}$$

as in Corollary 2, n. 1 and a Dolbeault representative

$$\theta = q\,! \sum \theta_\alpha \wedge \psi_{\alpha+1}\,. \tag{2}$$

For $\varepsilon > 0$ let us consider the set Ξ_ε of all graphs W of affine linear maps $g : \mathbf{C}^{q+1} \to \mathbf{C}^{n-q-1}$ of the form

$$g : \begin{cases} w_j - w_j^0 = \sum_1^{q+1} a_{ji}\, z_i & \sum |w_j^0| < \varepsilon \\ 1 \leq j \leq n - q - 1. \end{cases}$$

Let $\widehat{\Xi}_\varepsilon$ be the corresponding set of isomorphisms

$$\gamma : \mathbf{C}^{q+1} \to \mathbf{C}^{q+1} \times \mathbf{C}^{n-q-1}$$

where

$$\gamma = 1 \times g$$

of $\mathbf{C}^{q+1}$ onto the graph of g.

For any γ we denote by $\widehat{\Sigma}_{G_\gamma}$ the set of functions

$$f = t + a_1 z_1 + \dots + a_{q+1} z_{q+1} + b\, G_\gamma (z_1, \dots, z_{q+1})$$

where G_γ is a holomorphic function in D^{q+1} vanishing at the origin of order ≥ 2. The set of divisors F of the functions $f \in \widehat{\Sigma}_{G_\gamma}$ will be denoted by Σ_{G_γ}.

Let $U = D^{q+1} \times D^{n-q-1}$ and let ϱ be a C^∞ function with compact support in U, equal 1 in a neighborhood of the origin in $\mathbf{C}^n$.

LEMMA 8. *If for every* $\gamma \in \widehat{\Xi}_\varepsilon$ *we have*

$$\sup_{\gamma \in \Sigma_{G_\gamma}} \left| \int_F \gamma^* \varrho\theta \right| < \infty$$

then

$$d\check{\theta} \equiv 0.$$

PROOF. We may assume $\varrho = 1$ in the ε neighborhood $\{\Sigma |z_i| + \Sigma |w_i| < \varepsilon\}$ of the origin. The condition of the uniform boundedness of the integral $F \in \Sigma_{G_\gamma}$ implies, by virtue of Lemma 7, the conditions:

$$\text{for every } W \in \Xi_\varepsilon, \operatorname{Res}_0^{q+1}(z^\gamma d\check{\theta} \mid W) = 0 \qquad \forall\, \gamma \in \mathbf{N}^{q+1}:$$

We have to show that these conditions imply $d\check{\theta} \equiv 0$.

Given the nature of the forms

$$\check{\theta} = \sum_{\alpha \in \mathbf{N}^{q+1}} \frac{1}{z^{\alpha+1}} \theta_\alpha$$

where

$$\theta_\alpha = \sum_{\substack{i_1 < \dots < i_p \\ j_1 < \dots < j_{q-p}}} C_\alpha(w)_{i_1 \dots i_p, j_1 \dots j_{q-p}} dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge dw^{j_1} \wedge \dots \wedge dw^{j_{q-p}}.$$

It follows that $d\check{\theta}$ has an expression of the form

$$d\check{\theta} = \left(\sum_{\alpha \in \mathbf{N}^{q+1} - \{0\}} \frac{1}{z^{\alpha+1}} A_\alpha(w) \right) dz_1 \wedge \dots \wedge dz_{q+1}$$

$$+ \sum_{1}^{n-q-1}{}_{j}\, dw_j \sum_{1}^{q+1}{}_{i} \left(\sum_{\alpha \in \mathbf{N}^{q+1} - \{0\}} \frac{1}{z^{\alpha+1}} A^j_{\alpha, i}(w)\, dz_1 \wedge \dots \wedge \widehat{dz_i} \wedge \dots \wedge dz_{q+1} \right)$$

$$+ \sum_{\substack{1 \\ j < \varrho}}^{n-q-1}{}_{j\varrho}\, dw_j \wedge dw_\varrho \sum_{\substack{1 \\ i < k}}^{q+1}{}_{ik} \left(\sum_{\alpha \in \mathbf{N}^{q+1} - \{0\}} \frac{1}{z^{\alpha+1}} A^{j\varrho}_{\alpha, ik}(w) \right.$$

$$dz_1 \wedge \dots \wedge \widehat{dz_i} \wedge \dots \wedge \widehat{dz^k} \wedge \dots \wedge dz_{q+1}\Big)$$

$$+ \dots$$

$$= \omega_0 + \Sigma\, dw_j\, \omega_j + \Sigma\, dw_j \wedge dw_\varrho\, \omega_{j\varrho} + \dots .$$

If we first specialize W to the spaces

$$W = \{w_1 = a_1, \dots, w_{n-q-1} = a_{n-q-1}\} \qquad \Sigma\, |a_j| < \varepsilon$$

we get

$$\mathrm{Res}_0^{q+1}\left(z^\gamma \left\{\Sigma \frac{1}{z^{\alpha+1}} A_\alpha(a)\, dz_1 \wedge \dots \wedge dz_{q+1}\right\}\right) = 0 \qquad \forall\, \gamma \in \mathbf{N}^{q+1}$$

i. e.,

$$A_\alpha(a) = 0 \qquad \text{for all } a \text{ with} \qquad \Sigma\, |a_j| < \varepsilon.$$

Therefore $\omega_0 \equiv 0$.

Let us now specialize W to the spaces

$$W = \{w_1 = b_1 z_i + a_1,\ w_2 = a_2, \dots, w_{n-q-1} = a_{n-q-1}\} \qquad \Sigma\, |a_j| < \varepsilon.$$

We get the conditions $\forall\, \gamma \in \mathbf{N}^{q+1}$:

$$0 = \mathrm{Res}_0^{q+1}\left(z^\gamma \left\{b_1\, dz_i\, \Sigma \frac{1}{z^{\alpha+1}} A^1_{\alpha, i}(b_1 z_i + a_1, \dots, a_{n-q-1})\right.\right.$$

$$\left.\left. dz_1 \wedge \dots \wedge \widehat{dz_i} \wedge \dots \wedge dz_{q+1}\right\}\right),$$

i. e., setting to zero the coefficient of b_1 in the Taylor expansion of these conditions with respect to b_1, we get

$$A^1_{\alpha, i}(a_1, \dots, a_{n-q-1}) = 0 \quad \text{for all } a,\ \Sigma\, |a_j| < \varepsilon.$$

Therefore

$$\omega_1 \equiv 0.$$

Analogously

$$\omega_j \equiv 0 \qquad \text{for} \qquad j = 1, \dots, n-q-1.$$

Specializing W to the spaces

$$W = \{w_1 = b_1 z_i + a_1,\ w_2 = b_2 z_k + a_2,\ w_3 = a_3, \dots, w_{n-q-1} = a_{n-q-1}\},$$

$$\Sigma\, |a_j| < \varepsilon$$

by analogous argument we get

$$A^{12}_{\alpha, ik}(a) \equiv 0$$

thus

$$\omega_{12} \equiv 0$$

and analogously

$$\omega_{ij} \equiv 0.$$

Proceeding in this way we get the result.

We can now prove the following

THEOREM 1. *Let* φ *be a* C^∞ *form of type* (q, q) *defined on* V *with* $\bar{\partial}\varphi = 0$. *The following conditions are equivalent :*

i) *for every* $\gamma \in \Xi_\varepsilon$ *we have*

$$\sup_{F \in \Sigma_{G_\gamma}} \left| \int_F \gamma^* \varrho\varphi \right| < \infty ;$$

ii) *there exist* C^∞ *forms* ψ *of degree* $(q-1, q)$ *and* α *of degree* $(q, q-1)$ *on* V *such that*

$$\varphi = \partial\psi + \bar{\partial}\chi \quad \textit{with} \quad \bar{\partial}\psi = 0.$$

PROOF. ii) $\Longrightarrow$ i). Indeed we have

$$\int_F \gamma^* \varrho\varphi = \int_F \gamma^* \varrho\, (\partial\psi + \bar{\partial}\chi)$$

$$= \int_F \gamma^* \varrho\, d(\psi + \chi) \quad \text{because } \bar{\partial}\psi \text{ is of type } (q-1, q+1) \text{ and } \partial\chi \text{ is of type } (q+1, q-1)$$

$$= - \int_F \gamma^* d\varrho \wedge (\psi + \chi) \quad \text{by Stokes theorem.}$$

Now

$$\gamma^* d\varrho \wedge (\psi + \chi) = \eta_\gamma$$

is a C^∞ form on D^{q+1} (if ε is sufficiently small so that $\gamma^*\varrho$ equals 1 near the origin).

