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Regularity results for non-linear elliptic systems in two dimensions

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The purpose of this paper is to prove the regularity of the weak solution of Dirichlet problem for non linear elliptic systems in two dimensions. This problem is considered in the following form:

Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, $u$ be a weak solution of the system

$\sum_{|\alpha| \leq \nu} (-1)^{|\alpha|} D^\alpha \alpha(x, Du(x)) = f_r(x) ; r = 1, \ldots, m, Du = \{Du_s|_{\gamma} = u_s\} \in \mathbb{R}^n$

with a boundary condition $u_0$ i.e.

1) $u = (u_1, \ldots, u_m), u_0 = (u_0^1, \ldots, u_0^m); u_r \in \tilde{W}_k^{\nu r}(\Omega); r = 1, \ldots, m$

2) $\int_{\Omega} \sum_{r=1}^{m} \sum_{|\alpha| \leq \nu} \alpha(x, Du(x)) D^\alpha \varphi_r(x) dx = 0$

for every $\varphi_r \in \tilde{W}_k^{\nu r}(\Omega)$.

The regularity means that $u_r$ belongs to $C_{\nu r, \mu}(\Omega)$ for $r = 1, \ldots, m$.

This result was proved by

1. Ch. B. Morrey (1937) for $N = 2, m = 1, \nu = 1, k = 2$,
2. E. De Giorgi (1957) for $N \geq 2, m = 1, \nu = 1, k = 2$,
3. O. A. Ladyženskaja - N. N. Uralceva (1959) for $N \geq 2, m = 1, \nu = 1, 1 < k < \infty$,
4. Ch. B. Morrey (1960) $N \geq 2, m = 1, \nu = 1, 1 < k < \infty$,
5. J. Nečas (1966) $N = 2, m = 1, \nu \geq 1, k = 2$,
6. J. Nečas (1967) $N = 2, m = 1, \nu \geq 1, 1 < k < \infty$.

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In this paper, the regularity is proved for \( N = 2, m \geq 1, \kappa_r \geq 1, k \geq 2 \).

For \( N \geq 2 \) there was proved a partial regularity (see Morrey, [12]) as follows: for every \( \Omega_0, \Omega_0 \subset \Omega \) there exists an \( \Omega_1 \) so that \( u \) is regular on \( \Omega_1 \) and the Lebesgue's measure of \( \Omega_0 - \Omega_1 \) is equal to zero. A stronger result concerning the Hausdorff measure of \( \Omega_0 - \Omega_1 \) and under weaker conditions, was proved by E. Giusti, M. Miranda (see [7]). The regularity in this case \( (N > 2) \) cannot be proved; there exist counter-examples (De Giorgi [4], Giusti-Miranda [6]) of non-regular solutions of the equations with coefficients analytical in \( u \). For the present we do not know a counter-example satisfying the stronger Morrey's conditions of the growth of coefficients.

Let us put the problem considered here in the following way:

\( \Omega \) is a bounded domain in \( E_N \) with infinitely smooth boundary \( \partial \Omega ; \overline{\Omega} = \Omega \cup \partial \Omega \). \( N_i \) are linear differential operators with constant coefficients.

\[
(0) \quad N_i u = \sum_{r=1}^{m} N_{ir} u_r = \sum_{r=1}^{m} \sum_{|a| \leq \kappa_r} a_{ira} \partial_{a} u_r; \quad i = 1, \ldots, h.
\]

Let us denote \( \xi^a = \xi_{i_1}^{a_1} \cdots \xi_{i_h}^{a_h} \); \( N_{ir} \xi^a = \sum_{|a| = \kappa_r} a_{ira} \xi^a \) and suppose

\[
\text{rang } N_\xi = \text{rang } (N_{ir} \xi^a)_{i=1, \ldots, m} = m
\]

for every \( \xi \in E_N ; \xi = (0, \ldots, 0) \).

As a special case we may take

\[
N_i u = N_{ra} u = \partial_{a} u_r
\]

for every \( r = 1, \ldots, m ; |a| \leq \kappa_r \).

The functions \( F_i(x, \xi) \) (for \( i = 1, \ldots, h \)) are defined and continuous with all their first derivatives on \( \overline{\Omega} \times E_h \) (*) and are nonlinear with polynomial growth (of the order \( k - 1 \)) in \( \xi \).

Let us denote \( \theta_\xi = \left( 1 + \sum_{i=1}^{h} |\xi_i|^2 \right)^{1/2} \);

\[
\theta u(x) = \left( 1 + \sum_{i=1}^{h} |N_i u(x)|^2 \right)^{1/2}
\]

(*) They are differentiable on \( \Omega \times E_h \) and the derivatives may be continously extended on \( \overline{\Omega} \times E_h \).
for non-linear elliptic systems in two dimensions

Let us suppose that there exists $C > 0$ such that for every $x \in \Omega$, $\xi \in E_h$

\begin{align}
(1) & \quad \sum_{i=1}^{h} \left| F_i(x, \xi) \right| + \sum_{i=1}^{N} \left| \frac{\partial F_i}{\partial x_i}(x, \xi) \right| \leq C \cdot \theta^{k-1}, \\
(2) & \quad \sum_{j=1}^{h} \left| F_{ij}(x, \xi) \right| \leq C \cdot \theta^{k-2}, \\
(3) & \quad F_{ij}(x, \xi) = F_{ji}(x, \xi).
\end{align}

We shall consider a weak solution of the equation

\begin{align}
\sum_{|a| \leq \alpha_r} (-1)^{|a|} \left( \sum_{i=1}^{h} a_{ira} D^a F_i(x, [N_j u(x)]_{i-1}^{h}) \right) &= \sum_{|a| \leq \alpha_r} (-1)^{|a|} \left( \sum_{i=1}^{h} a_{ira} D^a f_i(x) \right)
\end{align}

for $r = 1, \ldots, m$, which may be written in a divergent form

\begin{align}
(1.1) & \quad \int_{\Omega} \sum_{i=1}^{h} \left[ F_i(x, [N_j u(x)]_{i-1}^{h}) - f_i(x) \right] N_i \varphi(x) \, dx = 0.
\end{align}

We shall suppose that the operator on the left represents a monotone operator from $\bigotimes_{r=1}^{m} \mathcal{W}_k^{e_r}(\Omega)$ into $\left( \bigotimes_{r=1}^{m} \mathcal{W}_k^{e_r}(\Omega) \right)'$.

In Case $A$ operators $N_i$ which consist only of their main parts, i.e.

\begin{align}
N_i u = \sum_{r=1}^{m} \sum_{|a| = \alpha_r} a_{ira} D^a u_r; \quad i = 1, \ldots, h
\end{align}

will be considered. In this case it is sufficient to suppose that

(4) there exist two positive constants $\gamma_1, \gamma_2$ so that

\begin{align}
\gamma_1 \theta^{k-2} |\eta|^2 \leq \sum_{i,j=1}^{h} F_{ij}(x, \xi) \eta_i \eta_j \leq \gamma_2 \theta^{k-2} |\eta|^2
\end{align}

for every $\eta \in E_h$, $x \in \Omega$, $\xi \in E_h$. | $\eta|^2 = \sum_{i=1}^{h} |\eta_i|^2$. 

Case B: Let us decompose $N_i$ in the main part $N'_i$ and

$$N'_i = N_i - N_i'$$

(the corresponding notation $v' = (v'_1, ..., v'_h); v'' = (v''_1, ..., v''_h); v = (v', v'') \in E_{2h}$).

$F_i(x, v)$ are defined and continuous with all their first derivatives on $\tilde{Q} \times E_{2h}$. The conditions of growth are the same as in A. Instead (4) let us suppose

(4') there exist $C_1, C_2$ positive so that

$$\sum_{i=1}^{2h} F_i(x, v) (v'_i + v''_i) \geq C_1 \sum_{i=1}^{2h} |v'_i| - C_2 \sum_{i=1}^{2h} F_{ij}(x, v) \mu_j (\mu'_i + \mu''_i) > 0$$

$\forall \mu \in E_{2h}; \mu \not= 0$

(4'') there exist two positive constants $\gamma_1, \gamma_2$ so that

$$\gamma_1 \theta^{-k-2}_v |\eta|^2 \leq \sum_{i,j=1}^{h} F_{ij}(x, v) \eta_i \eta_j \leq \gamma_2 \theta^{-k-2}_v |\eta|^2$$

for every $x \in \tilde{Q}; \mu \in E_{2h}; \eta \in E_h$.

Suppose that for the regularity conditions (0) — (4) (in case A) or (0) — (4'') (in case B) are satisfied uniformly with regard to an orthonormal transformation of a coordinate system in $E_N$.

