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COHOMOLOGY AND HOLOMORPHIC DIFFERENTIAL FORMS ON COMPLEX ANALYTIC SPACES (*)

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Introduction.

The concept of holomorphic differential forms on a complex analytic manifold is well known. The problem of its extension to complex analytic spaces has been already discussed by Reiffen and Vetter (see [2]). They gave four inequivalent definitions of sheaves of germs of holomorphic differential forms on a space X. It seems that actually no one of these possible definitions is privileged, as any method offers advantages according to the aims wished. For instance, using the first of these definitions, Reiffen proved, under an appropriate hypothesis of local contractibility, a « Poincaré Lemma » (see [1]). But is not easy to apply this lemma in order to study the cohomology with constant coefficients of a complex analytic space X, since the sheaves considered by Reiffen in [1] do not necessarily vanish above the complex dimension of X.

In this work, starting from the foregoing considerations, we consider another sheaf $\widetilde{\Omega}^p(X)$ of germs of holomorphic differential forms on X, in order to get a suitable tool for the study of the cohomology with constant coefficients of X.

The sheaf $\widetilde{\Omega}^p(X)$ which we introduce turns out to be coherent; furthermore $\widetilde{\Omega}^p(X)$ is zero for p larger than the complex dimension of the space X, and under the same assumptions of Reiffen, a Poincaré Lemma can be proved for $\widetilde{\Omega}^p(X)$ (see § 3).

From these properties of $\widetilde{\Omega}^p(X)$, we can deduce the following results:

1) A De Rham theorem for « contractible » Stein spaces.

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2) A criterion for the vanishing (resp. for the finite dimensionality) of certain cohomology groups in the case of cohomologically q-complete (resp. q-convex) spaces which are « contractible ».

3) A relative De Rham theorem for « contractible » Stein spaces. By differents methods, a relative De Rham theorem had been obtained in some particular cases by Succi (see [5], [6]).

After completing this work, I learned that, in a forthcoming paper; Bloom and Herrera have considered sheaves, analogous to $\widetilde{\Omega}^p(X)$, in the differential case.

I should like to thank Prof. Villani for help and encouragement given to me during this research.

§ 1. Sheaves of germs of holomorphic differential forms on a complex analytic space.

In the following we shall consider analytic sets rather than analytic spaces. There will be no loss of generality, since every local statement, which is true for analytic sets, is also true for analytic spaces.

Let G denote a complex analytic manifold or, more simply, an open connected subset of $\mathbb{C}^n$ with structure sheaf $\mathcal{O} = \mathcal{O}_G$; let $\Omega^p(G)$ be the sheaf of germs of holomorphic differential forms of degree p defined on G; $d = d_p$ the external differential operator; A a closed analytic subset of G and $\mathcal{I} = \mathcal{I}(A)$ the coherent sheaf of germs of all holomorphic functions defined on G and vanishing on A.

Let A_{reg} be the set of regular points of A; the natural injection $j: A_{\text{reg}} \longrightarrow G$, induces, for every open subset $U \subset G$, a homorphism $j_U^*: \Gamma(U, \Omega^p(G)) \longrightarrow \Gamma(U \cap A, \Omega^p(A_{\text{reg}}))$.

DEFINITION (1.1). The restriction $\omega \mid A$ of a differential form $\omega \in \Gamma(U, \Omega^p(G))$ to A is the form $j_U^*(\omega)$ defined on the set of regular points of A contained in U. We say ω is zero (or vanishes) on A when $j_U^*(\omega) = 0$.

Let $\mathcal{H}^p(G)$, $(p \geq 0)$, be the subsheaf of $\Omega^p(G)$ of the germs of holomorphic differential forms vanishing on A.

It follows that $\mathcal{H}^0(G) = \mathcal{I}$. The sheaf $\Omega^p(G)/\mathcal{H}^p(G)$ is zero outside of A. Let $\widetilde{\Omega}^p(A)$ be the restriction of $\Omega^p(G)/\mathcal{H}^p(G)$ to A and $\mathcal{O}_A = \mathcal{O}_G/\mathcal{I} \mid A$ the structure sheaf of A.

DEFINITION (1.2). $\widetilde{\Omega}^p(A)$ will be called the sheaf of germs of holomorphic differential forms of degree p defined on the analytic space $(A, \mathcal{O}_A)$.