Using Lemma 6 and Lemma 9 of [1] we get an inequality of the form

$$\left| \int_F \gamma^* \varrho\varphi \right| \leq C_\gamma \, \| \eta_\gamma \| ,$$

with C independent of F.

i) $\Longrightarrow$ ii). We can write in V:

$$\varphi = \theta + \bar{\partial}\eta$$

where θ is of the form described in (2) and η is a C^∞ form of type $(q, q-1)$ on V.

The argument given above for the implication ii) $\Longrightarrow$ i) shows that condition i) is satisfied by θ in place of φ.

Applying the previous Lemma 8 we find that ii) $\Longrightarrow d\check{\theta} = 0$. By the same argument of Proposition 5 we then get that the cohomology class

$$\xi \in H^q(V, \Omega^q)$$

represented by φ and θ (and in Čech cohomology by $\check{\theta}$) has the property:

$$\xi \in \operatorname{Im}\{H^q(V, \Omega^{q-1}) \xrightarrow{d} H^q(V, \Omega^q)\}$$

(use Lemma 3 and Lemma 1). This condition is equivalent to condition ii).

8. *The case of $\partial\bar{\partial}$-cohomology groups.* We maintain the notations of the previous section.

We recall Proposition 2 asserting the exactness of the sequence:

$$H^q(V, \Omega^{q-1}) \oplus H^q(V, \bar{\Omega}^{q-1}) \xrightarrow{\alpha} H^q(V, \Omega^q) \oplus$$

$$\oplus H^q(V, \bar{\Omega}^q) \xrightarrow{\beta} V^{qq}(V) \xrightarrow{\lambda} \mathbf{C} \to 0$$

where α is induced by $d: \Omega^{q-1} \to \Omega^q$, $\beta(a \oplus b) = a + b$, $\lambda = \partial - \bar{\partial}$ and $\mathbf{C}$, represent $H^{2q+1}(V, \mathbf{C})$.

We therefore define the *reduced group*

$$\widetilde{V}^{qq}(V) = \operatorname{Ker} \lambda.$$

We then have the following:

THEOREM 2. *Let ϱ be a C^∞ function compactly supported in $U = D^{q+1} \times \times D^{n-q-1}$ and equal 1 near the origin. Let φ be a C^∞ (q, q)-form on V with*

$$\partial\bar{\partial}\varphi = 0, \qquad \lambda\{\varphi\} = \{\partial\varphi - \bar{\partial}\varphi\} = 0.$$

The following conditions are equivalent:

i) *for every $\gamma \in \widehat{\Xi}_s$ we have*

$$\sup_{F \in \Sigma_G} \left| \int_F \gamma^* \varrho\varphi \right| < \infty$$

ii) *there exist C^∞ forms ψ of degree $(q, q-1)$ and χ of degree $(q-1, q)$ on V such that*

$$\varphi = \partial\psi + \bar{\partial}\chi.$$

PROOF. We can find two C^∞ forms of degree (q, q) on V φ_1, φ_2 with

$$\bar{\partial}\varphi_1 = 0, \ \bar{\partial}\varphi_2 = 0, \qquad \varphi = \varphi_1 + \bar{\varphi}_2 .$$

This is because of Proposition 2 and the fact that $\{\varphi\} \in \widetilde{V}^{qq}(V)$.

Condition ii) $\Longrightarrow$ i) by the usual argument. Conversely if

$$\sup_{F \in \Sigma_G} \left| \int_F \gamma^* \varrho\varphi_1 + \int_F \gamma^* \varrho\bar{\varphi}_2 \right| < \infty .$$

We can write

$$\varphi_1 = \varphi_1^0 + \bar{\partial}\mu , \qquad \varphi_2 = \varphi_2^0 + \bar{\partial}\nu$$

where φ_1^0 and φ_2^0 are of the canonical form given for θ in n. 7 and in Carollary 2 n. 1.

The usual argument given for the implication ii) $\Longrightarrow$ i) shows that i) is valid for φ_1^0 and φ_2^0 in place of φ_1 and φ_2. These in turn reduce by

Proposition 3 and Corollaries to the separate conditions

$$\mathrm{Res}_0^{q-1}(z^\gamma\, d\check{\varphi}_1^0) = 0 \qquad \forall\, \gamma \in \mathbf{N}^{q+1}$$

and

$$\overline{\mathrm{Res}_0^{q+1}(z^\sigma\, d\check{\varphi}_2^0)} = 0 \qquad \forall \sigma \in \mathbf{N}^{q+1}\,.$$

These by Lemma 8 imply $d\check{\varphi}_1^0 = 0$, $\overline{d\check{\varphi}_2^0} = 0$, or by Theorem 1

$$\varphi_1^0 = \partial\psi_1 + \bar{\partial}\chi_1$$

$$\bar{\varphi}_2^0 = \bar{\partial}\,\bar{\psi}_2 + \partial\,\bar{\chi}_2$$

and in conclusion

$$\varphi = \partial\psi + \bar{\partial}\,\chi.$$

Remark. For the class $\{\varphi\} \in \widetilde{V}^{qq}(V)$ condition i) of Theorem 2 is thus equivalent to $\{\varphi\} = 0$.

§ 3. Applications to Algebraic Manifolds.

9. *Preliminaries. a)* Let Z be a compact complex manifold of pure complex dimension n, purely dimensional. Let Y be a complex submanifold of Z, of pure dimension $n - d - 1$.

Proposition 7. *The manifold $Z - Y = X$ is a strongly $(n - d - 1)$-concave manifold.*

Proof. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be a covering of Z by coordinate patches where holomorphic coordinates $z_i^1, \dots, z_i^{n-d-1}, w_i^1, \dots, w_i^{d+1}$ are so chosen that

$$Y \cap U_i = \{z \in U_i \mid w_i^1 = 0, \dots, w_i^{d+1} = 0\}.$$

Let ϱ_i be a C^∞ partition of unity subordinate to $\mathcal{U}$ and set

$$\varphi = \sum_i \varrho_i \left(\sum_{j=1}^{d+1} | w_i^j |^2\right).$$

Clearly φ is a C^∞ function on Z, $\varphi|_Y = 0$ and moreover the Levi form of φ, $\mathcal{L}(\varphi)_y \geq 0$, $\forall\, y \in Y$, and has at least $d + 1$ positive eigenvalues.

Let ψ be a C^∞ function on Z vanishing with first and second derivatives on Y and such that $\psi > 0$ on X. Set

$$\Phi = \varphi + \psi.$$

Then the sets $\{\Phi > \varepsilon\}$ are relatively compact in X. Moreover one can find a $\delta > 0$ so small that if $\Phi(x) < \delta$ then the Levi form $\mathcal{L}(\Phi)$ has at least $d + 1$ positive eigenvalues.

If q is the concavity of X, $n - q = d + 1$, i.e., $q = n - d - 1$.

Let us assume now that we have on Z a holomorphic line bundle F, which is metrically pseudoconcave in the following sense: there exists a Hermitian metric $h = \{h_i\}$ (h_i being the local expression of the metric on a covering $\mathcal{U} = \{U_i\}$ of Z) on the fibers of F such that

$$-\partial\bar{\partial} \log h_i$$

is a positive definite Hermitian form at any point of Z on the complex tangent space to Z.

PROPOSITION 8. *Assume that* $s_1, \dots, s_{d+1} \in \Gamma(Z, F)$ *are holomorphic sections of* F *such that*

$$Y = \{z \in Z \mid s_1(z) = \dots = s_{d+1}(z) = 0\}.$$

Then under the above assumption $X = Z - Y$ *is a strongly d-complete space.*

PROOF. Let $s_1^{(i)}, \dots, s_{d+1}^{(i)}$ be the local representation of these sections as holomorphic functions. If $\{g_{ij}\}$ are the transition functions of F then

$$h_i = |g_{ji}|^2 h_j \quad \text{and} \quad s_\alpha^{(i)} = g_{ij}\, s_\alpha^{(j)}, \qquad 1 \leq \alpha \leq d+1.$$

Thus (as natural by the fact that h is a metric on the fibers of F)

$$g = h_i \Sigma \mid s_j^{(i)} \mid^2$$

is a C^∞ function on Z. This function vanishes only on Y.

