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HARDY SPACES OF ALMOST PERIODIC FUNCTIONS

by J. P. MILASZEWICZ

1. Introduction.

In the classic context, Hardy spaces can be defined in two equivalent ways: first as the subspaces of L^p of the unit circle with vanishing negative Fourier coefficients; second as spaces of holomorphic functions in the unit disc such that the p-th means in each circle of radius less than one are uniformly bounded.

One departure point for a generalization is to substitute Z, the integers, by any subgroup g of the real line with the discrete topology. This leads to the construction of the so called « big disc » (the closed unit disc if $g = Z$) and of a generalized Poisson kernel. As in the classic context two directions can be chosen to define the Hardy classes; one dealing with the « boundary » of this generalized disc, that is the dual group of g (see [X]), and another dealing with its « interior » (see [VIII]).

This note deals mainly with the second definition. In the first part we investigate in which measure some classic results can be extended as Herglotz's theorem ((2.3.8)). Afterwards we give a description of the Hardy classes as spaces of analytic almost periodic functions on a half plane, following an idea that can be found in [III] ((4.2.5)).

Finally, we show that the two constructions mentioned above lead to different concepts unless $g = Z$ ((5.2.1)).

2. Basic notations and preliminaries.

(2.1) g will indicate a subgroup of the real line $\mathbb{R}$ with the discrete topology; $g_+ = \{x \in G / x \geq 0\}$; $L^1(g)$ is the group algebra of g of complex

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valued integrable functions with respect to Haar measure of g, that is the absolutely summable series with indexes in g.

$$A_1 = L^1(g_+) = \{f \in L^1(g)/f(x) = 0 \quad \forall\, x \notin g_+\}$$

$$\Gamma = \widehat{g} = \{\chi : g \rightarrow T/\chi(x+y) = \chi(x)\cdot\chi(y)\}$$

$$T = \{z \in C/\,|\,z\,| = 1\}.$$

A_1 is a closed ideal of $L^1(g)$. Γ is the dual group of g; if we give Γ the compact-open topology it is a locally compact group which can be indentified with the maximal ideal space of $L^1(g)$ ([IX]-§ 34). With $*$ we shall indicate the convolution product in Γ. In the present situation, the existence of the identity element in $L^1(g)$ implies that Γ is compact.

$$\bar{D} = \{z \in C/|z| \leq 1\}, \quad D = \{z \in C/|\,z\,| < 1\}$$

$$\bar{\Delta} = \{\chi : g_+ \rightarrow \bar{D}/\chi(x+y) = \chi(x)\cdot\chi(y),\ \chi \neq 0\}$$

$\bar{\Delta}$ is a semigroup with respect to pointwise product; if we endow $\bar{\Delta}$ with the compact-open topology, it can be identified with the maximal ideal space of the algebra A_1; the identification is established in the following way: if we call $H(A_1)$ the maximal ideal space of A_1, we define

$$h : \bar{\Delta} \rightarrow H(A_1) \quad \text{by} \quad \langle h(\xi), f\rangle = \int_g f(x)\langle \xi, x\rangle\, dx$$

dx indicates the Haar measure of g and $\langle \xi, x\rangle =$ value of ξ on x. The function h is a homeomorphism when we give $H(A_1)$ the weak* topology induced by the Gelfand transform. The group $\Gamma = H(L'(g))$ is imbedded homeomorphically in $\bar{\Delta}$ in the natural way. We shall call $\Delta = \bar{\Delta} - \Gamma$. If we denote $\widehat{f}$ the Gelfand transform of $f \in L^1(g)$ we have

$$\widehat{f}(\xi) = \langle h(\xi), f\rangle = \int_G f(x)\langle \xi, x\rangle\, dx \tag{2.1.1}$$

$\widehat{f} : \bar{\Delta} \rightarrow C$ is continuous by the definition of the topology in $\bar{\Delta}$; the Shilov boundary of the algebra $\widehat{A}_1$ of Gelfand transforms in $\bar{\Delta}$ is exactly Γ. For each $\xi \in \bar{\Delta}$ there is a unique real regular Baire measure representing it ([I]-

7.1 and [II]·4.8)

(2.1.2) $$\widehat{f}(\xi) = \langle h(\xi), f \rangle = \int \widehat{f}(\alpha)\, dm_\xi(\alpha),\ f \in L^1(g_+).$$

The convolution of representing measures is the representing measure of the product i. e.

(2.1.3) $$m_\xi * m_\eta = m_{\xi \cdot \eta}\ \ \xi, \eta \in \bar{\Delta}.$$

In $\bar{\Delta}$ there is a distinguished point, namely

$$\langle \xi_0, x \rangle = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0. \end{cases}$$

This point plays the role of the center of $\bar{\Delta}$; its representing measure is the Haar normalized measure of Γ. $\xi \in \bar{\Delta}, \xi \neq \xi_0 \Longrightarrow \xi = r \cdot \alpha$ in a unique way; $r : g_+ \longrightarrow [0, 1]$ $r \in \bar{\Delta}$ and $\alpha \in \Gamma$. If $f \in C(\Gamma)$ we have

(2.1.4) $$\int f(\beta)\, dm_\xi(\beta) = \iint f(\beta_1 \cdot \beta_2)\, dm_r(\beta_1)\, dm_\alpha(\beta_2) =$$
$$= \int f(\beta_1 \alpha)\, dm_r(\beta_1) = (f * m_r)(\alpha)$$

because of the symmetry of m_r and the fact that m_α is the unit mass measure concentrated on α. Remark that formula (2.1.4) is still valid if $\xi = \xi_0$. For the proofs of the foregoing statements see [II]. The mapping $i : [0,1] \longrightarrow \bar{\Delta}$ defined by $\langle i(r), x \rangle = r^x$ for $r > 0$ and $i(0) = \xi_0$ is a homeomorphism between $[0, 1]$ and the set of positive elements of $\bar{\Delta}$.

(2.1.5) We have $\widehat{f}(\xi_0) = \langle h(\xi_0), f \rangle = \int \widehat{f}(\alpha)\, d\alpha\ (f \in A_1)$ $d\alpha$ indicates the Haar normalized measure of Γ; we also note it dm. The mapping $\xi \longrightarrow m_\xi$ from $\bar{\Delta}$ into $C(\Gamma)^* =$ dual space of $C(\Gamma)$, is a continuous homeomorphism with its image when $C(\Gamma)^*$ is endowed with the weak* topology.

Thus calling $m_{i(r)} = m_r$ $r \in [0, 1]$ we have that the family of measures $(m_r)_{0 \le r < 1}$ is such that

(2.1.6) $$\langle m_r, f \rangle = \int f(\alpha)\, dm_r(\alpha) \xrightarrow[r \to 1]{} f(e_1)$$

$e_1 =$ identity element of Γ $(f \in C(\Gamma))$.

In the case $g = Z$ we get the Poisson kernel.

As a consequence of the preceeding remarks we get that if U is a neighborhood of e_1 then $m_r(U) \xrightarrow[r \to 1]{} 1$.

(2.1.7) We also have for $f \in C(\Gamma)$ that $\langle m_r, f \rangle \xrightarrow[r \to 0]{} \int f(\alpha)\, d\alpha$.

(2.2) Let $f \in A_1$, $\widehat{f}: \overline{\Delta} \to C$ be its Gelfand transform $\widehat{f}(\xi) = \int \widehat{f}(\alpha\beta)\, dm_r(\beta)$ where $\xi = r\alpha$ (see (2.1.4)).

(2.2.1) LEMMA: if $x \in g$ then $\int \langle \alpha, x \rangle\, dm_r(\alpha) = r^{|x|}$ $0 < r < 1$.

PROOF: consider first $x \geq 0$ and define

$$f_x \in A_1 \quad f_x(y) = \begin{cases} 1 & \text{if} \quad y = x \\ 0 & \quad\;\; y \neq x \end{cases}$$

then $\widehat{f}(\alpha) = \int_g f(y) \langle \alpha, y \rangle\, dy = \langle \alpha, x \rangle$ (see (2.1.1)). Applying (2.1.3) we get

$$\widehat{f}(r) = \int \langle \alpha, x \rangle\, dm_r(\alpha)$$

using again (2.1.1) we obtain

$$\widehat{f}(r) = \int_g f(y) \langle r, y \rangle\, dy = r^x$$

The fact that m_r is a real measure gives

$$\overline{\int \langle \alpha, x \rangle\, dm_r(\alpha)} = \int \langle \alpha, -x \rangle\, dm_r(\alpha) = r^{-x}$$

if $x < 0$ and this completes the proof.

