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AN ABSTRACT LEBESGUE-NIKODYM THEOREM

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The development of the theory of Radon measures, continuous linear forms on the vector space $\mathcal{K}(T)$ of continuous real-valued functions with compact supports on the locally compact space T , endowed with a suitable inductive-limit topology [B₃], leans heavily on the order structure of $\mathcal{K}(T)$. We have shown elsewhere [F_{1,2}] how some aspects of the theory of integration for Radon measures can equally be developed by availing oneself of the algebraic structure of $\mathcal{K}(T)$ rather than its order structure. We note, however, that such a treatment does not yield a theory of integration in that, for example, the objects which there correspond to the integrable functions of the more familiar development are elements of an abstract completion of $\mathcal{K}(T)$, i.e. are not functions. Our object in investigating the rôle of the algebraic structure was to explore the possibility of its application to

(i) vector measures and the integration of vector — valued functions, where rarely is there an order structure which arises in a natural way from the structure of the space which we are concerned, and

(ii) the theory of distributions, where utilisation of a naturally-occurring order imposes excessive restrictions — a positive distribution is a measure.

In [F₂] we attempted to establish a Lebesgue-Nikodym theorem within our order-free structure, but were unable to do so without the aid of a supplementary hypothesis (loc cit Prop 2.3) which is not verified for Radon measures, but which holds in the theory of distribution [F₃].

In the present paper we obtain a Lebesgue-Nikodym in a form applicable to the theory of integration. § 1.1 gives an account of some ideas from the theory of duality for topological vector spaces, in particular, of an extension of Grothendieck's completion theorem. § 1.2 introduces the pre-

hilbert structure defined by a positive linear form on a normed algebra. § 1.3 describes dualities defined by a linear form on an algebra. §§ 1.4-5 formulate a concept of absolute continuity of one linear form on a normed algebra with respect to another such form and apply the completion theorem of § 1.1 to establish an abstract Lebesgue-Nikodym theorem. In § 1.6 this theorem is applied to give the Lebesgue-Nikodym theorem for Radon measures on a compact space. § 2.1 describes an extension of the results of §§ 1.1-5 to the inductive limit (as a topological vector space) of a family of normed algebras, and in § 2.2 we obtain the Lebesgue-Nikodym theorem on a locally compact space.

§ 1.1 We make use of the ideas of the theory of duality for topological vector spaces. We summarize here, briefly, the definitions and results which we shall need.

Let $\langle E_1, E_2 \rangle$ be a dual system (pairing) of vector spaces E_1, E_2 over the complex field C .

A family of subsets of $E_i (i = 1, 2)$ is said to be saturated with respect to $\langle E_1, E_2 \rangle$ if it contains

- (1) the subsets of each of its members,
- (2) the scalar multiples of its members,
- (3) the absolutely convex, weakly closed hulls of finite unions of its members.

\mathcal{F}_i will denote the family obtained by saturating the family of finite subsets of E_i .

A saturated family \mathcal{C}_1 of weakly bounded subsets of E_1 defines a locally convex topology $\sigma_{\mathcal{C}_1}$ on E_2 (the topology of uniform convergence on the sets of \mathcal{C}_1). If $\langle E_1, E_2 \rangle$ is separated then $\sigma_{\mathcal{C}_1}$ is Hausdorff (separated) if $\mathcal{F}_1 \subset \mathcal{C}_1$. (The corresponding statements obtained by interchanging the suffixes 1 and 2 also apply). With this notation we write $\sigma_{\mathcal{F}_2}$ for the weak topology $\sigma(E_1, E_2)$ on E_1 .

\mathcal{K}_2 denotes the family obtained by saturating the family of absolutely convex weakly compact sets in E_2 . $\mathcal{F}_2 \subset \mathcal{K}_2$.

Let $(\mathcal{F}_1 \subset) \mathcal{M}_1$ be a saturated family in E_1 , and let $(\mathcal{F}_2 \subset) \mathcal{N}_2 (\subset \mathcal{K}_2)$ be a saturated family in E_2 , both families being of weakly bounded sets.