Let

$$\Phi = -\log g.$$

Then $\{\Phi < \text{const}\}$ are relatively compact sets in X. Moreover

$$\partial\bar{\partial}\, \Phi = -\partial\bar{\partial} \log h_i - \partial\bar{\partial} \log \Sigma \mid s_j^{(i)} \mid^2.$$

A direct calculation yields:

$$\partial\bar{\partial} \log \Sigma\, s_j\, \bar{s}_j = \frac{1}{(\Sigma\, s_j\, \bar{s}_j)^2} \sum_{ij} \left| \det \begin{pmatrix} s_i & s_j \\ ds_i & ds_j \end{pmatrix} \right|^2.$$

Let $x \in X$ and assume that $s_h(x) \neq 0$ for instance. This last Hermitian form vanishes on the set

$$\left\{ \frac{s_1}{s_h} = \frac{s_2}{s_h} = \ldots = \widehat{\frac{s_h}{s_h}} = \ldots = \frac{s_{d+1}}{s_h} \right\},$$

which is a set of dimension $\geq n - d$. Therefore on X, $\partial\bar{\partial}\, \Phi$ has at *each point* at least $n - d$ positive eigenvalues.

EXAMPLE. The above situation is realized when Z is a projective algebraic variety and F is the line bundle of the hyperplanes of the projective space, restricted to Z.

COROLLARY. *Let* $\mathcal{F}$ *be a locally free sheaf on* X,

under assumptions of Proposition 7	$H^j(X, \mathcal{F}) \simeq_\Phi 0$	for $j < d$
under assumptions of Proposition 8	$H^j(X, \mathcal{F}) \simeq_\Phi 0$	for $j < d$
and	$H^j(X, \mathcal{F}) = 0$	for $j > d$.

b) Let Z be compact, Y a compact submanifold of Z and $X = Z - Y$. We have the relative cohomology sequence:

$$0 \to H^0_Y(Z, \Omega^r) \to H^0(Z, \Omega^r) \to H^0(X, \Omega^r) \xrightarrow{\delta}$$

$$\to H^1_Y(Z, \Omega^r) \to H^1(Z, \Omega^r) \to H^1(X, \Omega^r) \xrightarrow{\delta} \ldots$$

$$\to H^q_Y(Z, \Omega^r) \to H^q(Z, \Omega^r) \to H^q(X, \Omega^r) \xrightarrow{\delta} \ldots$$

Here $H^0_Y(Z, \Omega^r) = 0$ (Ω^r is locally free) and for any q, $H^q(Z, \Omega^r) \in \Phi$ (i. e., is finite dimensional). Thus

LEMMA 9. *The homomorphism*

$$\delta : H^q(X, \Omega^r) \to H^{q+1}_Y(Z, \Omega^r)$$

is a Φ*-isomorphism* ($\forall\, q \geq 0$ *and* $\forall\, r \geq 0$).

REMARK 1. If for Y we make the assumption of Propositions 7 and 8, then Lemma 9 gives explicitly

$$0 \xrightarrow{\sim}_{\Phi} H^q(X, \Omega^r) \xrightarrow{\sim}_{\Phi} H_Y^{q+1}(Z, \Omega^r), \qquad \text{if} \quad q \neq d$$

so that the really significant fact is

$$\delta : H^d(X, \Omega^r) \xrightarrow{\sim}_{\Phi} H_Y^{d+1}(Z, \Omega^r).$$

For the local cohomology groups $H_Y^*(Z, \Omega^r)$ we have a spectral sequence:

$$E_2^{p,q} = H^p(Z, \mathcal{H}_Y^q(\Omega^r)) \Longrightarrow H_Y^m(Z, \Omega^r), \qquad (m = p + q)$$

where $\mathcal{H}_Y^q(\Omega^r)$ denotes the sheaf of local cohomology of dimension q, with supports on Y and values in the sheaf Ω^r.

From Lemma 3 we deduce in particular

LEMMA 10. *If Y is a submanifold of Z of codimension $d+1$ at each one of its points, then*

$$\mathcal{H}_Y^q(\Omega^r) = 0 \qquad \textit{if} \quad q \neq d+1.$$

PROOF. Locally at a point $y \in Y$ we can find a neighborhood isomorphic to $U = D^{d+1} \times D^{n-d-1}$ such that $Y \cap U = \{0\} \times D^{n-d-1}$. We have the cohomology sequence:

$$0 \to H^0(U, \Omega^r) \to H^0(V, \Omega^r) \to$$

$$H_Y^1(U, \Omega^r) \to H^1(U, \Omega^r) \to \dots$$

$$H_Y^q(U, \Omega^r) \to H^q(U, \Omega^r) \to H^q(V, \Omega^r) \to$$

$$H_Y^{q+1}(U, \Omega^r) \to \dots$$

If $d \geq 1$ then $H^0(U, \Omega^r) \simeq H^0(V, \Omega^r)$ thus

$$H_Y^1(U, \Omega^r) = 0.$$

Also

$$H^q(V, \Omega^r) \simeq H_Y^{q+1}(U, \Omega^r) = 0 \qquad \text{if} \quad q \neq d.$$

Since U can describe a fundamental system of neighborhoods of y in Z, we get the conclusion of the lemma for $d \geq 1$. For $d = 0$, a simpler argument gives the same conclusion.

COROLLARY. *If Y is a submanifold of Z of codimension $d+1$ at each one of its points, then:*

$$H^i_Y(Z, \Omega^r) = 0 \qquad \text{if} \quad j \leq d$$

$$H^{d+1}_Y(Z, \Omega^r) = H^0(Z, \mathcal{H}^{d+1}_Y(\Omega^r))$$

$$H^{d+1+r}_Y(Z, \Omega^r) = H^r(Z, \mathcal{H}^{d+1}_Y(\Omega^r)), \qquad \forall\, r \geq 0.$$

In particular we get

$$H^j(Z, \Omega^r) \xrightarrow{\sim} H^j(X, \Omega^r) \qquad \text{for} \quad 1 \leq j < d$$

and the exact sequence:

$$0 \to H^d(Z, \Omega^r) \to H^d(X, \Omega^r) \xrightarrow{\mu} H^0(Z, \mathcal{H}^{d+1}_Y(\Omega^r)) \to H^{d+1}(Z, \Omega^r).$$

Where μ is the composition of the homomorphism δ of the relative cohomology sequence and the isomorphisms coming from the spectral sequence. Thus μ is a Φ-isomorphism.

To make explicit the Φ-isomorphism μ, we proceed as follows: We consider a covering $\mathcal{U} = \{U_i\}$ of Z by coordinate patches of the form

$$U_i \approx D^{d+1} \times D^{n-d-1}$$

with the condition:

$$U_i \cap Y \neq \varnothing \qquad \text{then} \qquad Y \cap U_i = \{0\} \times D^{n-d-1}.$$

This covering can be chosen finite since Z is compact.

We set $V_i = U_i - U_i \cap Y$ and consider the commutative diagram:

$$\text{(I)} \qquad \begin{array}{ccc} & & 0 \\ & & \downarrow \\ H^d(X, \Omega^d) & \xrightarrow{\mu} & H^0(Z, \mathcal{H}^{d+1}_Y(\Omega^d)) \\ \downarrow \lambda & & \downarrow i \\ \prod_i H^d(V_i, \Omega^d) & \xrightarrow[\sim]{\pi\delta_i} & \prod_i H^{d+1}_Y(U_i, \Omega^d) \\ \downarrow \delta & & \downarrow \delta \\ \prod_{ij} H^d(V_i \cap V_j, \Omega^d) & \xrightarrow[\sim]{\pi\delta_{ij}} & \prod_{ij} H^{d+1}_Y(U_i \cap U_j, \Omega^d) \end{array}$$

where the right hand vertical maps exhibit the definition of the group $H^0(Z, \mathcal{K}_Y^{d+1}(\Omega^d))$ as $H^0(\mathcal{U}, \mathcal{K}_Y^{d+1}(\Omega^d))$, i. e.,

$$H^0(\mathcal{U}, \mathcal{K}_Y^{d+1}(\Omega^d)) = \operatorname{Ker}\{\mathcal{C}^0(\mathcal{U}, \mathcal{K}_Y^{d+1}(\Omega^d)) \xrightarrow{\delta} \mathcal{C}^1(\mathcal{U}, \mathcal{K}_Y^{d+1}(\Omega^d))\}.$$

Since

$$H^d(U_i, \Omega^d) = 0 = H^{d+1}(U_i, \Omega^d)$$

$$H^d(U_i \cap U_j, \Omega^d) = 0 = H_Y^{d+1}(U_i \cap U_j, \Omega^d),$$

it follows that the natural maps

$$\delta_i : H^d(V_i, \Omega^d) \longrightarrow H_Y^{d+1}(U_i, \Omega^d)$$

$$\delta_{ij} : H^d(V_i \cap V_j, \Omega^d) \longrightarrow H^{d+1}(U_i \cap U_j, \Omega^d)$$

are isomorphisms.

In the previous diagram λ is defined as the product of the restriction maps

$$\lambda_i : H^d(X, \Omega^d) \longrightarrow H^d(V_i, \Omega^d).$$

We have

$$\mu \circ i = (\pi\delta_i) \circ \lambda$$

which shows that λ is itself Φ-injective.