If we consider the uniform closure of $\widehat{A_1}$ we obtain a Banach subalgebra of $C(\overline{\Delta})$ which we shall call A_0. The Shilov boundary for A_0 is still Γ; formula (2.1.4) also holds for functions in A_0; it is possible to extend it slightly as follows:

(2.2.2) If $r_1 < r_2 \leq 1$ then $f(r_1 \beta) = \int f(r_2 \alpha\beta)\, dm_{r_1/r_2}(\alpha)$. Following [II] we shall call functions in A_0 generalized analytic functions in $\bar{\Delta}$.

(2.2.3) LEMMA: Let $f: \Gamma \to C$ be a continuous function; f is the restriction to Γ of a function F in A_0 iff

$$\int f(\alpha)\, \overline{\langle \alpha, x \rangle}\, d\alpha = 0 \qquad \forall\, x < 0.$$

Besides,

$$F(r\alpha) = \int f(\alpha \cdot \beta)\, dm_r(\beta).$$

PROOF: [VIII]-(6.1).

(2.2.4) DEFINITION: given $F: \Delta \to C$ we shall call $F_r: \Gamma \to C$ the restriction of F to $r.\ \Gamma = \{r \cdot \alpha \mid \alpha \in \Gamma\}$ $(r < 1)$.

(2.2.5) DEFINITION: given $F: \Delta \to C$ we say that $F_r \in A_0$ if F_r is the restriction to Γ of some function in $A_0\,(r < 1)$.

(2.2.6) DEFINITION: for $F: \Delta \to C$ we say that F is generalized harmonic (in the following g. h.) if

(i) F is continuous

(ii) $r_1 < r_2 < 1 \Longrightarrow F_{r_1} = F_{r_2} * m_{r_1/r_2}$.

We say that F is generalized analytic (g. a. in the following) if

(iii) F is g. h. and for some $0 < r < 1$ $F_r \in A_0$. Remark that if (ii) holds and F_r is continuous for each $r < 1$ then F is continuous.

(2.2.7) LEMMA: $F: \Delta \to C$ is g. a. iff $F_{r'} \in A_0 \quad \forall\, r' < 1$.

PROOF: suppose $F_r \in A_0$; if $r' \leq r$.

$$\widehat{F}_{r'}(x) = \widehat{F_r * m_{r'/r}}(x) = \widehat{F}_r(x) \cdot \widehat{m}_{r'/r}(x) = \widehat{F}_r(x) \left(\frac{r'}{r}\right)^{|x|} \qquad \text{(see (2.2.1))}$$

if $r' > r$, analogously

$$\widehat{F}_r(x) = \widehat{F}_{r'}(x) \cdot \widehat{m}_{r/r'}(x) = \widehat{F}_{r'}(x) \cdot \left(\frac{r}{r'}\right)^{|x|}$$

(2.2.3) yields the desired conclusion.

(2.2.8) Corollary: Let $F: \Delta \longrightarrow C$ be g. h.; then if $r_1 < r_2 < 1$ we have

$$\widehat{F}_{r_1}(x) \cdot r_2^{|x|} = \widehat{F}_{r_2}(x)\, r_1^{|x|} .$$

(2.3) Let be $r_0 = e^{-1}$ and $1 \le p \le \infty$ $\left(e = \sum_{n \le 0} \frac{1}{n!}\right)$.

(2.3.1) Definition: if $p < \infty$

$$h^p(\Delta) = \left\{ F: \Delta \longrightarrow C/F \text{ is g. h. and } \sup_{\substack{r<1 \\ \beta \in \Gamma}} \int |F(r\alpha\beta)|^p \, dm_{r_0}(\alpha) < \infty \right\}$$

$$H^p(\Delta) = \{F \in h^p(\Delta)/F \text{ is g. a.}\}$$

if $p = \infty$

$$h^\infty(\Delta) = \{F: \Delta \longrightarrow C/F \text{ is g. h. and bounded}\}$$

$$H^\infty(\Delta) = \{F \in h^\infty(\Delta)/F \text{ is g. a.}\}.$$

(2.3.2) Theorem: Let $F \in h^p(\Delta)$ with $1 < p \le \infty$ or $F \in H^1(\Delta)$; then $\lim_{r \to 1} F(r \cdot \alpha)$ exists except on a set of zero m_ξ-measure for each $\xi \in \Delta$. The limit function F_1 belongs to $\bigcap_{\xi \in \Delta} L^p(\Gamma, m_\xi)$ and for $p < \infty$

$$F_r \xrightarrow[r \to 1]{L^p(\Gamma, m_\xi)} F_1 \qquad \forall\, \xi \in \Delta$$

for

$$p = \infty \quad \|F_r\|_{L^\infty(\Gamma, m_\xi)} \xrightarrow[r \to 1]{} \|F_1\|_{L^\infty(\Gamma, m_\xi)} .$$

We have also the Poisson integral formula

$$F(r\alpha) = \int F_1(\alpha \cdot \beta)\, dm_r(\beta) = (F_1 * m_r)(\alpha). \tag{2.3.3}$$

Proof: [VIII] 5.1.

(2.3.4) Lemma: Let F be g. h. and $1 \le p < \infty$; then if $r_1 < r_2 < 1$

$$\sup_{\substack{r<1 \\ \beta \in \Gamma}} \int |F(r\,\alpha\beta)|^p \, dm_{r_1}(\alpha) \le \sup_{\substack{r<1 \\ \beta \in \Gamma}} \int |F(r\,\alpha\beta)|^p \, dm_{r_2}(\alpha)$$

Proof: fix $r^1 < 1$ and $\beta^1 \in \Gamma$

$$\int |F(r^1\,\alpha\beta)|^p \, dm_r(\alpha) = \iint |F(r^1\,\alpha_1\,\alpha_2\,\beta)|^p \, dm_{r_1/r_2}(\alpha_1) \cdot dm_{r_2}(\alpha_2) =$$

$$= \iint | F(r^1 \alpha_1 \alpha_2 \beta) |^p \, dm_{r_2}(\alpha_2) \, dm_{r_1/r_2}(\alpha_1) \leq$$

$$\leq \sup_{\substack{r<1 \\ \beta \in \Gamma}} \int | F(r \alpha\beta) |^p \, dm_{r_2}(\alpha) .$$

(2.3.5) Corollary: Let F be g. h. and $1 \leq p < \infty$, $r_2 < 1$; then

$$\sup_{r<1} \int | F(r\alpha) |^p \, d\alpha \leq \sup_{\substack{\beta \in \Gamma \\ r<1}} \int | F(r \alpha\beta) |^p \, dm_{r_2}(\alpha)$$

Proof: apply (2.3.4) with $r_1 = 0$.

(2.3.6) Definition: Let $F \in h^p(\Delta)$; if $p < \infty$

$$\| F \|_p^p = \sup_{\substack{r<1 \\ \beta \in \Gamma}} \int | F(r \alpha\beta) |^p \, dm_{r_0}(\alpha) \quad (r_0 = e^{-1})$$

$$\| F \|_\infty = \sup_{\xi \in \Delta} | F(\xi) | .$$

(2.3.7) Proposition: $(h^p(\Delta), \| \ \|_p)$ is a normed space

(2.3.8) Theorem: if $F \in h^1(\Delta)$ then there exists a Baire regular measure $\mu_F \in C(\Gamma)^*$ such that

$$F_r = \mu_F * m_r \text{ for } 0 \leq r < 1.$$

We intend equality in the sense of measures

Proof: Let $r_n \nearrow 1$, $r_1 \neq 0$; consider the measures associated to F_{r_n} i.e. if $f \in C(\Gamma)$

$$\langle F_{r_n}, f \rangle = \int F_{r_n}(\alpha) f(\alpha) \, d\alpha$$

(2.3.5) yields

$$| \langle F_{r_n}, f \rangle | \leq \| f \|_\infty \cdot \| F \|_1 .$$

A general compactness argument says that any closed ball of a dual space is weak* compact. Thus we have that there exists a subsequence n_k and a measure $\mu_F \in C(\Gamma)^*$ such that

$$F_{r_{n_k}} \xrightarrow[k \to \infty]{w^*} \mu_F \text{ i. e.}$$

$$\langle F_{r_{n_k}}, f \rangle \xrightarrow[k \to \infty]{} \langle \mu_F, f \rangle \quad \forall f \in C(\Gamma).$$

We clain that $F_r = \mu_F * m_r \quad 0 \leq r < 1$.
For this we shall prove that both members have the same Fourier-Stieltjes transform and this will complete the proof for then, they take the same values on trigonometric polynomials in Γ, and then in all of $C(\Gamma)$ because trigonometric polynomials are dense in $C(\Gamma)$ (see [IX], § 38).

