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GIOVANNI VIDOSSICH

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ON THE REPRESENTATION OF UNIFORM STRUCTURES BY EXTENDED RETICLES

GIOVANNI VIDOSSICH

The extended reticles (see § 0) were introduced by G. AQUARO in [3] as a tool for representation of uniform structures; in that paper it is proved that some uniformities admit of a representation by suitable extended reticles and ([3, pp. 353-354]) it is left as an open question to find if every uniformity has such a representation. This problem has technically an analogy with the problem in [7, Res. Probl. B_3] — if every uniformity has a basis of σ-uniformly discrete coverings — but the two problems are different because the former has a topological premise — the definition of extended reticle — and a uniform conclusion, while the latter has an entirely uniform nature.

In the present note, existence and uniqueness theorems on the possibility to represent a uniformity by extended reticles are given by means of the notion of p-equivalence classes (see § 0) introduced in [1; 2]. Main result: Every compatible uniformity on a uniformizable topological space X is the uniformity associated to an extended reticle iff X is pseudocompact, which gives a negative answer to the question of G. AQUARIO — a result quite natural when one thinks to the topological premise of the problem.

The uniformities associated to the universal extended reticle (for its definition, as well as for those of other uncustomary terms used in this remark, see § 0) $\mathfrak{R}(X)$ of a topological space X have interesting applications as [4] and [5] show. But from the results of this paper and from our knowledge, it seems that we cannot say the same thing about extended reticles. The best matter to do it seems to be to define the uniformities $\mathfrak{U}_k(\mathfrak{R}(X))$ and $\mathfrak{U}_\infty(\mathfrak{R}(X))$ starting directly from locally finite or σ-locally finite $\mathcal{U}$-reducible ($= \mathfrak{R}(X)$-reducible) open covers and to make no use of the

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notion of extended reticle: this leads to many simplifications in the proofs when 0.4. (3) is employed in the place of the very complicate original definition of « to be $\mathcal{U}$-contained ».

0. Summary of known results.

In this section some definitions and results from [1; 2] and [3] are recalled, biginning from [1; 2], in order to make the paper self contained as much as possible. *We shall use freely — without any reference — the terminology and the notations of this section.*

Uniformity means a set of entourages and not of uniform covers.

Let $(X, \mathfrak{U})$ be a uniform space. For every $A, B \subseteq X$, B is said to be a $\mathfrak{U}$-*neighborhood* of A, written

$$A \subset\subset_{\mathfrak{u}} B,$$

iff there is $W \in \mathfrak{U}$ such that $W[A] \subseteq B$. A cover $(B_i)_{i \in I}$ of X is said to be $\mathfrak{U}$-*reducible* iff there is a cover $(A_i)_{i \in I}$ of X such that $A_i \subset\subset_{\mathfrak{u}} B_i$ for all $i \in I$; then every cover with the property of $(A_i)_{i \in I}$ is said a $\mathfrak{U}$-*reduction* of $(B_i)_{i \in I}$.

Two uniformities $\mathfrak{U}_1$, $\mathfrak{U}_2$ on X are said to be *p equivalent* iff

$$A \subset\subset_{\mathfrak{u}_1} B \Longleftrightarrow A \subset\subset_{\mathfrak{u}_2} B$$

for all $A, B \subseteq X$. In [1] it is shown that « $\mathfrak{U}_1$ is p-equivalent to $\mathfrak{U}_2$ » is an equivalence relation on the set of all uniformities compatible with a given topology on a set X and that every its equivalence class, called *p-equivalence class*, contains a coarsest uniformity which is also the *unique* precompact uniformity in the class. This uniformity has as basis all sets of the form

$$\bigcup_{i=1}^{n} B_i \times B_i,$$

where $(B_i)_{i=1}^n$ is a finite $\mathfrak{U}$-reducible open cover of X, $\mathfrak{U}$ being any member of the p-equivalence class.

Let X be a topological space, $\mathbb{C}(X)$ and $\mathbb{O}(X)$ the sets of all closed and respectively open substes of X. An *extended reticle* on X is a subset $\mathbb{R}$ of $\mathbb{C}(X) \times \mathbb{O}(X)$ such that:

(ER$_\mathrm{i}$) $\mathbb{R} \neq \emptyset$.

(ER$_\mathrm{ii}$) $(A, B) \in \mathbb{R} \Longrightarrow A \subseteq B$.

