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# SUMMABILITY FACTORS FOR GENERALIZED ABSOLUTE RIESZ SUMMABILITY I

By Z. U. AHMAD

1.1. Let  $\Sigma a_n$  be a given infinite series, and let  $\lambda_n$  be a sequence of positive, monotonically increasing numbers, diverging to infinity. We write

$$A_\lambda(t) = A_\lambda^0(t) = \sum_{\lambda_n \leq t} a_n,$$

$$A_\lambda(t) = 0, \text{ for } t \leq \lambda_1;$$

and for  $r > 0$ ,

$$A_\lambda^r(t) = \sum_{\lambda_n < t} (t - \lambda_n)^r a_n = r \int_{\lambda_1}^t (t - \tau)^{r-1} A_\lambda(\tau) d\tau = \int_{\lambda_1}^t (t - \tau)^r dA_\lambda(\tau).$$

Then  $R_\lambda^r(t) \equiv A_\lambda^r(t)/t^r$  is called the *Riesz mean of type  $\lambda_n$  and order  $r$* , while  $A_\lambda^r(t)$  is called the *Riesz sum of type  $\lambda_n$  and order  $r$* . We say that  $\Sigma a_n$  is absolutely summable by this Riesz mean, or summable  $|R, \lambda, r|$ ,  $r \geq 0$ , if  $R_\lambda^r(t)$  is a function of bounded variation in  $(h, \infty)$  for some positive number  $h$ ; or if

$$\int_h^\infty \left| \frac{d}{dt} \{R_\lambda^r(t)\} \right| dt < \infty \text{ ([10], [11]).}$$

We say that  $\Sigma a_n$  is summable  $|R, \lambda, r|_p$ ,  $p \geq 1$ ,  $r > 0$ ,  $rp' > 1$ , and  $1/p + 1/p' = 1$ , if

$$\int_h^\infty t^{p-1} \left| \frac{d}{dt} \{R_\lambda^r(t)\} \right|^p dt < \infty, \text{ [7],}$$

where  $h$  is some positive number as before. Evidently, for  $p = 1$ ,  $|R, \lambda, r|_p$  is the same as  $|R, \lambda, r|$ .

1.2. Suppose that  $h$  is some positive number, and unless or otherwise stated  $k$  is a positive integer. We suppose further that  $\Phi(t)$  and  $\Psi(t)$  are functions with absolutely continuous  $(k-1)$ th derivatives in every interval  $[h, W]$ , and that  $\Phi(t)$  is non-negative and non-decreasing function of  $t$  for  $t \geq h$ , tending to infinity with  $t$ .

Without any loss of generality we take  $\Phi(\lambda_1) = h = \lambda_1$ .

By  $B_k(t)$  we mean the Rieszian sum of type  $\lambda_n$  and order  $k$  of the series  $\sum a_n \lambda_n$ , and by  $E_k(t)$  we mean the Rieszian sum of type  $\Phi(\lambda_n)$  and order  $k$  of the series  $\sum a_n \Psi(\lambda_n) \Phi(\lambda_n)$ .

1.3. **Introduction.** Concerning  $|R, \lambda, k| \implies |R, \Phi(\lambda), k|$ -summability factors, when  $k$  is a positive integer, the following theorem is known.

**THEOREM A** [3]. *If there is a function,  $\gamma(t)$ , defined and positive for  $t \geq h$ , such that, for  $t \geq h$ ,*

$$(i) \quad \gamma(t) = O(t),$$

$$(ii) \quad t^n \Psi^{(n)}(t) = O \left[ \left\{ \frac{\gamma(t)}{t} \right\}^{k-n} \right], \text{ for } n = 0, 1, 2, \dots, k;$$

and

$$(iii) \quad \{\gamma(t)\}^n \Phi^{(n)}(t) = O\{\Phi(t)\}, \text{ for } n = 1, 2, \dots, k;$$

and if the series  $\sum a_n$  is summable  $|R, \lambda, k|$ , then the series  $\sum \Psi(\lambda_n) a_n$  is summable  $|R, \Phi(\lambda), k|$ .