10. *Some auxiliary lemmas.* Let us consider in $\mathbf{C}^n$ the resolution

$$0 \longrightarrow \mathbf{C} \longrightarrow \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \longrightarrow \dots \longrightarrow \Omega^n \longrightarrow 0$$

and let

$$\mathcal{Z}^r = \operatorname{Ker}(\Omega^r \xrightarrow{d} \Omega^{r+1}).$$

LEMMA 11. *If U is a domain of holomorphy we have*

$$H^r(U, \mathcal{Z}^s) \simeq H^{r+s}(U, \mathbf{C}) \qquad \text{if } r \geq 1.$$

PROOF.

$$0 \longrightarrow \mathcal{Z}^s \longrightarrow \Omega^s \xrightarrow{d} \Omega^{s+1} \longrightarrow \dots \longrightarrow \Omega^n \longrightarrow 0$$

is an acyclic resolution of $\mathcal{Z}^s$. Thus

$$H^r(U, \mathcal{Z}^s) = \frac{\operatorname{Ker}\{\Gamma(U, \Omega^{s+r}) \xrightarrow{d} \Gamma(U, \Omega^{s+r+1})\}}{d\,\Gamma(U, \Omega^{s+r+1})}$$

$$= H^{s+r}(U, \mathbf{C}).$$

Let U be the set $D^{d+1} \times D^{n-d-1}$ and $V = (D^{d+1} - \{0\}) \times D^{n-d-1}$.

LEMMA 12.

$$H^j(V, \mathcal{Z}^s) =_\Phi 0 \qquad \text{for } j \neq 0, d$$

precisely we have

$$\textit{for } 0 < j < d \quad H^j(V, \mathcal{Z}^s) = 0$$

$$\textit{for } d < j \qquad H^j(V, \mathcal{Z}^s) = \begin{cases} 0 \text{ if } j + s \neq 2d+1 \\ \mathbf{C} \text{ if } j + s = 2d+1 \end{cases}.$$

PROOF. From the exact sequence

$$0 \longrightarrow \mathcal{Z}^s \longrightarrow \Omega^s \xrightarrow{\ d\ } \mathcal{Z}^{s+1} \longrightarrow 0$$

we get if $j < d$ via Lemma 3

$$H^j(V, \mathcal{Z}^s) \simeq H^{j-1}(V, \mathcal{Z}^{s+1}) \simeq \dots$$

$$\simeq H^1(V, \mathcal{Z}^{s+j-1})$$

$$\simeq \frac{H^0(V, \mathcal{Z}^{s+j})}{dH^0(V, \Omega^{s+j+1})} \qquad \text{since } j \geq 1$$

$$\simeq \frac{H^0(U, \mathcal{Z}^{s+j})}{dH^0(U, \Omega^{s+j-1})} \qquad \text{since } d \geq 1$$

$$\simeq 0 \qquad \text{by Lemma 11 since } U \text{ is contractible.}$$

Analogously if $j > d$ we get from Lemma 3

$$H^j(V, \mathcal{Z}^s) \simeq H^{j+1}(V, \mathcal{Z}^{s-1}) \simeq \dots$$

$$\simeq H^{j+s}(V, \mathbf{C}) = \begin{cases} 0 \text{ if } j + s \neq 2d+1 \\ \mathbf{C} \text{ if } j + s = 2d+1. \end{cases}$$

Let $Y = \{0\} \times D^{n-d-1} \subset U$.

LEMMA 13. *With the same notations*

$$H^j(V, \mathcal{Z}^s) \xrightarrow{\sim} H_Y^{j+1}(U, \mathcal{Z}^s) \qquad \text{for } j \geq 1.$$

PROOF. The relative cohomology sequence gives

$$H^j(U, \mathcal{Z}^s) \to H^j(V, \mathcal{Z}^s) \xrightarrow{\delta} H_Y^{j+1}(U, \mathcal{Z}^s) \to H^{j+1}(U, \mathcal{Z}^s).$$

Use then Lemma 11 and the fact that U is contractible.

COROLLARY 1.

$$\text{for } j < d+1: \qquad H_Y^j(U, \mathcal{Z}^s) = 0$$

$$\text{for } j > d+1: \qquad H_Y^j(U, \mathcal{Z}^s) = \begin{cases} 0 & \text{if } j+s \neq 2d+2 \\ \mathbf{C} & \text{if } j+s = 2d+2. \end{cases}$$

PROOF. If $j > d+1$ then $j \geq 1$ and Lemma 13 and 12 give the second part of the lemma.

If $j < d+1$ and also if $j \geq 2$ then again Lemmas 13 and 12 give the result.

If $j = 1, d > 1$ and we have the exact sequence

$$0 \to H^0(U, \mathcal{Z}^s) \xrightarrow{\alpha} H^0(V, \mathcal{Z}^s) \to H_Y^1(U, \mathcal{Z}^s) \to 0$$

where α is an isomorphism. Thus

$$H_Y^1(U, \mathcal{Z}^s) = 0.$$

If $j = 0$ from the same sequence, we get

$$H_Y^0(U, \mathcal{Z}^s) = 0,$$

since α is always injective.

COROLLARY 2. *With the notations of the previous section we have the exact sequence of sheaves*

$$0 \to \mathcal{H}_Y^{d+1}(\mathcal{Z}^d) \to \mathcal{H}_Y^{d+1}(\Omega^d) \xrightarrow{d} \mathcal{H}_Y^{d+1}(\mathcal{Z}^{d+1}) \to \mathbf{C} \to 0$$

$$0 \to \mathcal{H}_Y^{d+1}(\mathcal{Z}^{d-1}) \to \mathcal{H}_Y^{d+1}(\Omega^{d-1}) \xrightarrow{d} \mathcal{H}_Y^{d+1}(\mathcal{Z}^d) \to 0.$$

PROOF. From the exact sequences of sheaves

$$0 \to \mathcal{Z}^d \to \Omega^d \xrightarrow{d} \mathcal{Z}^{d+1} \to 0$$

$$0 \to \mathcal{Z}^{d-1} \to \Omega^{d-1} \xrightarrow{d} \mathcal{Z}^d \to 0$$

we get the two exact cohomology sequences with supports:

$$\mathcal{H}_Y^d(\mathcal{Z}^{d+1}) \to \mathcal{H}_Y^{d+1}(\mathcal{Z}^d) \to \mathcal{H}_Y^{d+1}(\Omega^d) \to$$
$$\to \mathcal{H}_Y^{d+1}(\mathcal{Z}^{d+1}) \to \mathcal{H}_Y^{d+2}(\mathcal{Z}^d) \to \mathcal{H}_Y^{d+2}(\Omega^d)$$

and

$$\mathcal{H}_Y^d(\mathcal{Z}^d) \to \mathcal{H}_Y^{d+1}(\mathcal{Z}^{d-1}) \to \mathcal{H}_Y^{d+1}(\Omega^{d-1}) \to \mathcal{H}_Y^{d+1}(\mathcal{Z}^d) \to \mathcal{H}_Y^{d+2}(\mathcal{Z}^{d-1}).$$

From the previous lemmas we derive

$$\mathcal{H}_Y^d(\mathcal{Z}^d) \simeq \mathcal{H}_Y^d(\mathcal{Z}^{d+1}) = 0$$

$$\mathcal{H}_Y^{d+2}(\mathcal{Z}^{d-1}) \simeq 0, \quad \mathcal{H}_Y^{d+2}(\mathcal{Z}^d) \simeq \mathbf{C}, \quad \mathcal{H}_Y^{d+2}(\Omega^d) = 0.$$

Substituting in the exact sequences we get the statement of this corollary.

11. *The fundamental equation of an algebraic variety.* *a*) Let Z ben an irreducible algebraic variety in a projective space $P_N(\mathbf{C})$ where $z_0, \ldots, z_N$ are homogeneous coordinates. Let $n = \dim_{\mathbf{C}} Z$. Consider a system of $(n+2) \times (N+1)$ indeterminates $(s_{ij})_{\substack{0 \le i \le n+1 \\ 0 \le j \le N}} = S$ and set, for $z \in Z$,

$$x = Sz.$$

Over $\mathbf{C}(s), x_0, \ldots, x_{n+1}$ satisfy an irreducible equation

$$f_Z(s\,;x) = 0$$

which is homogeneous in the x's of a certain degree k. We may as well assume that the coefficients of f_Z are polynomials in the variables s, without common factors. The polynomial f_Z is thus determined up to a constant factor.

The polynomial f_Z is called the fundamental polynomial for the variety Z.

