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# THE SPECTRUM OF OPERATORS ON $L^p(E^n)$ <sup>(1)</sup>

MARTIN SCHECHTER

## 1. Introduction.

Let  $P(D)$  be a partial differential operator on  $E^n$  with complex constant coefficients. If  $C_0^\infty$  denotes the set of infinitely differentiable complex valued functions with compact supports on  $E^n$ , then  $P(D)$  on  $C_0^\infty$  is a closable operator in  $L^p = L^p(E^n)$  for  $1 \leq p \leq \infty$ . Let  $P_0 = P_{0p}$  denote its closure in  $L^p$ . It is the purpose of this note to describe the spectrum of  $P_0$  under certain assumption on  $P(D)$ .

In describing our results, we shall need to define the polynomial  $P(\xi)$  associated with  $P(D)$ . If we set

$$(1.1) \quad D_k = \partial/i\partial x_k, \quad 1 \leq k \leq n,$$

we can consider  $P(D)$  as a « polynomial » in  $D_1, \dots, D_n$ . If we replace  $D_1, \dots, D_n$  by real variables  $\xi_1, \dots, \xi_n$ , we obtain a polynomial  $P(\xi)$  called the polynomial associated with  $P(D)$ .

Our first result is

**THEOREM 1.1.** *Let  $P(D)$  be an operator of order  $m$  such that the associated polynomial  $P(\xi)$  satisfies*

$$(1.2) \quad 1/P(\xi) = O(1/|\xi|^b) \text{ as } |\xi| \rightarrow \infty,$$

where  $b > 0$  and  $|\xi|^2 = \xi_1^2 + \dots + \xi_n^2$ . Let  $p$  satisfy  $1 < p < \infty$  and

$$(1.3) \quad |1/p - 1/2| < b/n(m - b).$$

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Then

$$(1.4) \quad \sigma(P_0) = \{P(\xi), \xi \in E^n\},$$

where  $\xi = (\xi_1, \dots, \xi_n)$ . Thus (1.4) holds for all  $p$  satisfying  $1 < p < \infty$  when

$$(1.5) \quad b \geq mn/(n+2).$$

COROLLARY 1.2. *If  $P(D)$  is such that*

$$(1.6) \quad |P(\xi)| \rightarrow \infty \text{ as } |\xi| \rightarrow \infty,$$

then there is an  $\eta > 0$  such that (1.4) holds when

$$(1.7) \quad |1/p - 1/2| < \eta.$$

In describing our next result we let  $\mu = (\mu_1, \dots, \mu_n)$  denote a multi-index of non-negative integers and put

$$P^{(\mu)}(\xi) = \partial^{|\mu|} P(\xi) / \partial \xi_1^{\mu_1} \dots \partial \xi_n^{\mu_n},$$

where  $|\mu| = \mu_1 + \dots + \mu_n$ .

THEOREM 1.3. *Assume that  $1 < p < \infty$  and that  $P(\xi)$  satisfies (1.2) and*

$$(1.8) \quad P^{(\mu)}(\xi)/P(\xi) = O(1/|\xi|^{a|\mu|}) \text{ as } |\xi| \rightarrow \infty, |\mu| \leq l,$$

for real  $a \leq 1$  and

$$(1.9) \quad b > (1-a)n|1/p - 1/2|,$$

where  $l$  is the smallest integer greater than  $n|1/p - 1/2|$ . Then (1.4) holds.

Balslev [6] proved (1.4) for elliptic operators. Iha and Schubert [4, 5] proved that for  $1 < p < \infty$  and any operator satisfying (1.6) either  $\sigma(P_0)$  is the whole complex plane or (1.4) holds. They also proved a theorem slightly weaker than Theorem 1.1. They give an example of a fourth order operator for which (1.4) does not hold when

$$|1/p - 1/2| > 3/8.$$

Our Theorem 1.3 applied to this operator shows that (1.4) does hold for

$$|1/p - 1/2| < 2/9.$$

It would be of interest to know the exact point of demarcation. I would like to thank F.T. Iha and C.F. Schubert for informing me about their work.

**2. A. Multiplier Theorem.**

Let  $S$  denote the set of all complex valued functions  $v \in C^\infty(E^n)$  such that

$$|x|^k |D^\mu v|$$

is bounded on  $E^n$  for each integer  $k \geq 0$  and multi-index  $\mu$ , where

$$D^\mu = D_1^{\mu_1} \dots D_n^{\mu_n}.$$

If  $F$  denotes the Fourier transform, then both  $F$  and  $F^{-1}$  map  $S$  into itself. A function  $m(\xi)$  on  $E^n$  is called a multiplier in  $L^p$  if there exists a constant  $C$  such that

$$(2.1) \quad \|F^{-1}[m(\xi)Fu]\|_p \leq C \|u\|_p, u \in S,$$

where  $\xi$  is the argument of the Fourier transform and the norm is that of  $L^p$ .