(6) The right part $f_i \in W^3_p(\Omega)$ for $i = 1, ..., h, p > 2$, the boundary condition $u_r^0 \in W^{\kappa_r+1}_p(\Omega)$ for $r = 1, ..., m, \tilde{p} > \max(p, h)$.

§ 1 consists of some lemmas on $L_p$-estimates of solutions of the linear equations. Lemma 1.4 gives such estimates for an equation with measurable coefficients, whose bilinear form is the following

$$\int_{\tilde{Q}} \sum_{i,j=1}^{h} A_{ij} N_i u N_j v.$$

Here $A_{ij} \in L_{\infty}(\Omega); u_r, v_r \in \tilde{W}^{3,r}_p(\Omega); r = 1, ..., m$; for $\gamma_1, \gamma_2 > 0$ and every $\eta \in E_h$ is

i) $\gamma_1 |\eta|^2 \leq \sum_{i,j=1}^{h} A_{ij} \eta_i \eta_j \leq \gamma_2 |\eta|^2$.

This condition is weaker than the usually required condition of ellipticity which, in this case, has the form

ii) $\sum_{i,j=1}^{h} A_{ij} \left( \sum_{\alpha=1}^{m} \sum_{\alpha=1}^{n_r} a_{\alpha \alpha} \xi^\alpha_r \right) \left( \sum_{i=1}^{h} \sum_{j=1}^{h} a_{ij} \xi^\beta_s \right) \geq C \sum_{r=1}^{m} \sum_{\alpha=1}^{n_r} |\xi^\alpha_r|^2$. 

\[ \sum_{i,j=1}^{h} A_{ij} \left( \sum_{\alpha=1}^{m} \sum_{\alpha=1}^{n_r} a_{\alpha \alpha} \xi^\alpha_r \right) \left( \sum_{i=1}^{h} \sum_{j=1}^{h} a_{ij} \xi^\beta_s \right) \geq C \sum_{r=1}^{m} \sum_{\alpha=1}^{n_r} |\xi^\alpha_r|^2. \]
For example, for
\[ N_1 u = \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}, \quad N_2 u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}; \quad A_{ij} = \delta_{ij}. \]
is i) evidently satisfied,
\[ N\xi = \left( \begin{array}{c} \xi_1 \\ \xi_2 \end{array} \right) \]
is regular for non-vanishing \( \xi \). But ii) has the form
\[ (\xi_2 - \xi_1)^2 + (\xi_1 + \xi_2)^3 \geq C \sum_{i,j=1}^{2} |\xi_j|^2 \]
and such constant \( C \) does not exist.

§ 2 contains some remarks on existence of solution and continuous dependence on \( f \) and \( u_0 \) and a proof of the main theorem. A homotopy is used there between a linear equation with constant coefficients with well-known properties and the investigated non-linear equation.

The proof is based on a priori estimate denoted as «property \( \mathcal{A} \)» of the equation and having this form:

Let us suppose the solution \( u \) belongs to
\[ \Pi_{r=1}^m [ C_{s_r}(\Omega) \cap W^{s_r+1}_{p_F}(\Omega) ], \quad f \in \Pi_{i=1}^h W^1_p(\Omega). \]

Then \( u \) belongs to \( \Pi_{r=1}^m [ W^{s_r+1}_{p_F}(\Omega) ] \) and its norm is bounded by a constant which depends only on \( f, u_0 \).

Several cases of operators which possess the above property, are investigated in § 3.

The author is indebted to Professor J. Nečas for much valuable advice concerning the paper.

NOTATIONS. \( D^a \) denotes the partial derivative \( \frac{\partial^{|\alpha|}}{\partial x_1^{a_1} \cdots \partial x_N^{a_N}} \) where \( \alpha = (\alpha_1, \ldots, \alpha_N) \); all \( \alpha_i \) are integers, non-negative numbers, \( |\alpha| = \sum_{i=1}^{N} \alpha_i \).

The functional spaces \( D(\Omega), \mathcal{E}(\Omega), W^k_p(\Omega), W^{k}_{p_F}(\Omega), \widetilde{W}^k_{p_F}(\Omega), C_k(\Omega), C_{k, \mu}(\Omega) \) (with \( k \) integer, non-negative, \( p \geq 1, 0 \leq \mu \leq 1 \)) are denoted as usually (See for example [15]).
Let \( u \) be a vector-function \( u = (u_1, \ldots, u_m) \), \( \kappa = (\kappa_1, \ldots, \kappa_m) \) with \( \kappa_i \geq 0 \), integer for \( i = 1, \ldots, m \). Then \( u \in W^\kappa_p(\Omega) [\tilde{W}^\kappa_p(\Omega), C_\kappa(\tilde{\Omega}), C_{\kappa, \mu}(\tilde{\Omega})] \) means that each \( u_i \in W^\kappa_i p(\Omega) [\tilde{W}^\kappa_i p(\Omega), C_{\kappa_i}(\tilde{\Omega}), C_{\kappa_i, \mu}(\tilde{\Omega})] \) for \( i = 1, \ldots, m \).

\( \kappa + 1 \) denotes the vector \( (\kappa_1 + 1, \ldots, \kappa_m + 1) \), i.e. \( \kappa + (1, \ldots, 1) \).

### 1. Properties of the operators \( N_i \).

We shall be concerned with linear differential equation which may be written as follows:

\[
(1.2) \quad A(u, \varphi) = \sum_{i=1}^{m} N_i u \cdot N_i \varphi = \int_{\Omega} \sum_{r=1}^{m} \sum_{|a| = \kappa_r} f_{ra} D^a \varphi_r.
\]

\( N_i \) consist only of their principal parts \( N_i' \), i.e.

\[
(1.3) \quad N_i u = \sum_{r=1}^{m} \sum_{|a| = \kappa_r} a_{ira} D^a u_r
\]

and satisfy condition (0), \( \varphi, u \in \tilde{W}^\kappa_2(\Omega), f_{ra} \in L^p(\Omega) \).

**Lemmas 1.1**: The linear differential operator given by (1.2) is uniformly elliptic and strongly elliptic.

**Proof**: (1, 2) can be written in the form

\[
A(u, \varphi) = \sum_{s=1}^{m} \sum_{r=1}^{m} l_{rs} (u_r) = \int_{\Omega} \sum_{|a| = \kappa_s} D^a f_{sa}.
\]

where \( u, \varphi \in [D(\Omega)]^m \) and \( f_{sa} \in W^\kappa_2(\Omega) \),

\[
l_{rs} (u_r) = \sum_{|a| = \kappa_r} \left( \sum_{|\beta| = \kappa_s} a_{ira} a_{ib} \right) u_r.
\]

Let us denote for \( \xi \in E_N \):

\[
l_{rs} (\xi) = \sum_{|a| = \kappa_r} \left( \sum_{|\beta| = \kappa_s} a_{ira} a_{ib} \right) \xi^{a+b}.
\]
Now we shall prove the uniform ellipticity, i. e.

\[ \det | l_{rs}(\xi) | \geq C | \xi |^2 \prod_{i=1}^{m} \xi_i \]

and strong ellipticity, i. e.

\[ \sum_{i,j=1}^{m} l_{ij}(\xi) \eta_i \eta_j \geq C \sum_{i=1}^{m} \eta_i^2 \cdot | \xi |^{2n_i} \quad \text{for a positive } C. \]

Let us denote $N^*\xi$ the adjoint matrix to $N\xi$. Then $\det | l_{rs}(\xi) | = \det | N^*\xi \cdot N\xi |$ and so it is the Gramm's determinant of column vectors of $N^*\xi$. Therefore it is equal to zero only in the case of linear dependence of column vectors of $N^*\xi$, i. e. for $\xi = (0, \ldots, 0)$ and it is positive for non vanishing $\xi$.

The quadratic form in (1.5) is positively defined if and only if all the main subdeterminants of its coefficients are positive (according to Silvestr's theorem). But they have the same form as $\det | l_{rs}(\xi) |$.

Let us suppose that for every $n$ there exists a real vector $\xi^n \in E_N$; $\xi^n \neq (0, \ldots, 0)$ such that

\[ \det | l_{rs}(\xi^n) | < \frac{1}{n} \cdot | \xi^n |^2 \prod_{i=1}^{m} n_i. \]

Let us consider the sequence $[\eta_n]_{n=1}^{\infty}$; $\eta^n = \frac{\xi^n}{| \xi^n |}$. We may choose a convergent subsequence (let us denote also $\eta^n$) such that

1) $| \eta^n | = 1$; $n = 1, 2, \ldots$

2) $\eta^n \rightarrow \eta$ for $n \rightarrow \infty$

3) $0 < \det | l_{rs}(\eta^n) | < \frac{1}{n}$.

Then $\det | l_{rs}(\eta) | = 0$ for non vanishing vector $\eta$ and that is a contradiction with condition (0). In the same way (1.5) may be proved.