(ER$_\mathrm{iii}$) $(A, B) \in \mathbb{R} \Longrightarrow (\complement B, \complement A) \in \mathbb{R}$.

(ER_{iv}) $(A, B) \in \mathfrak{R} \Longrightarrow$ there are $B^* \in \mathfrak{O}(X)$ and $A^* \in \mathfrak{C}(X)$ such that $(A, B^*) \in \mathfrak{R}$, $B^* \subseteq A^*$ and $(A^*, B) \in \mathfrak{R}$.

(ER_v) $((A_i, B_i))_{i \in I}$ is in $\mathfrak{R}$ and $(B_i)_{i \in I}$ is locally finite $\Longrightarrow (\bigcup_{i \in I} A_i, \bigcup_{i \in I} B_i) \in \mathfrak{R}$.

For every infinite cardinal k, a *k-extended reticle* is defined analogously substituting (ER_v) with

$(ER_v)_k$ $((A_i, B_i))_{i \in I}$ is in $\mathfrak{R}$, $(B_i)_{i \in I}$ is locally finite and $|I| \leq k \Longrightarrow$ $\Longrightarrow (\bigcup_{i \in I} A_i, \bigcup_{i \in I} B_i) \in \mathfrak{R}$.

Clearly every extended reticles is a k-extended reticle for all infinite cardinal; taking $k = |\mathfrak{P}(X)| + \aleph_0$, ($ER_v$) becames equivalent to $(ER_v)_k$ for *this* k. Therefore the following definitions and results apply also to extended reticles taking $k = |\mathfrak{P}(X)| + \aleph_0$ or $k \geq |I|$.

Let $\mathfrak{R}$ be a k-extended reticle on X. If $(B_i)_{i \in I}$ is an open cover of X, then a closed cover $(A_i)_{i \in I}$ is said to be an $\mathfrak{R}$-*reduction* of $(B_i)_{i \in I}$ iff $(A_i, B_i) \in \mathfrak{R}$ for all $i \in I$. An open cover of X is said to be $\mathfrak{R}$-*reducible* iff it has some $\mathfrak{R}$-reduction.

For any cover $(A_i)_{i \in I}$ of X and every $A \subseteq X$, we pose

$$\mathrm{St}(A, (A_i)_{i \in I}) = \bigcup \{A_i \mid i \in I, A_i \cap A \neq \emptyset\}.$$

0.1 ([3, § 2, Lemma 8]): *Let $\mathfrak{R}$ be a k-extended reticle on a topological space X, k being any infinite cardinal. Then for every locally finite $\mathfrak{R}$-reducible open cover $(B_i)_{i \in I}$ such that $|I| \leq k$ and every its $\mathfrak{R}$-reduction $(A_i)_{i \in I}$, there is a locally finite $\mathfrak{R}$-reducible open cover $(U_j)_{j \in J}$ of X such that $|J| \leq |I|$ and* $\mathrm{St}(A_i, (U_j)_{j \in J}) \subseteq B_i$ *for all $i \in I$.*

0.2. ([3, § 2, Lemma 10]): *In the hypothesis of* 0.1, *if $(B_n)_{n=1}^{\infty}$ is an $\mathfrak{R}$-reducible open cover of X, then there is a locally finite $\mathfrak{R}$-reducible open cover $(B_n^*)_{n=1}^{\infty}$ of X such that $B_n^* \subseteq B_n$ for all $n \in \mathbb{N}$.*

Let $\mathfrak{R}$ be a k-estended reticle on X and k_0 any infinite cardinal $\leq k$. As [3, § 2, Lemma 9] shows, the set of all locally finite $\mathfrak{R}$-reducible open covers of X whose indexing set has cardinality $\leq k_0$ constitutes a uniform covering system in the sense of J. W. Tukey; the corresponding uniformity is called the k_0-*uniformity associated to* $\mathfrak{R}$ and denoted (here we change the original Aquaro's notations $\mathcal{A}_{k_0}(\mathfrak{R})$) by

$$\mathfrak{U}_{k_0}(\mathfrak{R}).$$

When $\mathbb{R}$ is an extended reticle, the *uniformity associated to* $\mathbb{R}$, written

$$\mathfrak{U}_\infty(\mathbb{R}),$$

is the k-uniformity essociated to $\mathbb{R}$ with $k = |\mathbb{P}(X)| + \aleph_0$.