The shaf $\widetilde{\Omega}^p(A)$ becomes, in natural way, a sheaf of $\mathcal{O}_A$-modules and coincides, on A_{reg}, with the sheaf of germs of holomorphic differential forms defined in the classical way for manifolds. Furthermore it is not difficult to check that $\widetilde{\Omega}^p(A)$ depends only on the complex analytic structure of A (and not on the imbedding of A in G).

Notice that $\widetilde{\Omega}^p(A)$ is the zero sheaf if p is larger than the complex dimension of A (in fact every p-form defined on G vanishes on A, in the sense of def. (1.1)).

It will be useful, for the following, to characterize the sheaf $\mathcal{H}^p(G)$ in two other ways.

Let $A^0 = A$, $A^1 = A_{\text{sing}} =$ set of singular points of A, $A^2 = A^1_{\text{sing}}$, etc., we give the two following definitions:

1) Let $\mathcal{H}^p_1$ be the subsheaf of $\Omega^p(G)$ consisting of the germs of differential forms ω satisfying $\omega \mid A^\nu_{\text{reg}} = 0$ for all $\nu \geq 0$

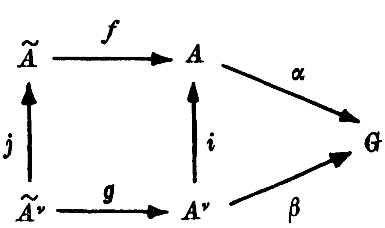
2) Let $\mathcal{H}^p_2$ be the subsheaf of $\Omega^p(G)$ consisting of the germs of differential forms ω, for which, for any complex manifold W and any holomorphic map $\varphi: W \longrightarrow G$, such that $\varphi(W) \subset A$, one has $\varphi^*(\omega) = 0$; where $\varphi^*: \Omega^p(G) \longrightarrow \Omega^p(W)$ is the induced map.

Lemma (1.1).

$$\mathcal{H}^p_1 = \mathcal{H}^p_2 = \mathcal{H}^p .$$

Proof. Since the inclusions $\mathcal{H}^p_1 \subset \mathcal{H}^p$, $\mathcal{H}^p_2 \subset \mathcal{H}^p$ are trivial, it suffices to prove that $\mathcal{H}^p_1 \supset \mathcal{H}^p$ and $\mathcal{H}^p_2 \supset \mathcal{H}^p$.

For the first, consider the following commutative diagram:



where $(\widetilde{A}, f)$ is a Hironaka resolution of A (locally a Hironaka resolution always exists, see [3]); $\widetilde{A}^\nu = f^{-1}(A^\nu)$, $g = f \mid \widetilde{A}^\nu$ and i, j, α, β, are canonical injections.

It is easy to see that $\omega \in \mathcal{H}^p(G)$ if and only if $(\alpha \circ f)^*(\omega)$ vanishes on $\widetilde{A}$. Therefore if $\omega \mid A = 0$, by commutativity of the diagram, also $(\beta \circ g)^*(\omega) = 0$. Now this implies $\beta^*(\omega) = 0$. In fact, if $\beta^*(\omega) \neq 0$ on A^ν, it follows that $\beta^*(\omega) \neq 0$ on some connected component, say $(A^{\nu}_{\rm reg})_1$, of $A^{\nu}_{\rm reg}$. From the surjectivity of $g \mid g^{-1}(A^{\nu}_{\rm reg})_1 : g^{-1}(A^{\nu}_{\rm reg})_1 \longrightarrow (A^{\nu}_{\rm reg})_1$ and from the rank theorem (see [4]), we get injectivity of g^*, and hence $g^*(\beta^*(\omega)) \neq 0$, a contradiction.

We have thus proved that $\omega \mid A = 0$ implies $\omega \mid A^{\nu} = 0$.

To prove that $\mathcal{H}^p_2 \supset \mathcal{H}^p$ we proceed as follows: let W be a complex manifold an $\varphi : W \longrightarrow A$ a holomorphic map. There exists an integer ν_0 such that $\varphi(W) \subset A^{\nu_0}$ and $\varphi(W) \cap A^{\nu_0}_{\rm reg} \neq \emptyset$.

If $\omega \in \mathcal{H}^p$, we have $\varphi^*(\omega) \mid \varphi^{-1}(A^{\nu_0}_{\rm reg}) = \varphi^*(\omega \mid A^{\nu_0}_{\rm reg}) = \varphi^*(0) = 0$. Now, by the assumptions, $\varphi^{-1}(A^{\nu_0}_{\rm reg})$ is an open subset of W, which is dense in W. Therefore $\varphi^*(\omega) \mid \varphi^{-1}(A^{\nu_0}_{\rm reg}) = 0$ implies $\varphi^*(\omega) = 0$ on W.