Let $x \in g$; we want to show that

$$\widehat{F}_r(x) = \int \langle \overline{\alpha, x} \rangle F_r(\alpha)\, d\alpha = \int \langle \overline{\alpha, x} \rangle\, d(\mu_F * m_r)(\alpha) = (\widehat{\mu_F * m_r})(x).$$

We shall use that for $s_1 < s_2 < 1$

$$\widehat{F}_{s_1}(x)\, s_2^{|x|} = \widehat{F}_{s_2}(x)\, s_1^{|x|} \qquad \text{(see (2.2.8))}.$$

Consider $0 < r < 1$

$$(\widehat{m_r * \mu_F})(x) = \iint \langle \overline{\beta_1, x} \rangle \langle \overline{\beta_2, x} \rangle\, dm_r(\beta_1)\, d\mu_F(\beta_2) =$$

$$= r^{|x|} \int \langle \overline{\beta_2, x} \rangle\, d\mu_F(\beta_2) = r^{|x|} \operatorname*{limit}_{k \to \infty} \int F_{r_{n_k}}(\beta) \langle \overline{\beta, x} \rangle\, d\beta =$$

$$= r^{|x|} \operatorname{limit} F_{r_{n_1}} \left(\frac{r_{n_k}}{r_{n_1}}\right)^{|x|} = r^{|x|} \widehat{F}_{r_{n_1}}(x) \left(\frac{1}{r_{n_1}}\right)^{|x|} = r^{|x|} \widehat{F}_r(x) \left(\frac{1}{r}\right)^{x} = \widehat{F}_r(x).$$

Take now $r = 0$

$$(\widehat{m_0 * \mu_F})(x) = \widehat{m}_0(x) \cdot \widehat{\mu}_F(x) = \operatorname*{limit}_{r \to 0} \widehat{m}_r(x) \cdot \widehat{\mu}_F(x) =$$

$$= \operatorname*{limit}_{r \to 0} (\widehat{m_r * \mu_F})(x) = \operatorname*{limit}_{r \to 0} \widehat{F}_r(x) = F(\xi_0) \int \langle \overline{\beta, x} \rangle\, d\beta = \widehat{F(\xi_0)}(x).$$

3. The Banach algebra structure for $h^p(\Delta)$.

(3.1) We have already observed that $h^p(\Delta)$ is a normed linear space in (2.3.7)

(3.1.1) THEOREM: $(h^p(\Delta), \| \ \|_p)$ is a Banach space; $H^p(\Delta)$ is a closed subspace of $h^p(\Delta)$.

PROOF: suppose first that $p = \infty$ and let F_n be a Cauchy sequence in $h^\infty(\Delta)$. It is clear that there exists $F: \Delta \to \mathbb{C}$ continuous such that

$$\| F_n - F \|_\infty \xrightarrow[n \to \infty]{} 0 .$$

Let $r_1 < r_2 < 1$

$$F_{r_2} * m_{r_1/r_2} = \operatorname*{limit}_{n \to \infty} (F_n)_{r_2} * m_{r_1/r_2} = \operatorname*{limit}_{n \to \infty} (F_n)_{r_1} = F_{r_1} .$$

In the case $(F_n)_r \in A_0 \ \forall\, r < 1$ it is clear that $F_r \in A_0 \ \forall\, r < 1$.

Suppose now that $p < \infty$; let $r_1 < r_2 < 1$ and $r \leq r_1$

$$| F_n(r\alpha) - F_m(r\alpha) | = \left| \int \left[F_n\left(\frac{r}{r_2}\alpha\beta\right) - F_m\left(\frac{r}{r_2}\alpha\beta\right) \right] dm_{r_2}(\beta) \right| \leq$$

$$\leq \left[\int \left| F_n\left(\frac{r}{r_2}\alpha\beta\right) - F_m\left(\frac{r}{r_2}\alpha\beta\right) \right|^p dm_{r_2}(\beta) \right]^{1/p} .$$

We know that this converges to zero as m and n tend to infinity if $r_2 = r_0$ by hypothesis. To see that this holds in this case independently of $r_1 \leq r_2$ we observe that m_{r_2} and m_{r_0} are mutually absolutely continuous and that the respective Radon-Nykodym derivatives are bounded everywhere (see [VIII] 2.23). Thus there exists a constant $K(r_0; r_2)$ such that

(3.1.2) $$\int |f(\beta)| \, dm_{r_2}(\beta) \leq K(r_0, r_2) \int |f(\beta)| \, dm_{r_0}(\beta) \ \forall f \in C(\Gamma)$$

and then

$$| F_n(r\alpha) - F_m(r\alpha) | \leq K(r_0, r_2)^{1/p} \| F_n - F_m \|_p .$$

Thus F_n restricted to $r_1 . \overline{\Delta} = \{ r_1 \xi : \xi \in \overline{\Delta} \}$ is uniformly fundamental, and this implies that F_n converges uniformly on compact subsets of Δ. So we obtain $F: \Delta \to C$ which is continuous; analogously to the case $p = \infty$ it can be shown that F is g. h. (g. a. if the F_n are g. a.).

Let us see that F_n converges to F in $h^p(\Delta)$; given $\varepsilon > 0$ there exists $n_0(\varepsilon)$ such that

$$n, m \geq n_0(\varepsilon) \Longrightarrow \| F_n - F_m \|_p < \varepsilon ;$$

suppose that $r < 1$ and $\beta \in \Gamma$ are fixed

$$\left[\int | F(r\alpha\beta) - F_n(r\alpha\beta) |^p \, dm_{r_0}(\alpha) \right]^{1/p} =$$

$$= \underset{m \to 0}{\text{limit}} \left[\int | F_m(r\alpha\beta) - F_n(r\alpha\beta) |^p \, dm_{r_0}(\alpha) \right]^{1/p} \leq \varepsilon$$

if $n \geq n_0(\varepsilon)$: this implies that $F \in h^p(\Delta)$ and $F_n \xrightarrow{\| \, \|_p} F$.

3.2. We proceed now to define the product between g.h. functions

(3.2.1) LEMMA: let $F, G : \Delta \to C$ be g. h. functions; if $r < 1$ and

$$r = r_1 \cdot r_2 = s_1 \cdot s_2$$

we have

$$(F_{r_1} * G_{r_2})(\alpha) = (F_{s_1} * G_{s_2})(\alpha).$$

PROOF: suppose $r_1 < s_1$

$$F_{r_1} * G_{r_2} = (F_{s_1} * m_{r_1/s_1}) * G_{r_2} = F_{s_1} * G_{r_2 \cdot r_1/s_1} = F_{s_1} * G_{s_2} .$$

This equality holds almost every where with respect to Haar measure on Γ but because of the continuity of the functions involved, equality holds everywhere.

(3.2.2) DEFINITION: given F, G g. h. functions we define their Hadamard product $F \circledast G : \Delta \to C$ in the following way

$$(F \circledast G)(r\alpha) = (F_{r_1} * G_{r_2})(\alpha) \text{ where } r_1 \cdot r_2 = r.$$

It is clear that, because of (3.2.1), this definition does not depend on the decomposition of r as product of two numbers less than 1.

(3.2.3) LEMMA: if F and G are g.h. then $F \circledast G$ is g.h.; if G is g.a. then $F \circledast G$ is g.a.

PROOF: we need only to prove that (2.2.6)-(ii) holds because of the remark in (2.2.6)

$$(F \circledast G)_{r_2} * m_{r_1/r_2} = (F_{\sqrt{r_2}} * G_{\sqrt{r_2}}) * \left(m_{\sqrt{\frac{r_1}{r_2}}} * m_{\sqrt{\frac{r_1}{r_2}}}\right) =$$

$$= \left(F_{\sqrt{r_2}} * m_{\sqrt{\frac{r_1}{r_2}}}\right) * \left(G_{\sqrt{r_2}} * m_{\sqrt{\frac{r_1}{r_2}}}\right) = F_{\sqrt{r_1}} * G_{\sqrt{r_1}} = (F \circledast G)_{r_1}.$$

In the case G is g.a. then

$$(\widehat{F \circledast G})_r(x) = (\widehat{F_{\sqrt{r}} * G_{\sqrt{r}}})(x) = \widehat{F}_{\sqrt{r}}(x) \cdot \widehat{G}_{\sqrt{r}}(x)$$

vanishes whenever $G_{\sqrt{r}}(x)$ vanishes; (2.2.3) implies that $F \circledast G$ is g.a.