0.3 ([3, § 2, Th. 1 and Prop. 4]): *In the hypothesis of* 0.1, *we have*:

(1) $W \in \mathfrak{U}_k(\mathbb{R})$ *iff there is a locally finite* (resp.: *σ-locally finite*) $\mathbb{R}$*-reducible open cover* $(B_i)_{i \in I}$ *of* X *such that* $|I| \leq k$ *and* $\bigcup_{i \in I} B_i \times B_i \subseteq W$.

(2) *For every* $((A_i, B_i))_{i \in I}$ *in* $\mathbb{R}$ *such that* $|I| \leq k$ *and* $(B_i)_{i \in I}$ *is locally finite, there is* $W \in \mathfrak{U}_k(\mathbb{R})$ *such that* $W[A_i] \subseteq B_i$ *for all* $i \in I$.

If A, B are two subsets of a topological space X, then A is said to be *$\mathcal{U}$-contained* in B iff there is a sequence — called *Uryshon sequence* — $((U_m^n)_{n=0}^{2^m})_{m=0}^{\infty}$ of finite families of open subsets of X such that:

(i) $U_m^n = U_{m+1}^{2n}$ for every $m \in \mathrm{N}$ and every $n \in \{0, \dots, 2^m\}$.

(ii) $\overline{U_m^n} \subseteq U_m^{n+1}$ for every $m \in \mathrm{N}$ and every $n \in \{0, \dots, 2^m - 1\}$.

(iii) $A \subseteq U_0^0$ and $U_0^1 \subseteq B$.

The set $\{(A, B) \in \mathbb{C}(X) \times \mathbb{O}(X) \mid A \textit{ is } \mathcal{U}\textit{-contained in } B\}$ is an extended reticle on X, called the *universal extended reticle of the topological space* X and denoted by $\mathbb{R}(X)$. From 0.1. (2), from [3, § 3, Lemma 1 and § 8, Lemma 2] it follows the following proposition:

0.4. *For every topological space* X *and every pair* $(A, B) \in \mathbb{C}(X) \times \mathbb{O}(X)$, *the following statements are pairwise equivalent:*

(1) A *is $\mathcal{U}$-contained in* B.

(2) *There is an entourage* W *of a uniformity on* X *with topology less fine than that of* X *for which we have* $W[A] \subseteq B$.

(3) *There is a continuous* $f: X \to [0, 1]$ *such that* $f(A) \subseteq \{1\}$ *and* $f(\complement B) \subseteq \{0\}$.

0.5. ([3, § 6, Prop. 1, Def. 1, Lemma 2 and § 8, Cor. 1 to Prop. 2]): *For every topological space* X *the following statements are pairwise equivalent*:

(1) X *is pseudocompact.*

(2) *Every uniformity on* X *having topology less fine than that of* X *is precompact.*

(3) *Every locally finite* $\mathbb{R}(X)$*-reducible open cover of* X *has a finite subcover.*

(4) *Every countable* $\mathbb{R}(X)$*-reducible open cover of* X *has a finite subcover.*

1. Uniqueness theorems.

1.1. LEMMA: *Let k be an infinite cardinal, X a topological space and $\mathfrak{R}$ a k-extended reticle on X. Then for every subsets A, B of X, from $A \subset\subset_{\mathfrak{U}_k(\mathfrak{R})} B$ it follows the existence of $(A^*, B^*) \in \mathfrak{R}$ such that $A \subseteq A^*$ and $B^* \subseteq B$.*

PROOF: Let $W \in \mathfrak{U}_k(\mathfrak{R})$ be such that $W[A] \subseteq B$. By 0.3. (1), there is a locally finite $\mathfrak{R}$-reducible open cover $(B_i)_{i \in I}$ of X with $|I| \leq k$ and $\bigcup_{i \in I} B_i \times B_i \subseteq W$. Let $(A_i)_{i \in I}$ be any $\mathfrak{R}$-reduction of $(B_i)_{i \in I}$ and $I^* = \{i \in I \mid B_i \subseteq B\}$. For every $x \in X$ there is $i_x \in I$ such that $x \in A_{i_x}$: then $\{x\} \times B_{i_x} \subseteq W$ and $B_{i_x} \subseteq W[x] \subseteq B$, which implies $A \subseteq \bigcup_{i \in I^*} A_i$. From this and $(ER_\nabla)_k$ it follows that it suffices to choose $A^* = \bigcup_{i \in I^*} A_i$ and $B^* = \bigcup_{i \in I^*} B_i$. ▮

