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AN EXISTENCE THEOREM FOR BOUNDED VECTOR-VALUED FUNCTIONS

S. ZAIDMAN (*)

Introduction.

In professor's L. Amerio paper [1], supposing existence of bounded solutions for $t \geq 0$ (t -time), of non-linear almost-periodic differential equations, one proves existence of bounded solutions which are defined on the whole time axis, $-\infty < t < \infty$.

In our paper [2] we proved a very similar result for solutions of the heat equation, with almost-periodic known term. We shall see below that this situation can be extended to a certain class of Banach-space valued functions admitting a certain representation through a given semi-group of class O^0 .

§ 1. Let us consider first a reflexive Banach space X ; then, a one-parameter semi-group of operators in $L(X, X): T_t, t \geq 0$; such that $T_0 = I$, $T_{t+\zeta} = T_t T_\zeta$; $T_t \in L(X, X) \forall t \geq 0$ and $T_t x$ is continuous from $0 \leq t < \infty$ to X .

Consider also a continuous function $-\infty < t < \infty$ to X , which is almost-periodic in Bochner's sense, that is:

Each sequence $(f(t + a_n))_{n=1}^\infty$ contains a subsequence $(f(t + a_{n_p}))_{p=1}^\infty$ which is uniformly convergent on $-\infty < t < \infty$, in strong topology of X .

Let now $u(t)$ be a continuous function: $0 \leq t < \infty$ to X , admitting representation

$$(1.1) \quad u(t) = T_t u(0) + \int_0^t T_{t-\zeta} f(\zeta) d\zeta, \quad \forall t \geq 0$$

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and let us assume

$$(1.2) \quad M = \sup_{t \geq 0} \|u(t)\|_X < \infty.$$

Then we have

THEOREM. *There exists a continuous function $W(t)$, $-\infty < t < \infty$ to X , such that*

$$(1.3) \quad W(t) = T_{t-t_0} W(t_0) + \int_{t_0}^t T_{t-\zeta} f(\zeta) d\zeta, \quad \forall t \geq t_0$$

$$(1.4) \quad \sup_{-\infty < t < \infty} \|W(t)\| < \infty.$$

PROOF. Let us consider the sequence of translates

$$u_n(t) = u(t+n)$$

They are defined for $t \geq -n$, and we have

$$(1.5) \quad \sup_{t \geq -n} \|u_n(t)\| = M = \sup_{t \geq 0} \|u(t)\|$$

As well known, reflexivity implies weak sequential compactness of bounded sets in X . Then, using almost-periodicity of $f(t)$ and the diagonal procedure, we obtain a sequence of positive integers $(n_k)_1^\infty$ with following properties:

$$(1.6) \quad \lim_{k \rightarrow \infty} f(t+n_k) = g(t), \text{ uniformly on } -\infty < t < \infty \text{ (and consequently } g(t) \text{ is an almost-periodic function)}$$

$$(1.7) \quad \text{For each } N = 0, 1, 2, \quad u_{n_k}(-N) \text{ is defined for } k > N,$$

and

$$(1.8) \quad (w) \lim_{k \rightarrow \infty} u_{n_k}(-N) = W_N \text{ exists and belongs to } X \text{ (here } (w) \text{ — means weak topology in } X; \text{ remember that a reflexive space is weakly sequentially complete).}$$

Remark now, that for each real t , $u_{n_k}(t)$ is defined for $k \geq k_t$. Then we shall see that, $\forall t \in (-\infty, \infty)$

$$(1.9) \quad (w) \lim_{k \rightarrow \infty} u_{n_k}(t) = V(t) \text{ exists.}$$

$$(1.10) \quad \sup_{-\infty < t < \infty} \|V(t)\| < \infty$$

$$(1.11) \quad V(t) = T_{t-t_0} V(t_0) + \int_{t_0}^t T_{t-\sigma} g(\sigma) d\sigma, \quad \forall t \geq t_0, \quad \forall t_0 \in \mathbb{R}^1.$$

In fact (1.10) is a consequence of (1.9) and (1.5). To prove (1.9) we use following

LEMMA 1. Let $t \in (-\infty, \infty)$ be given, and N a positive integer such that $t + N > 0$. Then, $\forall k > N$, we have

$$(1.12) \quad u_{n_k}(t) = T_{t+N} u_{n_k}(-N) + \int_{-N}^t T_{t-\tau} f(\tau + n_k) d\tau.$$

This Lemma is a Corollary of a slightly more general result

LEMMA 2. Let $u(t), t \geq 0 \rightarrow \mathcal{X}$ (arbitrary Banach space), be a continuous function; $T_t; t \geq 0 \rightarrow L(\mathcal{X}, \mathcal{X})$ be a strongly continuous one parameter semi-group of linear bounded operators in \mathcal{X} ; $f(t), -\infty < t < \infty \rightarrow \mathcal{X}$ be a continuous function.