(3.2.4) THEOREM: $h^1(\Delta) * h^p(\Delta) \subset h^p(\Delta)\ 1 \leq p \leq \infty$; moreover

$$\| F \circledast G \|_p \leq \| F \|_1 \, \| G \|_p \quad \text{for} \quad F \in h^1(\Delta),\ G \in h^p(\Delta)$$

PROOF: the case $p = \infty$ offers no difficulty so we can suppose $p < \infty$ Let $0 < r^2 < 1$ and $\beta \in \Gamma$

$$\int |(F \circledast G)(r^2 \alpha\beta)|^p \, dm_{r_0}(\alpha) = \int \left| \int F(r\alpha\beta\gamma^{-1}) \, G(r\gamma) \, d\gamma \right|^p dm_{r_0}(\alpha) \leq$$

$$\leq \int \left[\int | F(r\alpha\beta\gamma^{-1}) \, G(r\gamma) \, K(r)^{-1} \, d\gamma \right]^p K(r)^p \, dm_{r_0}(\alpha)$$

where $K(r) = \int | F(r\alpha\beta\gamma^{-1}) | \, d\gamma = \int | F(r\gamma) | \, d\gamma$; we shall show later that $K(r) \neq 0$ for $0 < r < 1$ unless $F = 0$;

$$\int |(F \circledast G)(r^2 \alpha\beta)|^p \, dm_{r_0}(\alpha) \leq$$

$$\leq \int \left[\int | F(r\alpha\beta\gamma^{-1}) | \, | G(r\gamma) |^p \, K(r)^{-1} \, d\gamma \right] K(r)^p \cdot dm_{r_0}(\alpha) =$$

$$= \iint K(r)^{p-1} \, | F(r\alpha\beta\gamma^{-1}) | \, | G(r\gamma) |^p \, d\gamma \, dm_{r_0}(\alpha) =$$

$$= K(r)^{p-1} \int \left[\int | F(r\alpha\beta\gamma^{-1}) | \, dm_{r_0}(\alpha) \right] | G(r\gamma) |^p \, d\gamma \leq$$

$$\leq K(r)^{p-1} \| F \|_1 \int | G(r\gamma) |^p \, d\gamma \leq K(r)^{p-1} \| F \|_1 \| G \|_p^p \leq \| F \|_1^p \, \| G \|_p^p.$$

The last two inequalities hold because of (2.3.5). The validity of the inequaliy for each $0 < r < 1$ and $\beta \in \Gamma$ leads to the desired conclusion.

Suppose now that fos some $r_1 > 0$ $K[r_1) = 0$; this implies that $F_{r_1} = 0$ a.e. respect to Haar measure on Γ. But as F_{r_1} is continuous it vanishes everywhere. Since on the other hand we have (see (2.2.8))

$$\widehat{F}_{r_1}(x)\, r_2^{|X|} = \widehat{F}_{r_2}(x)\, r_1^{|X|} = 0 \quad \forall\ r_2 < 1$$

then $F_{r_2} = 0\ \forall\ 0 < r_2 < 1$ and this gives $F = 0$.

(3.2.5) COROLLARY : $h^p(\Delta)$ is a Banach algebra and $H^p(\Delta)$ is a. closed ideal.

PROOF : apply (3.2.3) and (3.2.4).

(3.2.6) LEMMA : given $F \in h^1(\Delta)$ the measure μ_F obtained in (2.3.8) such that $F_r = \mu_F * m_r$ is unique.

PROOF : $\widehat{\mu_F}(x) = \dfrac{\widehat{F}_r(x)}{r^{|X|}}\ \forall\ 0 < r < 1$ and $x \in g$. The fact that two measures which have the same Fourier-Stieltjes transform must coincide leads us to the conclusion.

(3.2.7) LEMMA : suppose $F, G \in h^1(\Delta)$ and μ_F, μ_G are the boundary measures given by (2.3.8); then

$$(F \circledast G)_r = (\mu_F * \mu_G) * m_r.$$

PROOF : $(F \circledast G)_r = F_{\sqrt{r}} * G_{\sqrt{r}} = (\mu_F * m_{\sqrt{r}}) * (\mu_G * m_{\sqrt{r}}) = (\mu_F * \mu_G) * m_r$.

(3.2.8) THEOREM : the mapping $F \to \mu_F$ from $h^1(\Delta)$ into $C(\Gamma)^*$ is a continuous injective homomorphism.

PROOF : it is clear that the mapping is linear and (3.2.6) (3.2.7) imply that it is multiplicative; the argument used in (3.2.6) gives the injectivity part; the continuity follows because μ_F is taken in the closed ball of radius $\| F \|_1$.

We shall prove in another section that the mapping in (3.2.8) has a continuous inverse iff the departure group g is Z.

(3.2.9) LEMMA: let Γ be a compact group; if $p \geq \frac{2^n}{2^n - 1}$ $(n \in N)$ and $f_i \in L^p(\Gamma)$ $1 \leq i \leq 2^h$ then

$$f_1 * \dots * f_{2^n} \in C(\Gamma)$$

($L^p(\Gamma)$ is the space L^p associated to Haar measure on Γ).

PROOF: the proof is by induction.

For $n = 1$ we have $p \geq 2$; if $f_1, f_2 \in L^p(\Gamma)$ then $f_2 \in L^q(\Gamma)$ where $\frac{1}{p} + \frac{1}{q} = 1$ because of the compactness of the group. But as $L^p * L^q \subset C(\Gamma)$ ([VI] (20.19)) we have that $f_1 * f_2$ is continuous. Suppose now that the statement is true for $n = k_0$ and let

$$p \geq \frac{2^{k_0+1}}{2^{k_0+1} - 1} = p_1 \quad f_i \in L^p(\Gamma) \quad i \leq 2^{k_0+1}.$$

Consider $g_j = f_j * f_{2^{k_0}+j}$ $1 \leq j \leq 2^{k_0}$; $\frac{1}{p_1} + \frac{1}{p_1} - \frac{1}{p_2} = 1$ has the solution $p_2 = \frac{2^{k_0}}{2^{k_0} - 1}$ then we have that ([VI] (20.19)) $g_j \in L^{p_2}(\Gamma)$; thus

$$f_1 * \dots * f_{2^{k_0+1}} = g_1 * \dots * g_{2^{k_0}} \in C(\Gamma)$$

because of the inductive hypothesis.

(3.2.10) DEFINITION: given $F \in h^p(\Delta)$ $1 \leq p \leq \infty$ we say that $F \in C(\Gamma)$ if the boundary measure is a continuous function; we say that $F \in A_0$ if $F \in C(\Gamma)$ and $F(x) = 0$ $\forall\, x < 0$. It is clear that such an F together with its boundary values is a continuous function on.

(3.2.11) PROPOSITION: (1) $h^p * h^q \subset C(\Gamma)$ for $\frac{1}{p} + \frac{1}{q} = 1$ $1 < p < \infty$

(1′) $H^p * h^q \subset C(\Gamma)$ $\quad \frac{1}{p} + \frac{1}{q} = 1$ $1 \leq p < \infty$

(2) $h^p * h^q \subset h^r$ $\quad \frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$ $1 < p < \infty$, $1 < q < \infty$

(2′) $H^p * h^q \subset H^r$ $\quad \frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$ $1 < p < \infty$, $1 < q < \infty$

(3) $h^1 * h^p \subset h^p$ $1 \leq p \leq \infty$

(3′) $h^1 * H^p \subset H^p$ $1 \leq p \leq \infty$

(4) For $p \geq \dfrac{2^n}{2^n - 1}$ $\underbrace{h^p * \ldots * h^p}_{2^n \text{ times}} \subset C(\Gamma)$

(4′) For $p \geq \dfrac{2^n}{2^n - 1}$ $H^p * \underbrace{h^p * \ldots * h^p}_{2^n - 1 \text{ times}} \subset A_0$.

Proof: (1) Consider $F \in h^p$ and $G \in h^q$; then their boundary functions F_1, G_1 are in $L^p(\Gamma)$ and $L^q(\Gamma)$ respectively; but $F_1 * G_1$ is continuous (see [VI] (20.19)); (3.2.7) makes the rest.

(1′) We proceed as in (1) using also (3.2.3)

(2) and (2′). We repeat the technique of (1) and (1′) observing that

$$L^p(\Gamma) * L^q(\Gamma) \subset L^r \qquad ([\text{VI}]\ (20.19))$$

(3) and (3′) have been prooved in (3.2.4) and (3.2.5).

(4) Apply (3.2.9) to the boundary functions and then (3.2.7).

(4′) Apply (3.2.5), (3.2.9) and (3.2.7).