This is a generalization of a number of previously known results (See [3], [1], [2]). In particular, in the special cases in which (i)  $\Psi(t) = 1, \gamma(t) = t$ , (ii)  $\Phi(t) = e^t, \Psi(t) = t^{-k}, \gamma(t) = 1$ , it reduces respectively to the following theorems.

**THEOREM B** [4]. *If the series  $\sum a_n$  is summable  $|R, \lambda, k|$  and*

$$t^k \Phi^{(k)}(t) = O\{\Phi(t)\},$$

for  $t \geq \lambda_1$ , then the series  $\sum a_n$  is summable  $|R, \Phi(\lambda), k|$ .

**THEOREM C** [12]. *If  $k \geq 0$ , and the series  $\sum a_n$  is summable  $|R, \lambda, k|$ , then the series  $\sum a_n \lambda_n^{-k}$  is summable  $|R, l, k|$ , where  $l_n = e^{\lambda_n}$ .*

Recently Mazhar has extended these theorems (Theorems B and C) for generalized absolute Riesz summability (defined in 1.1) in the form of

**THEOREM D** [8]. *If, for  $p \geq 1$ , and  $t \geq \lambda_1$ ,*

- (i)  $t^k \Phi^{(k)}(t) = O\{\Phi(t)\},$
- (ii)  $\{\Phi(t)/t \Phi^{(1)}(t)\}^{p-1} = O(1),$

*then any infinite series  $\sum a_n$  which is summable  $|R, \lambda, k|_p$ , is also summable  $|R, \Phi(\lambda), k|_p$ .*

**THEOREM E** [9]. *If  $p \geq 1$ , and  $\sum a_n$  is summable  $|R, \lambda, k|_p$ , then  $\sum a_n \lambda_n^{-k + \frac{1}{p'}}$  is summable  $|R, l, k|_p$ , where  $l_n = e^{\lambda_n}$  and  $1/p + 1/p' = 1$ .*

The object of the present paper is to generalize Theorem A for generalized absolute Riesz summability so as to include Theorems D and E.

2.1. We establish the following theorem.

**THEOREM.** *If there is a function,  $\gamma(t)$ , defined and positive for  $t \geq h$ , such that, for  $t \geq h$ ,*

- (i)  $\gamma(t) = O(t);$
- (ii)  $t^n \Psi^{(n)}(t) = O\left[\left\{\frac{\gamma(t)}{t}\right\}^{k-n}\right],$  for  $n = 0, 1, 2, \dots, k;$
- (iii)  $\{\gamma(t)\}^n \Phi^{(n)}(t) = O\{\Phi(t)\},$  for  $n = 1, 2, \dots, k;$
- (iv)  $\{\Phi(t)/\gamma(t) \Phi^{(1)}(t)\}^{p-1} = O(1),$

*and if the series  $\sum a_n$  is summable  $|R, \lambda, k|_p$ , then the series  $\sum \Psi(\lambda_n) a_n$  is summable  $|R, \Phi(\lambda), k|_p$ .*

2.2. The following lemmas will be required for the proof of our theorem.

**LEMMA 1** [6]. *For  $k > 0$ ,*

$$w^{k+1} \frac{d}{dw} \{R_\lambda^k(w)\} = k B_{k-1}(w) = \frac{d}{dw} \{B_k(w)\}.$$

**LEMMA 2** [5]. *If  $k$  is a positive integer, then*

$$A_\lambda(t) = \frac{1}{k!} \left(\frac{d}{dt}\right)^k A_\lambda^k(t).$$

LEMMA 3 ([13], p. 89). *If  $n$  is a positive integer and  $m \neq 0$ , then the  $n$ th derivative of  $\{f(x)\}^m$  is a sum of constant multiples of a finite number of terms of the form :*

$$\{f(x)\}^{m-r} \prod_{s=1}^n \{f^{(s)}(x)\}^{\alpha_s},$$

where  $1 \leq r \leq n$  and  $\alpha$ 's are zeros or positive integers such that

$$\sum_{s=1}^n \alpha_s = r \text{ and } \sum_{s=1}^n s \alpha_s = n.$$

If  $m$  is a positive integer,  $1 \leq r \leq \min(m, n)$ .