PROPOSITION (1.1). Let X, Y be analytic spaces, and $\varphi : X \longrightarrow Y$ a holomorphic map. Then for every open subset $U \subset Y$, φ induces a natural homomorphism $\varphi^* : \Gamma(U, \widetilde{\Omega}^p(Y)) \longrightarrow \Gamma(\varphi^{-1}(U), \widetilde{\Omega}^p(X))$.

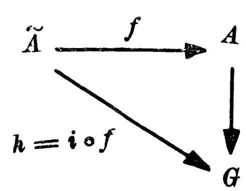
PROOF. We may assume X (resp. Y) locally imbedded in $\mathbb{C}^m$ (resp. $\mathbb{C}^n$). Locally the map φ is the restriction of a holomorphic map $\widetilde{\varphi} : \mathbb{C}^m \longrightarrow \mathbb{C}^n$. By lemma (1.1) it is clear that if $\omega \in \Omega^p(\mathbb{C}^n)$ vanishes on Y, then $\widetilde{\varphi}^*(\omega)$ vanishes on X. Then the proposition follows.

§ 2. The coherence theorem.

LEMMA (2.1). Let G be a complex analytic manifold with structure sheaf $\mathcal{O}_G$, and A an analytic subset of G. Then $\mathcal{H}^p(G)$ is $\mathcal{O}_G$-coherent.

PROOF. Since this is a local statement, we may consider a Hironaka resolution $(\widetilde{A}, f)$ of A.

Consider the following diagram:



Since f is a proper map, so is h. By a theorem of Grauert, (see [5]), the sheaf $h_0(\Omega^p(\widetilde{A}))$, zero direct image of $\Omega^p(\widetilde{A})$, is $\mathcal{O}_G$-coherent.

By proposition (1.1), $\mathcal{H}^p(G)$ is the kernel of an analytic homomorphism. Explicitly, define $\psi : \Omega^p(G) \longrightarrow h_0(\Omega^p(\widetilde{A}))$ as follows: $\psi(\omega) = h^*(\omega)$; where

$\omega \in \Omega^p(G)$, and $h^*(\omega) \in \Omega^p(\widetilde{A})$ is regarded as belonging to $h_0(\Omega^p(\widetilde{A}))$, by direct image.

This proves the lemma, since the kernel of an analytic homorphism between coherent sheaves is coherent.

THEOREM (2.1). Let G be a complex analytic manifold and A an analytic subset of G. Then the sheaf $\widetilde{\Omega}^p(A)$ is $\mathcal{O}_A$-coherent.

PROOF. Consider the exact sequence

$$0 \longrightarrow \mathcal{H}^p(G) \longrightarrow \Omega^p(G) \longrightarrow \Omega^p(G)/\mathcal{H}^p(G) \longrightarrow 0.$$

Since $\mathcal{H}^p(G)$ and $\Omega^p(G)$ are $\mathcal{O}_G$ coherent sheaves, so is $\Omega^p(G)/\mathcal{H}^p(G)$.

$\Omega^p(G)/\mathcal{H}^p(G) \mid A$ is a sheaf of $\mathcal{O}_A$-modules; hence our assertion follows.

§ 3. The Poincaré lemma.

Let G be a complex analytic manifold and A an analytic subset of G. By the classical Poincaré lemma, the sequence

$$0 \longrightarrow \mathbb{C} \xrightarrow{\varepsilon} \Omega^0(G)_x \xrightarrow{d} \Omega^1(G)_x \xrightarrow{d} \dots .$$

(where ε is the canonical injection) is exact. Since d maps $\mathcal{H}_x^p(G)$ into $\mathcal{H}_x^{p+1}(G)$, this sequence induces the following sequence of sheaves on A:

$$(*) \qquad 0 \xrightarrow{\dot{\varepsilon}} \mathbb{C} \longrightarrow \widetilde{\Omega}^0(A) \longrightarrow \widetilde{\Omega}^1(A) \longrightarrow \dots .$$

In this section we shall be concerned with the study of the exactness of the sequence $(*)$.

In the following, we write $\widetilde{\Omega}^p$ for $\widetilde{\Omega}^p(A)$.

LEMMA (3.1) (see [1] Hilfssatz 1). Let $F_1 \xrightarrow{d_1} F_2 \xrightarrow{d_2} F_3 \xrightarrow{d_3} \dots$ be an exact sequence of abelian groups and, for each integer n, let L_n be subgroup of F_n with $d_n(L_n) \subset L_{n+1}$.

