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THE CARTAN-THULLEN THEOREM FOR BANACH SPACES

by SEÁN DINEEN

The object of this paper is to give a characterization of domains of holomorphy which is similar to that given by Cartan-Thullen for finite dimensional spaces in [4]. In § 1 we recall some results from the theory of holomorphic functions on a Banach space, define domains of holomorphy and prove a number of fundamental lemmas. In § 2 we prove our main results. Our main reference for the theory of holomorphic functions on a Banach space is [16] and for domains of holomorphy in finite dimensions we refer to [2], [3], [4], [10], [13] and [15]. In [1], [2], [3], [6], [11] and [12] there are a number of interesting results concerning domains of holomorphy in infinite dimensions.

SECTION 1. E will represent a complex Banach space with unit ball B_1 . For each positive integer m let $\mathcal{L}^{(m)E}$ denote the set of all continuous m -linear mappings from $E^m = E \times E \times \dots \times E$ (m times) into C (the complex numbers). Let Δ_m denote the mapping from E into E^m which takes x into (x, x, \dots, x) (m times). A continuous m -homogeneous polynomial is a mapping from E into C which is the composition of Δ_m and an element of $\mathcal{L}^{(m)E}$. We denote by $\mathcal{P}^{(m)E}$ the set of all continuous polynomials on E and we note that it forms a Banach space under the norm

$$\|P\| = \sup_{\substack{\|x\| \leq 1 \\ x \in E}} |P(x)|.$$

A complex valued function f defined on an open subset U of E is said to be holomorphic at $\xi \in U$ if there exists a sequence $(P_n(\xi))_{n=0}^{\infty}$ (where $P_n(\xi) \in$

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$\in \mathcal{P}({}^n E)$ for each n such that

$$(1) \quad \limsup_{m \rightarrow \infty} \|P_m(\xi)\|^{1/m} < \infty$$

$$(2) \quad f(x) = \sum_{m=0}^{\infty} P_m(\xi)(x - \xi)$$

for all x in some neighbourhood of ξ . f is said to be holomorphic on U if it is holomorphic at all points of U . Since the expansion in (2) is unique

(see [16]) we write $\frac{\widehat{d}^m f(\xi)}{m!} = P_m(\xi)$ and call (2) the Taylor series expansion

of f at ξ . For the remainder of this paper U will denote a connected open subset of E with a nonempty boundary δU (the questions we consider are trivial for $U = E$). $\mathcal{C}A$ will denote the complement with respect to E of the subset A of E . If A_1 and A_2 are subsets of E we denote by $d(A_1, A_2)$ the distance between A_1 and A_2 , i. e.

$$d(A_1, A_2) = \inf_{\substack{x \in A_1 \\ y \in A_2}} \|x - y\|$$

DEFINITION 1. $B \subset E$ is U -bounded if B is a bounded subset of E and $d(B, \mathcal{C}U) > 0$. We note if E is finite dimensional that the closed U -bounded sets are exactly the compact subsets of U .

DEFINITION 2. (a) $\mathcal{H}(U)$ is the set of all complex-valued holomorphic functions on U .

(b) $\mathcal{H}_b(U)$ is the set of all holomorphic functions on U which are bounded on all U -bounded sets.

DEFINITION 3. Let $B \subset U$ then \widehat{B}_U (resp. $\widehat{B}_{U,b}$) is the set of all $\zeta \in U$ such that

$$|f(\zeta)| \leq \sup_{x \in B} |f(x)|$$

for all $f \in \mathcal{H}(U)$ (resp. $\mathcal{H}_b(U)$) and is called the U (resp. U_b)-holomorphic hull of B .

DEFINITION 4. A connected open subset U of E is a domain of holomorphy (resp. b domain of holomorphy) if there exists $f \in \mathcal{H}(U)$ (resp. $\mathcal{H}_b(U)$) which cannot be extended to an element of $\mathcal{H}(U_1)$ (resp. $\mathcal{H}_b(U_1)$) where U_1 is a connected open subset of E which properly contains U .

For a subset V of $(\mathcal{P}({}^m E))'$ (the dual of $\mathcal{P}({}^m E)$) we let $\|P\|_V = \sup_{\Phi \in V} |\Phi(P)|$ for all $P \in \mathcal{P}({}^m E)$.