### 2.3. PROOF OF THE THEOREM :

Under the hypothesis of the theorem we have by Lemma 1, for  $p > 1$  (\*),

$$(2.3.1) \quad \int_{\lambda_1}^{\infty} t^{-(kp+1)} |B_{k-1}(t)|^p dt < \infty,$$

and we have to establish that

$$(2.3.2) \quad \int_{\Phi(\lambda_1)}^{\infty} w^{-(kp+1)} |E_{k-1}(w)|^p dw < \infty.$$

By writing  $w = \Phi(t)$  in the above integral we find that the required inequality can be written in the form of

$$(2.3.3) \quad \int_{\lambda_1}^{\infty} \frac{\Phi^{(1)}(t)}{\{\Phi(t)\}^{kp+1}} |E_{k-1}(\Phi(t))|^p dt < \infty.$$

Now, we have

$$E_{k-1}(\Phi(t)) = \int_{\Phi(\lambda_1)}^{\Phi(t)} (\Phi(t) - u)^{k-1} dE(u)$$

---

(\*) For the case  $p = 1$ , the theorem is known (Theorem A).

$$\begin{aligned}
 &= \int_{\lambda_1}^t (\Phi(t) - \Phi(u))^{k-1} dE(\Phi(u)) \\
 &= \int_{\lambda_1}^t (\Phi(t) - \Phi(u))^{k-1} \frac{\Psi(u) \Phi(u)}{u} dB(u) \\
 &= \left[ (\Phi(t) - \Phi(u))^{k-1} \frac{\Psi(u) \Phi(u)}{u} B(u) \right]_{\lambda_1}^t - \\
 &\quad - \int_{\lambda_1}^t B(u) \frac{d}{du} \left\{ (\Phi(t) - \Phi(u))^{k-1} \frac{\Psi(u) \Phi(u)}{u} \right\} du \\
 &= - \int_{\lambda_1}^t B(u) \frac{d}{du} \left\{ (\Phi(t) - \Phi(u))^{k-1} \frac{\Psi(u) \Phi(u)}{u} \right\} du.
 \end{aligned}$$

Applying Lemma 2 and integrating  $(k - 1)$ -times we get

$$\begin{aligned}
 E_{k-1}(\Phi(t)) &= \frac{(1)^{k-1}}{(k-1)!} \left[ B_{k-1}(u) \left( \frac{d}{du} \right)^{k-1} \left\{ (\Phi(t) - \Phi(u))^{k-1} \frac{\Psi(u) \Phi(u)}{u} \right\} \right]_{\lambda_1}^t + \\
 &\quad + \frac{(-1)^k}{(k-1)!} \int_{\lambda_1}^t B_{k-1}(u) \left( \frac{d}{du} \right)^k \left\{ (\Phi(t) - \Phi(u))^{k-1} \frac{\Psi(u) \Phi(u)}{u} \right\} du \\
 &= \frac{(-1)^{k-1}}{(k-1)!} B_{k-1}(t) \frac{\Psi(t) \Phi(t)}{t} \{\Phi^{(1)}(t)\}^{k-1} + \\
 &\quad + \frac{(-1)^k}{(k-1)!} \int_{\lambda_1}^t B_{k-1}(u) \left( \frac{d}{du} \right)^k \left\{ (\Phi(t) - \Phi(u))^{k-1} \frac{\Psi(u) \Phi(u)}{u} \right\} du \\
 &= \frac{(-1)^{k-1}}{(k-1)!} (\varepsilon_1(t) - \varepsilon_2(t)).
 \end{aligned}$$

Thus, by virtue of Minkowski's inequality, it is sufficient to prove that

$$(2.3.4) \quad \int_{\lambda_1}^{\infty} \frac{\Phi^{(1)}(t)}{(\Phi(t))^{kp+1}} |\varepsilon_1(t)|^p dt < \infty,$$

and

$$(2.3.5) \quad \int_{\lambda_1}^{\infty} \frac{\Phi^{(1)}(t)}{(\Phi(t))^{kp+1}} |\varepsilon_2(t)|^p dt < \infty.$$