1.2. THEOREM: *Let X be a uniformizable topological space, $\mathcal{P}$ a p-equivalence class on X and k an infinite cardinal. Then $\mathcal{P}$ may contain at most one uniformity which is the k-uniformity associated to a k-extended reticle.*

PROOF: Assume that $\mathfrak{U}_1, \mathfrak{U}_2 \in \mathcal{P}$ are the k-uniformities associated to two (suitable) k-extended reticles, respectively to $\mathfrak{R}_1$ and $\mathfrak{R}_2$. From $(A, B) \in \mathfrak{R}_1$ it follows, by 0.3. (2), $A \subset\subset_{\mathfrak{U}_1} B$ and hence $A \subset\subset_{\mathfrak{U}_2} B$. Thus by 1.1 there is $(A^*, B^*) \in \mathfrak{R}_2$ such that $A \subseteq A^*$ and $B^* \subseteq B$. Then for every locally finite $\mathfrak{R}_1$-reducible open cover $(B_i)_{i \in I}$ of X, there is an $\mathfrak{R}_2$-reducible open cover $(B_i^*)_{i \in I}$ with $B_i^* \subseteq B_i$ for all $i \in I$, which consequently is locally finite. This implies, by 0.3. (1), that $\mathfrak{U}_1$ is less fine than $\mathfrak{U}_2$. The same reasons imply the converse and so $\mathfrak{U}_1 = \mathfrak{U}_2$. ▮

The above theorem implies that in $\mathcal{P}$ we may find only one member which is the uniformity associated to an extended reticle. This result is improved by the following Theorem 1.4, whose proof needs the following lemma which is essentially a famous step in the proof of the paracompactness of metrizable spaces.

1.3. LEMMA: *Let $(X, \mathfrak{U})$ be a uniform space. For every $W \in \mathfrak{U}$ there is a σ-discrete $\mathfrak{U}$-reducible open cover $(A_i)_{i \in I}$ of X refining $(W[x])_{x \in X}$.*

PROOF: Let d be a pseudometric on X whose uniformity is less fine than $\mathfrak{U}$ and contains W. Because $(W[x])_{x \in X}$ is refined by a d-open cover $(U_i)_{i \in I}$ of X, we may repeat exactly the proof of [6, § 4, Lemma 4] to obtain our goal. ▮

The converse of the following theorem fails by virtue of 2.5.

1.4. THEOREM: *Let X be a uniformizable space and $\mathcal{P}$ a p-equivalence class on X. If $\mathcal{P}$ contains an element $\mathfrak{U}$ which is the uniformity associated to an extended reticle $\mathfrak{R}$ on X, then $\mathfrak{U}$ is the finest uniformity in $\mathcal{P}$.*

PROOF: Let $\mathfrak{U}'$ be in $\mathcal{P}$ and $W_0, W \in \mathfrak{U}'$ such that $W \circ W \subseteq W_0$ and W is symmetric. By 1.3, there is a σ-discrete $\mathfrak{U}'$-reducible open cover $(B_i)_{i \in I}$ of X finer than $(W[x])_{x \in X}$. Let $(A_i)_{i \in I}$ be a $\mathfrak{U}'$-reduction, consequently a $\mathfrak{U}$-reduction, of $(B_i)_{i \in I}$. By 1.1, there is $(A_i^*, B_i^*)_{i \in I}$ in $\mathfrak{R}$ such that $A_i \subseteq A_i^*$ and $B_i^* \subseteq B_i$ for all $i \in I$. By 0.3. (1), $\bigcup_{i \in I} B_i^* \times B_i^* \in \mathfrak{U}$. From this and $\bigcup_{i \in I} B_i^* \times B_i^* \subseteq \bigcup_{x \in X} W[x] \times W[x] \subseteq W_0$ it follows that $\mathfrak{U}$ is finer than $\mathfrak{U}'$. ▌

2. Existence theorems.

The following proposition shows that all uniformities defined in [3, § 3] by means of extended reticles — i. e. $\mathfrak{U}_\infty(\mathfrak{R}(X))$ and $\mathfrak{U}_k(\mathfrak{R}(X))$ (k an infinite cardinal) — belong to the same p-equivalence class.