If the sequence $L_p \longrightarrow L_{p+1} \longrightarrow L_{p+2} \longrightarrow \dots$ is exact, then the induced sequence

$$F_{p-1}/L_{p-1} \longrightarrow F_p/L_p \longrightarrow F_{p+1}/L_{p+1} \longrightarrow \dots (p \geq 2) \quad \text{is exact.}$$

COROLLARY (3.1). Let x be a point of A. If the sequence $\mathcal{H}_x^p \to \to \mathcal{H}_x^{p+1} \to \mathcal{H}_x^{p+2} \to \dots$ is exact, then the sequence $\tilde{\Omega}_x^{p-1} \to \tilde{\Omega}_x^p \to \tilde{\Omega}_x^{p+1} \to \dots$ is exact.

By corollary (3.1) it suffices to study the exactness of the sequence $\mathcal{H}_x^p \to \mathcal{H}_x^{p+1} \to \mathcal{H}_x^{p+2} \to \dots$.

We assume that A is locally contractible (in the following sense):

DEFINITION (3.1) (see [1] def. 2).

Let A, B be analytic subsets of a domain $G \subset \mathbb{C}^n$, and $O \in A \cap B$ be a regular point of B. The germ of analytic set A_0 is said to be $\mathbb{C}^n$-holomorphically contractible to the germ B_0 if, for each open subset V of $\mathbb{C}^n$, $V \subset G$ and V containing O; there exists an open subset $U \subset \mathbb{C}^n$ containing O, a domain $W \subset \mathbb{C}^1$ containing 0 and 1, and a holomorphic map $\Phi : U \times \times W \to V$ satisfying the following conditions:

1) $\Phi(z, 1) = z$ for each $z \in U$
2) $\Phi(z, 0) \in B$ for each $z \in U$
3) $\Phi((A \cap U) \times W) \subset A$.

LEMMA (3.2). Let A, B be analytic subsets of G and O a regular point of B, $O \in A \cap B$. If the germ A_0 is $\mathbb{C}^n$-holomorphically contractible to the germ B_0, then the sequence

$$\mathcal{H}_0^k \to \mathcal{H}_0^{k+1} \to \mathcal{H}_0^{k+2} \to \dots .$$

is exact ($k = \dim_0 B$).

PROOF. Let ω be a p-form ($p \geq k + 1$) defined on a neighbourhood V of O, vanishing on A, and such that $d\omega = 0$. We must show that there exists a $(p - 1)$-form ϑ, vanishing on A, and such that $d\vartheta = \omega$.

Let $A_U = A \cap U$, $A_V = A \cap V$. Consider the commutative diagram

(a)
$$\begin{array}{ccc} A_U \times W & \xrightarrow{i_1} & U \times W \\ \Psi \downarrow & & \downarrow \Phi \\ A_V & \xrightarrow{i_2} & V \end{array}$$

where i_1, i_2 are canonical injections, and $\Psi = \Phi \mid A_U \times W$. The induced diagram

(b)
$$\begin{array}{ccc} \Gamma(V, \Omega^p(V)) & \xrightarrow{i_2^*} & \Gamma(A_V, \tilde{\Omega}^p(A_V)) \\ \Phi^* \downarrow & & \downarrow \Psi^* \\ \Gamma(U \times W, \Omega^p(U \times W)) & \xrightarrow{i_1^*} & \Gamma(A_U \times W, \tilde{\Omega}^p(A_U \times W)) \end{array}$$

is also commutative.

By hypothesis $\omega \mid A_V = i_2^*(\omega) = 0$; it follows that

$$\Phi^*(\omega) \mid A_U \times W = i_1^*(\Phi^*(\omega)) = \Psi^*(i_2^*(\omega)) = \Psi^*(0) = 0.$$

The form $\Phi^*(\omega)$ can be written as $\Phi^*(\omega) = \beta \wedge dt + \eta$, where β and η do not contain the differential dt, and we get

$$(\beta \wedge dt) \mid A_U \times W + \eta \mid A_U \times W = 0.$$

Therefore $\eta \mid A_U \times W = 0$ and $(\beta \wedge dt) \mid A_U \times W = 0$.

The last equality implies $\beta(z, t_0) \mid A_U = 0$ for every $t_0 \in W$.

Now let $j_0 : U \to U \times W$ and $j_1 : U \to U \times W$ be the maps defined by $j_0(z) = (z, 0)$ and $j_1(z) = (z, 1)$.