We shall need the following form of *Grothendieck's completion theorem*: if $\langle E_1, E_2 \rangle$ is separated,

$G = \{f \in E_1^* : f \text{ } \sigma_{\mathcal{N}_2}\text{-continuous on the sets of } \mathcal{M}_1\}$ ⁽¹⁾ is the (separated) completion $(E_2, \sigma_{\mathcal{M}_1})^\wedge$ of E_2 for the topology $\sigma_{\mathcal{M}_1}$.

⁽¹⁾ E_1^* denotes the algebraic dual of E_1 .

This result may be established by showing

(i) (as in [K] p. 272) that G is complete, and separated for $\sigma_{\mathcal{N}_1}$ ($\mathcal{F}_1 \subset \mathcal{N}_1$ and $\langle E_1, E_1^* \rangle$ is separated), and

(ii) that E is $\sigma_{\mathcal{N}_1}$ -dense in G .

The argument establishing the necessity of [S] Th. 6.2, p. 148 proves (ii) when we put $(E, \tau) = (E, \sigma_{\mathcal{N}_2})$ since $(E, \tau)' = (E_1, \sigma_{\mathcal{N}_2})' = E_2$ (where $(E, \tau)'$ denotes the topological dual of E for the topology τ) by the Mackey-Arens theorem ([S] Th. 3.2, p. 131), since $\mathcal{F}_2 \subset \mathcal{N}_2 \subset \mathcal{N}_2$.

(A « proof » of this theorem given in [F₂] is false. It relies on the false inequality (3) of [K] p. 272).

§ 1.2. We shall apply this theorem to a situation where we take (essentially) for E_1, E_2 two copies of a $*$ -normed algebra A (i.e. a normed algebra over C on which there is defined an involution $*$ such that $\|\xi^*\| = \|\xi\|$ for all $\xi \in A$, (see [R] p. 180). Elements ξ for which $\xi = \xi^*$ are said to be self-adjoint.

We consider a linear form μ on A having the properties

$$F_1) \quad \mu(\xi^*) = \overline{\mu(\xi)},$$

(this is equivalent to: μ is real on self-adjoint elements of A , and implies that the sesqui-linear forms

$$A \times A \rightarrow C$$

$$(\xi, \eta) \rightarrow \mu(\xi \eta^*)$$

and

$$(\xi, \eta) \rightarrow \mu(\eta^* \xi)$$

are hermitian, it is an easy matter to construct an example for which they do not coincide), and

$$F_2) \quad \mu \text{ is positive,}$$

$$\text{i. e.} \quad \mu(\xi \xi^*) \geq 0 \quad \forall \xi \in A,$$

$$(\text{and so} \quad \mu(\xi^* \xi) = \mu(\xi^* \xi^{**}) \geq 0 \quad \forall \xi \in A),$$

so that the sesqui-linear forms are positive-definite.

We shall henceforth consider only the first of these two forms defined by μ , and shall write it $\mu(\cdot, \cdot)$. It defines a prehilbert structure on A with corresponding semi-norm $p_0(\xi) = (\mu(\xi \xi^*))^{\frac{1}{2}}$. If $Z = \{\xi : \mu(\xi \xi^*) = 0\} =$

$\{\xi: \mu(\xi\eta) = 0 \ \forall \eta \in A\}$ the positive hermitian form on A/Z associated with $\mu(\cdot, \cdot)$, (which we shall denote by $\dot{\mu}(\cdot, \cdot)$) defines a separated prehilbert structure. The norm p for this structure is that associated with the semi-norm $p_0 \cdot \dot{\mu}(\cdot, \cdot)$ extends uniquely to the separated completion $(A/Z, p_0)^\wedge$ (which we write $(A/Z)^\wedge$) of A/Z for p_0 , and makes it a Hilbert space. We can equally construct the separated completion $(A, p_0)^\wedge$ (or A^\wedge) of A for p_0 , and ([B₁] II § 3.7) $(A/Z)^\wedge$ is isomorphic with A^\wedge , to which we may therefore transport the extended form $\dot{\mu}(\cdot, \cdot)$, and if i is the map of [B₁] II § 3.7, Th. 3, $i(A)$ is dense in A^\wedge (loc. cit. § 3.8). i is here a linear map, and is in fact the canonical map $A \rightarrow A/Z$, so that $i(A)$ is isomorphic with A/Z .