2.1. PROPOSITION: *Let X be a uniformizable space and $\mathcal{P}$ a p-equivalence class on X. If $\mathcal{P}$ contains an element which is the k-uniformity associated to a k-extended reticle $\mathfrak{R}$ on X for an infinite cardinal k, then $\mathcal{P}$ contains also the k'-uniformity associated to $\mathfrak{R}$, for every infinite cardinal $k' \leq k$. In particular, if $\mathcal{P}$ contains the uniformity associated to an extended reticle $\mathfrak{R}$, then $\mathcal{P}$ contains the k uniformity associated to $\mathfrak{R}$ for every infinite cardinal k.*

PROOF: Let k' be any infinite cardinal $\leq k$. From the definitions it follows that $\mathfrak{U}_{k'}(\mathfrak{R})$ is less fine than $\mathfrak{U}_k(\mathfrak{R})$ and so it is sufficient to show that $A \subset\subset_{\mathfrak{U}_k(\mathfrak{R})} B$ implies $A \subset\subset_{\mathfrak{U}_{k'}(\mathfrak{R})} B$. By 1. 1, $A \subset\subset_{\mathfrak{U}_k(\mathfrak{R})} B$ implies the existence of $(A^*, B^*) \in \mathfrak{R}$ such that $A \subseteq A^*$ and $B^* \subseteq B$. By 0.3. (2), $A^* \subset\subset_{\mathfrak{U}_{k'}(\mathfrak{R})} B^*$ and so $A \subset\subset_{\mathfrak{U}_{k'}(\mathfrak{R})} B$. The latter assertion follows from the former by taking $k = |P(X)| + \aleph_0$. ▌

2.2 THEOREM: *Let X be a pseudocompact uniformizable space. For every uniformity $\mathfrak{U}$ compatible with the topology of X, the set*

$$\mathfrak{R}(\mathfrak{U}) = \{(A, B) \in \mathfrak{P}(X) \times \mathfrak{P}(X) \mid A \text{ closed, } B \text{ open and } A \subset\subset_{\mathfrak{U}} B\}$$

is an extended reticle on X and $\mathfrak{U}$ is — at the same time — the uniformity and, for every infinite cardinal k, the k-uniformity associated to $\mathfrak{R}(\mathfrak{U})$.

PROOF: It is easily proved (or, alternatively, see [1, Th. 1]) that $\mathfrak{R}(\mathfrak{U})$ fulfils axioms (ER_i), ..., (ER_{iv}). To prove (ER_v), let $((A_i, B_i))_{i \in I}$ be in $\mathfrak{R}(\mathfrak{U})$ such that $(B_i)_{i \in I}$ is locally finite. Let I_0 be the set of all $i \in I$ such that $A_i \neq \emptyset$. In order to show that I_0 is finite, assume I_0 to be infinite and argue for a contradiction. By 0.4, to each $i \in I_0$ there corresponds a continuous map $f_i: X \to \mathbf{R}$ such that $f_i(A_i) = \{1\}$ and $f_i(X \setminus B_i) \subseteq \{0\}$. Since I_0 is infinite, there is an injective sequence $(i_n)_{n=1}^{\infty}$ in I_0. Since $(B_{i_n})_n$ is locally finite, $f = \sum_{n=1}^{\infty} n \cdot f_{i_n}$ is a continuous map $X \to \mathbf{R}$. But f is unbounded since $f_{i_n}(A_{i_n}) = \{1\}$, what is impossible by the pseudocompactness of X. Hence I_0 must be finite. Then we have

$$\bigcup_{i \in I} A_i = \bigcup_{i \in I_0} A_i \subset\subset_{\mathfrak{U}} \bigcup_{i \in I_0} B_i \subseteq \bigcup_{i \in I} B_i .$$

