Annali della
Scuola Normale Superiore di Pisa
*Classe di Scienze*

HADI M. HADDAD

# Chain sequence preserving linear transformations

*Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 3e série*, tome 24, nº 1 (1970), p. 79-84

<http://www.numdam.org/item?id=ASNSP_1970_3_24_1_79_0>

# CHAIN SEQUENCE PRESERVING LINEAR TRANSFORMATIONS

by HADI M. HADDAD

## 1. Introduction.

Let $T = (\alpha_{nk})$ be an infinite matrix defining a sequence to sequence linear transformation by

$$(Ta)_n = \Sigma_k \, \alpha_{nk} \, a_k \, .$$

The purpose of this paper is to study such linear transformations which preserve chain sequences. A number sequence $a$ is a *chain sequence* means that there exists a number sequence $g$ such that for every positive integer $r$, $0 \leq g_{r-1} \leq 1$ and $a_r = (1 - g_{r-1})\, g_r$. The following main result gives sufficient conditions that a sequence to sequence linear transformation is chain sequence preserving.

THEOREM 1. Let $T = (\alpha_{nk})$ be an infinite matrix such that for every positive integer $n$,

(i) $\alpha_{nk} = 0$ for $k < n$, and $\alpha_{nk} \geq 0$ for $k \geq n$,

(ii) $\Sigma_k \, \alpha_{nk} = \beta_n \leq \dfrac{1}{2}$, and

(iii) either $\alpha_{nk} \leq \alpha_{n+1,\, k+1}$ for $k \geq n$ or $\alpha_{nk} \geq \alpha_{n+1,\, k+1}$ for $k \geq n$.

Then $Ta$ is a chain sequence for every chain sequence $a$. If in addition, the sequence $\{\beta_n\}$ converges to $b$, then the chain sequence $Ta$ converges to $bc$ for every chain sequence $a$ which converges to $c$.

The proof of this theorem is based on a sufficient condition for a number sequence to be a chain sequence. We will first develop this sufficient condition.

---

Pervenuto alla Redazione il 25 Settembre 1969.

## 2. Sufficient condition for chain sequences.

The following theorem is found to be useful as well as it is interesting in its own right.

THEOREM 2. If $a$ is a sequence of non-negative numbers such that for each positive integer $r$, $\sqrt{a_r} + \sqrt{a_{r+1}} \leq 1$, then $a$ is a chain sequence.

To prove this theorem, the following well known lemma is needed. For a proof see [3, p. 86].

LEMMA 1. Let $a$ be a number sequence. The following two statements are equivalent

(1) $a$ is a chain sequence.

(2) For every number sequence $\xi$, and every positive integer $n$,

$$\sum_{r=1}^{n} \xi_r^2 - 2 \sum_{r=1}^{n-1} \sqrt{a_r}\, \xi_r\, \xi_{r+1} \geq 0.$$

PROOF OF THEOREM 2. Let $\xi$ be any number sequence, and $n$ any positive integer. Since $2\xi_r\, \xi_{r+1} \leq \xi_r^2 + \xi_{r+1}^2$, we have

$$\sum_{r=1}^{n} \xi_r^2 - 2 \sum_{r=1}^{n-1} \sqrt{a_r}\, \xi_r\, \xi_{r+1} \geq \sum_{r=1}^{n} \xi_r^2 - \sum_{r=1}^{n-1} \sqrt{a_r}\, (\xi_r^2 + \xi_{r+1}^2).$$

The right hand side of this inequality can be written in the form

$$(1 - \sqrt{a_1})\, \xi_1^2 + \sum_{r=1}^{n-2} (1 - \sqrt{a_r} - \sqrt{a_{r+1}})\, \xi_{r+1}^2 + (1 - \sqrt{a_{n-1}})\, \xi_n^2,$$

which is non-negative, since each term is non-negative. Consequently the condition of statement (2) of Lemma 1 is satisfied, and therefore $a$ is a chain sequence.