For each integer $r \geq 0$ let $K_{r+1} : \Omega^{r+1}(U \times W) \to \Omega^r(U)$ be the following operator: for $\alpha \in \Omega^{r+1}(U \times W)$

$$\alpha = \Sigma\, a_{i_1 \dots i_{r+1}}(z, t)\, dz_{i_1} \wedge \dots \wedge dz_{i_{r+1}} + \Sigma\, b_{i_1 \dots i_r}(z, t)\, dt \wedge \dots \wedge dz_{i_r}$$

let

$$K_{r+1}(\alpha) = \Sigma \left(\int_\gamma b_{i_1 \dots i_r}(z, t)\, dt \right) dz_{i_1} \wedge \dots \wedge dz_{i_r}$$

(where γ is a regular curve in W joining 0 and 1).

For each $\sigma \in \Omega^r(U \times W)$ the following identity holds: $K_{r+1} d\sigma + dK_r \sigma = j_1^* \sigma - j_0^* \sigma$. Since $d\omega = 0$ and $(\Phi \circ j_1)$ is the canonical injection of U into V, for $r = p$ and $\sigma = \Phi^*(\omega)$ we get $j_1^*(\Phi^*(\omega)) = \omega \mid U$ and $d(K_p(\Phi^*(\omega))) = \omega \mid U - j_0^*(\Phi^*(\omega))$.

We claim that $j_0^*(\Phi^*(\omega)) = 0$. In fact, O is a regular point of B; so we may assume that $U \cap B \subset \{(w_1, w_2, \dots, w_n) \in \mathbb{C}^n : w_{k+1} = \dots = w_n = 0\}$ $(k = \dim_0 B)$. Let $\Phi_1, \Phi_2, \dots, \Phi_n$ be the components of Φ, by definition (3.1) $(\Phi_1(z, 0), \dots, \Phi_n(z, 0)) = (w_1, \dots, w_k, 0, \dots, 0)$ for every $z \in U$. Consequently $\Phi_{k+1}(z, t) = t\chi_{k+1}(z, t), \dots, \Phi_n(z, t) = t\chi_n(z, t)$ (where $\chi_j(z, t)$ are suitable holomorphic functions on $U \times W$). If $p \geq k + 1$, in each p-tuple $(\nu_1, \dots, \nu_p)$ $1 \leq \nu_1 < \nu_2 < \dots \nu_p \leq n$, there is a $\nu_\varrho = k + j, (j \geq 1)$. Hence $j_0^*(\Phi^*(\omega)) = 0$. This proves that $\vartheta = K_p(\Phi^*(\omega))$ is a primitive of ω, i. e. one has $d\vartheta = \omega$.

In order to end the proof we have to show that ϑ vanishes at each regular point of A_U. Now this is true because the coefficients of ϑ are obtained by integration of the coefficients of β along γ, and $\beta(z, t_0) \mid A_U = 0$ for each $t_0 \in W$.

REMARK. Lemma (3.2) may be proved also if we assume that W is the real interval $[0, 1]$. In this case W has no complex analytic structure;

so we assume that Φ is a holomorphic map in the variables $z_1, z_2, \dots, z_n$ (local coordinates for U) and is of class C^∞ in the variable t (coordinate for W); moreover we suppose that $\Psi^{-1}(A_V)_{\text{sing}}$ is nowhere dense in $A_U \times W$.

Sketch of proof. The coefficients of $\Phi^*(\omega)$ are holomorphic in $z_1, \dots, z_n$ and C^∞ in t. However, the point is that their integrals appearing in the form ϑ are still holomorphic functions of $z_1, \dots, z_n$. In fact let $\alpha(z_1, \dots, z_n, t)$ be one of such coefficients; set $z_k = x_k + iy_k$, $\alpha(z, t) = a(x, y, t) + ib(x, y, t)$. Then

$$\int_0^1 \alpha(z, t)\, dt = \int_0^1 a(x, y, t)\, dt + i \int_0^1 b(x, y, t)\, dt.$$

Let

$$G(x, y) = \int_0^1 a(x,y,t)\ dt \qquad \text{and} \qquad H(x, y) = \int_0^1 b(x, y, t)\, dt.$$

The Cauchy-Riemann conditions $\partial G/\partial x_r = \partial H/\partial y_r$; $\partial G/\partial y_r = -\partial H/\partial x_r$ ($r = 1, 2, \dots$) are satisfied, because they are satisfied for the functions $a(x, y, t)$, $b(x, y, t)$.