The equation (1.2) has a solution $u \in \tilde{W}^{\mu}_{2}$ for $f_{ra} \in L_{2\mu}$. Using the estimates of Agmon, Douglas, Nirenberg (see [1]) and continuous dependence on the right part, we see that $u \in \tilde{W}^{\mu}_{p}$ for $f_{ra} \in L_{p}$ and there exists $C > 0$ so that

\[ \| u \|_{W^{\mu}_{p}} \leq C \cdot \sum_{r=1}^{m} \sum_{| \alpha | = \mu_r} \| f_{ra} \|_{L_{p}}. \]
The functions \( g_i = N_i u \) satisfy the equation

\[
\int_\Omega \sum_{i=1}^h g_i N_i \varphi = \int_\Omega \sum_{r=1}^m \sum_{a=\alpha}^{n_{pr}} f_{ra} D^{\alpha} \varphi_r
\]

for every \( \varphi \in [D(\Omega)]^m \) and there exists \( C > 0 \) so that

\[
\| g_i \|_{L_p} \leq C \sum_{r=1}^m \sum_{a=\alpha}^{n_{pr}} \| f_{ra} \|_{L_p}.
\]

That means that the right part of any equation may be written in the form \( \int_\Omega \sum_{i=1}^h g_i \cdot N_i \varphi \) and the \( L_p \) norms of \( f \) and \( g \) are equivalent.

Next, let us write (1.2) in the form

\[
\int_\Omega \sum_{i=1}^h N_i u \cdot N_i \varphi = \int_\Omega \sum_{i=1}^h g_i N_i \varphi
\]

and let us interest in the dependence of the estimates of \( u \) on \( p \).

**Lemma 1.2:** Let \( u \in \overset{\circ}{W}_2^p(\Omega) \) be a solution of (1.7), \( 2 \leq p \leq 2 + q \). Then there exists a positive constant \( C_1(q) \) such that

\[
\left( \sum_{i=1}^h \| N_i u \|_{L^p}^{2+q} \right)^{1/2+q} \leq C_1(q) \left( \sum_{i=1}^h \| g_i \|_{L^p}^{2+q} \right)^{1/2+q}.
\]

**Proof:** According to the foregoing remarks

\[
\left( \sum_{i=1}^h \| N_i u \|_{L^{2+q}}^{2+q} \right)^{1/2+q} \leq C_1(q) \left( \sum_{i=1}^h \| g_i \|_{L^{2+q}}^{2+q} \right)^{1/2+q}.
\]

From (1.7) we obtain immediately

\[
\left( \sum_{i=1}^h \| N_i u \|_{L^2}^2 \right)^{1/2} \leq \left( \sum_{i=1}^h \| g_i \|_{L^2}^2 \right)^{1/2}.
\]

The result follows according to the interpolation theorem of Riesz-Thorin (see [22]).

**Lemma 1.3:** Let \( 1 < p < \infty \); \( N_i \) satisfy condition (0). Then

\[
\left( \sum_{i=1}^h \| N_i u \|_{L^p(\Omega)}^p \right)^{1/p} \text{ is an equivalent norm in } \overset{\circ}{W}_p^\infty(\Omega).
\]
PROOF: For \( p = 2 \) the result is an immediate consequence of Lemma 1.1. For \( p \neq 2 \) we may use the method of J. Nečas (see [14]), consisting of applying the Lizorkin's theorem on multiplicators (see [10]) to this special case. \( F(f) \) denotes the Fourier transformation of the function \( f \in L_p \) (in the sense of distributions).

**Theorem 1 (Lizorkin):** Let \( \Phi(\xi) \) be a function defined and continuous with all its derivatives \( D^a \Phi(\alpha = (\alpha_1, \ldots, \alpha_N), \alpha_i = 0 \text{ or } 1) \) for every \( \xi = (\xi_1, \ldots, \xi_N) ; \xi_j \neq 0 \) for \( j = 1, \ldots, N \).

Let all such derivatives satisfy condition

\[
|\xi^a D^a \Phi(\xi)| \leq M < \infty \quad \text{on} \quad [\xi; \xi_j \neq 0, j = 1, \ldots, N].
\]

Then \( T f = F^{-1}(\Phi(\xi) \cdot F(f)(\xi)) \) is a linear and bounded mapping from \( L_p(E_N) \) into \( L_p(E_N) \) for \( 1 < p < \infty \).

We shall use the equivalent norm in \( \hat{W}^*_p(\Omega) \);

\[
\| u \|_{\hat{W}^*_p(\Omega)} = \left( \sum_{r=1}^m \sum_{|a| = n_r} \| D^n u_r \|_{L_p} \right)^{1/p}.
\]

We obtain immediately

\[
\left( \sum_{i=1}^h \| N_i u \|_{L_p} \right)^{1/p} \leq C \| u \|_{\hat{W}^*_p}.
\]

On the contrary, we may suppose \( u \in [D(\Omega)]^m \). Using Fourier transform we may write

\[
f_i(\xi) = F(N_i u)(\xi) = \sum_{r=1}^m N_{ir} \xi (-i)^r F(u_r)(\xi).
\]

Let us denote \( f = (f_1, \ldots, f_h) \), \( \varphi = (\varphi_1, \ldots, \varphi_m) \) with \( \varphi_j = (-i)^j F(u_j) \). Then \( f = N \cdot \varphi \). Moreover, let \( \{A_j\} \) be set of all the determinants \( (m \times m) \) of \( N \xi \).

For arbitrary \( \xi \equiv (0, \ldots, 0) \) there exists (at least one) \( A_j(\xi) \equiv 0 \). Let \( \{A_{jre}\} \) be its subdeterminants of the orders \( (m - 1) \times (m - 1) \). (We may define \( A_{jre} = 0 \) if \( N_{re} \) does not belong to \( A_j \)). Let us write (from Cramer's rule)

\[
q_r = \frac{\sum_{e=1}^h A_{jre} f_e}{A_j}
\]

and also

\[
\sum_{j=1}^h A_j^e \quad q_r = \sum_{j=1}^h \sum_{e=1}^h A_j \cdot A_{jre} f_e.
\]
But $\sum_{j=1}^{h} A_j^2 = 0$ for every $\xi = (0, \ldots, 0)$, therefore

$$\varphi_r = \frac{\sum_{j=1}^{h} \sum_{e=1}^{h} A_j A_{jre} f_e}{\sum_{j=1}^{h} A_j^2}.$$  \hfill (1.11)

It remains to prove the estimate (1.5) for

$$\Phi (\xi) = \frac{A_j (\xi) \cdot A_{jre} (\xi) \cdot \xi^a}{\sum_{j=1}^{h} A_j^2 (\xi)}, \quad \text{where } |\alpha| = \kappa_r.$$  \hfill (1.12)

In the same way as in the proof of Lemma (1.1) it may be shown that

$$\sum_{j=1}^{h} A_j^2 (\xi) \geq C \cdot |\xi|^2 \sum_{i=1}^{m} \kappa_i^2.$$  

Thus the assumptions of Theorem 1 are satisfied and hence

$$\left( \sum_{i=1}^{m} \sum_{|\alpha| = \kappa_i} \| D^{\alpha} u_i \|^2_{L_p^p} \right)^{1/p} \leq C \left( \sum_{i=1}^{h} \| N_i u \|^2_{L_p^p} \right)^{1/p}. \hfill (1.13)$$

Let us now consider the equation

$$\sum_{i,j=1}^{h} A_{ij} (x) N_i u (x) N_j \varphi (x) dx = \sum_{j=1}^{h} g_j (x) N_j \varphi (x) dx, \hfill (1.14)$$

where $\varphi, u \in \tilde{W}_2^\infty (\Omega)$, $g_j \in L_p (\Omega)$ for $j = 1, \ldots, h$, $p \geq 2$ and $A_{ij} \in L_\infty (\Omega)$ for $i, j = 1, \ldots, h$. Let us suppose

$$A_{ij} = A_{ji}; \quad i, j = 1, \ldots, h$$

and

$$\gamma_1 \| \eta \|^2 \leq \sum_{i,j=1}^{h} A_{ij} \eta_i \eta_j \leq \gamma_2 \| \eta \|^2 \quad \text{for some } \gamma_1, \gamma_2 > 0$$

and every $\eta \in E_h$.