REMARK 1. We would like to note here that the condition stated in Theorem (2) is not necessary. To show this, consider the chain sequence $a$, generated by the sequence given by

$$g_n = \frac{1}{2} - \left(\frac{1}{2}\right)^{n+2}, \qquad n = 0, 1, 2, \ldots .$$

The chain sequence $a$ is then given by

$$a_n = \left[1 - \left\{\frac{1}{2} + \left(\frac{1}{2}\right)^{n+1}\right\}\right]\left[\frac{1}{2} - \left(\frac{1}{2}\right)^{n+2}\right],$$

which can be written as

$$a_n = \frac{1}{4}\left[1 + \left(\frac{1}{2}\right)^{n+1}\left\{1 - \left(\frac{1}{2}\right)^{n}\right\}\right],$$

which shows that $a_n > 1/4$ for every positive integer $n$. So the sum of the square roots of every two consecutive terms is greater than 1.

REMARK 2. In [1], T. S. Chihara proved that the condition of Theorem (2) is necessary and sufficient for periodic number sequences with period 2.

The following lemma, whose proof is obvious, is frequently used and it will be used in the proof of Theorem 1.

LEMMA 2. If $a$ is a chain sequence, then the sum of any two consecutive terms is not greater than 1.

## 3. Proof of Theorem 1.

To prove this theorem, let $a$ be a chain sequence. $Ta$ is given by

$$(Ta)_n = \Sigma_k \, \alpha_{nk} \, a_k .$$

Let us observe first that for each positive integer $n$, $(Ta)_n$ is an absolutely convergent series.

For each positive integer $n$, we have

$$(Ta)_n + (Ta)_{n+1} = \sum_{k=1}^{\infty} \alpha_{nk} \, a_k + \sum_{k=1}^{\infty} \alpha_{n+1,\,k+1} \, a_k$$

$$= \sum_{k=n}^{\infty} (\alpha_{nk} \, a_k + \alpha_{n+1,\,k+1}).$$

This last expression is either not greater than

$$\sum_{k=n}^{\infty} \alpha_{n+1,\,k+1} \, (a_k + a_{k+1}), \tag{3.1}$$

or not greater than

$$\sum_{k=n}^{\infty} \alpha_{nk} (a_k + a_{k+1}). \tag{3.2}$$

This is true because of condition (iii).

Each of the expressions (3.1) and (3.2) is not greater than 1/2, since $a_k + a_{k+1} \leq 1$ for every $k$. So we have

$$(Ta)_n + (Ta)_{n+1} \leq \frac{1}{2}. \tag{3.3}$$

Now

$$\begin{aligned} \sqrt{(Ta)_n} + \sqrt{(Ta)_{n+1}} &\leq [2 \{(Ta)_n + (Ta)_{n+1}\}]^{\frac{1}{2}} \\ &\leq \sqrt{\left[2\left(\frac{1}{2}\right)\right]} \\ &\leq 1, \end{aligned}$$

which proves that $Ta$ is a chain sequence.

To prove the second part of this theorem, let $a$ be a chain sequence which converges to a number $c$. Let us note first that $0 \leq c \leq 1/4$. Let $\varepsilon > 0$ and let $M$ be a positive integer such that $|a_k - c| < \varepsilon$ for all $k \geq M$, and let $N$ be a positive integer greater than $M$ and such that $|\beta_n - b| < \varepsilon$ for all $n \geq N$.

For $n \geq N$, we have

$$\begin{aligned} |(Ta)_n - bc| &\leq \left| \sum_{k=1}^{N-1} \alpha_{nk} (a_k - c) \right| + \sum_{k=N}^{\infty} \alpha_{nk} |a_k - c| + c |\beta_n - b| \\ &< 3\varepsilon/4, \end{aligned} \tag{3.4}$$

for the first term on the right hand side is zero, the second is less than $\varepsilon/2$, and the third is less than $\varepsilon/4$.