4. The $h^p(\Delta)$ spaces and harmonic almost periodic functions on a half plane.

4.1. In the following we shall use the terminology concerning almost periodic functions; Dirichlet coefficients and Dirichlet series as stated in [V] with the only modification that we write the Dirichlet series of f: $\mathbb{R} \to \mathbb{C}$ in the form

$$f(t) \backsim \sum_{x \in \mathbb{R}} A_x e^{-ixt}.$$

We have (see [II], [VIII]) a mapping $\exp^{-1}: \{w \in C : \operatorname{Re} w \geq 0\} = \bar{P} \to \bar{\Delta}$ that restricted to $\operatorname{Re} w = 0$ is a continuous homomorphism with values in Γ and such that the image subgroup is dense in Γ, so that the image of $\{\operatorname{Re} w = u > 0\}$ is dense in $e^{-u} \cdot \Gamma$ and consequently $\exp^{-1}(\bar{P})$ is dense in $\bar{\Delta}$. Let $f \in A_0$, then $f_0 \exp^{-1}$ is a bounded continuous almost periodic function in $\bar{P}$, analytic in $P = \{\operatorname{Re} w > 0\}$. More precisely we have

(4.1.1) Theorem: the mapping $f \to f_0 \exp^{-1}$ defines an isomorphism between A_0 and the bounded continuous almost periodic functions in $\bar{P}$

that are analytic in P and such that the exponents of their Dirichlet series lie in g_+. The vector space isomorphism is an isometry if in both spaces we consider the supremum norm.

PROOF: see [III]-(2.6); see also (4.1.5).

(4.1.2) Consider $f \in C(\Gamma)$; we know that f is the restriction to Γ of a function F in A_0 iff $\widehat{f}(x) = 0 \ \forall\, x < 0$ and that if this is the case

$$F(r\alpha) = \int f(\alpha\beta)\, dm_r(\beta) = (f * m_r)(\alpha)\,.$$

For any $f \in C(\Gamma)$ we can define a g. h. function F in which f as boundary function, namely

$$F(r\,\alpha) = \int f(\alpha\beta)\, dm_r(\beta). \tag{4.1.3}$$

This is the only g. h. function in $\bar{\Delta}$ whose restriction to Γ coincides with f. For $f \in C(\Gamma)$ we define (4.1.3) to be the harmonic extension of f to $\bar{\Delta}$ or the Poisson integral of f. We note it also $\mathcal{P}f$.

We also have $\sup\limits_{\substack{r \leq 1 \\ \alpha \in \Gamma}} |(\mathcal{P}f)(r\alpha)| = \sup\limits_{\alpha \in \Gamma} |f(\alpha)|$.

(4.1.4) Consider now $\mathcal{P}f_0 \exp^{-1} : \bar{P} \to C$; in the case f is a trigonometrical polynomial i. e. $f(\alpha) = \sum\limits_{x \in g} a_x \langle \alpha, x\rangle$ where $a_x = 0$ except for a finite number of $x \in g$ then

$$F(r\alpha) = \sum_{x \in g} a_x\, r^{|x|} \langle \alpha, x \rangle$$

and

$$F(e^{-w}) = \sum_{x \geq 0} a_x\, e^{-xw} + \sum_{x < 0} a_x\, e^{-|x|\,\bar{w}}$$

So that $F_0 \exp^{-1}$ is a harmonic function in P, continuous and bounded in $\bar{P}$, almost periodic in the whole half plane such that

$$\sup_{w \in \bar{P}} |F(e^{-w})| = \sup_{\substack{r \leq 1 \\ \alpha \in \Gamma}} |F(r\alpha)| = \sup_{\alpha \in \Gamma} |F(\alpha)| = \sup_{v \in \mathbb{R}} |F(e^{-iv})|\,.$$

Observe that in this case we have that the Dirichlet coefficients of $\tilde{F} = F_0 \exp^{-1}$ are the Fourier coefficients of f.

(4.1.5) THEOREM: the mapping $f \longrightarrow \widetilde{F} = (\mathcal{P}f)_0 \exp^{-1}$ is a isometric isomorphism between the space $C(\Gamma)$ and the space of bounded continuous uniformly almost periodic functions in $\bar{P}$ that are harmonic in P and such that the exponents of their Dirichlet series lie in g. Moreover the Dirichlet coefficients of F are the Fourier coefficients of f. A_0 is carried onto the analytic ones.

PROOF: For trigonometric polynomials in $C(\Gamma)$ the stated properties have been already established in (4.1.4). Let $f \in C(\Gamma)$, we can approximate it uniformly in Γ by a sequence of trigonometric polynomials (see [IX] § 38) f_n, so that $\mathcal{P}f_n$ converges uniformly to $\mathcal{P}f$ and $\widetilde{F}_n$ to $\widetilde{F}$. We want to see that the Dirichlet development of $\widetilde{F}$ is

$$\sum_{x \geq 0} a_x e^{-xw} + \sum_{x<0} a_x e^{-|x|\bar{w}}$$

where

$$a_x = \widehat{f}(x) = \int f(\alpha) \langle \overline{\alpha, x} \rangle \, d\alpha \;\; \forall \; x \in g.$$

Consider $f_0 \exp^{-1} = \widetilde{f}: \{\operatorname{Re} w = 0\} \longrightarrow \mathbb{C}$; it is uniformly approximated by $f_{n^0} \exp^{-1} = \widetilde{f}_n$ so that the Dirichlet coefficients of $\widetilde{f}$ are the limit of the Dirichlet coefficients of $\widetilde{f}_n$ and thus (see [V] § 9 - chap. II) we get that the Dirichlet coefficients of $\widetilde{f}$ and the Fourier coefficients of f are the same. Now we apply proposition 31, chap. IV of [V] and obtain that the function

$$\text{(i)} \qquad G(u + iv) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{u}{u^2 + (v-t)^2} \widetilde{f}(it) \, dt$$

has as Dirichlet series the extended of that of $\widetilde{f}$ i. e.

$$G(w) \backsim \sum_{x \geq 0} a_x e^{-xw} + \sum_{x<0} a_x e^{-|x|\bar{w}}$$

but (see [VII] page 123 or [VIII] (2.22)) formula (i) reproduces $\widetilde{F}$ from its boundary function f so that $G = \widetilde{F}$.

Let now be $F' : P \longrightarrow C$ with the stated properties and let $F'(w) \backsim$ $\backsim \sum_{x \geq 0} a_x e^{-xw} + \sum_{x<0} a_x e^{-|x|\bar{w}}$ be its Dirichlet series ($x \in g$).

The restriction of F' to the imaginary axis can be approximated uniformly by trigonometric polynomials of the type

$$F_n'(iv) = \sum_{x \geq 0} a_{x,n} e^{-ixv} + \sum_{x<0} a_{x,n} e^{-ixv} \qquad \text{(see [V] § 12)}$$

with $a_{x,n} \neq 0$ only when $a_x \neq 0$ i. e. for $x \in g$.

The natural extensions of F_n' to the half plane $\bar{P}$ defined by

$$F_n'(w) = \sum_{x \geq 0} a_{x,n} e^{-xw} + \sum_{x<0} a_{x,n} e^{-|x|\bar{w}}$$

approximate F' uniformly because oe the harmonicity and boundedness of the functions involved. As

$$F_n' = \mathcal{P} f_{n^\circ} \exp^{-1} = \widetilde{F}_n$$

where

$$f_n(\alpha) = \sum_{x \geq 0} a_{x,n} \langle \alpha, x \rangle + \sum_{x<0} a_{x,n} \langle \alpha, x \rangle$$

we get that there exists $f \in C(\Gamma)$ such that $f_n \to f$ uniformly. It is clear then that $\mathcal{P} f_0 \exp^{-1} = F'$; observe that if $a_x = 0$ whenever $x < 0$ then $\mathcal{P} f \in A_0$ (i. e. when F' is analytic in P).

4.2 We turn now to see how the mapping $\exp^{-1} : P \to \Delta$ behaves when composed with g. h. functions.

(4.2.1) Theorem: The mapping $\exp^{-1} : P \to \Delta$ induces a linear isomorphism between the space of g. h. functions in Δ and the space of harmonic functions in P, which are bounded and uniformly almost periodic on each half plane $\{\operatorname{Re} w \geq u > 0\}$ and whose Dirichlet series have its exponents in g. This statement holds if we change g. h. and harmonic functions by g. a. and analytic respectively.

Proof: Let $F : \Delta \to \mathbb{C}$ be g. h. Suppose that $\widetilde{F} = F_0 \exp^{-1} : P \to \mathbb{C}$ has Dirichlet series

$$\widetilde{F}(w) \backsim \sum_{x \geq 0} a_x e^{-xw} + \sum_{x<0} a_x e^{-|x|\bar{w}}.$$

Consider $u > 0$ and define $F_{e^{-u}} : \bar{\Delta} \to \mathbb{C}$ by $F_{e^{-u}}(\xi) = F(e^{-u}.\xi)$; it is clear that $F_{e^{-u}}$ is g. h. and continuons in all of $\bar{\Delta}$ and that $F_{e^{-u}}$ is g. a., i. e. belongs to A_0 iff F is g. a. Because of (4.1.5), as $\widetilde{F}_{e^{-u}}(w) = F_{e^{-u}}(e^{-w}) =$

$\widetilde{F}(u+w)$ we obtain that $\widetilde{F}$ is bounded and uniformly almost periodic in $\{w : \mathrm{Re}\, w \geq u\}$, harmonic in $\{w : \mathrm{Re}\, w > u\}$ and analytic iff F is g. a. On the other and we have that

$$\widetilde{F}_{e^{-u}}(iv) = F(e^{-(u+iv)}) \sim \sum_{x\geq 0} a_x\, e^{-xu}\, e^{-ixv} + \sum_{x<0} a_x\, e^{-|x|u} e^{-ixv}$$

and then working as in (4.1.5) we have

$$(\widetilde{F}_{e^{-u}})(w) \sim \sum_{x\geq 0} a_x\, e^{-xu}\, e^{-xw} + \sum_{x<0} a_x\, e^{-|x|u} e^{-|x|\bar{w}}.$$

But then (4.1.5) gives us that if $a_x \neq 0$ then $x \in g$. To see that if $\widetilde{F} = 0$ it is enough to remeber that $\exp^{-1}(P)$ is dense in Δ.