The following is a special case of a theorem due to Littman [1].

**THEOREM 2.1.** *Let  $m_1, m_2$  be non-negative integers and suppose  $\theta, p$  satisfy  $0 \leq \theta \leq 1, 1 < p < \infty$  and*

$$(2.2) \quad (1 - \theta)m_1 + \theta m_2 > n |1/p - 1/2|.$$

*Suppose  $w(\xi)$  is a function in  $C^m(E^n)$  satisfying*

$$(2.3) \quad \max_{R < |\xi| < 2R} |D^\mu w|^{1-\theta} \max_{R < |\xi| < 2R} |D^\nu w|^\theta \leq C/R^{(1-\theta)|\mu| + \theta|\nu|}$$

*for all  $\mu, \nu$  such that  $|\mu| \leq m_1$  and  $|\nu| \leq m_2$ , where  $m = \max(m_1, m_2)$ . Then  $w$  is a multiplier in  $L^p$ .*

We shall use this theorem to prove

**THEOREM 2.2.** *Suppose  $1 < p < \infty$ , and let  $l$  be the smallest integer greater than  $n |1/p - 1/2|$ . Assume that  $w(\xi)$  is a function in  $C^l(E^n)$  such that*

$$(2.4) \quad |D^\mu w(\xi)| \leq C |\xi|^{a|\mu|+b}, \quad |\mu| \leq l,$$

for real  $a \leq 1$  and

$$(2.5) \quad b > (1-a)n \left| \frac{1}{p} - \frac{1}{2} \right|.$$

Then  $w$  is a multiplier in  $L^p$ .

PROOF. First assume that  $b \geq (1-a)l$ . In this case we take  $m_1 = m_2 = l$  and  $\theta = 0$ . Inequality (2.4) implies

$$(2.6) \quad \max_{R < |\xi| < 2R} |D^\mu w(\xi)| \leq C/R^{a|\mu|+b}.$$

Since  $b \geq (1-a)l$ ,  $b \geq (1-a)|\mu|$  for  $|\mu| \leq l$ , and consequently  $a|\mu| + b \geq |\mu|$ . Thus (2.6) implies (2.3). Since all of the hypotheses are satisfied, we obtain the desired conclusion from Theorem 2.1.

Next assume  $b < (1-a)l$ . In this case we must have  $a < 1$ . By (2.5) and the definition of  $l$  we have

$$0 < l - \frac{b}{1-a} < l - n \left| \frac{1}{p} - \frac{1}{2} \right| \leq 1,$$

and consequently there is a  $\theta$  satisfying  $0 < \theta < 1$  such that

$$0 < l - \frac{b}{1-a} < \theta < l - n \left| \frac{1}{p} - \frac{1}{2} \right| \leq 1.$$

In particular we have

$$(2.7) \quad n \left| \frac{1}{p} - \frac{1}{2} \right| < l - \theta < \frac{b}{1-a}.$$

We now take  $m_1 = l$ ,  $m_2 = l - 1$  and  $\theta$  satisfying (2.7). Note that (2.6) implies

$$(2.8) \quad \max_{R < |\xi| < 2R} |D^\mu w|^{1-\theta} \max_{R < |\xi| < 2R} |D^\nu w|^\theta \leq C/R^{a[(1-\theta)|\mu| + \theta|\nu|] + b}.$$

If  $|\mu| \leq m_1 = l$  and  $|\nu| \leq m_2 = l - 1$ , then

$$(1-\theta)|\mu| + \theta|\nu| \leq l - \theta < b/(1-a)$$

by (2.7). This is equivalent to

$$a[(1-\theta)|\mu| + \theta|\nu|] + b \geq (1-\theta)|\mu| + \theta|\nu|$$

for such  $\mu$  and  $\nu$ . Thus (2.8) implies (2.3) and all of the hypotheses of Theorem 2.1 are satisfied. This completes the proof.

The connection between multipliers and the spectrum of a partial differential operator is given by

**THEOREM 2.3.** *For  $1 \leq p < \infty$ , a point  $\lambda$  is in  $\rho(P_{0p})$  if and only if  $1/[P(\xi) - \lambda]$  is a multiplier in  $L^p$ .*

**PROOF.** We may take  $\lambda = 0$ . If  $0 \in \rho(P_{0p})$ , then there is a constant  $C$  such that

$$(2.9) \quad \|u\|_p \leq C \|P_{0p} u\|_p, \quad u \in D(P_{0p}).$$

In particular we have

$$(2.10) \quad \|v\|_p \leq C \|P(D)v\|_p, \quad v \in S.$$

Now (2.10) implies that

$$(2.11) \quad |P(\xi)| \geq 1/C, \quad \xi \in E^n.$$

To see this, let  $\xi$  be any vector in  $E^n$  and let  $\psi$  be any function in  $C_0^\infty$  such that  $\|\psi\|_p = 1$ . Set

$$(2.12) \quad \varphi_k(x) = \exp\{i(x_1\xi_1 + \dots + x_n\xi_n)\} \psi(x/k)/k^{n/p}, \quad k = 1, 2, \dots$$