This proves that the chain sequence $Ta$ is convergent and it converges to $bc$.

## 4. Linear combinations of chain sequences.

In a previous paper [2], the author of this paper obtained conditions sufficient for a linear combination of two chain sequences to be a chain sequence. The following theorem is an extension of that result to linear combinations of a sequence of chain sequences.

THEOREM 3. Let $\{\alpha_n\}$ be a sequence of non-negative numbers with $\Sigma\, \alpha_n \leq \frac{1}{2}$. Then each of the following statements is true.

(1) If for each positive integer $n$, $\{a_r^n\}_{r=1}^{\infty}$ is a chain sequence, then $\{\Sigma_n\, \alpha_n\, a_r^n\}_{r=1}^{\infty}$ is a chain sequence.

(2) If for each positive integer $n$, $\{a_r^n\}_{r=1}^{\infty}$ is a convergent chain sequence and converges to $c_n$, then $\{\Sigma_n\, \alpha_n\, a_r^n\}_{r=1}^{\infty}$ is a convergent chain sequence, and it converges to $\Sigma\, \alpha_n\, c_n$.

PROOF. In proving this theorem we will make use of Theorem 2. Let us first observe that for each positive integer $r$, $\Sigma_n\, \alpha_n\, a_r^n$ converges absolutely.

For each positive integer $r$, we have

$$(\Sigma_n\, \alpha_n\, a_r^n)^{\frac{1}{2}} + (\Sigma_n\, \alpha_n\, a_{r+1}^n)^{\frac{1}{2}} \leq [2\,(\Sigma_n\, \alpha_n\, a_r^n + \Sigma_n\, \alpha_n\, a_{r+1}^n)]^{\frac{1}{2}}$$
$$\leq [2\, \Sigma\, \alpha_n]^{\frac{1}{2}}$$
$$\leq 1.$$

This proves that $\{\Sigma_n\, \alpha_n\, a_r^n\}_{r=1}^{\infty}$ is a chain sequence.

Now let $\varepsilon > 0$ and let $K$ be a positive integer such that

$$\sum_{n=m+1}^{\infty} \alpha_n < \frac{\varepsilon}{2} \tag{4.1}$$

for all $m \geq K$.

Let $M \geq K$ and let $N$ be a positive integer such that $r \geq N$ implies that

$$|\, a_r^n - c_n\,| < \varepsilon, \quad \text{for} \quad n = 1, 2, \ldots, M. \tag{4.2}$$

Now for $r \geq N$, we have

$$|\, \Sigma_{n=1}^{\infty}\, \alpha_n\, a_r^n - \Sigma_{n=1}^{\infty}\, \alpha_n\, c_n\,| \leq \Sigma_{n=1}^{\infty}\, |\, a_r^n - c_n\,|$$
$$< \Sigma_{n=1}^{M}\, \alpha_n\, |\, a_r^n - c_n\,| + \frac{\varepsilon}{2}$$
$$< \varepsilon\, \Sigma_{n=1}^{M}\, \alpha_n + \frac{\varepsilon}{2}$$
$$< \varepsilon,$$

which proves that $\{\Sigma_{n=1}^{\infty}\, \alpha_n\, a_r^n\}_{r=1}^{\infty}$ converges and it converges to $\Sigma_{n=1}^{\infty}\, \alpha_n\, c_n$.

## REFERENCES

[1] T. S. CHIHARA, *On recursively defined orthogonal polynomials,* Proc. Amer. Math. Soc., vol. 16 (1965), 702-710.

[2] H. M. HADDAD, *Linear combinations of chain sequences,* Bulletin of the College of Science, Baghdad, Iraq, vol. 8 (1965), 27-30.

[3] H. S. WALL, *Analytic theory of continued fractions,* Van Nostrand, New York, 1948.

*University of Colorado*