LEMMA 1. Let $B \subset U$ and $V \subset (\mathcal{P}({}^m E))'$ then for all $\zeta \in \widehat{B}_U$ (resp. $\widehat{B}_{U,b}$) we have

$$\|\widehat{d}^m f(\zeta)\|_V \leq \sup_{x \in B} \|\widehat{d}^m f(x)\|_V$$

for each integer m and all $f \in \mathcal{H}(U)$ (resp. $\mathcal{H}_b(U)$).

PROOF. Let $\Phi \in V$ and $f \in \mathcal{H}(U)$ (resp. $\mathcal{H}_b(U)$) then by the definition of holomorphic hull

$$|\Phi(\widehat{d}^m f(\zeta))| \leq \sup_{x \in B} |\Phi(\widehat{d}^m f(x))| \leq \sup_{x \in B} \|\widehat{d}^m f(x)\|_V.$$

Hence

$$\sup_{\Phi \in V} |\Phi(\widehat{d}^m f(\zeta))| \leq \sup_{x \in B} \|\widehat{d}^m f(x)\|_V$$

i. e.

$$\|\widehat{d}^m f(\zeta)\|_V \leq \sup_{x \in B} \|\widehat{d}^m f(x)\|_V.$$

LEMMA 2. Let $B \subset U$ and $\alpha > 0$ be such that $B + \alpha B_1 \subset U$. If $f \in \mathcal{H}(U)$ (resp. $\mathcal{H}_b(U)$) is bounded on $B + \alpha B_1$ then f is holomorphic (by analytic continuation if necessary) on $\widehat{B}_U + \alpha B_1$ (resp. $\widehat{B}_{U,b} + \alpha B_1$) and

$$|f(x + y)| \leq \sup_{w \in B + \alpha B_1} |f(w)| \cdot \frac{\alpha}{\alpha - \|y\|}$$

where $x \in \widehat{B}_U$ (resp. $\widehat{B}_{U,b}$) and $\|y\| < \alpha$.

PROOF. By Cauchy's inequalities (see [16], p. 22) we get

$$\sup_{x \in B} \left\| \frac{\widehat{d}^m f(x)}{m!} \right\| \leq \frac{M}{\alpha^m} \text{ for } m = 0, 1, \dots$$

where $M = \sup_{x \in B + \alpha B_1} |f(x)|$. By lemma 1 this implies (taking $V = B_1$)

$$\sup_{x \in \widehat{B}_U \text{ (resp. } \widehat{B}_{U,b})} \left\| \frac{\widehat{d}^m f(x)}{m!} \right\| \leq \frac{M}{\alpha^m}.$$

By the Cauchy-Hadamard formula ([16], P. 10) f is holomorphic (by analytic continuation if necessary) on $\widehat{B}_U + \alpha B_1$ (resp. $\widehat{B}_{U,b} + \alpha B_1$) and for $x \in \widehat{B}_U$

(resp. $\widehat{B}_{U,b}$) and $\|y\| < \alpha$ we have

$$|f(x + y)| \leq \sum_{n=0}^{\infty} \left\| \frac{d^n f(x)}{n!} \right\| \cdot \|y\|^n \leq \sup_{x \in B + \alpha B_1} |f(x)| \cdot \frac{\alpha}{\alpha - \|y\|}.$$

If α is a positive real number and $A \subset E$ we say $A + \alpha B_1$ is an α neighbourhood of A .

LEMMA 3. Let U be a domain of holomorphy (resp. b -domain of holomorphy) and let B be U -bounded. If each $f \in \mathcal{H}(U)$ (resp. $\mathcal{H}_b(U)$) is bounded on some $\alpha(f)$ neighbourhood of B then \widehat{B}_U (resp. $\widehat{B}_{U,b}$) is U -bounded.