The equation

$$f_Z(s, x) = 0$$

represents the equation of the hypersurface of $P_{n+1}(\mathbf{C}(s))$ obtained by projection of Z from the subspace of $P_N(\mathbf{C}(s))$ defined by the equations

$$\{Sz = 0\}$$

This center of projection is a subspace of dimension $N - n - 2$.

b) We now want to study the various possible specializations of the fundamental polynomials $f_Z(s, x)$ when we specialize the matrix S to a numerical matrix.

We thus represent the set of these specializations by the points of a numerical space $\mathbf{C}^{(n+2)\times(N+1)}$ for every $S_0 \in \mathbf{C}^{(n+2)(N+1)}$ we set:

$$Z_{S_0} = \{z \in P_{n+1}(\mathbf{C}) \mid f_z(S_0, x) = 0\}$$

$$\mathcal{E}_{S_0} = \{z \in P_N(\mathbf{C})\ S_0 z = 0\}$$

and we denote by

$$\pi_{S_0} : P_N(\mathbf{C}) - \mathcal{E}_{S_0} \to P_{n+1}(\mathbf{C})$$

the map defined by

$$x = S_0 z.$$

LEMMA 14. *Each of the following conditions is verified in a Zariski open non empty subset of* $\mathbf{C}^{(n+2)(N+1)}$.

i) $\mathcal{E}_{S_0}$ *is a projective subspace of* $P_N(\mathbf{C})$ *of dimension* $N - n - 2$.

ii) $\mathcal{E}_{S_0} \cap Z = \emptyset$ *so that* $\pi_{S_0} \mid Z$ *is a holomorphic map.*

iii) $\pi_{S_0} : Z \to Z_{S_0}$ *induces an isomorphism*

$$\pi^*_{S_0} : \mathcal{R}(Z_{S_0}) \to \mathcal{R}(Z)$$

between the fields of rational functions on Z_{S_0} *and* Z.

PROOF Condition i) is equivalent to rank $S_0 = n + 2$. Let Ω_1 be the Zariski open set where this condition is verified.

Then we get a holomorphic map $g : \Omega_1 \to P_{N-n-2}(P_N(\mathbf{C}))$ of Ω_1 into the Grassmann manifold of $P_{N-n-2} \subset P_N(\mathbf{C})$.

We know that the set of $P_{N-n-2} \subset P_N(\mathbf{C})$ which meet Z forms a proper algebraic subvariety of $P_{N-n-2}(P_N(\mathbf{C}))$ (cf. [2] § 2). Let $\widetilde{\Omega}_2$ be the complement. Then $\Omega_2 = g^{-1}(\widetilde{\Omega}_2)$ is a non empty Zariski open set where condition ii) is satisfied.

Condition iii) can be formulated as follows. Suppose $Z \not\subset \{z_0 = 0\}$ and set $Z_i = z_i/z_0 \,|_Z$, $X_j = x_j/x_0$ then

$$\mathcal{R}(Z) = \mathbf{C}(Z_1, \dots, Z_N) \qquad \mathcal{R}(Z_{S_0}) = \mathbf{C}(X_1, \dots, X_{n+1})$$

where

$$X_i = \frac{s_{i0} + s_{i1} Z_1 + \dots + s_{iN} Z_N}{s_{00} + s_{01} Z_1 + \dots + s_{0N} Z_N}.$$

The condition iii) expresses the fact

$$(*) \qquad \mathcal{R}(Z_{S_0}) = \mathcal{R}(Z).$$

We may assume $Z_1, \dots, Z_n$ to be algebraically independent so that

$$\mathcal{R}(Z_1, \dots, Z_N) = \mathcal{R}(Z_1, \dots, Z_n, \theta)$$

where θ is algebraic over $\mathbf{C}(Z_1, \dots, Z_n)$ satisfying a minimal equation of a certain degree ν,

$$(\alpha) \qquad A_0(Z_1, \dots, Z_n)\,\theta^\nu + \dots + A_\nu(Z_1, \dots, Z_n) = 0.$$

Then $Z_{n+j} = R_j(Z_1, \dots, Z_n, \theta)$ are rational functions of $Z_1, \dots, Z_n$ and θ.

Condition $(*)$ is then equivalent to

$$\det \frac{\partial(X_1, \dots, X_n)}{\partial(Z_1, \dots, Z_n)} \neq 0$$

and

$$x_{n+1}(Z_1, \dots, Z_n, \theta_i) \neq x_{n+1}(Z_1, \dots, Z_n, \theta_j) \qquad \text{for } i \neq j$$

where $x_{n+1}(Z_1, \dots, Z_n, \theta)$ is the expression of x_{n+1} obtained by replacing Z_{n+j} by R_j and where $\theta_1, \dots, \theta_\nu$ are the roots of (α).

These conditions lead to inequalities among the s_{ij} which are satisfied in Zariski open and nonempty sets.

c) We will assume that Z is a manifold. We choose a submanifold Y of Z of pure codimension $d+1$ (i.e., of pure dimension $n-d-1$).

LEMMA 15. *By selecting a convenient projective imbedding of $Z \subset P_N(\mathbf{C})$, we may assume that $Y = Z \cap P_h$ when P_h is a linear projective subspace of codimension $\geq d+1$ in $P_N(\mathbf{C})$. Moreover for any point $y_0 \in Y$ we may find a P_{N-d-1} containing P_h such that*

i) *$Z \cap P_{N-d-1}$ is of pure codimension $d+1$ in Z.*

ii) *y_0 is a simple point of $Z \cap P_{N-d-1}$.*

PROOF. Consider the homogeneous ideal

$$\mathcal{P}_Y = \{g \in \mathbf{C}[z] \mid \quad g|_Y = 0\}$$

let ν be the maximal degree of the polynomials in a basis of $\mathcal{P}_Y$. Then Y can be written as the set of common zeros of all polynomials in $\mathcal{P}_Y$ of a fixed degree $\mu \geq \nu$.

If we take the imbedding of Z given by the homogeneous monomials of degree μ, then $Y = Z \cap P_h$ where P_h is a linear space.

Clearly if $Z \subset P_N(\mathbf{C})$ is this new imbedding, $h \leq N - (d+1)$. We can find an integer $\mu_0 \geq \nu$ so large that if $\mu \geq \mu_0$, for every point $y_0 \in Y$ we can find a set of $d+1$ homogeneous polynomials $f_0, \ldots, f_d$ in $\mathcal{P}_Y$, of degree μ, such that

$$(*) \qquad (\Sigma (-1)^i f_i \, df_0 \wedge \ldots \wedge \widehat{df_i} \wedge \ldots \wedge df_d)_{y_0} \neq 0.$$

Moreover $f_0, f_1, \ldots, f_d$ can be so chosen that

$$Z_1 = \{z \in Z \,|\, f_0(z) = 0\} \quad \text{is of codimension 1 in } Z$$

$$Z_2 = \{z \in Z_1 \,|\, f_1(Z) = 0\} \quad \text{is of codimension 1 in } Z_1$$

$$\ldots \qquad \ldots$$

$$Z_{d+} = \{z \in Z_d \,|\, f_d(z) = 0\} \quad \text{is of codimension 1 in } Z^d$$

so that

$$(**) \qquad \{z \in Z \,|\, f_0(z) = \ldots = f_d(z) = 0\} \text{ is of pure codimension } d+1 \text{ in } Z.$$

This is because the vector space of polynomials of degree μ_0 in $\mathcal{P}_Y$ has Y as the set of common zeros to all of its elements.

In the imbedding by monomials of degree μ_0 it follows then that $Z_{d+1} = Z \cap P_{N-(d+1)}$ and ii) is satisfied because of (*), and i) is satisfied because of (**)

LEMMA 16. *For any $y_0 \in Y$ we can choose S_0 satisfying the three conditions of Lemma 14 and such that*

a) $\pi_{S_0}(Y)$ is contained in a projective space P_{n-d} of codimension $d+1$ in $P_{n+1}(\mathbf{C})$,

b) π_{S_0} induces a biregular map between a neighborhood of y_0 in Z on a neighborhood of $\pi_{S_0}(y_0)$ in Z_{S_0}.

PROOF. We choose P_{N-d-1} as in Lemma 15. We now select a center of projection $\mathcal{E}_{S_0} = P_{N-n-2} \subset P_{N-d-1}$. We can assume that

i) $\mathcal{E}_{S_0} \cap (P_{N-d-1} \cap Z) = \varnothing$,

ii) $\mathcal{E}_{S_0}$ do not meet any projective line joining y_0 to another point of Z,

iii) $\mathcal{E}_{S_0}$ do not meet the tangent space to Z at y_0.