\textbf{Lemma 1.4 :} Let $u \in \tilde{W}_2^\infty$ be a solution of (1.14) with $A_{ij}$ satisfying (1.15), $2 \leq p \leq 2 + q$. Then there exist two positive constants $\gamma_3 (q) > 1$
and $\gamma_4(\varrho) > 1$ such that

$$
\lVert u \rVert_{W^k_p} \leq \gamma_5 \cdot \gamma_4(\varrho)^{\frac{1}{2} - \frac{2}{p}} \cdot \left( \sum_{i=1}^{h} \lVert g_i \rVert_{L^p} \right)^{\frac{1}{p}}
$$

for $p$ satisfying

$$
p \leq 2 \left( 1 - \log \left[ \frac{1 - \frac{1}{\gamma_2}}{1 - \frac{\gamma_1}{\gamma_2}} \right] \log \gamma_3 \right)^{-1}.
$$

**Proof:** It is sufficient to prove (1.16) for $A_{ij} \in \mathcal{C}(\overline{\Omega})$, $g_i \in \mathcal{C}(\overline{\Omega})$ ($i, j = 1, \ldots, h$) as Lemma 1.4 is the easy corollary of continuous dependence on $A_{ij}$, $g_i$.

Such a solution $u \in W^k_2$ is also a solution of

$$
\int_{\overline{\Omega}} \sum_{i=1}^{h} N_i u N_i \varphi = \int_{\overline{\Omega}} \left( \delta_{ij} - \frac{1}{\gamma_2} A_{ij} \right) N_j u N_i \varphi + \frac{1}{\gamma_2} \sum_{i=1}^{h} g_i \cdot N_i \varphi.
$$

We shall estimate the $L_p$-norm of the first term on the right.

$$
\left( \sum_{i=1}^{h} \left( \sum_{j=1}^{h} \left( \delta_{ij} - \frac{1}{\gamma_2} A_{ij} \right) N_j u \right)^{\frac{1}{p}} \right)^{p} \leq \sup_{\varphi \in \mathcal{I}} \int_{\overline{\Omega}} \left( \sum_{j=1}^{h} N_j u \right)^{\frac{1}{2}} \cdot \left( \sum_{j=1}^{h} \lVert \varphi \rVert_{L^p} \right)^{\frac{1}{2}} \leq \left(1 - \frac{\gamma_1}{\gamma_2} \right) \cdot C^{1 - \frac{2}{p}} \cdot \left( \sum_{j=1}^{h} \lVert N_j u \rVert_{L^p} \right)^{\frac{1}{p}}.
$$

Now, according to Lemma 1.2 we obtain

$$
\left( \sum_{i=1}^{h} \lVert N_i u \rVert_{L^p} \right)^{\frac{1}{p}} \leq

\leq (C \cdot C_4(\varrho))^{1 - \frac{2}{p}} \cdot \left( \sum_{i=1}^{h} \lVert g_i \rVert_{L^p} \right)^{\frac{1}{p}} \leq C_4(\varrho)^{1 - \frac{2}{p}} \cdot \left( \sum_{i=1}^{h} \lVert g_i \rVert_{L^p} \right)^{\frac{1}{p}}.
$$

Let us denote $\gamma_3(\varrho) = C \cdot C_4(\varrho)$, $\gamma_4(\varrho) = C_4(\varrho)$. 

for non-linear elliptic systems in two dimensions
The $p$ satisfying (1.17) satisfies also

\[ \frac{1}{\gamma_0} - \frac{2}{p} \left( 1 - \frac{\gamma_1}{\gamma_2} \right) \leq 1 - \frac{1}{2} \frac{\gamma_1}{\gamma_2}. \]

Therefore it follows from (1.19)

\[ \left( \sum_{i=1}^{k} \| N_i u \|_{L^p} \right)^{1/p} \leq \gamma_0^{1 - \frac{2}{p}} \gamma_1 \left( \sum_{i=1}^{k} \| g_i \|_{L^p} \right)^{1/p}. \]

The equivalence of the norms in $\overset{0}{W}_p^\infty (\Omega)$ (Lemma 1.3) implies the result.

§ 2.

The existence and unicity of the solution of (1.1) follows immediately from a special case of the Lerray-Lions Theorem:

**Theorem 2.** Let $V$ be a reflexive Banach-space, $A(v)$ a bounded operator from $V$ to $V'$ which is weakly continuous from all finitely dimensional subspaces of $V$ to $V'$. (Let us denote $(F, \varphi)$ the value of the functional $F$ in the point $\varphi$). Let the following assumptions be satisfied:

1) \[ \lim_{\| \varphi \| \to \infty} \frac{(A(\varphi), \varphi)}{\| \varphi \|} = \infty, \]

2) \[ A(\varphi) \text{ is strictly monotone, i.e.} \]

\[ (A(\varphi) - A(\psi), \varphi - \psi) \geq 0 \quad \text{for} \quad \varphi \neq \psi; \quad \varphi, \psi \in V. \]

Then $A$ is a one-to-one mapping on $V'$ and $A^{-1}$ is a bounded mapping from $V'$ to $V$.

In our notation, there is $V = \overset{0}{W}_p^\infty (\Omega)$ and

\[ (A(\varphi), \varphi) = \int_{\Omega} \sum_{i=1}^{k} F_i, (x, [N_i (\varphi + u_0)]_{j=1}^{k}) N_i \varphi (x) \, dx. \]

The boundedness and continuity is proved in [2], [20], [21] in the theorem of Nemyckij's operators. Conditions 1,2 follow immediately from assumptions $A$, $B$ (see [21], [16]). Moreover, we obtain 2) in the form

\[ (A(\varphi) - A(\psi), \varphi - \psi) \geq C \cdot \| \varphi - \psi \|_{\overset{0}{W}_p^\infty (\Omega)}^{k}. \]
in case $A$ and therefore $A^{-1}$ is a continuous mapping; the solution depends continuously on the right part and the boundary condition.

In the case $B$ there is $A^{-1}$ only demi continuous, i.e. it is continuous from the strong topology in $(\tilde{W}_k^p)'$ into the weak topology in $\tilde{W}_k^p$ (see [2], [15], [16]).

Let us denote $\mathcal{B}$ a bounded mapping from $W = \{u \in W_k^p, u - u_0 \in \tilde{W}_k^p\}$ into $(\tilde{W}_k^p)'$ such that $\mathcal{B}(u) = A(u - u_0)$ and consider the following equation

$$(\mathcal{B}(u), \varphi) = \int_{\Omega} \sum_{i=1}^{h} f_i \cdot N_i \varphi,$$

where $f_i \in L^p_k$, $\varphi \in \tilde{W}_k^p$, $u \in W$.

Let us say $\mathcal{B}$ has the property $\mathcal{A}$ if and only if $f = (f_1, \ldots, f_h) \in [W_k^1]^h$, $u = \mathcal{B}^{-1}(f) \in M$ implies $u \in W_p^{k+1}$ and

$$\|u\|_{W_p^{k+1}} \leq C(\|f\|_{W_k^1}^h),$$

where $C$ is bounded uniformly for $k \in (2, k_0)$.

Let us denote $F_i(\xi, s) = \xi_i \cdot \theta_{\xi}^{i-2}$; $i = 1, \ldots, h$, $2 \leq s \leq k$

and $F_i(x, \xi, t) = t \cdot F_i(\xi, k) + (1 - t) \cdot F_i(x, \xi)$; $i = 1, \ldots, h$, $0 \leq t \leq 1$.

Let us define

$$(\mathcal{B}_a(u), \varphi) = \int_{\Omega} \sum_{i=1}^{h} F_i([N_j u]_{j=1}^{h}, s) \cdot N_i \varphi$$

and

$$(\mathcal{B}_t(u), \varphi) = \int_{\Omega} \sum_{i=1}^{h} F_i(x, [N_j u]_{j=1}^{h}, t) \cdot N_i \varphi$$

analogously to $\mathcal{B}$.

**Theorem 3 (on regularity):** Let $\mathcal{B}$ satisfy $A$ or $B$ and let $\mathcal{B}, \mathcal{B}_a, \mathcal{B}_t$ have property $\mathcal{A}$.

Then there exists $\mathcal{B}^{-1}$ and it is a bounded mapping from $[W_k^1(\Omega)]^h$ into $W_p^{k+1}(\Omega)$.