PROOF. It is easy to check that \widehat{B}_U (resp. $\widehat{B}_{U,b}$) is always a bounded subset of E so there remains only to show that $d(\widehat{B}_U, \mathcal{C}U) > 0$ (resp. $d(\widehat{B}_{U,b}, \mathcal{C}U) > 0$). If not there exists a sequence, $(\xi_n)_{n=0}^{\infty}$, of elements belonging to B_U (resp. $B_{U,b}$) such that $d(\xi_n, \mathcal{C}U) \rightarrow 0$ as $n \rightarrow \infty$. Lemma 2 implies that every $f \in \mathcal{H}(U)$ (resp. $\mathcal{H}_b(U)$) has an extension to a holomorphic function in some $\alpha(f)$ neighbourhood of each ξ_n . Since U is a domain of holomorphy (resp. b -domain of holomorphy) and $d(\xi_n, \mathcal{C}U) \rightarrow 0$ as $n \rightarrow \infty$ this is a contradiction.

LEMMA 4. Let U be a connected open subset of E such that it is impossible to find two open connected subsets U_1 and U_2 of E with the following properties.

- (1) $U \cap U_1 \supset U_2 \not\equiv \emptyset$ and $U_1 \not\subset U$
- (2) For every $f \in \mathcal{H}_b(U)$ there exists an $f_1 \in \mathcal{H}_b(U_1)$ such that $f = f_1$ on U_2

then $\widehat{B}_{U,b}$ is U -bounded for each U -bounded set B and $d(\widehat{B}, \mathcal{C}U) = d(B_{U,b}, \mathcal{C}U)$.

PROOF. Since $B \subset \widehat{B}_{U,b}$ we get immediately that $d(B, \mathcal{C}U) \geq d(\widehat{B}_{U,b}, \mathcal{C}U)$. Suppose $d(B, \mathcal{C}U) > d(\widehat{B}_{U,b}, \mathcal{C}U)$. Choose $\xi_1 \in \widehat{B}_{U,b}$, $\xi_2 \in \mathcal{C}U$ and $\alpha > 0$ such that $B + \underline{\xi_2 - \xi_1} + \alpha B_1 \subset U$ and is U -bounded where $\underline{\xi_2 - \xi_1}$ denotes the convex balanced hull of $\xi_2 - \xi_1$ (choose ξ_1, ξ_2 such that $\|\xi_2 - \xi_1\| < d(B, \mathcal{C}U)$ and take $0 < \alpha < 1/2 [d(B, \mathcal{C}U) - \|\xi_2 - \xi_1\|]$). For $z \in \widehat{B}_{U,b}$ we get by Cauchy's inequalities and lemma 1 that

$$\left\| \frac{\widehat{d}^n f(z)}{n!} \right\|_{\underline{\xi_2 - \xi_1} + \alpha B_1} \leq \sup_{y \in B} \left\| \frac{\widehat{d}^n f(y)}{n!} \right\|_{\underline{\xi_2 - \xi_1} + \alpha B_1} \leq \sup_{y \in B + \underline{\xi_2 - \xi_1} + \alpha B_1} |f(y)|$$

(this is meaningful when we identify E with a subset of $(\mathcal{P}({}^n E))'$ by means of the mapping $P \rightarrow P(x)$ for all $P \in \mathcal{P}({}^n E)$). Lemma 2 implies that f is holomorphic on the set $\widehat{B}_{U,b} + \underline{\xi_2 - \xi_1} + \alpha B_1$ and for $\alpha_1 < \alpha$ we have

$$\sup_{y \in \widehat{B}_{U,b} + \underline{\xi_2 - \xi_1} + \alpha_1 B_1} |f(y)| < \infty.$$

But $\xi_2 \in (\widehat{B}_{U,b} + \underline{\xi_2 - \xi_1}) \cap \mathcal{C}U$ and this contradicts the hypothesis for U and hence proves the lemma.

The analogous lemma for $\mathcal{H}(U)$ can be proved in a similar fashion and we get

LEMMA 5. Let U be an open connected subset of E such that for each $\xi \in \delta U$ there exists $f \in \mathcal{H}(U)$ which cannot be extended to a holomorphic function in a neighbourhood of ξ . Then the holomorphic hull of each compact subset K of U , \widehat{K}_U , is a compact subset of U and $d(K, \mathcal{C}U) = d(\widehat{K}_U, \mathcal{C}U)$.

In the next two lemmas we concern ourselves with topologies on $\mathcal{H}(U)$ (see [5], [7] and [16]). Let T_0 denote the compact open topology on $\mathcal{H}(U)$ and let T denote the bornological topology associated with T_0 (see [7]). Since T_0 is complete (see [16]), T is also complete and barrelled.