Let us see the onto part; suppose we have $G : P \longrightarrow C$ harmonic with the properties stated in the theorem. Let us consider for each $u > 0$ $G^u : \overline{P} \longrightarrow \mathbb{C}$ defined as follows

$$G^u(w) = G(u+w).$$

We have that G^u is harmonic in P, continuous bounded and almost periodic in $\overline{P}$ and such that the exponents of its Dirichlet series lie in g. It is also true that G^u is analytic whenever G is. By (4.1.5) there exists $F^u : \overline{\Delta} \longrightarrow C$ continuous g. h. such that $\widetilde{F}^u(w) = G^u(w) = F^u(e^{-w})$.

Define $F : \Delta \longrightarrow \mathbb{C}$ in the following way $F(r\alpha) = F^u(e^u r\alpha)$, $r \leq e^{-u}$. It is necessary to see that this definition does not depend on $u > 0$. Let $r \leq e^{-u_1}$ $r \leq e^{-u_2}$ and $r \cdot \alpha = e^{-w}$, $w = u + iv$

$$F^{u_1}(e^{u_1} r\alpha) = F^{u_1}(e^{u_1} e^{-w}) = G^{u_1}(w-u_1) = G^{u_2}(w-u_2) = F^{u_2}(e^{u_2} e^{-w}).$$

The continuity of F^{u_1} and F^{u_2} together with the density of $\{r \cdot e^{-iv}; v \in \mathbb{R}\}$ in $r \cdot \Gamma$ imply that $F^{u_1}(e^{u_1} r\alpha) = F^{u_2}(e^{u_2} e\alpha)$ $\forall\, \alpha \in \Gamma$ and F is well defined.

F is continuous on Δ because it is continuous on each «disc» $r \cdot \overline{\Delta}$ $(r < 1)$.

Let us see that F is g. h.; suppose

$$r_1 < r_2 < 1 \; r_1 = e^{-u_1} \; r_2 = e^{-u_2} \quad r_2 \leq e^{-u} \,(u > 0)$$

$$(F_{r_2} * m_{r_1/r_2})(\alpha) = \int F(r_2\, \alpha\beta)\, dm_{r_1/r_2}(\beta) = \int F^u(e^u\, r_2\, \alpha\beta)\, dm_{r_1/r_2}(\beta) =$$

$$\iint F^u(\alpha\beta\gamma)\, dm_{e^u \cdot r_2}(\gamma)\, dm_{r_1/r_2}(\beta) = \int F^u(\alpha\beta)\, dm_{e^u \cdot r_1}(\beta) = F_u(e^u\, r_1\, \alpha) = F(r_1\alpha) = F_{r_1}(\alpha).$$

Let us see now that
$$F(e^{-w}) = G(w)$$
$$F(e^{-w}) = F(e^{-u}\,e^{-iv}) = F^u(e^{-iv}) = G^u(iv) = G(u+iv) = G(w).$$

If G is analytic fixing $u_0 > 0$ we should get that F^{u_0} is g. a. on $\bar{\Delta}$ but as F^{u_0} is the restriction of F to $e^{-u_0}\,\bar{\Delta}$ so that because of (2.2.7) F should be g.a.

(4.2.2) Let us see now how are mapped the spaces $h^p(\Delta)$ by means of composition with $\exp^{-1}$.

LEMMA : if $F \in h^p(\Delta)$ then
$$\|F\|_p^p = \sup_{\substack{v \in \mathbb{R} \\ 0 \le r < 1}} \int |F(r\alpha\, e^{-iv})|^p\, dm_{r_0}(\alpha) \qquad \text{for } p < \infty$$
$$\|F\|_\infty = \sup_{\substack{v \in \mathbb{R} \\ r<1}} |F(r \cdot e^{-iv})|$$

PROOF : $\{r \cdot e^{-iv} : v \in \mathbb{R}\}$ is dense in $r \cdot \Gamma = \{\xi \in \Delta / \xi = r\alpha\}$; the continuity of F_r yields the conclusion.

(4.2.4) COROLLARY : through the correspondence $F \to F_0\ \exp^{-1}$ the space $h^p(\Delta)$ is carried isomorphically onto the space of functions $\widetilde{F} : P \to C$ that verify the conditions stated in (4.2.1) and such that
$$\frac{1}{\pi} \int_{-\infty}^{+\infty} |\widetilde{F}(u+it)|^p \frac{1}{1+(v-t)^2}\, dt$$
is bounded uniformly for $u > 0$, $v \in \mathbb{R}$ in the case $p < \infty$; in the case $p = \infty$ such an $\widetilde{F}$ is bounded in the whole half plane P.

Moreover we have
$$\|F\|_p^p = \sup_{\substack{v \in \mathbb{R} \\ u>0}} \frac{1}{\pi} \int_{-\infty}^{+\infty} |F(u+it)|^p \frac{1}{1+(v-t)^2}\, dt.$$

(4.2.5) THEOREM : Let $\widetilde{F}, \widetilde{G}$ be almost periodic bounded functions in every proper half plane $\{\operatorname{Re} w \geq u > 0\}$ and harmonic in P; if
$$F(w) \backsim \sum_{x \ge 0} a_x\, e^{-xw} + \sum_{x<0} a_x\, e^{-|x|\bar{w}}$$
$$G(w) \backsim \sum_{x \ge 0} b_x\, e^{-x} + \sum_{x<2} b_x\, e^{-|x|\bar{w}}$$

are their Dirichlet series then the series

$$\sum_{x \geq 0} a_x b_x e^{-xw} + \sum_{x<0} a_x b_x e^{-|x|\bar{w}}$$

is the Dirichlet series of a function with the same properties.

PROOF: consider $\overbrace{F, G : \Delta \longrightarrow C}$ g.h. such that $F_0 \exp^{-1} = \widetilde{F}$ and $G_0 \exp^{-1} = \widetilde{G}$; then $F \circledast G = (F \circledast G)_0 \exp^{-1}$ is the required function; observe that (4.2.1) gives us that

$$e^{-|x|u} a_x = \widehat{F}_{e^{-u}}(x) \qquad \forall\, u > 0$$

so that

$$(\widehat{F \circledast G})_{e^{-u}}(x) = \widehat{F}_{e^{-u/2}}(x) \cdot \widehat{G}_{e^{-u/2}}(x) = e^{-|x|u} a_x \cdot b_x .$$

(4.2.6) Calling $\widetilde{h^p}$ and $\widetilde{H^p}$ the image by composition with $\exp^{-1}$ of $h^p(\Delta)$ and $H^p(\Delta)$ respectively we have the analogous of (3.2.11) for P.

(4.2.7) THEOREM: Let $\widetilde{F} \in \widetilde{h^p}$ and $\widetilde{G} \in \widetilde{h^q}$ with Dirichlet series

$$\widetilde{F}(w) \sim \sum_{x \geq 0} a_x e^{-xw} + \sum_{x<0} a_x e^{-|x|\bar{w}}$$

$$\widetilde{G}(w) \sim \sum_{x \geq 0} b_x e^{-xw} + \sum_{x<0} b_x e^{-|x|\bar{w}}$$

then

$$\sum_{x \geq 0} a_x \cdot b_x e^{-xw} + \sum_{x<0} a_x b_x e^{-|x|\bar{w}}$$

is the Dirichlet series of a function which is

(1) continuous bounded almost periodic in P and harmonic in P if $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and $1 < p < \infty$

(1′) continuous bounded almost periodic in P and analytic in P if $\widetilde{F} \in \widetilde{H^p}$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$ with $1 \leq p < \infty$.

PROOF: apply (4.2.1) as in (4.2.5), then (3.2.11) and (4.1.5). For the analogues of (2), (2′), (3) and (3′) the proof is the same changing (4.1.5) by (4.2.1).

For the analogues of (4) and (4′) we proceed as with (1) and (1′).