Suppose

$$u(t) = T_t u(0) + \int_0^t T_{t-\tau} f(\tau) d\tau, \quad \forall t \geq 0.$$

Then, if $t \in (-\infty, \infty)$, is given and $b > a > 0, a + t > 0$, we have

$$(1.13) \quad u(t+b) = T_{t+a} u(b-a) + \int_{-a}^t T_{t-\zeta} f(\zeta + b) d\zeta$$

REMARK. Lemma 1 follows from Lemma 2 if we take $b = n_k, a = N$.

PROOF OF LEMMA 2.

As $t+b > t+a > 0$, we have using (1.1)

$$(1.14) \quad u(t+b) = T_{t+b} u(0) + \int_0^{t+b} T_{t+b-\zeta} f(\zeta) d\zeta = T_{t+a} T_{b-a} u(0) + \int_0^{t+b} T_{t+b-\sigma} f(\sigma) d\sigma.$$

Next remark, again by (1.1), the representation

$$u(b-a) = T_{b-a} u(0) + \int_0^{b-a} T_{b-a-\sigma} f(\sigma) d\sigma.$$

Introducing in (1.14) the value of $T_{b-a} u(0)$ we get

$$(1.15) \quad u(t+b) = T_{t+a} \left(u(b-a) - \int_0^{b-a} T_{b-a-\sigma} f(\sigma) d\sigma \right) + \int_0^{t+b} T_{t+b-\sigma} f(\sigma) d\sigma = \\ T_{t+a} u(b-a) + \int_{b-a}^{t+b} T_{t+b-\sigma} f(\sigma) d\sigma.$$

Now, set $\sigma = \zeta + b$; it follows $\int_{b-a}^{t+b} T_{t+b-\sigma} f(\sigma) d\sigma = \int_{-a}^t T_{t-\zeta} f(\zeta + b) d\zeta$ which proves our Lemma, and consequently Lemma 1 too.

Actually we see that (1.9) is true in the following way: Fix an arbitrary real t ; then take N a positive integer, such that $t + N > 0$, and take $k > N$. We use then (1.12); as $f(t + n_k) \rightarrow g(t)$ uniformly on $(-\infty, \infty)$ and in X strong, we have obviously

$$\lim_{k \rightarrow \infty} \int_{-N}^t T_{t-\zeta} f(\zeta + n_k) d\zeta = \int_{-N}^t T_{t-\zeta} g(\zeta) d\zeta.$$

Then we have also

$$(w) \lim_{k \rightarrow \infty} T_{t+N} u_{n_k}(-N) = T_{t+N} W_N.$$

because a linear continuous operator in a B -space is continuous also in respect to the weak convergence.

Now, we shall see that for function $V(t)$, $-\infty < t < \infty \rightarrow X$ defined by (1.9), the representation formula (1.11) holds for each semi axis $t \geq t_0$.

Take in fact two reals $t \geq t_0$, and choose an integer N such that $-N < t_0$. Apply Lemma 1 to t, t_0, N . We have

$$u_{n_k}(t) = T_{t+N} u_{n_k}(-N) + \int_{-N}^t T_{t-\zeta} f(\zeta + n_k) d\zeta \\ u_{n_k}(t_0) = T_{t_0+N} u_{n_k}(-N) + \int_{-N}^{t_0} T_{t_0-\zeta} f(\zeta + n_k) d\zeta.$$

Then, reasoning as above, we obtain

$$V(t) = T_{t+N} W_N + \int_{-N}^t T_{t-\zeta} g(\zeta) d\zeta$$

$$V(t_0) = T_{t_0+N} W_N + \int_{-N}^{t_0} T_{t_0-\zeta} g(\zeta) d\zeta$$

Trying now to get (1.11) we write

$$T_{t-t_0} V(t_0) = T_{t-t_0} \left(T_{t_0+N} W_N + \int_{-N}^{t_0} T_{t_0-\zeta} g(\zeta) d\zeta \right) = T_{t+N} W_N +$$

$$+ \int_{-N}^{t_0} T_{t-\zeta} g(\zeta) d\zeta.$$

Hence

$$T_{t-t_0} V(t_0) + \int_{t_0}^t T_{t-\zeta} g(\zeta) d\zeta = T_{t+N} W_N + \int_{-N}^t T_{t-\zeta} g(\zeta) d\zeta = V(t)$$

that is (1.11).

Now we give the idea of the final step in the proof. Using uniform (on real axis) convergence of sequence $f(t + n_k)$ to $g(t)$, we obtain uniform convergence of sequence $g(t - n_k)$ to $f(t)$. Starting now with $V(t)$ and repeating above procedure, we obtain function $W(t)$, $-\infty < t < \infty$ which is continuous, bounded, and admits representation (1.3) $\forall t \geq t_0$.

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