LEMMA 6. Let $x_n \in U$ for $n = 1, 2, \dots$ and suppose $\sup_n |f(x_n)| < \infty$ for all $f \in \mathcal{H}(U)$ then $p(f) = \sup_n |f(x_n)|$ defines a continuous semi-norm on $(\mathcal{H}(U), T)$.

PROOF. Let $B_p = \{f; p(f) \leq 1\}$. Then B_p is absorbing (by hypothesis) and since $B_p = \bigcap_p \{f; |f(x_n)| \leq 1\}$ it is closed and convex. Since $(\mathcal{H}(U), T)$ is barrelled this completes the proof.

LEMMA 7. Let p be a continuous semi-norm on the space $(\mathcal{H}(U), T)$ and suppose $(V_n)_{n=1}^\infty$ is an increasing sequence of open subsets of U such that $\bigcup_{n=1}^\infty V_n = U$ then there exists a positive integer n_0 and $C > 0$ such that

$$p(f) \leq C \sup_{x \in V_{n_0}} |f(x)|.$$

PROOF. Suppose the result is not true. Then for each positive integer n we can choose $f_n \in \mathcal{H}(U)$ such that $p(f_n) \geq n$ and $\sup_{x \in V_n} |f_n(x)| \leq 1/n$.

The sequence $(f_n)_{n=1}^\infty$ is a bounded subset of $(\mathcal{H}(U), T_0)$ and consequently of $(\mathcal{H}(U), T)$ but $p(f_n) \rightarrow \infty$ as $n \rightarrow \infty$ which is a contradiction.

SECTION 2.

THEOREM (Cartan-Thullen I). *Let U be a connected open subset of the Banach space E . For the properties listed below we have (1) \implies (2) \iff (3) \iff (4) \iff (5) \iff (6). (2) \implies (1) if E is separable but this does not hold for arbitrary E .*

- (1) U is a b -domain of holomorphy
- (2) For each $\xi \in \delta U$ there exists $f \in \mathcal{H}_b(U)$ which cannot be extended analytically to a neighbourhood of ξ .
- (3) It is impossible to find two open connected subsets U_1 and U_2 of E such that:

$$(a) U \cap U_1 \supset U_2 \neq \emptyset \text{ and } U \not\supset U_1$$

- (b) For every $f \in \mathcal{H}_b(U)$ there exists an $f_1 \in \mathcal{H}_b(U_1)$ such that $f = f_1$ on U_2

(4) If B is U -bounded then $\widehat{B}_{U,b}$ is U -bounded and $d(B, \mathcal{C}U) = d(\widehat{B}_{U,b}, \mathcal{C}U)$.

(5) If B is U -bounded then $\widehat{B}_{U,b}$ is U -bounded.

(6) For each sequence $(\xi_n)_{n=1}^\infty$ of elements of U such that $\xi_n \rightarrow \xi \in \delta U$ as $n \rightarrow \infty$ there exists $f \in \mathcal{H}_b(U)$ such that $\sup_n |f(\xi_n)| = \infty$.

PROOF. (1) \implies (2) \implies (3), (4) \implies (5) and (6) \implies (2) are obvious.

Suppose (5) holds and (6) does not. Then there exists a sequence, $(\xi_n)_{n=0}^\infty$, of elements of U such that $\xi_n \rightarrow \xi \in \delta U$ as $n \rightarrow \infty$ and $\sup_n |f(\xi_n)| < \infty$ for all $f \in \mathcal{H}_b(U)$.

Now $\mathcal{H}_b(U)$ endowed with the topology of uniform convergence on U -bounded sets is a Frechet space and the mapping $f \rightarrow f(\xi_n)$ ($f \in \mathcal{H}_b(U)$) is continuous with respect to this topology for each n . Hence $p(f) = \sup_n |f(\xi_n)|$ defines a continuous semi-norm on $\mathcal{H}_b(U)$ and thus there exists $B \subset U$, U -bounded and $C > 0$ such that

$$\sup_n |f(\xi_n)| \leq C \sup_{x \in B} |f(x)|$$

for all $f \in \mathcal{H}_b(U)$. By using the fact that $\mathcal{H}_b(U)$ is an algebra we easily show that C can be taken equal to 1.