Thus (ER_v) holds, and $\mathfrak{R}(\mathfrak{U})$ is an extended reticle on X. Let $\mathfrak{U}_0$ be the uniformity associated to $\mathfrak{R}(\mathfrak{U})$. From $A \subset\subset_{\mathfrak{U}} B$ it follows $\bar{A} \subset\subset_{\mathfrak{U}} \mathring{B}$, hence $(\bar{A}, \mathring{B}) \in \mathfrak{R}(\mathfrak{U})$ and so, by 0.3. (2), $\bar{A} \subset\subset_{\mathfrak{U}_0} \mathring{B}$, which implies $A \subset\subset_{\mathfrak{U}_0} B$; vice-versa, from $A \subset\subset_{\mathfrak{U}_0} B$ it follows, by 1.1, the existence of $(A^*, B^*) \in \mathfrak{R}(\mathfrak{U})$ such that $A \subseteq A^*$ and $B^* \subseteq B$, what implies $A \subset\subset_{\mathfrak{U}} B$ by definition of $\mathfrak{R}(\mathfrak{U})$ Thus $\mathfrak{U}$ and $\mathfrak{U}_0$ belong to the same p-equivalence class $\mathcal{P}$, which — by 2.1 — contains also the k-uniformity associated to $\mathfrak{R}(\mathfrak{U})$ for all infinite cardinal k. By 0.5 all members of $\mathcal{P}$ are precompact and consequently they must coincide because, by a theorem of [1] recalled in § 0, $\mathcal{P}$ contains a unique precompact uniformity. ▌

2.3 THEOREM: *The topological space X is pseudocompact iff every uniformity on X whose topology is less fine than that of X is the $\aleph_0$-uniformity associated to an $\aleph_0$-extended reticle on X. In order that X be pseudocompact, it is sufficient and necessary that the finest precompact uniformity on X having topology less fine than that of X is the $\aleph_0$-uniformity associated to an $\aleph_0$-extended reticle on X.*

PROOF: The necessity follows from 2.2, since a topology less fine than a pseudocompact topology is pseudocompact. Sufficiency: Let $\mathfrak{R}(X)$ be the universal extended reticle on X, $\mathcal{P}$ the p equivalence class containing $\mathfrak{U}_\infty(\mathfrak{R}(X))$ and $\mathfrak{U}(\mathcal{P})$ the unique precompact uniformity of $\mathcal{P}$. Because $\mathfrak{U}_\infty(\mathfrak{R}(X))$ is the finest uniformity on X having topology less fine than that of X, $\mathfrak{U}(\mathcal{P})$ is the finest precompact uniformity on X having topology less fine than that of X. Therefore we prove both statements of the theorem at the same time assuming, as we do, that $\mathfrak{U}(\mathcal{P})$ is the $\aleph_0$-uniformity associated to an $\aleph_0$-extended reticle on X. By 2.1, $\mathfrak{U}_{\aleph_0}(\mathfrak{R}(X)) \in \mathcal{P}$ and hence, by 1. 2, $\mathfrak{U}_{\aleph_0}(\mathfrak{R}(X) = \mathfrak{U}(\mathcal{P})$. Let $(A_n)_{n=1}^{\infty}$ be a countable $\mathfrak{R}(X)$-reducible open covering of X. By 0.2 there is a locally finite $\mathfrak{R}(X)$-reducible open covering $(G_n)_{n=1}^{\infty}$

of X such that $G_n \subseteq A_n$ for all $n = 1, \dots, \infty$. Let $(F_n)_{n=1}^{\infty}$ be an $\mathfrak{R}(X)$-reduction of $(G_n)_{n=1}^{\infty}$. By 0.3. (2), there is $W \in \mathfrak{U}_{\aleph_0}(\mathfrak{R}(X)) = \mathfrak{U}(\mathcal{P})$ such that $W[F_n] \subseteq G_n$ for every n. Since $\mathfrak{U}(\mathcal{P})$ is precompact, there is $\{x_1, \dots, x_k\} \subseteq X$ such that $X = \bigcup_{i=1}^{k} W[x_i]$. If $x_i \in F_{n_i}$ for all i, then $X \subseteq \bigcup_{i=1}^{k} W[F_{n_i}] \subseteq \bigcup_{i=1}^{k} G_{n_i} \subseteq \subseteq \bigcup_{i=1}^{k} A_{n_i} \subseteq X$. Therefore X is pseudocompact by 0.5 « (4) $\rightarrow$ (1) ». ▌

2.4. COROLLARY : *Let X be a topological space and k an infinite cardinal. Then X is pseudocompact iff every uniformity on X having topology less fine than that of X is the k-uniformity associated to a k-extended reticle* (resp. : *the uniformity associated to an extended reticle on X*). *In order that X be pseudocompact, it is sufficient* (and necessary) *that this fact happens for the finest precompact uniformity on X having topology less fine than that of X.*