PROOF OF (2.3.4). We have

$$\begin{aligned} & \int_{\lambda_1}^{\infty} \frac{\Phi^{(1)}(t)}{(\Phi(t))^{kp+1}} |\varepsilon_1(t)|^p dt \\ &= \int_{\lambda_1}^{\infty} \frac{\Phi^{(1)}(t)}{(\Phi(t))^{kp+1}} \left| \frac{\Psi(t)\Phi(t)}{t} \right|^p (\Phi^{(1)}(t))^{(k-1)p} |B_{k-1}(t)|^p dt \\ &\leq K \int_{\lambda_1}^{\infty} \left\{ \frac{\gamma(t)\Phi^{(1)}(t)}{\Phi(t)} \right\}^{pk} \left\{ \frac{\Phi(t)}{\gamma(t)\Phi^{(1)}(t)} \right\}^{p-1} \left( \frac{\gamma(t)}{t} \right)^{p-1} t^{-(kp+1)} |B_{k-1}(t)|^p dt \\ &\leq K \int_{\lambda_1}^{\infty} t^{-(kp+1)} |B_{k-1}(t)|^p dt \leq K(*), \end{aligned}$$

by hypotheses.

PROOF OF (2.3.5).

Since, by Leibnitz's formula and Lemma 3,

$$\begin{aligned} \vartheta(t, u) &\equiv \left( \frac{d}{du} \right)^k \left\{ (\Phi(t) - \Phi(u))^{k-1} \frac{\Psi(u)\Phi(u)}{u} \right\} \\ &= \sum_{j=0}^k \binom{k}{j} (-1)^{k-j} \Gamma(k-j+1) u^{-(k-j+1)} \times \\ &\quad \times \left( \frac{d}{du} \right)^j \{ (\Phi(t) - \Phi(u))^{k-1} \Psi(u)\Phi(u) \} \end{aligned}$$

---

(\*) Throughout  $K$ 's denote absolute constants, not necessarily the same at each occurrence.

$$\begin{aligned}
 &= \sum_{j=0}^k \sum_{r=0}^j (-1)^{k-j} \frac{\Gamma(k+1)}{\Gamma(r+1)\Gamma(j-r+1)} u^{-(k-j+1)} \times \\
 &\quad \times \Phi^{(j-r)}(u) \left(\frac{d}{du}\right)^r \{(\Phi(t) - \Phi(u))^{k-1} \Psi(u)\} \\
 &= \sum_{j=0}^k \sum_{r=0}^j \sum_{s=0}^r (-1)^{k-j} \frac{\Gamma(k+1)}{\Gamma(s+1)\Gamma(r-s+1)\Gamma(j-r+1)} \times \\
 &\quad \times u^{-(k-j+1)} \Phi^{(j-r)}(u) \Psi^{(r-s)}(u) \left(\frac{d}{du}\right)^s \{(\Phi(t) - \Phi(u))^{k-1}\} \\
 &= \sum_{j=0}^k \sum_{r=0}^j (-1)^{k-j} \frac{\Gamma(k+1)}{\Gamma(r+1)\Gamma(j-r+1)} u^{-(k-j+1)} \times \\
 &\quad \times \Phi^{(j-r)}(u) \Psi^{(r)}(u) (\Phi(t) - \Phi(u))^{k-1} + \\
 &\quad + \sum_{i=1}^k \sum_{r=1}^j \sum_{s=1}^r \sum_{m=1}^{\min(s, k-1)} K_{j, r, s, m} u^{-(k-j+1)} \times \\
 &\quad \times \Phi^{(j-r)}(u) \Psi^{(r-s)}(u) (\Phi(t) - \Phi(u))^{k-1-m} \prod_{i=1}^s (\Phi^{(i)}(u))^{\alpha_i}
 \end{aligned}$$