Therefore $\xi_n \in \widehat{B}_{U,b}$ for each n which contradicts the fact that $\widehat{B}_{U,b}$ is U -bounded. Hence (5) \implies (6). (3) \implies (4) by lemma 4.

In [12] there is an example of a Banach space whose unit ball is not a b -domain of holomorphy. This gives an example in which (2) \Rightarrow (1).

We complete the proof by showing (4) \Rightarrow (1) if E is separable. Since E is separable, U contains a countable dense subset M . Let $(\xi_n)_{n=2}^\infty$ be a sequence of elements in M containing each point in M infinitely often. For each ξ_n let A_n be the open ball with centre ξ_n and radius $d(\xi_n, \mathcal{C}U)$. If (4) holds we can construct by induction a sequence $(B_n)_{n=2}^\infty$ of U -bounded sets and a sequence $(z_n)_{n=2}^\infty$ of points of U with the following properties:

- 1) $B_2 = \xi_2 = z_2$
- 2) $z_{n+1} \in A_n \cap \widehat{B}_{n, U, b}$ $z_{n+1} \in B_{n+1}$
- 3) B_n is an increasing sequence and each U -bounded set is contained

in some B_n . (This is possible since $d(A_n, \mathcal{C}U) = 0$ and $d(\widehat{B}_{n, U, b}, \mathcal{C}U) > 0$). By construction we can choose $f_n \in \mathcal{H}_b(U)$ such that $\sup_{x \in B_n} |f_n(x)| \leq 1/2^n$ and $|f_n(z_{n+1})| > 2^n + |\sum_{i=1}^{n-1} f_i(z_{n+1})|$. Let $f = \sum_{n=2}^\infty f_n$. Since $f_n \in \mathcal{H}_b(U)$ for all n and $\sum_{n=m+1}^\infty \|f_n\|_{B_m} \leq \sum_{n=m+1}^\infty 1/2^n < \infty$ for each m , we have $f \in \mathcal{H}_b(U)$. Also

$$|f(z_n)| \geq |f_{n-1}(z_n)| - |\sum_{i=1}^{n-2} f_i(z_n)| - |\sum_{i=n}^\infty f_i(z_n)| \geq 2^{n-1} - \sum_{i=n}^\infty 1/2^i \geq 2^{n-1}.$$

Hence $f(z_n) \rightarrow \infty$ as $n \rightarrow \infty$. This implies that if $\xi \in \delta U$ and $\varepsilon > 0$ is arbitrary

$$\sup_{x \in (\xi + \varepsilon B_1) \cap U} |f(x)| = \infty.$$

Thus f has no extension as a holomorphic function to a larger subset of E than U . This completes the proof.

As regards $\mathcal{H}(U)$ we do not know if the converse to lemma 5 is true even if E is separable. We now prove some results about $\mathcal{H}(U)$ similar to theorem 1.

THEOREM (Cartan-Thullen II).

Let U be an open connected subset of the separable Banach space E then the following are equivalent

- (1) U is a domain of holomorphy
- (2) There exists an increasing sequence of U -bounded sets, $(B_n)_{n=2}^\infty$, such that $B_n = \widehat{B}_{n, U}$ and each compact set is contained in the interior of some B_n .

PROOF. Suppose (1) is true; then there exists $f \in \mathcal{H}(U)$ which has U as its natural domain of existence. For each compact subset K of U choose $\alpha(K) > 0$ such that f is bounded on $K + 2\alpha(K) B_1$. The sets $K + \alpha(K) B_1$

cover U as K ranges over all compact subsets of U . Since E is separable we can choose a sequence $(K_n)_{n=2}^\infty$ of compact subsets of U such that $U = \bigcup_{n=2}^\infty \{K_n + \alpha(K_n) B_1\}$. Now let $\beta_n = \inf_{i \leq n} \alpha(K_i)$ and define $B_n = \bigcup_{i=2}^n \{K_i + \alpha(K_i) B_1\}$. Then B_n is an increasing sequence of U -bounded sets and f is bounded on $B_n + \beta_n B_1$. Lemma 2 implies that f is holomorphic (by analytic continuation if necessary) on $\widehat{B}_{n,U} + \beta_n B_1$. Since f has U as its natural domain of existence this implies $d(\widehat{B}_{n,U}, \mathcal{C}U) \geq \beta_n > 0$. Since B_n is U -bounded for each n this means that $\widehat{B}_{n,U}$ is U -bounded for each n . Each compact subset of U is easily seen to be contained in the interior of some B_n . To complete the proof we note that the sequence $\widehat{B}_{n,U}$ has all the required properties.