DEFINITION (3.2). (see [1] def. 3).

Let X be a complex analytic space, Y an analytic subset of X and O a regular point of Y.

The germ X_0 is said to be holomorphically contractible to the germ Y_0, if for every $U \in \mathfrak{U}$, where $\mathfrak{U}$ is a base of open neighbourhoods of O, there exists: $V \in \mathfrak{U}$, a domain $W \subset \mathbb{C}^1$ containing 0 and 1, and a holomorphic map $\varphi : V \times W \to U$ satisfying the following conditions:

1) $\varphi(z, 1) = z$ for each $z \in V$
2) $\varphi(z, 0) \in Y$ for each $z \in V$

POINCARÉ LEMMA. Let X be a complex analitic space, O a point of X. If in a neighbourhood of O there exists an analytic set $Y \ni O$ such that emb. $\dim_0 Y = k$ and such that the germ X_0 is holomorphically contractible to the germ Y_0, the sequence

$$\widetilde{\Omega}_0^{k-1} \to \widetilde{\Omega}_0^{k} \to \widetilde{\Omega}_0^{k+1} \to \dots$$

(for $k = 0$ we set $\widetilde{\Omega}_0^{-1} = \mathbb{C}$)

is exact.

PROOF. As in [1] (see Lemma von Poincaré) we may assume that U is equivalent, by an isomorphism $\tilde{\varphi}$, to an analitic set A defined on a domain G of $\mathbb{C}^n$. We may suppose that Y is contained in the analytic set B so defined:

$$B = \{(w_1, \dots, w_n) : w_{k+1} = \dots = w_n = 0\} \cap G ;$$

and we may assume that $\tilde{\varphi}(O)$ is the origin of $\mathbb{C}^n$.

A_0 is holomorphically contractible to $\tilde{\varphi}(Y)_0$ and so A_0 is $\mathbb{C}^n$-holomorphically contractible to B_0 (see [1]).

By lemma (3.2) the sequence

$$\mathcal{H}_0^k \to \mathcal{H}_0^{k+1} \to \mathcal{H}_0^{k+2} \to \dots$$

s exact and by lemma (3.1) so is

$$\tilde{\Omega}_0^{k-1} \to \tilde{\Omega}_0^k \to \tilde{\Omega}_0^{k+1} \to \dots \qquad \text{Q.E.D.}$$

§. 4. Cohomology of complex analytic spaces.

Let X be a complex analytic space and set:

$$Z^q(X) = \ker \{d^q : H^0(X, \tilde{\Omega}^q)\} \to H^0(X, \tilde{\Omega}^{q+1})\}$$

$$B^q(X) = d^{q-1}(H^0(X, \tilde{\Omega}^{q-1}).$$

THEOREM (4.1) (De Rham's theorem).

Let X be a Stein space, holomorphically contractible to each of its points (see def. (3.2)). Then

$$H^q(X, \mathbb{C}) = Z^q(X)/B^q(X) \qquad q \geq 0.$$

PROOF. Because of the Poincaré lemma and the coherence of $\tilde{\Omega}^p(X)$, the sequence

$$0 \to \mathbb{C} \to \tilde{\Omega}^0 \to \tilde{\Omega}^1 \to \dots$$

is an acyclic resolution of the constant sheaf $\mathbb{C}$.

COROLLARY. Under the above hypothesis it follows that $H^q(X, \mathbb{C}) = 0$ for each $q > n$, where $n = \dim_{\mathbb{C}} X$.

PROOF. For $q > n$, $\widetilde{\Omega}^q(X) = 0$ (see § 1).

DEFINITION (4.1). We shall call the space X cohomologically q_0-complete if $H^q(X, \mathcal{F}) = 0$ for each $q > q_0$ and for each coherent analytic sheaf $\mathcal{F}$ on X.

DEFINITION (4.2). We shall call the space X cohomologically q_0-convex if $\dim_{\mathbb{C}} H^q(X, \mathcal{F}) < +\infty$ for each $q > q_0$ and for each coherent analytic sheaf $\mathcal{F}$ on X.

THEOREM (4.2). Let X be a cohomologically q_0-complete complex analytic space, which is holomorphically contractible to each of its points. Let $n = \dim_{\mathbb{C}} X$. Then $H^q(X, \mathbb{C}) = 0$ for each $q > q_0 + n$.