where  $\alpha_i$ 's are zeros or positive integers, such that

$$\sum_{i=1}^s \alpha_i = m; \quad \sum_{i=1}^s i\alpha_i = s,$$

we have

$$\begin{aligned}
 \vartheta(t, u) &= \sum_{j=0}^k \sum_{r=0}^j K_{j, r} F_1(u) u^{-(k+1)} \Phi(u) (\Phi(t) - \Phi(u))^{k-1} + \\
 &\quad + \sum_{j=1}^k \sum_{r=1}^j \sum_{s=1}^r \sum_{m=1}^{\min(s, k-1)} K_{j, r, s, m} F_2(u) \times \\
 &\quad \times u^{-(k+1)} (\Phi(u))^m (\Phi(t) - \Phi(u))^{k-m-1},
 \end{aligned}$$

where

$$F_1(u) = \frac{\{\gamma(u)\}^{j-r} \Phi^{(j-r)}(u)}{\Phi(u)} \frac{u^r \Psi^{(r)}(u)}{\{\gamma(u)/u\}^{k-r}} \left\{ \frac{\gamma(u)}{u} \right\}^{k-},$$

$$0 \leq r \leq j \leq k,$$

$$F_2(u) = \frac{\{\gamma(u)\}^{j-r} \Phi^{(j-r)}(u) u^{r-s} \Psi^{(r-s)}(u)}{\Phi(u)} \frac{1}{\{\gamma(u)/u\}^{k-r+s}} \times \\ \times \left( \frac{\gamma(u)}{u} \right)^{k-j} \prod_{i=1}^s \left[ \frac{\{\gamma(u)\}^i \Phi^{(i)}(u)}{\Phi(u)} \right]^{a_i},$$

$$1 \leq s \leq r \leq j \leq k \quad \text{and} \quad 1 \leq m \leq \min(s, k-1),$$

are bounded functions in  $(\lambda_1, \infty)$ , by hypotheses.

Therefore, in order to establish (2.3.5), by virtue of Minkowski's inequality we only need to show that, for  $0 \leq r \leq j \leq k$ ,

$$J_1 \equiv \int_{\lambda_1}^{\infty} \frac{\Phi^{(1)}(t)}{\{\Phi(t)\}^{kp+1}} dt \left| \int_{\lambda_1}^t B_{k-1}(u) F_1(u) u^{-(k+1)} \times \right. \\ \left. \times \Phi(u) (\Phi(t) - \Phi(u))^{k-1} du \right|^p < \infty,$$

and, for  $1 \leq s \leq r \leq j \leq k$ ,  $1 \leq m \leq \min(s, k-1)$ ,

$$J_2 \equiv \int_{\lambda_1}^{\infty} \frac{\Phi^{(1)}(t)}{\Phi(t)^{kp+1}} dt \left| \int_{\lambda_1}^t B_{k-1}(u) F_2(u) u^{-(k+1)} \times \right. \\ \left. \times \{\Phi(u)\}^m (\Phi(t) - \Phi(u))^{k-m-1} du \right|^p < \infty.$$

Now, applying Hölder's inequality, we observe that, for  $0 \leq r \leq j \leq k$ ,

$$J_1 \leq \int_{\lambda_1}^{\infty} \frac{\Phi^{(1)}(t)}{\Phi(t)^{kp+1}} dt \left( \int_{\lambda_1}^t |B_{k-1}(u)| |F_1(u)| u^{-(k+1)} \Phi(u) (\Phi(t) - \Phi(u))^{k-1} du \right)^p \\ < \int_{\lambda_1}^{\infty} \frac{\Phi^{(1)}(t)}{\{\Phi(t)\}^{kp+1}} dt \left( \int_{\lambda_1}^t |B_{k-1}(u)|^p |F_1(u)|^p \times \right. \\ \left. \times u^{-(k+1)p} (\Phi(u))^k (\Phi^{(1)}(u))^{-(p-1)} (\Phi(t) - \Phi(u))^{k-1} du \right) \times \\ \times \left( \int_{\lambda_1}^t (\Phi(t) - \Phi(u))^{k-1} \Phi^{(1)}(u) du \right)^{p-1}$$