PROOF : The necessity follows as for 2.3. Sufficiency: We use the notations of the proof of 2.3. Because $\mathfrak{U}(\mathcal{P})$ is the k-uniformity (resp. : uniformity) associated to a k-extended (resp.: extended) reticle $\mathfrak{R}$ on X, by 2.1 $\mathcal{P}$ contains $\mathfrak{U}_{\aleph_0}(\mathfrak{R})$. From the definitions it follows that $\mathfrak{U}_{\aleph_0}(\mathfrak{R})$ is less fine than $\mathfrak{U}(\mathcal{P})$ and hence $\mathfrak{U}(\mathcal{P}) = \mathfrak{U}_{\aleph_0}(\mathfrak{R})$ since $\mathfrak{U}(\mathcal{P})$ is the coarsest uniformity in $\mathcal{P}$. Thus the conclusion follows from 2.3. ▌

The following proposition, as well as 2.3 and 2.4, gives a negative answer to the problem of G. AQUARO explained in the introduction of the, paper; it also shows that the converse of 1.4 does not hold.

2.5. PROPOSITION : *There is a p-equivalence class $\mathcal{P}$ on* **R** *having a finest uniformity such that no element of it is the uniformity nor the k-uniformity associated to an extended reticle or, respectively, a k-extended reticle on X, k being an arbitrary infinite cardinal.*

PROOF : Let $\mathfrak{U}_0$ be the uniformity defined by the euclidean metric on **R** and $\mathcal{P}$ the p equivalence class containing $\mathfrak{U}_0$. By a result of YU-M. SMIRNOV cited in [2, p. 204], $\mathcal{P}$ contains a finest uniformity. By 2.1, to prove our assertion it suffices to show that no element of $\mathcal{P}$ may be the $\aleph_0$-uniformity associated to an $\aleph_0$-extended reticle on **R**. Assume that there is an $\aleph_0$-extended reticle $\mathfrak{R}$ on **R** such that $\mathfrak{U}_{\aleph_0}(\mathfrak{R}) \in \mathcal{P}$. For every $n \in \mathrm{N} \setminus \{0\}$ define $A_n = \{n\}$ and $B_n = \{x \in \mathbf{R} \mid |n - x| < 1/n\}$. For every $n \in \mathrm{N} \setminus \{0\}$, $A_n \subset\subset_{\mathfrak{n}_0} B_n$ and so $A_n \subset\subset_{\mathfrak{n}_{\aleph_0}(\mathfrak{R})} B_n$. Then, by 1.1, there is $(A_n^*, B_n^*) \in \mathfrak{R}$ such that $A_n \subseteq A_n^*$ and $B_n^* \subseteq B_n$ $(n = 1, \dots, \infty)$. Clearly $(B_n^*)_{n=1}^{\infty}$ is locally finite and hence, by $(\mathrm{ER_v})_k$, $(\bigcup_{n=1}^{\infty} A_n^*, \bigcup_{n=1}^{\infty} B_n^*) \in \mathfrak{R}$. Thus, defining $A = \bigcup_{n=1}^{\infty} A_n^*$ and $B = \bigcup_{n=1}^{\infty} B_n^*$, 0.3. (2) implies that $A \subset\subset_{u\,\aleph_0(\mathfrak{R})} B$. Then $A \subset\subset_{u_0} B$ and consequently there is $a \in]\,0, \frac{1}{2}\,[$ such that, defining $W_a = \{(x, y) \in R^2 \mid |x - y| < a\}$, $W_a[A] \subseteq B$. Let $n \in \mathrm{N}$ be such that $1/n < a$, and let $x \in W_a[A_n]$. Since $W_a[A_n] \subseteq W_a[A] \subseteq B \subseteq \bigcup_{n=1}^{\infty} B_n$, there is $n' \in \mathrm{N}$ such that $x \in B_{n'}$. From this and the definitions of W_a, A_n and $B_{n'}$ it follows:

$$|n - n'| \leq |n - x| + |x - n'| < a + 1/n'.$$

Since $n > 2$ and $a + 1/n' < 2$, necessarily $n' \geq 2$. Therefore $a + 1/n' < 1$, which implies $n = n'$. Then $W_a[A_n] \subseteq B_n$, which is impossible: if $t \in]\,1/n, a\,[$, then $n + t \in W_a[A_n] \setminus B_n$. ∎

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