Indeed:

i) $Y^1 \cup Y = P_{N-d-1} \cap Z$ is of pure dimension $n - d - 1$ and $\mathcal{E}_{S_0}$ is in P_{N-d-1} of codimension $n - d + 1$ so i) can be satisfied by almost all choice of $\mathcal{E}_{S_0}$.

ii) The cords joining y_0 to the other points of Z fill up a variety whose intersection with P_{N-d-1} consists of the cords joining y_0 to another point of $Y^1 \cup Y$. These then fill an algebraic subvariety of P_{N-d-1} of dimension $\leq n - d - 1 + 1 = n - d$ (cf. [2] Lemma 9). Since $\mathcal{E}_{S_0}$ is of codimension $n - d + 1$ we can satisfy ii).

iii) By construction P_{N-d-1} is transversal to Z at y_0, i. e., the tangent space to Z at y_0 cut out in P_{N-d-1} the tangent space P_{n-d-1} to Y at y_0. Again by the same dimension argument we see that $\mathcal{E}_{S_0}$ can be chosen to satisfy iii).

Now if $\mathcal{E}_{S_0}$ satisfies the previous requirements, it corresponds to a choice of S_0 which satisfies condition i) and ii) of Proposition 14. Moreover $\pi_{S_0}^{-1}\pi_{S_0}(y_0) = y_0$ and because of iii), π_{S_0} is biregular at y_0, $\pi_{S_0}(y_0)$. This implies that π_S must be generally one to one and thus also condition iii) of Proposition 14 is verified.

12. Let Z be a projective manifold of dimension n, and let Y be a submanifold of Z of pure codimension $d+1$. Let us consider the space $\mathcal{C}_d^+(X)$ of compactly supported d-dimensional cycles of $X = Z - Y$ and the evaluation map

$$\varrho_0 : H^d(X, \Omega^d) \longrightarrow \Gamma(\mathcal{C}_d^+(X), \mathcal{O}).$$

This evaluation map is of special interest when $H^d(X, \Omega^d)$ is an infinite dimensional space. This is certainly the case when X is strictly d-pseudoconvex, in particular when Y is a complete intersection.

The map ϱ_0 may have a large kernel. The description of this kernel up to finite dimensional spaces is provided by the following.

THEOREM 3. *Assume that X is d-pseudoconvex, then the sequence of homomorphisms*:

$$H^d(X, \Omega^{d-1}) \xrightarrow{\ d\ } H^d(X, \Omega^d) \xrightarrow{\ \varrho_0\ } \Gamma(\mathcal{C}_d^+(X), \mathcal{O})$$

is Φ exact.

PROOF. (0) Let φ^{dd} be a C^∞ form of type (dd), $\bar{\partial}\varphi = 0$ representing a cohomology class

$$\xi \in H^d(X, \Omega^d).$$

We first of all remark that if

$$\xi \in \operatorname{Im}\{H^d(X, \Omega^{d-1}) \xrightarrow{\ d\ } H^d(X, \Omega^d)\}$$

then we can write

$$\varphi^{dd} = \partial\psi^{d-1, d} + \bar{\partial}\eta^{d, d-1}$$

with $\psi^{d-1,d}$, $\eta^{d,d-1}$, C^∞ forms of the corresponding degrees on X and with $\bar\partial\psi^{d-1,d}=0$. Then

$$\varrho_0(\xi)(c)=\int_c\varphi^{dd}=\int_c d\psi+d\eta=0$$

for $\forall\, c\in C_d^+(X)$. This shows that $\varrho_0\circ d=0$.

The rest of the proof will be divided in several steps.

(α) For every $y_0\in Y$ we select $S=S(y_0)$ as in Lemma 16 and choose coordinates $x_0,\dots,x_{n+1}$ in $P_{n+1}(\mathbf{C})$ so that

$$\pi_{S(y_0)}(y_0)=(1,0,\dots,0),$$

the tangent space to $Z_{S(y_0)}$ at $\pi_{S(y_0)}(y_0)$ is the space $x_{n+1}=0$.

We select nonhomogeneous coordinates $y_i=x_i/x_0$ so that in a neighborhood of the origin in $\mathbf{C}^{n+1}$ where $y_1,\dots,y_{n+1}$ are the coordinates, $Z_{S(y_0)}$ has an equation of the form

$$(*)\qquad y_{n+1}=G(y_1,\dots,y_n)$$

where G vanishes at the origin of order ≥ 2. We may also assume that equation $(*)$ is valid in the disc

$$U^1=D^{d+1}\times D^{n-d-1}=D^n$$

in the space $\mathbf{C}^n$ of the variables $y_1,\dots,y_n$. The parallel projection to the y_{n+1} axis gives a coordinate patch U on $Z_{S(y_0)}$ and thus on Z near y_0.

Moreover we may suppose that $\pi_{S(y_0)}(Y)$ is contained in the space

$$y_1=\dots=y_{d+1}=0$$

so that in the coordinate patch U, $U\cap Y$ corresponds to $\{0\}\times D^{n-d-1}$. We set $V=U-U\cap Y$.

(β) We select a finite number of coordinate patches of the type described in (α) $\{U_j\}_{1\leq j\leq t}$ so that $\cup\, U_j$ covers a neighborhood of Y in Z. With the choice of this covering we set up the diagram (I) of n. 9.

(γ) Let U be one of the coordinate patches U_j. To simplify the notations we will denote by

Y_0 the projection of Y in P_{n+1}

Z_0 the projection of Z in P_{n+1}

y_0 the projection of $y\in Y$ in P_{n+1}

and as in (α), we select y_0 in $(1, 0, \dots, 0)$ with tangent space to Z_0 at y_0 the space $x_{n+1} = 0$.

We may also assume that the axis of $y_{n+1} = x_{n+1}/x_0$ cuts Z_0 in a finite number of points

$$Q_1 = 0, \quad Q_2, \dots, Q_k$$

all outside $x_0 = 0$ and such that in a neighborhood of each of these points the hypersurface Z_0 has an equation

$$(*) \qquad y_{n+1} = f_i(y_1, \dots, y_n) \qquad \text{near } Q_i,\ 1 \leq i \leq k,$$

with f_i holomorphic for $\Sigma\ |y_i|^2 < r$.

Let W_i denote the neighborhoods given by the equations $(*)$ and let π_{n+1} denote the parallel projection to the y_{n+1} axis on the space $y_1, \dots, y_n$, so that $\pi_{n+1}(W_i) = \{(y_1, \dots, y_n) \mid \Sigma\ |y_i|^2 < r\} = \Delta$. We may also assume $D^n = D^{d+1} \times D^{n-d-1} \subset \Delta$. We set

$$z = (z_1, \dots, z_{d+1}) = (y_1, \dots, y_{d+1}); \quad w = (w_1, \dots, w_{n-d-1}) = (y_{d+1}, \dots, y_n); \quad y_{n+1}$$

as a convenient new notation for the nonhomogeneous coordinates. Let

$$f(z, w, y_{n+1}) = 0$$

be the equation of Z_0 (obtained by specializing S and passing to nonhomogeneous coordinates).

By the above assumption we have

$$f \equiv y_{n+1}^k + A_1(z, w)\, y_{n+1}^{k-1} + \dots + A_k(z, w)$$

up to a constant factor $\neq 0$.

Let us consider the family of $P_{d+1} \subset P_{n+1}$ given by nonhomogeneous equations by

$$P_{d+1} \equiv \begin{cases} w = w^0 + Az & w^0 = (w_1^0, \dots, w_{n-d-1}^0); \\ by_{n+1} = t + a_1 z_1 + \dots + a_{d+1} z_{d+1} & A = (a_{ji}),\ 1 \leq j \leq n-d-1, \\ & 1 \leq i \leq d+1. \end{cases}$$

We claim that if $-t/b \neq f_i(0, w^0)$ for $1 \leq i \leq k$ then the variety

$$V = P_{d+1} \cap Z_0$$

is pure d-dimensional.

Indeed obviously each component of V has dimension $\geq d$ since Z_0 is a hypersurface.

Since $\pi_{n+1} \mid Z_0$ is proper with discrete fibers it is enough to show that $\pi_{n+1}(V)$ has each component of dimension $\leq d$.

Now the variety $\pi_{n+1}(V)$ is given by

$$\begin{cases} f\left(z, w, -\left(\frac{t}{b} + \frac{a_1}{b} z_1 + \dots + \frac{a_{d+1}}{b} z_{d+1}\right)\right) = 0 \\ w = w^0 + Az \end{cases}$$

which leads to the equation in $z_1, \dots, z_{d+1}$:

$$f\left(z, w_0 + Az, -\left(\frac{t}{b} + \frac{a_1}{b} z_1 + \dots + \frac{a_{d+1}}{b} z_{d+1}\right)\right) = 0.$$

This equation is not identically satisfied since for $z = 0$, $-t/b$ is not a root of $f(0, w_0, y_{n+1})$ (these roots being represented by $(*)$ for $z = 0$, $w = w_0$).