Using Sobolev embedding theorem, it follows immediately:

**Corollary:** Let $u$ be a solution of (1.1), where $\mathcal{B}$ satisfies $A$ or $B$ and $\mathcal{B}, \mathcal{B}_a, \mathcal{B}_t$ have property $\mathcal{A}$. Then $u \in C_{a, \mu}(\Omega)$ with $\mu = 1 - \frac{2}{p}$ and

$$\|u\|_{C_{a, \mu}} \leq C(\|f\|_{W_k^1}^h).$$
Proof: \( \mathcal{B}_s \) satisfy conditions \((0) - (4)\) or \((0) - (4')\) with \( s \) instead of \( k \). Let us denote \( \mathcal{P} \) the subset of \( s \in (2, k) \) such that for \( u \in \mathcal{B}_{s^{-1}}(f) \)

\[
\| u \|_{W^{s+1}_p} \leq C(\| f \|_{W^1_p})
\]

holds with \( C \) independent on \( s \). \( \mathcal{P} = \emptyset \) for \( 2 \in \mathcal{P} \) (see results of Agmon, Douglis, Nirenberg [1]). \( \mathcal{P} \) is closed:

Let \( s_n \in \mathcal{P} \) converge to \( s \); \( \mathcal{B}_{s_n}(u_{s_n}) = f \) then

\[
(2.1) \quad \| u_{s_n} \|_{W^{s+1}_p} \leq C(\| f \|_{W^1_p})
\]

and there exists a subsequence (let us denote it also \( u_{s_n} \)) such that \( u_{s_n} \to u_s \) in \( W^s_p \). But such \( u_s \) belongs to \( W^{s+1}_p \) (use \( 2.1 \)), therefore it solves \( \mathcal{B}_s(u_s) = f \) and according to \( \mathcal{A} \)

\[
\| u_s \|_{W^{s+1}_p} \leq C(\| f \|_{W^1_p})
\]

\( \mathcal{P} \) is open: Let \( s_0 \in \mathcal{P} \); \( \mathcal{B}^{-1} \) be an inverse operator to \( \mathcal{B}_{s_0} \) and

\[
C_s(u) = \mathcal{B}_{s_0}(u) - \mathcal{B}_s(u) + \mathcal{B}_s(u_{s_0})
\]

then \( \mathcal{B}^{-1} \cdot C_s \) is defined on

\[
\mathcal{V} = \{ u \in W^{s+1}_p ; \| u - u_{s_0} \|_{W^{s+1}_p} \leq 1 ; u - u_0 \in W^s_2 \},
\]

is weakly continuous on \( \mathcal{V} \) (see remarks before Theorem 3) and \( \mathcal{B}^{-1} (C_s(\mathcal{V})) \subset \subset \mathcal{V} \) for sufficiently small \( s - s_0 \). According to Schauder's fixed point theorem (see [19]) there exists \( u \in \mathcal{V} \), \( u = \mathcal{B}^{-1} C_s(u) \). Then \( \mathcal{B}_s(u) = \mathcal{B}_{s_0}(u_{s_0}) \) and according to \( \mathcal{A} \)

\[
\| u \|_{W^{s+1}_p} \leq C
\]

We may conclude \( \mathcal{P} = (2, k) \). All the proceedings may be repeated for \( \mathcal{B}_k \), which completes the proof.

Operators which satisfy \( \mathcal{A} \).

Theorem 4: Let \( N_r \) be all the highest derivatives, i.e. \( N_r = N_{ra} = D^a u_r \) for \( r = 1, \ldots, m \), \( |a| = \kappa_r \), let \( \mathcal{B} \) satisfy \( \mathcal{A} \). Then \( \mathcal{B} \) has property \( \mathcal{A} \).

The proof is based on the following two estimates

\[
\| \theta^k_{u^*} \|_{W^1_p(u)} \leq C(f) \cdot \| \theta^k_{u^*} \|_{C(u)}^{k-1}
\]
and
\[
\| \theta_u^k \|_{W^1_p(\Omega)} \leq C(f) \cdot \| \theta_u \|_{C_0(\overline{\Omega})}^{\frac{3}{2}(k-2)}, \quad p > 2.
\]

They may be obtained in this way:

All first derivatives of \( u \) solve a linear equation with measurable coefficients. In the interior of \( \Omega \) or in the directions « parallel with boundary \( \partial \Omega \) it is sufficient to use the theorems about linear equations (see § 1) or to choose suitable test-functions. In the normal directions, more precise theorems about dual norms must be used.

To this purpose, the following description of boundary will be considered : [see [15]] a neighborhood of every point of \( \partial \Omega \) is described by an infinitely differentiable function \( a \) which is defined on the cube

\[
K_r = [x' \mid |x'| < r] \quad x' = (x_1, ..., x_{N-1}) \quad a(x') = x_N
\]

and \( \sum_{i=1}^{N-1} \left| \frac{\partial a}{\partial x_i}(0) \right| = 0 \) in a corresponding coordinate system. The boundary is covered by a finite number \( P \) of such systems.

Let us suppose

\[
V_r^i = \{ x \mid |x'| < r ; a^i(x') < x_N < a^i(x') + r \} \subset \Omega
\]

\[
U_r^i = \{ x \mid |x'| < r ; a^i(x') - r < x_N < a^i(x') \} \cap \Omega = \emptyset
\]

for every sufficiently small \( r \) and \( i = 1, ..., P \). Let us denote \( V_r^0 \) the domain with infinitely smooth boundary

\[
\overline{V}_r^0 \subset \Omega \quad \bigcup_{i=0}^P V_r^i \supset \Omega
\]

for every \( r > 0 \).

In [13], the existence of the functions \( \gamma_r^i ; i = 0, ..., P \) \( \gamma_r^i \in C(V_r^i) \cap C(\overline{V}_r^i) \) is proved, having the following properties:

1) \( \gamma_r^i = 0 \) on \( \Omega - V_r^i \),

2) \( \gamma_r^0 \) is equivalent to \( \delta_0(x) = \text{dist}(x, \partial V_r^0) \),

\( \gamma_r^i \) is equivalent to \( \delta_i(x) = \text{dist}(x, \partial (\Omega - V_r^i)) ; \quad i = 1, ..., P, \)

3) \( |D^\alpha \gamma_r^i| \leq C \cdot |\gamma_r^i|^{-|\alpha|} ; \quad i = 0, 1, ..., P. \)

In the next lemmas the right part \( f \) is supposed in \( [W_p^1(\Omega)]^m \),

\[
K \text{ denotes } \max_{i=1, ..., m} x_i.
\]
Lemma 3.1: Let \( u = \varphi_{\beta - 1} (f) \in M \). Then

\[
\int_{\Omega} \theta_{u}^{k} \leq C(f),
\]

(3.1)

\[
\int_{\Omega} \gamma_{\alpha}^{2k} \theta_{u}^{k-2} \sum_{i=1}^{h} \sum_{i=1}^{n} N_{i} \left( \frac{\partial u}{\partial x_{i}} \right) \leq C(f) .
\]

(3.2)

The constant \( C \) depends on \( \| f \|_{W_{p}^{-1}} \) and does not depend on \( k \in \langle 2, k_{0} \rangle \).

Proof:

1) \( \int_{\Omega} \theta_{u}^{k} \leq C \cdot \left\{ 1 + \| u_{0} \|_{W_{k}^{\infty}}^{k} + \int_{\Omega} \sum_{i=1}^{h} N_{i} (u - u_{0}) \right\} \leq \]

\[
\leq C \cdot \left\{ 1 + \| u_{0} \|_{W_{k}^{\infty}}^{k} + \int_{\Omega} \sum_{i=1}^{h} N_{i} (u - u_{0}) \cdot [F_{i} (x, [N_{j} u_{0}]) - F_{i} (x, [N_{j} u_{0}]) \right\} .
\]

The last term on the left is equal to

\[
\int_{\Omega} \sum_{i=1}^{h} N_{i} (u - u_{0}) \cdot (f_{i} - F_{i} (x, [N_{j} u_{0}])) \leq
\]

\[
\leq C \cdot \left\{ \| f \|_{L_{2}(\Omega)^{h}} + \| u_{0} \|_{W_{k}^{\infty}}^{k} \cdot \| u \|_{W_{k}^{\infty}} \right\} .
\]

Then

\[
\int_{\Omega} \theta_{u}^{k} \leq C_{1} + C_{2} \cdot \left( \int_{\Omega} \theta_{u}^{k} \right)^{1/k}
\]

and

\[
\int_{\Omega} \theta_{u}^{k} \leq C.
\]

2) Let us take \( \varphi = \frac{\partial \psi}{\partial x_{i}} \in [D(V^{0})]^{m} \) for \( \psi \in [D(V^{0})]^{m} \). Integrating in parts (1.1), we find

\[
\int_{\Omega} \sum_{i=1}^{h} N_{i} \psi \cdot \left\{ \frac{\partial F_{i}}{\partial x_{i}} + \sum_{j=1}^{h} F_{ij} \cdot N_{j} \left( \frac{\partial u}{\partial x_{i}} \right) \right\} = \int_{\Omega} \sum_{i=1}^{h} \frac{\partial f_{i}}{\partial x_{i}} \cdot N_{i} \psi .
\]
According to the assumptions on $F_i$ and $N_i$, this equation is satisfied for all $\psi \in W_2^\infty$, thereby also for $\psi = \gamma_0^{2K} \frac{\partial u}{\partial x_1}$.

\[
\int_\Omega \gamma_0^{2K} \sum_{i,j=1}^h F_{ij} \cdot N_i \frac{\partial u}{\partial x_1} \cdot N_j \frac{\partial u}{\partial x_1} = \int_\Omega \sum_{i=1}^h N_i \left( \gamma_0^{2K} \frac{\partial u}{\partial x_1} \right) \left[ \frac{\partial f_i}{\partial x_1} - \frac{\partial F_i}{\partial x_1} \right] + R,
\]

where $R$ consists of the terms $a_{\alpha} F_{ij} \cdot D^\alpha u_k \cdot D^\beta u_l \gamma_0^K$ with smooth $a_{\alpha}$; $|\alpha| = \kappa + 1$; $|\beta| \leq \kappa$.