(2) \implies (1) This is quite similar to the proof that (4) \implies (1) in the previous theorem. Let $(B_n)_{n=2}^\infty$ be the sequence of U -bounded given by hypothesis. Let M be a countable dense subset of U and take $(\xi_n)_{n=2}^\infty$ as a sequence of elements in M containing each point in M infinitely often. For each ξ_n let A_n be the open ball with centre ξ_n and radius $d(\xi_n, \mathcal{C}U)$. Let $C_2 = B_2$ and choose $z_2 \in A_2 \cap \mathcal{C}B_2$ (this is possible since $d(A_2, \mathcal{C}U) = 0$ and B_2 is U -bounded).

Suppose C_2, \dots, C_n and z_2, \dots, z_n have been chosen. Choose k_{n+1} such that $B_{k_{n+1}} \supset C_n, z_n \in B_{k_{n+1}}$. Let $C_{n+1} = B_{k_{n+1}}$ and choose $z_{n+1} \in A_{n+1} \cap \mathcal{C}C_{n+1}$. For each n there exists $f_n \in \mathcal{H}(U)$ such that

$$\sup_{x \in B_n} |f_n(x)| < 1/2^n \text{ and } |f_n(z_n)| > 2^n + |\sum_{i=2}^n f_i(z_n)|.$$

The function $f = \sum_{n=2}^\infty f_n \in \mathcal{H}(U)$ and has U as its natural domain of existence. Hence (2) \implies (1).

The example quoted in theorem 1 also shows that the separability condition was essential in theorem II.

THEOREM (Cartan-Thullen III).

Let U be a connected open subset of a Banach space E then the following are equivalent:

(1) For each $\xi \in \delta U$ there exists $f \in \mathcal{H}(U)$ which cannot be extended to a holomorphic function in a neighbourhood of ξ .

(2) For each sequence, $(\xi_n)_{n=1}^\infty$, of elements of U which converges to some point in δU there exists $f \in \mathcal{H}(U)$ such that $\sup_n |f(\xi_n)| = \infty$.

PROOF. (2) \implies (1) is obvious. We now show if (2) is not true then (1) is not true.

If (2) is not true there exists a sequence, $(\xi_n)_{n=1}^\infty$, of elements of U which converges to $\xi \in \delta U$ and such that $\sup_n |f(\xi_n)| < \infty$ for all $f \in \mathcal{H}(U)$.

By lemma 6 $p(f) = \sup_n |f(\xi_n)|$ is a continuous semi-norm on $(\mathcal{H}(U), T)$. Let $f \in \mathcal{H}(U)$ be arbitrarily chosen. For each positive integer n let

$$U_n = \{x, |f(x)| < n\} \text{ and take}$$

$$V_n = \{x; x \in U_n \text{ and } d(x, \delta U_n) < 1/n\}.$$

$(V_n)_{n=1}^\infty$ is an increasing sequence of subsets of U and $\bigcup_{n=1}^\infty V_n = U$. By lemma 7 there exists n_0 and C such that

$$p(f) = \sup_n |f(\xi_n)| \leq C \sup_{x \in V_{n_0}} |f(x)|.$$

Since $\mathcal{H}(U)$ is an algebra we can take $C = 1$ and hence $\xi_n \in \widehat{V}_{n_0, U}$ for each n . By our choice of the V_n 's, f is bounded on a $1/n_0$ neighbourhood of V_{n_0} .

Lemma 2 implies f can be continued as a holomorphic function in a $1/n_0$ neighbourhood of ξ_n for each n and hence in some neighbourhood of ξ . This completes this proof.

If the closed bounded subsets of E' (the dual of E) are weak*-sequentially compact condition (2) of the last theorem can be replaced by the following equivalent condition (see [8]).

(2') For each sequence, $(\xi_n)_{n=1}^\infty$, of elements of U which has no limit point in U there exists $f \in \mathcal{H}(U)$ such that

$$\sup_n |f(\xi_n)| = \infty.$$

In particular (2') \iff (2) if E is separable or reflexive.

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