Let $\pi : Z \longrightarrow Z_0$ be the projection of Z onto Z_0 and let

$$c = c(t, a, b, w^0, A) = Z \cdot P_{N-(n-d)}$$

denote the cycles which project via π to cycles $P_{d+1} \cdot Z_0$.

Note that $P_{N-(n-d)}$ is the projective space which projects P_{d+1} from the center of projection $\mathcal{E}$.

(δ) Let us fix w^0, sufficiently small so that $(0, w^0) \in D^n$, $\| w^0 \| = \Sigma \mid w_j^0 \mid < \varepsilon$, and let us also fix the matrix $A = A^0$. Then P_{d+1} intersects the space $z = 0$ in the point

$$\left(0, w^0, -\frac{t}{b}\right)$$

and this will not be a point of Y_0 if

$$(**) \qquad -\frac{t}{b} \neq f_i(0, w^0) \qquad 1 \leq i \leq k.$$

Therefore, if condition $(**)$ is satisfied

$$\operatorname{supp} c(t, a, b, w^0, A^0) \subset X.$$

As t, a, b varies, the cycle $c(t, a, b)$ varies, and it will meet Y only at a point Q with

$$\pi(Q) = (0, w^0, f_i(0, w^0)) \in W_i.$$

We set

$$\Delta_\varepsilon = \{(y_1, \dots, y_n) \mid \Sigma \mid y_i \mid \leq \varepsilon\}$$

$$W_i(\varepsilon) = \pi_{n+1}^{-1} |_{Z_0} (\Delta_\varepsilon) \quad \text{and} \quad \widetilde{W}_i(\varepsilon) = \pi^{-1} |_Z (W_i(\varepsilon)).$$

Then by the above remark, we have with respect to any fixed metric in P_N that there exists a $\delta(\varepsilon) > 0$ such that:

$$(***) \qquad \text{dist}(\text{supp } c(t, a, b), Y - \cup Y \cap \widetilde{W}_i(2\varepsilon)) > \delta(\varepsilon) > 0 \quad t, a, b \text{ not all zero.}$$

(ε) Let ϱ be a C^∞ function on Z with compact support

$$\text{supp } \varrho \subset \cup \widetilde{W}_i(3\varepsilon)$$

$$\varrho = 1 \quad \text{on} \quad \cup \widetilde{W}_i(2\varepsilon)$$

and let

$$\{\varphi^{dd}\} \in H^d(X, \Omega^d) \quad \text{with} \quad \{\varphi^{dd}\} \in \text{Ker } \varrho_0 .$$

We consider then

$$\int_{c(t, a, b)} \varphi^{dd} = \int_{c(t, a, b)} \varrho \varphi^{dd} + \int_{c(t, a, b)} (1 - \varrho) \varphi^{dd} .$$

Since

$$\int_{c(t, a, b)} \varphi^{dd} = 0$$

and because of (***), we deduce that we can find a constant $M > 0$ such that

$$\left| \int_{c(t, a, b)} (1 - \varrho) \varphi^{dd} \right| < M$$

and therefore also

$$\left| \int_{c(t, a, b)} \varrho \varphi^{dd} \right| < M .$$

(η) We are interested in the values of the integrals

$$I \equiv \int_{c(t, a, b) \cap \widetilde{W}_1} \varrho \varphi^{dd} .$$

From the above condition we get

$$\left| \sum_{1}^{k}{}_i \int\limits_{c(t,\,a,\,b)\,\cap\,\widetilde{W}_i} \varrho\, \varphi^{dd} \right| < M .$$

We want to show that a similar condition can be inferred for the integral I itself

Set

$$f(z, w^0 + A^0 z) = f(0, w^0) + \sum_{1}^{d+1} k_i z_i + g(z)$$

where $g(z)$ vanishes for $z = 0$ of order ≥ 2.

In local coordinates the integral I reduces to an integral of the form

$$I = \int\limits_F \varrho\, \psi$$

where ψ is a form of type (dd) defined and C^∞ on $D^{d+1} - \{0\}$, where ϱ is a compactly supported function in D^{d+1} equal to 1 near the origin, and where F is the divisor of the function

$$\theta + \sum_{1}^{d+1} \alpha_i z_i + b\, g(z)$$

where $\theta = t - b f(0, w^0)$, $\alpha_i = a_i - b k_i$, $1 \leq i \leq d + 1$.

Moreover one sees that i) $I = I(\theta, a, b)$ is defined for all values of α, b and $\theta \neq 0$, ii) as a function of θ, a, b, I is homogeneous of degree 0:

$$I(\lambda\theta, \lambda\alpha, \lambda b) = I(\theta, \alpha, b) \qquad \forall\, \lambda \neq 0.$$

Now if ε is sufficiently small and $\sigma = \sigma(\varepsilon)$ is accordingly chosen sufficiently small, in the region

$$0 < \left| \frac{\theta}{b} \right| < \sigma, \quad \Sigma \left| \frac{\alpha_i}{b} \right| = \sigma$$

the corresponding integrals I are over cycles $c(t, a, b)$ whose supports do not meet the neighborhoods

$$\widetilde{W}_2(3\,\varepsilon) \cup \ldots \cup \widetilde{W}_k(3\,\varepsilon)$$

Therefore from the inequality above all terms except the first disappear and we get

$$|I(\theta, \alpha, b)| < M$$

for

$$0 < \left|\frac{\theta}{b}\right| < \sigma \quad \text{and} \quad \Sigma\left|\frac{\alpha_i}{b}\right| < \sigma .$$

Now as in Corollary 3 of Proposition 4 we consider the entire function

$$E_I\left(\frac{\alpha_1}{\theta}, \dots, \frac{\alpha_{d+1}}{\theta}, \frac{b}{\theta}\right)$$

there defined and remark that there exists a constant M_1 such that for:

$$\left|\frac{b}{\theta}\right| > \frac{1}{\sigma}; \quad \Sigma\left|\frac{\alpha_i}{\theta}\right| < \sigma\left|\frac{b}{\theta}\right|$$

we have

$$\left|E_I\left(\frac{\alpha_1}{\theta}, \dots, \frac{\alpha_{d+1}}{\theta}, \frac{b}{\theta}\right)\right| < M_1 .$$

Consider the polycylinders for $R > 1/\sigma$

$$P_R = \left\{ \left|\frac{b}{\theta}\right| < R ; \ \left|\frac{\alpha_i}{\theta}\right| < \frac{\sigma}{2(d+1)} R \right\}$$

whose Shilov boundaries are

$$S_R = \left\{ \left|\frac{b}{\theta}\right| = R, \ \left|\frac{\alpha_i}{\theta}\right| = \frac{\sigma}{2(d+1)} R \right\}$$

on S_R, $|E_I| < M_1$, $\forall R > 1/\sigma$ thus we get that on the whole space $\mathbf{C}^{d+2}$ where $\alpha_1/\theta, \dots, \alpha_{d+1}/\theta$, b/θ are the coordinates we must have

$$\left|E_I\left(\frac{\alpha_1}{\theta}, \dots, \frac{\alpha_{d+1}}{\theta}, \frac{b}{\theta}\right)\right| < M_1 .$$

This implies the following condition:

For every divisor of the form $F = (\theta + \Sigma\, \alpha_i z_i + b g(z))$ we have

$$\sup \left| \int_F \varrho\psi \right| < \infty.$$

By Theorem 1 we conclude with the following assertion:

Let us consider a covering $\mathcal{U} = \{U_j\}_{1 \leq j \leq t}$ of Y as defined in (α) and a cohomology class

$$\xi \in \operatorname{Ker} \{H^d(X, \Omega^d) \xrightarrow[\varrho_0]{} \Gamma(W, \mathcal{O})\}$$

where W represents the component of $\mathcal{C}_d^+(X)$ of the cycles of the form

$$Z \cdot P_{N-(n-d)}\,.$$

Let φ^{dd} be a Dolbeault representative of ξ. Then for each j, $1 \leq j \leq t$ we can find C^∞ forms ψ_j, χ_j of types $(d-1, d)$ and $(d, d-1)$ defined on $V_j = U_j - U_j \cap Y$ such that

$$\varphi\,|_{V_j} = \partial \psi_j + \bar{\partial} \chi_j\,, \qquad \bar{\partial} \psi_j = 0.$$

(θ) The previous assertion has the following meaning: Let us consider the diagram (I) of n. 9 then

$$\lambda(\xi) \in \operatorname{Im} \left\{ \prod_i H^d(V_i, \Omega^{d-1}) \xrightarrow{d} \prod_i H^d(V_i, \Omega^d) \right\}.$$