5. $H^p(\Delta)$ and the Hardy spaces in the Dirichlet algebra context.

5.1. In [X] it is developed a theory of Hardy spaces when the departure notion is that of a Dirichlet algebra A_0 on a compact space Γ, i. e.

(i) $A_0 \subset C(\Gamma)$ is uniformly closed

(ii) A_0 contains the constants and separates points

(iii) Re $A_0 = \{\mathrm{Re} f / f \in A_0\}$ is dense in Re $C(\Gamma)$.

So if λ is a representative real measure i. e. a Baire regular measure on Γ such that its restriction to A_0 is multiplicative then if $1 \leq p < \infty$ $H^p(\lambda)$ is the closure of A_0 in $L^p(\lambda)$.

In our situation the restriction of the elements of A_0 to Γ is a Dirichlet algebra; then given any $\xi \in \Delta$ the question arises in how are related $H^p(\Delta)$ and $H^p(m_\xi)$ where m_ξ in the unique real representing measure for ξ.

(5.1.2) Let $1 \leq p < \infty$, $\xi \in \Delta$; we define

$$i_{p,\xi}\colon H^p(\Delta) \longrightarrow H^p(m_\xi)$$

in the following way; (2.3.2) gives us that if $F \in h^p(\Delta)$ there is a boundary function F_1 such that $F_r = F_1 * m_r$ and $F_r \xrightarrow[r \to 1]{L^p(\Gamma, m_\xi)} F_1$ so that we define

$$i_{p,\xi}(F) = F_1 .$$

(5.1.3) THEOREM: for any $\xi \in \Delta$ and $1 \leq p < \infty$ $i_{p,\xi}$ is linear, continuous and injective.

PROOF: we shall deal with $\xi = r$ where $0 \leq r < 1$; the linearity of $i_{p,r}$ is clear from the context and (2.3.2); suppose first that $r = 0$; then m_0 is Haar measure m in Γ. As $F_r = F_1 * m_r$, then

$$\int |F_r(\alpha)|^p \, dm(\alpha) = \int \left| \int F_1(\alpha\beta) \, dm_r(\beta) \right|^p dm(\alpha) \leq$$

$$\leq \iint |F_1(\alpha\beta)|^p \, dm(\alpha) \, dm_r(\beta) =$$

$$= \int \left[\int |F_1(\alpha)|^p \, dm(\alpha) \right] dm_r(\beta) = \int |F_1(\alpha)|^p \, dm(\alpha).$$

Thus we get

$$\| F_r \|_{L^p(\Gamma, m)} \leq \| F_1 \|_{L^p(\Gamma, m)} .$$

This means that F_1 cannot vanish almost everywhere respect to Haar measure unless F vanishes in all of Δ. To prove the continuity of i_{p,ξ_0} recall that (see (2.3.2))

$$\| F_1 \|_{L^p(\Gamma, m)} = \operatorname*{limit}_{r \to 1} \| F_r \|_{L^p(\Gamma, m)}$$

but (see (2.3.4))

$$\sup_{\substack{r<1 \\ \beta \in \Gamma}} \int | F(r\alpha\beta)^p \, | \, dm(\alpha) \leq \| F \|_p^p$$

and then

$$\| F_1 \|_{L^p(\Gamma, m)} \leq \| F \|_p \, .$$

We remark that it is possible to see automatically that i_{p,ξ_0} is continuous by means of the theorem that says that if A and B are Banach algebras, B semi-simple $T: A \longrightarrow B$ a homomorphism such that $T(A)$ is dense in B then is continuous (see [VIII] § 24). Consider now the case $r > 0$; suppose then that $F_1 = 0$ almost everywhere respect to m_r; the absolute continuity of m_{r^1} respect to m_r for any $0 < r^1 < 1$ gives us that $F_1 = 0$ almost everywhere respect to any measure m_r with $0 < r < 1$; but then $F(r \cdot e_1) = (F_1 * m_r)(e_1) = 0 \;\; \forall \, 0 < r < 1$ where e_1 is the identity element of Γ; so that $F_0 \exp^{-1}: P \longrightarrow C$ is an analytic function that vanishes on the real axis (see (4.1.5)) but this implies that $F = F_0 \exp^{-1}$ vanishes in the whole half plane P. The density of $\{e^{-w}; w \in P\}$ in Δ and the continuity of F imply that F vanishes in all of Δ; this completes the injectivity part. We have that if $0 < r_1 < 1$ there exists a constant M such that

$$\sup_{\substack{r<1 \\ \beta \in \Gamma}} \int | F(r\,\alpha\beta) |^p \, dm_{r_1}(\alpha) \leq M \| F \|_p^p \qquad \text{(see (3.1.2))}$$

letting r tend to 1 and taking $\beta = e_1$

$$\int | F_1(\alpha) |^p \, dm_{r_1}(\alpha) \leq M \| F \|_p^p \, .$$

This completes the proof.

5.2. At this point it is of interest to determine under what conditions $H^p(\Delta)$ coincides with $H^p(m_\xi)$ for some $\xi \in \Delta$.

(5.2.1) Theorem: Let $1 \leq p < \infty$ and $m_r\, r < 1$ be fixed; then $i_{p,r}$ is surjective iff $g = Z$.

PROOF: we prove the only if part; for the if one see [VI] chapter 3. We recall that in [II]-(6.5) it is proved that m_r with $r > 0$ is non singular respect to Haar measure in Γ iff $g = Z$. Suppose that $i_{p,r}$ is surjective; we deal first with $r = 0$. The open mapping theorem togetheir with the fact that $i_{p,r}$ is injective imply that there exists a constant M sucht that

$$\| F \|_p \leq M \| F_1 \|_{L^p(\Gamma, m)} .$$

This means that for $f \in A_0$ (we are considering its boundary values)

$$\int | e^f |^p \, dm_{r_0}(\alpha) \leq M^p \int | e^f |^p \, d\alpha \quad (d\alpha = dm(\alpha))$$

i. e.

$$\int e^{p \operatorname{Re} f} \, dm_{r_0}(\alpha) \leq M^p \int e^{p \operatorname{Re} f} \, d\alpha$$

or

$$\int e^{\operatorname{Re} f} \, dm_{r_0}(\alpha) \leq M^p \int e^{\operatorname{Re} f} \, d\alpha \quad \forall f \, A_0 .$$

The density of $\operatorname{Re} A_0$ in $\operatorname{Re} C(\Gamma)$ leads us to

$$\int e^f \, dm_{r_0}(\alpha) \leq M^p \int e^f \, d\alpha \quad \forall f \in \operatorname{Re} C(\Gamma) = C_{\mathbb{R}}(\Gamma).$$

Let $K \subset \Gamma$ be a compact set and $f \geq 1$ on K, $f \geq 0$ on Γ. Then if $\varepsilon > 0$ $\log(f + \varepsilon) \in C_{\mathbb{R}}(\Gamma)$ and then

$$\int (f + \varepsilon) \, dm_{r_0} \leq M^p \int (f + \varepsilon) \, d\alpha$$

so that

$$\int f \, dm_{r_0} \leq \int f \, d\alpha.$$

As

$$m_{r_0}(K) \leq \int f \, dm_{r_0} \leq M^p \int f \, d\alpha$$

because of the regularity of m_{r_0}, the regularity of Haar measure yields

$$m_{r_0}(K) \leq M^p \, m(K)$$

but this means that m_{r_0} is absolutely continuous respect to Haar measure. The theorem mentioned at the beginning of the proof implies that $g = Z$.

Suppose now that $r > 0$; there is no loss of generality in considering $r = r_0 = e^{-1}$ because $H^p(m_{r_0})$ is topologically isomorphic to $H^p(m_r)$ being m_{r_0} and m_r mutually absolutely continuous. Thus we have

$$\| F \|_p \leq M \left[\int | F_1(\alpha) |^p \, dm_{r_0}(\alpha) \right]^{\frac{1}{p}}$$

in particular for $\beta \in \Gamma$

$$\int | F(r\,\alpha\beta) |^p \, dm_{r_0}(\alpha) \leq M^p \int | F_1(\alpha) |^p \, dm_{r_0}(\alpha) \quad \forall\, r < 1$$

so that

$$\int | F_1(\alpha\beta) |^p \, dm_{r_0}(\alpha) \leq M^p \int | F_1(\alpha) |^p \, dm_{r_0}(\alpha).$$

If $F \in A_0$ we have

$$\int | F_1(\alpha) |^p \, dm_{r_0 \cdot \beta}(\alpha) \leq M^p \int | F_1(\alpha) |^p \, dm_{r_0}(\alpha).$$

This inequality leads us to the fact that $m_{r \cdot \beta}$ is absolutely continuous respect to m_{r_0} for all $\beta \in \Gamma$ but this occurs only if $\beta \in \{e^{-iv} : v \in \mathbb{R}\}$ (see [II] (5.7)) and $\{e^{-iv} : v \in \mathbb{R}\} = \Gamma$ iff $g = Z$.