Let us denote

\[
j = \int_\Omega \gamma_0^{2K} \cdot \theta^{k-2u} \sum_{i=1}^h N_i^2 \left( \frac{\partial u}{\partial x_i} \right).\]

From the ellipticity

\[
\gamma_1 \cdot j \leq \int_\Omega \gamma_0^{2K} \sum_{i,j=1}^h F_{ij} \cdot N_i \frac{\partial u}{\partial x_1} \cdot N_j \frac{\partial u}{\partial x_1}.
\]

Let us estimate the right part of (3.4).

Now,

i) \[
\left| \int_\Omega N_i \left( \gamma_0^{2K} \frac{\partial u}{\partial x_1} \right) \cdot \frac{\partial f_i}{\partial x_1} \right| \leq C \cdot \| f \|_{W_2^{1/2}} \left( \int_\Omega \sum_{\alpha, \tau} D^\alpha \left( \gamma_0^{2K} \frac{\partial u_\tau}{\partial x_1} \right) \right)^{1/2} \leq C \cdot \| f \|_{W_2^{1/2}} \cdot \left( \int_\Omega \sum_{\alpha, \tau} \left. D^\alpha \left( \frac{\partial u_\tau}{\partial x_1} \right) \right| \right)^{1/2} \leq C_1 \cdot j^{1/2} + C_2.
\]

ii) \[
\left| \int_\Omega N_i \left( \gamma_0^{2K} \frac{\partial u}{\partial x_1} \right) \cdot \frac{\partial F_i}{\partial x_1} \right| \leq C \cdot \left\{ \int_\Omega \theta^{k-1u} \left( \gamma_0^{2K} \left| N_i \left( \frac{\partial u}{\partial x_1} \right) \right| \right) + \theta^{k u} \right\} \leq C_1 \cdot j^{1/2} \cdot \left( \int_\Omega \theta^{k u} \right)^{1/2} + C_2.
\]

iii) \[
\left| \int_\Omega \gamma_0^{2K} a_{\alpha} F_{ij} \cdot D^\alpha u_k \cdot D^\beta u_l \right| \leq C \cdot \int_\Omega \theta^{k-2u} \cdot \gamma_0^K \cdot | D^\alpha u_k | \cdot | D^\beta u_l | \leq C \cdot j^{1/2} \cdot \| u \|_{W_k^{1/2}}.
\]
Therefore, from i), ii) and iii) it follows

\[ j \leq C_1 j^{1/2} + C_2 \]

which implies the boundedness of \( j \).

Let us define on \( V_r^i \) the derivatives «parallel with the boundary», i.e. the derivatives in the plane orthogonal to the direction

\[
\left( -\frac{\partial a^i}{\partial x_1}, -\frac{\partial a^i}{\partial x_2}, \ldots, -\frac{\partial a^i}{\partial x_{N-1}}, 1 \right).
\]

The index of the coordinate system will be omitted.

If \( \hat{\partial}^l = \frac{\partial}{\partial x_i} + \frac{\partial a}{\partial x_i} \cdot \frac{\partial}{\partial x_N} \) denotes such a derivative for \( l = 1, \ldots, N - 1 \), then the following conclusion is true:

\[
(3.5) \quad u \in \bar{W}^r_k \cap W^{r+1}_k \implies \hat{\partial}^l u \in \bar{W}^r_k.
\]

It allows us to prove the analogue of the foregoing lemma for \( \hat{\partial}^l \). Let us denote

\[ H(u) = \sum_{i=1}^{k} N_i^2 (\hat{\partial}^l u). \]

**Lemma 3.2**: Let \( u = \Omega_3^{-1} (f) \in M \), then

\[
(3.6) \quad \int_{\hat{\Omega}} \gamma^2 K \theta^{k-2} u H(u) \leq C(f).
\]

**Proof**: Let us take a test-function \( \varphi \) in (1.1) in the form \( \varphi = \hat{\partial}^l \psi \); \( \psi \in [D(V^i)]^m \). Then

\[
\int_{\hat{\Omega}} \sum_{i=1}^{k} F_i (a, [N_i(u)]) N_i \hat{\partial}^l \psi = \int_{\hat{\Omega}} \sum_{i=1}^{k} f_i \cdot N_i \hat{\partial}^l \psi
\]

and

\[
\int_{\hat{\Omega}} \sum_{i=1}^{k} F_i \cdot \hat{\partial}^l (N_i \psi) = \int_{\hat{\Omega}} \sum_{i=1}^{k} f_i \hat{\partial}^l (N_i \psi) + (f_i - F_i) \cdot R_i,
\]

where \( R_i \) involves the derivatives of \( \psi_i \) up to the order \( \kappa_i \). After integrating in parts, there holds
where $R_2$ involves the derivatives of $u_i$ up to the order $\xi_i$.

According to remark (3.5) we may take

$$\psi = \gamma^2_K \cdot \delta_i (u - u_0) \in \dot{W}_2^k$$

and conclude

$$j = \int_{\Omega} \gamma^{2K} \theta^{k-2} u \cdot H(u) \leq \frac{1}{\gamma_1} \int_{\Omega} \gamma^{2K} \sum_{i,j=1}^{h} F_{ij} \cdot N_i \delta^i u \cdot N_j \delta^j u =$$

$$= \frac{1}{\gamma_1} \left( \int_{\Omega} \sum_{i=1}^{h} N_i \left( \gamma^{2K} \delta^i (u - u_0) \right) (\delta^i f_i + R_2) + R_1 \left( f_i - F_i \right) + \right.$$

$$+ \sum_{j=1}^{h} F_{ij} N_j \delta^j u \left( R_3 - N_j \left( \gamma^{2K} \cdot \delta^j u_0 \right) \right) \right),$$

where $R_3$ involves the derivatives of $\delta^i u$ up to the order $\xi_i$. The right part may be estimated as in Lemma 3.1 and it implies $j \leq C(f)$.

**Lemma 3.3:** Let $u = \gamma^{\frac{1}{2}} (f) \in M$ then

$$I = \int_{\Omega} \gamma^{2K} \cdot \theta^{2(k-2)} u \cdot \sum_{i=1}^{h} N_i^2 \frac{\partial u}{\partial x_i} \leq C(s) \cdot V^{k-2}$$

for sufficiently small $s$; \( V = \left\| 1 + \sum_{i=1}^{h} N_i u \right\|_{\partial \Omega}. \)

**Proof:** Let us denote $\alpha_r = (0, 0, \ldots, \alpha_r)$, $N_{\beta_r} u = D^{\nu_r} u_r$. We shall estimate the $L^2$-norm of the functions

$$g_r = \frac{\partial}{\partial x_N} \left( \gamma^{2K} \cdot F_{\beta_r} (x, \{N_j u(x)\}) \right),$$

using the following theorem (see [14], [15]).

Let $f \in W^{l}_{p} (\Omega)$; $l$ entire, $\nu$ entire, non-negative. Then

$$\left\| f \right\|_{W^{l}_{p} (\Omega)} \leq C \left( \sum_{|\alpha| = \nu} \left\| \frac{\partial^\alpha f}{\partial x^\alpha} \right\|_{W^{l-\nu}_{p} (\Omega)} + \left\| f \right\|_{W^{l-\nu}_{p} (\Omega)} \right).$$
Let us set \( p = 2, \quad l = 0, \quad v = \kappa_r - 1 \) for \( \kappa_r > 1 \) and \( v = 1 \), otherwise. The second case is quite analogous.

(3.10) \[
\| g_r \|_{L^2} \leq C \left\| \frac{\Sigma}{a = \kappa_r - 1} \right\| D^p g_r \|_{W^{1,\kappa_r}_2} + \| g_r \|_{W^{1,\kappa_r}_2}.
\]

1) \[
\| g_r \|_{W^{1,\kappa_r}_2} = \sup_{\varphi \in W^{1,\kappa_r}_2, \| \varphi \| = 1} \left| \int g_r \varphi \right| = \sup_{\varphi \in W^{1,\kappa_r}_2, \| \varphi \| = 1} \left| \int \frac{\partial \varphi}{\partial x_N} \gamma^2 K \cdot F_{\beta r} \right| \leq C \| F_{\beta r} \|_{L^2} \leq C \| \theta^{k-1} u \|_{L^2} \leq CV^{\frac{k}{2}-1}.
\]

Let us denote \( \bar{\kappa}_r = (0, 0, \ldots, \kappa_r - 1) \). Let \( \alpha = \bar{\kappa}_r \); \( |\alpha| = \kappa_r - 1 \).