Since $\pi\delta_i$ is an isomorphism we can also say that

$$(*) \qquad i \circ \mu(\xi) \in \operatorname{Im} \left\{ \prod_i H_Y^{d+1}(U_i, \Omega^{d-1}) \xrightarrow{d} \prod_i H_Y^{d+1}(U_i, \Omega^d) \right\}.$$

Now from Corollary 2 to Lemma 13 of n. 10 we deduce that we have an injective map of sheaves:

$$0 \to \mathcal{H}_Y^{d+1}(\mathcal{Z}^d) \to \mathcal{H}_Y^{d+1}(\Omega^d),$$

therefore applying the functor $H^0(\mathcal{U},\ \ )$ we get an injection

$$0 \to H^0(Z, \mathcal{H}_Y^{d+1}(\mathcal{Z}^d)) \xrightarrow{t} H^0(Z, \mathcal{H}_Y^{d+1}(\Omega^d)).$$

Thus from (*) we deduce that

$$\mu(\xi) \in t\{H^0(Z, \mathcal{H}_Y^{d+1}(\mathcal{Z}^d))\}.$$

We now consider the second exact sequence of Corollary 2 quoted before and from that we obtain the exact cohomology sequence

$$0 \to H^0(Z, \mathcal{H}_Y^{d+1}(\mathcal{Z}^{d-1})) \to H^0(Z, \mathcal{H}_Y^{d+1}(\Omega^{d-1})) \to$$

$$\to H^0(Z, \mathcal{H}_Y^{d+1}(\mathcal{Z}^{d})) \to H^1(Z, \mathcal{H}_Y^{d+1}(\mathcal{Z}^{d-1})).$$

Via the spectral sequence, we get the Φ-isomorphism

$$(**) \qquad H^1(Z, \mathcal{H}_Y^{d+1}(\mathcal{Z}^{d-1})) \simeq_\Phi H_Y^{d+2}(Z, \mathcal{Z}^{d-1}).$$

Indeed we have

$$\mathcal{H}_Y^{j}(\mathcal{Z}^{d-1}) =_\Phi 0 \qquad \text{if} \quad j \neq d+1$$

as we have seen in Corollary 1 to Lemma 13 n. 10.

Now for a compact manifold Z we have

$$H^j(Z, \mathcal{Z}^s) \simeq_\Phi H^{j+1}(Z, \mathcal{Z}^{s-1}) \simeq_\Phi \ldots$$

$$\simeq_\Phi H^{j+s}(Z, \mathbf{C}) \simeq_\Phi 0.$$

Therefore from the local cohomology sequence for the sheaf $\mathcal{Z}^{d-1}$ we get

$$(***) \qquad H_Y^{d+2}(Z, \mathcal{Z}^{d-1}) \overset{\delta}{\underset{\sim}{\longleftarrow}}_\Phi H^{d+1}(X, \mathcal{Z}^{d-1}).$$

Finally assuming X to be d-pseudoconvex we get

$$(****) \qquad H^{d+1}(X, \mathcal{Z}^{d-1}) \simeq_\Phi H^{d+2}(X, \mathcal{Z}^{d-2}) \simeq_\Phi \ldots$$

$$\simeq_\Phi H^{2d}(X, \mathbf{C}) \simeq_\Phi 0$$

(since X is obtained by removing a submanifold from a compact manifold $H^{2d}(X, \mathbf{C})$ is finite dimensional).

From (*) to (****) we therefore obtain that we have a Φ exact sequence

$$0 \to H^0(Z, \mathcal{H}_Y^{d+1}(\mathcal{Z}^{d-1})) \to H^0(Z, \mathcal{H}_Y^{d+1}(\Omega^{d-1})) \xrightarrow{d} H^0(Z, \mathcal{H}_Y^{d+1}(\mathcal{Z}^{d})) \to 0.$$

Therefore

$$\operatorname{Im}\{H^0(Z, \mathcal{H}_Y^{d+1}(\mathcal{Z}^{d})) \xrightarrow{t} H^0(Z, \mathcal{H}_Y^{d+1}(\Omega^{d}))\}$$

$$=_\Phi \operatorname{Im}\{H^0(Z, \mathcal{H}_Y^{d+1}(\Omega^{d-1})) \xrightarrow{d} H^0(Z, \mathcal{H}_Y^{d+1}(\Omega^{d}))\}.$$

Therefore taking into account the fact that $\varrho_0 \circ d = 0$ we get

$$\mu\,\{\mathrm{Ker}\,\varrho_0\} =_{\Phi} \mathrm{Im}\,\{H^0(Z, \mathcal{K}_Y^{d+1}(\Omega^{d-1})) \xrightarrow{\ d\ } H^0(Z, \mathcal{K}_Y^{d+1}(\Omega^d))\}$$

or finally since μ is a Φ-isomorphism

$$\mathrm{Ker}\,\varrho_0 =_{\Phi} \mathrm{Im}\,\{H^d(X, \Omega^{d-1}) \xrightarrow{\ d\ } H^d(X, \Omega^d)\}.$$

This completes the proof of the theorem.

Corollary 1. *Let W be the component of $\mathcal{C}_d^+(X)$ of those cycles of dimension d of the form*

$$Z \cdot P_{N-(n-d)}$$

compactly supported in X (i. e., the complete intersection with a linear space of the proper dimension). Then under the assumption that X be d-pseudoconvex we get the Φ-exact sequence,

$$H^d(X, \Omega^{d-1}) \xrightarrow{\ d\ } H^d(X, \Omega^d) \xrightarrow{\ \varrho_0\ } \Gamma(W, \mathcal{O}).$$

This is a corollary of the proof we have given, indeed exactly what we have proved.

13. Let us now consider for a d-pseudoconvex manifold X the group

$$V^{dd}(X) = \frac{\mathrm{Ker}\,\{A^{dd}(X) \xrightarrow{\ \partial\bar\partial\ } A^{d+1,\,d+1}(X)\}}{\partial A^{d-1,\,d}(X) + \bar\partial A^{d,\,d-1}(X)}.$$

By integration from every class

$$\xi \in V^{dd}(X)$$

we obtain a complex valued pluriharmonic function

$$\xi(c) = \int_c \varphi^{dd} \qquad \text{for } \varphi^{dd} \in \xi,\ c \in \mathcal{C}_d^+(X)$$

(which is independent of the choice of the representative $\varphi^{dd} \in \xi$) on the space $\mathcal{C}_d^+(X)$).

Let us denote by

$$\varrho_0 : V^{dd}(X) \longrightarrow \Gamma(\mathcal{C}_d^+(X), \mathcal{H})$$

$\mathcal{H}$ being the sheaf of germs of pluriharmonic functions in $\mathcal{C}_d^+(X)$ (1). Let W be the component of $\mathcal{C}_d^+(X)$ defined in Corollary 1 of Theorem 3. We have:

THEOREM 4. *If X is a d-pseudoconvex manifold the map*

$$\varrho_0 : V^{dd}(X) \longrightarrow \Gamma(W, \mathcal{H})$$

is Φ injective.

PROOF. By Corollary to Proposition 2 of n. 2, we have

$$V^{dd}(X) \simeq_{\Phi} \operatorname{Coker}\{H^d(X, \Omega^{d-1}) \xrightarrow{d} H^d(X, \Omega^d)\}$$

$$\oplus \operatorname{Coker}\{H^d(X, \bar{\Omega}^{d-1}) \xrightarrow{d} H^d(X, \bar{\Omega}^d)\}.$$

Also since W is a Zariski open set of a compact algebraic variety $H^1(W, \mathbf{C}) \in \Phi$ so that

$$\Gamma(W, \mathcal{H}) \simeq_{\Phi} \Gamma(W, \mathcal{O}) \oplus \Gamma(W, \bar{\mathcal{O}}).$$

Therefore ϱ_0, up to Φ equivalence, reduces to the couple of maps

$$\varrho_0 : \operatorname{Coker}\{H^d(X, \Omega^{d-1}) \xrightarrow{d} H^d(X, \Omega^d)\} \longrightarrow \Gamma(W, \mathcal{O})$$

$$\bar{\varrho}_0 : \operatorname{Coker}\{H^d(X, \bar{\Omega}^{d-1}) \xrightarrow{d} H^d(X, \bar{\Omega}^d)\} \longrightarrow \Gamma(W, \bar{\mathcal{O}})$$

and these are Φ-injective as follows from the Corollary of Theorem 3.

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(1) $\mathcal{H}$ is defined by the exact sequence of sheaves:

$$0 \longrightarrow \mathbf{C} \xrightarrow{\beta} \mathcal{O} \oplus \bar{\mathcal{O}} \xrightarrow{\alpha} \mathcal{H} \longrightarrow 0,$$

$\beta(c) = c \oplus c$, $\alpha(f \oplus g) = f - g$.

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