PROOF. Since the sequence

$$0 \longrightarrow \mathbb{C} \xrightarrow{\varepsilon} \widetilde{\Omega}^0 \xrightarrow{d_0} \widetilde{\Omega}^1 \xrightarrow{d_1} \dots \tag{1}$$

is exact, so is

$$0 \longrightarrow \ker d_k \longrightarrow \widetilde{\Omega}^k \longrightarrow \ker d_{k+1} \longrightarrow 0 \qquad (k = 1, 2, \dots) \tag{2}$$

Therefore we have the exact cohomology sequence

$$H^i(X, \widetilde{\Omega}^k) \longrightarrow H^i(X, \ker d_{k+1}) \longrightarrow H^{i+1}(X, \ker d_k) \longrightarrow H^{i+1}(X, \widetilde{\Omega}^k).$$

If we suppose $i > q_0$ we have

$$H^i(X, \widetilde{\Omega}^k) = H^{i+1}(X, \widetilde{\Omega}^k) = 0,$$

hence

$$H^i(X, \ker d_{k+1}) = H^{i+1}(X, \ker d_k).$$

Since, by (1), $\mathbb{C} = \ker d_0$ we get $H^q(X, \mathbb{C}) = H^q(X, \ker d_0)$.

Now, for $q > q_0$, we get by (2):

$$H^q(X, \mathbb{C}) = H^q(X, \ker d_0) \cong H^{q-1}(X, \ker d_1) \cong \dots \cong H^{q_0+1}(X, \ker d_{q-q_0-1}).$$

Finally for $k = q - q_0 - 1$ we have the exact sequence

$$H^{q_0}(X, \widetilde{\Omega}^{q-q_0-1}) \longrightarrow H^{q_0}(X, \ker d_{q-q_0}) \longrightarrow$$
$$\longrightarrow H^{q_0+1}(X, \ker d_{q-q_0-1}) \longrightarrow H^{q_0+1}(X, \widetilde{\Omega}^{q-q_0-1}).$$

Therefore

$$H^{q_0+1}(X, \ker d_{q-q_0-1}) =$$

$$= H^{q^0}(X, \ker d_{q-q_0})/\mathrm{Im}\,\{H^{q_0}(X, \widetilde{\Omega}^{q-q_0-1}) \longrightarrow H^{q_0}(X, \ker d_{q-q_0})\}$$

because of the equality $H^{q_0+1}(X, \Omega^{q-q_0-1}) = 0$.

Now, if $q - q_0 > n$ we get $\ker d_{q-q_0} = 0$; hence $H^{q_0}(X, \ker d_{q-q_0}) = 0$ and consequently $H^q(X, \mathbb{C}) = 0$.

THEOREM (4.3). Let X be a cohomologically q_0-convex complex analytic space, which is holomorphically contractible to each of its points. Let $n = \dim_{\mathbb{C}} X$. Then $\dim_{\mathbb{C}} H^q(X, \mathbb{C}) < +\infty$ for each $q > q_0 + n$.

PROOF. As in theorem (4.2) consider the exact sequence

$$H^{q-k-1}(X, \ker d_{k+1}) \xrightarrow{\alpha_k^q} H^{q-k}(X, \ker d_k) \xrightarrow{\beta_k^q} H^{q-k}(X, \widetilde{\Omega}^k) \ (3) \quad (k = 0, 1, 2, \ldots, n)$$

If $q > q_0 + n$ we have $\dim_{\mathbb{C}}(\mathrm{Im}\,\beta_k^q) < +\infty$, and, by (3), $\ker \beta_k^q = \mathrm{Im}\,\alpha_k^q$. Hence $\dim_{\mathbb{C}} \ker \beta_k^q \leq \dim_{\mathbb{C}} H^{q-k-1}(X, \ker d_{k+1})$. Now, set $m_k^q = \dim_{\mathbb{C}}(\mathrm{Im}\,\beta_k^q)$, we get

$$\dim_{\mathbb{C}} H^{q-k}(X, \ker d_k) \leq m_k^q + \dim_{\mathbb{C}} H^{q-k-1}(X, \ker d_{k+1})$$

and therefore

$$\dim_{\mathbb{C}} H^q(X, \ker d_0) \leq m_0^q + \dim_{\mathbb{C}} H^{q-1}(X, \ker d_1) \leq$$

$$\leq m_0^q + m_1^q + \ldots + m_n^q + \dim_{\mathbb{C}} H^{q-n-1}(X, \ker d_{n+1}) < +\infty.$$

Since $H^{q-n-1}(X, \ker d_{n+1}) = 0$. This proves the theorem.