2) \[
\| D^p g_r \|_{W^{1,\kappa_r}_2} = \sup_{\varphi \in W^{1,\kappa_r}_2, \| \varphi \| = 1} \left| \int D^p g_r \varphi \right| = \sup_{\varphi \in W^{1,\kappa_r}_2, \| \varphi \| = 1} \left| \int D^p g_r \varphi \cdot \frac{\partial}{\partial x_j} \left[ \gamma^2 K \cdot F_{\beta r} \right] \right|,
\]

where \( |\alpha'| = \kappa_r - 1 \); \( j = N \). Then

\[
\| D^p g_r \|_{W^{1,\kappa_r}_2} \leq C \left\| \frac{\partial}{\partial x_j} \left[ \gamma^2 K \cdot F_{\beta r} \right] \right\|_{L^2}
\]

\[
= \left\| \gamma^2 K \left( \frac{\partial F_{\beta r}}{\partial x_j} + \sum_{l=1}^k F_{\beta r,l} \cdot N_l \frac{\partial u}{\partial x} \right) + F_{\beta r} \cdot \frac{\partial \gamma^2 K}{\partial x_j} \right\|_{L^2}
\]

\[
\leq C \cdot \left\{ V^{\frac{k}{2}-1} + \left\| \gamma^2 K \theta^{k-2} u \sum_{l=1}^k N_l \left( \frac{\partial u}{\partial x} + \frac{\partial a}{\partial x} \frac{\partial u}{\partial x_N} \right) \right\|_{L^2} + \right.
\]

\[
+ \sup_{\beta_j \in K} \left| \frac{\partial a}{\partial x_j} (x') \right| \cdot \left| \gamma^2 K \theta^{k-2} u \sum_{l=1}^k N_l \left( \frac{\partial u}{\partial x} \right) \right|_{L^2}
\]

\[
\leq C \cdot V^{\frac{k}{2}-1} + C(s) \cdot l^{1/2},
\]

where \( C(s) \to 0 \) for \( s \to 0 \).
The first term is estimated by $C \| f \|_{W^1_2}$, the second one has the same form as 2).

Finally, from (3.10) and 1), 2), 3)

\begin{equation}
3) \left( \sum_{r=1}^{m} \left\| D^{\gamma^r} g_r \right\|_{L^1_2} \right)^{1/2} \leq \sup_{\| \varphi_r \|_{L^1_2} = 1} \left| \int_{\Omega} \sum_{r=1}^{m} \varphi_r \cdot D^{\gamma^r} g_r \right| = \sup_{\| \varphi_r \|_{L^1_2} = 1} \left| \int_{\Omega} \sum_{r=1}^{m} \varphi_r \cdot D^{\gamma^r} g_r \right| = \sup_{\| \varphi_r \|_{L^1_2} = 1} \left| \int_{\Omega} \sum_{r=1}^{m} \varphi_r \cdot \gamma^r K F_{\beta_r} \right| \leq \sup_{\| \varphi_r \|_{L^1_2} = 1} \left| \int_{\Omega} \sum_{r=1}^{m} \varphi_r \cdot \gamma^r K F_{\beta_r} \right| + C V^{k-1}_2 \leq \sup \left\{ \left| \int_{\Omega} \sum_{i=1}^{h} f_i \cdot N_i \left( \varphi \cdot \gamma^2 K \right) \right| + \left| \int_{\Omega} \sum_{i=1}^{h} F_i \cdot N_i \left( \varphi \cdot \gamma^2 K \right) \right| + C V^{k-1}_2 \right\}.
\end{equation}

As in 2) all the terms

\begin{equation}
\left( \int_{\Omega} \gamma^2 K \theta^2 \sum_{r=1}^{m} N^2_{\beta_r} \left( \frac{\partial u}{\partial x_N} \right) \right)^{1/2} \text{ for } l = \beta_r; \ r = 1, \ldots, m
\end{equation}

may be estimated by $C V^{k-1}_2 + C(s) I^{1/2}$

\begin{equation}
j = \int_{\Omega} \gamma^2 K \theta^2 \sum_{r=1}^{m} N^2_{\beta_r} \left( \frac{\partial u}{\partial x_N} \right) \leq \int_{\Omega} \gamma^2 K \theta^2 \sum_{r=1}^{m} F_{\beta_r} \cdot N_{\beta_r} \left( \frac{\partial u}{\partial x_N} \right) \leq \left| \int_{\Omega} \sum_{r=1}^{m} \theta^{k-2} u \cdot g_r - \frac{\partial j}{\partial x_N} \right| \left| \int_{\Omega} \sum_{r=1}^{m} \theta^{k-2} u \cdot \gamma^2 K \left( \sum_{\beta \neq \beta_r} \gamma^2 K \right) \frac{\partial u}{\partial x_N} \right| + C V^{k-1}_2 + C(s) I^{1/2}.
\end{equation}

Therefore $I \leq C(s) \cdot V^{k-2}$ for sufficiently small $s$.

From these lemmas it follows immediately:

**Corollary 3.4**

\begin{equation}
\| \theta^{k-1} u \|_{W^1_2} \leq C V^{k-1}_2.
\end{equation}
Lp-estimates are based on the generalization of Meyer's theorem.

**Lemma 3.5:** Let \( u = \mathcal{C}^{-1} f \in M \). Then \( w = \frac{\partial u}{\partial x_i} \cdot \gamma^K_0 \) is a weak solution of

\[
\int_\Omega \sum_{i,j=1}^h F_{ij} \cdot N_i \cdot w \cdot N_j \varphi = \int_\Omega \sum_{i=1}^h g_i \cdot N_i \varphi,
\]
i.e. this equation is satisfied for every \( \varphi \in [D(D\Omega)]^m \). The right part \( g_j \in L_p(\Omega) \) with

\[
\| g \|_{L_p}^h \leq CV^{k/2-1}.
\]

**Proof:** Analogously to the proof of (3.1) it may be written

\[
\int_\Omega \sum_{i,j=1}^h F_{ij} \cdot N_i \left( \frac{\partial u}{\partial x_i} \right) \cdot N_j \psi = \int_\Omega \sum_{j=1}^h \left( f_j - F_j \right) \cdot N_j \psi.
\]

For \( w, \psi = \gamma^K_0 \cdot \varphi \) we obtain

\[
\int_\Omega \sum_{i,j=1}^h F_{ij} \cdot N_i \cdot w \cdot N_j \varphi = \int_\Omega \sum_{j=1}^h N_j \varphi \cdot \left( \gamma^K_0 \frac{\partial}{\partial x_i} \left( f_j - F_j \right) \right) + \sum_{i=1}^h F_{ij} \cdot \sum_{r, |\alpha| < \kappa_r} C_{\alpha} \cdot D^\alpha u \cdot \frac{\partial}{\partial x_i} \left( f_j - F_j \right) + R
\]

where \( C_{\alpha} \) are smooth and \( R \) involves the terms

\[
C \cdot D^\alpha \cdot D^\beta \left( \gamma^K_0 \right) \cdot \left( F_{ij} \frac{\partial u}{\partial x_i} + \frac{\partial f_j}{\partial x_i} \right)
\]

with \( |\alpha| < \kappa ; |\beta| \leq \kappa_r \).

The expression at \( N_j \varphi \) (let us denote it \( g_j \)) may be estimated by

\[
C \cdot \left( g^{k-1}u + \left| \frac{\partial f_j}{\partial x_i} \right| \right)
\]

According to corollary (3.4) \( \theta^{k-1}u \in L_q(\Omega) \) for every \( 1 \leq q < \infty \) and \( g_j \in L_p(\Omega) \) with the corresponding norm.