This completes the proof.

(5.2.2) COROLLARY: the mapping in (3.2.8) is sujective iff $g = z$.

(5.2.3) We remark that the mapping from $H^\infty(\Delta)$ into $L^\infty(\Gamma, m_\xi)$ given by (2.3.2) is an isometry.

(5.2.4) At this point we are led to the following question: suppose $F \in H^p(\Delta)$. $(p > \infty)$ is it true that F belongs to the closure of A_0 in $H^p(\Delta)$. We give here a partial answer.

(5.2.5) THEOREM: Let $F \in H^p(\Delta)\,(p < \infty)$; then F belongs to the closure of A_0 in H^p iff the function $\alpha \to F_\alpha$ is continuous as a function from Γ into $H^p(\Delta)\,(F_\alpha(r\beta) = F(r\beta\alpha^{-1}))$.

(5.2.6) Before proving (5.2.3) we make a brief account of a tool we shall use, namely the integration of coutinuous functions defined on Γ with values in a Banach space [1]. Given $K : \Gamma \to V$ continuous we define

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[1] A complete exposition of this subject can be found in Bourbaki's Integration (Chap. 6).

$\int K(\alpha)\, d\alpha$ as the limit of $\Sigma K(\alpha_i)\, m(E_i)$ where the E_i are Borel disjoint sets whose union is Γ and such that if $\alpha \in E_i$, $\alpha^{-1}.\ E_i \subset U(e_i)$ being a neighborhood of the identity element of Γ belonging to a fundamental system of neighborhoods of e_1; that is, for each $U_\lambda(e)$ we fix a finite family $E_{\lambda,i}$ and $\alpha_{\lambda;i} \in E_{\lambda,i}$ and consider $\sum_i K(\alpha_{\lambda,i})\, m(E_{\lambda,i})$;it is clear that the net so defined is a Cauchy net and then it has unique limit point which we call $\int K(\alpha)\, dm(\alpha)$ (The filter set is the fundamental system of neighborhoods of e_1). The integral so defined is linear and $\left\| \int K \right\| \leq \int \| K \|$.

(5.2.7) PROOF of (5.2.5)

<==) Let $0 < \varepsilon < 1$ be given and $G \in A_0$ such that

$$\| F - G \|_p < \frac{\varepsilon}{3}$$

then

$$\| F_\alpha - F_{\alpha_0} \|_p \leq \| F_\alpha - G_\alpha \|_p + \| G_\alpha - G_{\alpha_0} \|_p + \| G_{\alpha_0} - F_{\alpha_0} \|_p <$$

$$< \frac{\varepsilon}{3} + \| G^\alpha - G_{\alpha_0} \|_p + \frac{\varepsilon}{3}.$$

The uniform continuity of G implies that there exists a neighborhood $U_\varepsilon(e_1)$ such that

$$| G(\delta) - G(\delta_1) | < \frac{\varepsilon}{3} \text{ if } \delta^{-1}\cdot\delta_1 \in U_\varepsilon(e)$$

so that

$$\| G_\alpha - G_{\alpha_0} \|_p^p \leq \frac{\varepsilon}{3} \text{ if } \alpha\cdot\alpha_0^{-1} \in U_\varepsilon(e_1)$$

then

$$\| F_\alpha - F_{\alpha_0} \|_p < \varepsilon \text{ if } \alpha\cdot\alpha_0^{-1} \in U_\varepsilon(e_1)$$

and we have that $\alpha \longrightarrow F_\alpha$ is continous.

==>) Let $\varepsilon > 0$ be given and consider $U_\varepsilon(e_1)$ such that

$$\| F_\alpha - F_{\alpha_0} \| < \varepsilon \text{ if } \alpha\cdot\alpha_0^{-1} \in U_\varepsilon(e_1).$$

Let $G : \Gamma \longrightarrow (0, +\infty)$ be a continuous function such that

(i) $$G_{|\, CU_\varepsilon} = 0$$

(ii) $$\int G(\alpha)\, d(\alpha) = 1.$$

The continuous function $\alpha \longrightarrow F_\alpha G(\alpha)$ is integrable and

$$\left\| \int F_\alpha G(\alpha)\, d\alpha - F \right\|_p = \left\| \int [F_\alpha - F]\, G(\alpha)\, d\alpha \right\|_p \leq$$

$$\leq \int \| F_\alpha - F \|_p\, G(\alpha)\, d\alpha < \varepsilon \int_{U_\varepsilon} G(\alpha)\, d\alpha = \varepsilon.$$

To see that $\int F_\alpha G(\alpha)\, d\alpha \in A_0$ we observe that

$$\left[\int F_\alpha G(\alpha)\, d\alpha \right]_1 = \int (F_\alpha)_1\, G(\alpha)\, d\alpha \; ((F_\alpha)_1 = i_{p,0}(F_\alpha))$$

$\int (F_\alpha)_1\, G(\alpha)\, d\alpha$ means the integral in $L^1(\Gamma, m)$ of the continuos $\alpha \longrightarrow (F_\alpha)_1 \cdot G(\alpha)$.

The equality of the integrals follows from the continuity of

$$i_{p,0}: H^p(\Delta) \longrightarrow L^p(\Gamma, m).$$

As

$$\int (F_\alpha)_1\, G(\alpha)\, d\alpha = \int (F_1)_\alpha\, G(\alpha)\, d\alpha$$

because $(F_\alpha)_1 = (F_1)_\alpha$ almost everywhere and as

$$\int (F_1)_\alpha\, G(\alpha)\, d\alpha = F_1 * G \text{ i.e.}$$

$$\left[\int (F_1)_\alpha\, G(\alpha)\, d\alpha \right](\beta) = \int F(\beta\alpha^{-1})\, G(\alpha)\, d\alpha$$

we get that the boundary values of $\int F_\alpha G(\alpha)\, d\alpha$ are continuous because

$$\int F_1(\beta\alpha^{-1})\, G(\alpha)\, d\alpha = (F_1 * G)(\beta)$$

is continuous.

That $\int (F_1)_\alpha\, G(\alpha)\, d\alpha = F_1 * G$ can proved first for F_1 continuous and then using a density argument for every $F_1 \in L^1(\Gamma, m)$.

The continuity of the boundary values of $\int F_\alpha G(\alpha)\, d\alpha$ together with the fact that it belongs to $H^p(\Delta)$ gives us that $\int F_\alpha G(\alpha)\, d\alpha$ belongs to A_0, i.e. it is restriction to Δ of a function in A_0; we have got that

$$\left\| \int F_\alpha\, G(\alpha)\, d\alpha - F \right\|_p < \varepsilon$$

and so we are done.

5.3 We intend to give the corresponding characterization given in (5.2.7) for P. For this we use the definitions and theorems concerning almost periodic functions in $\mathbb{R}$ with values in a Banach space X as stated in [IV] chapter 6.

(5.3.1) PROPOSITION if $f\colon \Gamma \longrightarrow X$ is continuous then $f_0 \exp^{-1}\colon \mathbb{R} \longrightarrow X$ is continuous and almost periodic with exponents in g.

PROOF: $f_0 \exp^{-1}$ is cleary continuous; the fact that it is weakly almost periodic, i.e. when composed with $\varphi \in X^* =$ dual space of X it is almost periodic, implies that f is almost periodic (see [IV], theorem 6.18) with exponents in g.

(5.3.2) PROPOSITION; if $F\colon \mathbb{R} \longrightarrow X$ is continuous almost periodic with exponents in g then there exists a unique $f\colon \Gamma \longrightarrow X$ continuous such that

$$F = f_0 \exp^{-1}.$$

PROOF: it follows from the uniform approximation theorem by trigonometric polynomials (with exponents in g) with coefficients in X as stated in [IV] tehorem 6.15. The uniqueness follows from the density of $\exp^{-1}(\mathbb{R})$ in Γ.

(5.3.3) COROLLARY: let $F \in \widetilde{H}_p\,(p < \infty)$; then F belongs to the closure of $\widetilde{A}_0 = \{F : \overline{P} \longrightarrow \mathbb{C};\ \exists f \in A_0 \,/\, \mathcal{P} f_0 \exp^{-1} = F\}$ in $\widetilde{H}^p$ iff the function from $\mathbb{R}$ into $\widetilde{H}_p$ defined by $v \longrightarrow F_v\,(F_v(w) = F(w + iv))$ is continuous and almost periodic.

(5.3.4) COROLLARY: If $F \in \widetilde{H}^p$ and F is uniformly almost periodic in all of P then F belongs to the closure of $\widetilde{A}_0$ in $\widetilde{H}^p$.

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