Let X be an analytic subset of a domain G of $\mathbb{C}^n$, and let Y be an analytic subset of X.

Suppose that both X and Y are holomorphically contractible to each of their points. Let $\mathcal{H}^p(G, X)$ (resp. $\mathcal{H}^p(G, Y)$) be the subsheaf of germs of holomorphic differential forms defined on G and vanishing on X (resp. on Y). Notice that $\mathcal{H}^p(G, X)$ is a subsheaf of $\mathcal{H}^p(G, Y)$, as follows from prop. (1.1) § 1.

Therefore we can define the subsheaf $\mathcal{H}^p(X, Y)$ of $\widetilde{\Omega}^p(X)$, consisting of the germs of holomorphic differential forms vanishing on Y, as the quotient

$$\mathcal{H}^p(X, Y) = \mathcal{H}^p(G, Y)/\mathcal{H}^p(G, X).$$

Because of the Poincaré lemma, the following sequences are exact:

$$0 \longrightarrow \mathbb{C}_{G-X} \xrightarrow{\varepsilon} \mathcal{H}^0(G, X) \xrightarrow{d_0} \mathcal{H}^1(G, X) \xrightarrow{d_1} \dots$$

$$0 \longrightarrow \mathbb{C}_{G-Y} \xrightarrow{\varepsilon} \mathcal{H}^0(G, Y) \xrightarrow{d_0} \mathcal{H}^1(G, Y) \xrightarrow{d_1} \dots$$

where ε is the canonical injection and $\mathbb{C}_{G-X}$, (resp. $\mathbb{C}_{G-Y}$) is the constant sheaf on $G - X$ (resp. on $G - Y$), extended by zero outside. Passing to quotients, we get the sequence

$$(*) \qquad 0 \longrightarrow \mathbb{C}_{X-Y} \xrightarrow{\varepsilon} \mathcal{H}^0(X, Y) \xrightarrow{\tilde{d}_0} \mathcal{H}^1(X, Y) \xrightarrow{\tilde{d}_1} \dots$$

which is obviously independent of the imbedding of X into G. By lemma (2.1) we have the following:

THEOREM (4.4) (Relative Poincaré lemma).

Let X be a complex analytic space and Y a subspace of X both holomorphically contractible to each of their points. Then the sequence $(*)$ is exact.

Because of the coherence of $\mathcal{H}^p(X, Y)$ (see § 2) and the relative Poincaré lemma, if X is a Stein space, the sequence $(*)$ is an acyclic resolution of the sheaf $\mathbb{C}_{X-Y}$.

Hence if

$$Z^q(X, Y) = \ker \{d^q : H^0(X, \mathcal{H}^q(X, Y)) \longrightarrow H^0(X, \mathcal{H}^{q+1}(X, Y))\}$$

$$B^q(X, Y) = d^{q-1}(H^0(X, \mathcal{H}^{q-1}(X, Y)))$$

we may calculate the q^{th}-relative cohomology group of X modulo Y, i. e. $H^q(X \bmod Y, \mathbb{C}) = H^q(X, \mathbb{C}_{X-Y})$, by the following:

THEOREM (4.5) (Relative De Rham's theorem).

Let X be a Stein space, Y a subspace of X, both holomorphically contractible to each of their points. Then

$$H^q(X \bmod Y, \mathbb{C}) = Z^q(X, Y)/B^q(X, Y) \qquad q \geq 0.$$

Added — November 5, 1969.

By means of spectral sequences and results of Herrera and Bloom (see corollary 3.15 (*) applied to the sheaf $\widetilde{\Omega}^p$ (see def. (1.2)) we can prove theorems (4.2), (4.3) for arbitrary reduced analytic spaces; more precisely we have:

THEOREM (4.6). Let X be a cohomologically q_0-complete complex analytic space, let $n = \dim_{\mathbb{C}} X$. Then $H^q(X, \mathbb{C}) = 0$ for each $q > q_0 + n$.

THEOREM (4.7). Let X be a cohomologically q_0-convex complex analytic space, let $n = \dim_{\mathbb{C}} X$. Then $\dim_{\mathbb{C}} H^q(X, \mathbb{C}) < +\infty$ for each $q > q_0 + n$.

Theorems (4.6), (4.7) will be proved in a forthcoming paper.

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(*) Bloom and Herrera — De Rham cohomology of an analytic space. Meanwhile published on Inv. Math. 7 — 275-296 (1969).