\( R \) may be written in the form

\[
R = \sum_{r=1}^m \sum_{|\alpha| = \kappa_r} g_{ra} \cdot D^\alpha \varphi
\]

with \( \| g_{ra} \|_{L_p} \leq CV^{k/2-1} \) according to the result of Nečas (see [17], [18]):
Let
\[ g \in L^q(\Omega), \quad 1 < q < \infty; \quad 0 \leq |l| \leq k-1, \]
hence there exist the functions \( g_j \) such that
\[ \int_\Omega D^l \varphi \cdot g = \int_\Omega \sum_{|j| = k} g_j D^j \varphi \quad \text{for every} \quad \varphi \in D(\Omega). \]
Moreover,
\[ \| g_j \|_{L^p} \leq C \cdot \| g \|_{L^q} \]
where
\[ \frac{1}{p} \geq \frac{1}{q} - \frac{k - |l|}{N} \quad \text{for} \quad (k - |l|) \cdot q < N \]
and
\[ 1 \leq p < \infty \quad \text{for} \quad (k - |l|) \cdot q \geq N. \]

Applying Lemma 1.4 with \( \tilde{\gamma}_1 = \gamma_1, \tilde{\gamma}_2 = \gamma_2 \cdot V^{k-2} \) to (3.13), we obtain immediately

**Lemma 3.6**: Let \( u \in \mathcal{B}^{-1}(f) \in M \) then there exist a constant \( \gamma_5 > 0 \) and \( q = 2 + \gamma_5 V^{2-k} \) such that \( \gamma^K_0 u \in W^{s+1}_q(\Omega) \) and

\[
\| \gamma^K_0 u \|_{W^{s+1}_q} \leq C \cdot V^{\frac{1}{2} k-1} \tag{3.14}
\]

**Lemma 3.7**: Let \( u = \mathcal{B}^{-1}(f) \in M \) then there exist a constant \( \gamma_5 > 0 \) and \( q = 2 + \gamma_5 V^{2-k} \) such that \( \gamma^K u \in W^{s+1}_q(\Omega) \) and

\[
\| \gamma^K u \|_{W^{s+1}_q} \leq C(r) \cdot V^{\frac{3}{2} (k-2)}. \tag{3.15}
\]

**Proof**: For \( w = \gamma^K \partial^l (u - u_0) \) we may use Lemma 1.4 analogously to Lemmas 3.5, 3.6.

For the normal derivatives we obtain by repeating the estimates in the proof of 3.3 for \( q = 2 + \gamma_5 V^{2-k} \) instead 2
\[
\left\| \gamma^K \cdot \partial^{k-2} u \cdot \sum_{i=1}^h N_i \right\|_{L^q} \leq C(r) \cdot V^{\frac{3}{2} (k-2)}. \tag{3.14}
\]

From (3.14), (3.15) it follows:
COROLLARY 3.8:

\[(3.16) \quad \| \theta^k u \|_{W^1_q} \leq CV^{\frac{3}{2}(k-2)} \quad \text{with} \quad q = 2 + \gamma_0 V^{2-k}.\]

**Lemma 3.9:** Let \( u = \mathcal{B}^{-1} (f) \in M. \) Then there exist \( p_0 > 2 \) such that \( u \in W^{k+1} (\Omega) \) with

\[(3.17) \quad \| u \|_{W^{k+1}_p} \leq C \left( \| f \|_{W^1_p} \right),\]

which implies

\[\| u \|_{C_a (\Omega)} + \| u \|_{W^{k+1}_p (\Omega)} \leq C.\]

**Proof:** From (3.12), (3.16)

\[\| \theta^k u \|_{W^1_q} \leq CV^{\frac{k}{2} - 1}\]

\[\| \theta^k u \|_{W_q^1} \leq CV^{\frac{3}{2}(k-2)}\]

hold for \( q = 2 + \gamma_0 V^{2-k}. \)

Using the Riesz-Thorin interpolation theorem, we have

\[\| \theta^k u \|_{W^1_{p_0}} \leq C \cdot V^{a \frac{3}{2}(k-2) + (1-a)(\frac{k}{2} - 1)}\]

for

\[\frac{1}{p_0} = \frac{a}{q} + \frac{1 - a}{2} ; \quad a \in (0, 1).\]

Using embedding theorem (see [17], [18]) for \( N = 2 :\)

\[(3.18) \quad \| \theta^k u \|_{C_a (\Omega)} \leq C \cdot \left( \frac{p_0 - 1}{p_0 - 2} \right)^{1 - \frac{1}{p_0}} \| \theta^k u \|_{W^1_{p_0}} \leq C_1 \left( \frac{p_0 - 1}{p_0 - 2} \right)^{1 - \frac{1}{p_0}} \cdot V^{(2a + 1) \left( \frac{k}{2} - 1 \right)} + C_2 \cdot V^{\frac{k}{2} - 1}\]

But

\[1 - \frac{1}{p_0} \leq \frac{1}{2} (1 + a \gamma_0 V^{2-k}) \quad \text{and} \quad \frac{p_0 - 1}{p_0 - 2} \leq C \cdot \frac{V^{k-2}}{a} .\]

Then

\[\| \theta u \|_{C_a (\Omega)} \leq C_1 V^{\left( \frac{k}{2} - 1 \right) (2 + a (3 + \gamma_0))} + C_2 \cdot V^{\frac{k}{2} - 1} .\]
For
\[ a \cdot (2 + \gamma_0) \cdot \left( \frac{k}{2} - 1 \right) < \frac{1}{2} \]
the inequality may be satisfied only for bounded \( V \).

The proceedings of Lemmas (3.5)-(3.7) may be repeated with this result. The coefficients in the equations (3.13) are continuous, the right parts belong to \( L_p \). According to the result of Agmon, Douglis, Nirenberg (see [1]) or Schechter (see [23]) \( u \in W^{n+1}_p \) and its norm depends only on \( \| f \|, \| u_0 \| \).

This completes the proof of Theorem 4.

Only the dependence of the highest derivatives of \( u \) is important in this proof and it allows us to prove, quite analogously and without changes, the following

**Theorem 5.** Let \( N_i \) be all the derivatives, i.e.
\[ N_i u = N_{r_i} u = D^a u_r \quad \text{for} \quad r = 1, \ldots, m; \quad |a| \leq r; \]
let \( \mathcal{B} \) satisfies \( B \). Then \( \mathcal{B} \) has property \( \mathcal{A} \).

**Theorem 6.** Let \( \mathcal{B} \) satisfy \( A \) or \( B \) with \( h = m \). Then \( \mathcal{B} \) has property \( \mathcal{A} \).

**Proof.** The proof in case \( B \) is a slight modification of case \( A \). Lemmas (3.1), (3.2) may be proved without changes. Let us set
\[ g_r = \frac{\partial}{\partial x_n} \left( \sum_{i=1}^{h} a_{ir_i} \gamma_s^{2K} F_i (x_i \{ N_j u(x) \}) \right) . \]
The estimates of \( L_2 \)-norms of \( g_r \) are the same as in (3.3). The matrix
\[ N((0, \ldots, 0, 1)) = (a_{ir_i})_{r_i=1, \ldots, m} \]
is regular and
\[ h_r = \frac{\partial}{\partial x_n} (\gamma_s^{2K} F_i) = \sum_{r=1}^{m} C_r g_r \in L_2 \]
with
\[ \| h_r \|_{L_2} \leq CV^{k/2-1} + C(\delta) I^{1/2} . \]
But
\[ I \leq \frac{1}{\gamma_1} \int_{\Omega} \gamma_s^{2K} \theta^{k-2} u \cdot \sum_{i,j=1}^{m} F_{ij} N_i \frac{\partial u}{\partial x_N} \cdot N_j \frac{\partial u}{\partial x_N} = \]
\[ = \frac{1}{\gamma_1} \left( \theta^{k-2} u \cdot \frac{\partial u}{\partial x_N} \cdot \gamma_s^{2K} \left\{ h_r - \frac{\partial}{\partial x_N} \cdot \gamma_s^{2K} - F_i \cdot \frac{\partial}{\partial x_N} \right\} \right) \]
and it is bounded as in (3.3). Analogously Lemma (3.7) is proved and this completes the proof.

**Theorem 7.** Let $\mathcal{B}$ satisfy $A$ or $B$, $k < 4$. Then $\mathcal{B}$ has property $A^\prime$.

**Proof.** (3.1), (3.2) are proved in the same way. However, we are not able, under these conditions, to obtain from (3.1), (3.2) a better estimate for

$$\int_\Omega \theta^{k-2}u |D^\alpha u_r|^2$$

with

$$|\alpha| = \kappa_r + 1, \quad \alpha = (0, 0, \ldots, \kappa_r + 1)$$

than $C(f) \cdot V^{k-2}$. Therefore,

$$\| \theta^{k-1}u \|_{W_2^1} \leq CV^{k-2},$$

analogously

$$\| \theta^{k-1}u \|_{W_p^1} \leq CV^{2k-4}.$$

In Lemma (3.4) the boundedness of $V$ may be obtained only for $k < 4$.

The difficulties lie in the fact that for more precise estimate of

$$\int_\Omega \theta^{k-2}u \cdot D^\alpha u_r$$

a theorem on a very general class of multiplicators of the form $\theta^{k-2}u$ would be necessary — which is not known to us at present.
REFERENCES


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