$$\begin{aligned}
 &< K \int_{\lambda_1}^{\infty} \frac{\Phi^{(1)}(t)}{\{\Phi(t)\}^2} dt \int_{\lambda_1}^t |B_{k-1}(u)|^p u^{-(kp+1)} \times \\
 &\quad \times \left( \frac{\Phi(u)}{\gamma(u) \Phi^{(1)}(u)} \right)^p \left( \frac{\gamma(u)}{u} \right)^{p-1} \Phi(u) \left( 1 - \frac{\Phi(u)}{\Phi(t)} \right)^{k-1} du \\
 &\leq K \int_{\lambda_1}^{\infty} u^{-(kp+1)} |B_{k-1}(u)|^p du \Phi(u) \int_u^{\infty} \left( 1 - \frac{\Phi(u)}{\Phi(t)} \right)^{k-1} \frac{\Phi^{(1)}(t)}{\{\Phi(t)\}^2} dt \\
 &\leq K \int_{\lambda_1}^{\infty} u^{-(kp+1)} |B_{k-1}(u)|^p du \\
 &\leq K,
 \end{aligned}$$

by hypotheses.

Again, applying Hölder's inequality, we find that for  $1 \leq s \leq r \leq j \leq k$  and  $1 \leq m \leq \min(s, k-1)$ ,

$$\begin{aligned}
 J_2 &< \int_{\lambda_1}^{\infty} \frac{\Phi^{(1)}(t)}{\{\Phi(t)\}^{kp+1}} (\Phi(t))^{(k-1)p} dt \left( \int_{\lambda_1}^t |B_{k-1}(u)| \times \right. \\
 &\quad \left. \times |F_2(u)| \Phi(u) u^{-(k+1)} \left\{ \frac{\Phi(u)}{\Phi(t)} \right\}^m \left\{ 1 - \frac{\Phi(u)}{\Phi(t)} \right\}^{k-m-1} du \right)^p \\
 &< \int_{\lambda_1}^{\infty} \frac{\Phi^{(1)}(t)}{\{\Phi(t)\}^{kp+1}} dt \left( \int_{\lambda_1}^t |B_{k-1}(u)|^p |F_2(u)|^p \times \right. \\
 &\quad \left. \times u^{-(k+1)p} (\Phi(u))^p (\Phi^{(1)}(u))^{1-p} \left\{ 1 - \frac{\Phi(u)}{\Phi(t)} \right\}^{k-m-1} du \right) \times \\
 &\quad \times \left( \int_{\lambda_1}^t \left\{ 1 - \frac{\Phi(u)}{\Phi(t)} \right\}^{k-m-1} \Phi^{(1)}(u) \left\{ \frac{\Phi(u)}{\Phi(t)} \right\}^{\frac{mp}{1-p}} du \right)^{p-1} \\
 &\leq K \int_{\lambda_1}^{\infty} \frac{\Phi^{(1)}(t)}{\{\Phi(t)\}^{kp+1}} dt \int_{\lambda_1}^t u^{-(kp+1)} |B_{k-1}(u)|^p |F_2(u)|^p \times
 \end{aligned}$$

$$\begin{aligned}
& \times \left\{ \frac{\Phi(u)}{\gamma(u)\Phi^{(1)}(u)} \right\}^{p-1} \left\{ \frac{\gamma(u)}{u} \right\}^{p-1} \Phi(u) \left\{ 1 - \frac{\Phi(u)}{\Phi(t)} \right\}^{k-m-1} du \\
& \leq K \int_{\lambda_1}^{\infty} u^{-(kp+1)} |B_{k-1}(u)|^p du \int_u^{\infty} \left\{ 1 - \frac{\Phi(u)}{\Phi(t)} \right\}^{k-m-1} \Phi(u) \frac{\Phi^{(1)}(t)}{(\Phi(t))^2} dt \\
& \leq K \int_{\lambda_1}^{\infty} u^{-(kp+1)} |B_{k-1}(u)|^p du \\
& \leq K,
\end{aligned}$$

by hypotheses. This completes the proof of (2.3.5).

Thus the proof of our theorem is completed.

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