Then  $\varphi_k \in C_0^\infty$  for each  $k$  and  $\|\varphi_k\|_p = 1$ . Furthermore by Leibnitz's formula

$$P(D)\varphi_k = \sum_{\mu} P^{(\mu)}(\xi) \psi_{\mu}(x/k) \exp\{i(x_1\xi_1 + \dots + x_n\xi_n)\} / \mu! k^{|\mu|+(n/p)},$$

where  $\psi_{\mu}(x) = D^{\mu} \psi(x)$  and  $\mu! = \mu_1! \dots \mu_n!$ . Since

$$\|\psi_{\mu}(x/k)/k^{n/p}\|_p = \|\psi_{\mu}\|_p,$$

We see that

$$\|P(D)\varphi_k\|_p \rightarrow |P(\xi)| \quad \text{as } k \rightarrow \infty.$$

But by (2.10)

$$1 = \|\varphi_k\|_p \leq C \|P(D)\varphi_k\|_p, \quad k = 1, 2, \dots$$

This gives (2.11). Let  $f$  be any function in  $S$ , and set

$$(2.13) \quad u = F^{-1} \left[ \frac{1}{P} Ff \right].$$

By (2.11) we know that  $u \in S$ . Furthermore  $P(D)u = f$ . Thus (2.10) implies

$$(2.14) \quad \left\| F^{-1} \left[ \frac{1}{P} Ff \right] \right\|_p \leq C \|f\|_p.$$

This shows that  $1/P(\xi)$  is a multiplier in  $L^p$ .

Conversely, suppose  $1/P(\xi)$  is a multiplier. Thus (2.14) holds. Since all multipliers are bounded, (2.11) holds for some constant  $C$ . Thus for each  $f \in S$  there is a  $u \in S$  such that  $P(D)u = f$  and

$$\|u\|_p \leq C \|f\|_p$$

by (2.14). Since  $S$  is dense in  $L^p$ , this shows that  $0 \in \rho(P_0)$ . This completes the proof.

### 3. Proofs of the Theorems.

We first give the

**PROOF OF THEOREM 1.3.** It is well known that

$$(3.1) \quad \sigma(P_0) \supset \{P(\xi), \xi \in E^n\}$$

(see [3, 4]). Thus it suffices to show that each  $\lambda$  satisfying  $P(\xi) \neq \lambda$  for  $\xi \in E^n$  is in  $\rho(P_0)$ . We may take  $\lambda = 0$ . By Theorem 2.3 it suffices to show that  $1/P(\xi)$  is a multiplier in  $L^p$ . Since  $a \leq 1$ , we must have  $b > 0$  and consequently (1.6) holds. This means that there is a constant  $C$  such that (2.11) is satisfied. Now for each  $\mu$  the derivative  $D^\mu(1/P)$  consists of a sum of terms of the form

$$(3.2) \quad \text{constant } P^{(\mu^{(1)})}(\xi) \dots P^{(\mu^{(k)})}(\xi) / P(\xi)^{k+1},$$

where  $\mu^{(1)} + \dots + \mu^{(k)} = \mu$  (this is easily verified by a simple induction argument). Thus

$$(3.3) \quad |D^\mu(1/P)| \leq C / |\xi|^{\alpha|\mu|+b}, \quad |\mu| \leq l.$$

We may apply Theorem 2.2 to conclude that  $1/P(\xi)$  is an  $L^p$  multiplier. This completes the proof.

Next we give the

PROOF OF THEOREM 1.1. Since  $P(\xi)$  is of degree  $m$ ,  $P^{(\mu)}(\xi)$  is of degree at most  $m - |\mu|$ . Hence

$$(3.4) \quad P^{(\mu)}(\xi)/P(\xi) = 0 \quad (|\xi|^{m-|\mu|-b}) \text{ as } |\xi| \rightarrow \infty.$$

For  $|\mu| \geq 1$  we have  $m - |\mu| - b \leq (m - b - 1) |\mu|$ . Hence (3.4) gives

$$(3.5) \quad P^{(\mu)}(\xi)/P(\xi) = 0 \quad (|\xi|^{(m-b-1)|\mu|}) \text{ as } |\xi| \rightarrow \infty.$$

This shows that (1.8) holds with  $a = b + 1 - m$ . If we now apply Theorem 1.3 we see that (1.4) holds provided (1.3) is satisfied. If (1.5) holds, then the right hand side of (1.3) is  $\geq 1/2$ . This allows  $p$  to take on all values between 1 and  $\infty$ .

PROOF OF COROLLARY 1.2. We merely note that (1.6) implies (1.2) for some constant  $b > 0$ . Thus by Theorem 1.1 we may take  $\eta = b/n(m - b)$ .

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