We now impose upon μ the further requirement

F₃) μ is continuous on A .

(If A has a unit and is complete, i.e. is a Banach $*$ -algebra, the continuity of μ follows from the hypothesis that it is positive ([N], p. 200)).

If

$$U = \{\xi: \|\xi\| \leq 1\}, \quad V = \{\xi: p_0(\xi) \leq 1\},$$

since

$$(p_0(\xi))^2 = \mu(\xi\xi^*) \leq \|\mu\| \|\xi\xi^*\| \leq \|\mu\| \|\xi\| \|\xi^*\| = \|\mu\| \|\xi\|^2,$$

$$U \subset mV \quad \text{for some } m > 0,$$

and so

$$i(U) \subset mi(V).$$

Now $i(V)$ is contained in the unit ball of the Hilbert space A^\wedge , and the closed unit ball is compact for the weak topology $\sigma(A^\wedge, A^\wedge)$ defined by the canonical duality $\langle A^\wedge, A^\wedge \rangle$ of A^\wedge with its Hilbert-space dual A^\wedge , so that $i(V)$ and therefore $i(U)$ are relatively weakly compact in A^\wedge . The weak closure of $i(U)$ in A is compact for $\sigma(A^\wedge, A^\wedge)$ and is therefore compact for the coarser topology $\sigma(A^\wedge, i(A))$.

§ 1.3. Our form μ defines a duality (D_0) on $A \times A$ — which it is convenient to write $A_1 \times A_2$ — by

$$A_1 \times A_2 \rightarrow C$$

$$(\xi, \eta) \rightarrow \mu(\xi\eta).$$

It is clear that this duality will not in general be separated.

Let \mathcal{M} be the family obtained by saturating the family $\{U\}$ in A_1 for the duality (D_0) , and let \mathcal{N} be the same family considered as a family in A_2 .

The sets of \mathcal{M} and \mathcal{N} are weakly bounded for (D_0) since

$$\sup \{ |\mu(\xi\eta)| : \|\xi\| \leq 1 \} \leq \|\mu\| \|\eta\|.$$

The extension of $\dot{\mu}$ to $A_1 \times A_2$ defines a duality (D) on $i(A_1) \times A_2$.

$$i(A_1) \times A_2 \rightarrow \mathcal{U}$$

$$(\dot{\xi}, \eta) \rightarrow \dot{\mu}(\dot{\xi}\eta).$$

Let $\mathcal{M}_1 = i({}^c\mathcal{M})$. Since i is continuous for p_0 and p , $i(U)$ is bounded for p and so for the weak topology $\sigma(i(A_1), A_2)$, which is the weak topology defined by (D) . It follows that the sets of \mathcal{M}_1 are weakly bounded for (D) .

Since i is linear, to prove that \mathcal{M}_1 is saturated for (D) it suffices to prove that the i -images of absolutely convex weakly closed (for (D_0)) sets of \mathcal{M} are again weakly closed. Such a set M is closed for p_0 , and since i is the canonical map $A \rightarrow A/Z$ i is closed, so that $i(M)$ is closed for p . Since it is absolutely convex and $A_2^\wedge = (i(A_1), p)^\wedge$, $i(M)$ is also $\sigma(i(A_1), A_2^\wedge)$ -closed, i. e. is weakly closed for (D) .

Finally, we define \mathcal{N}_2 to be the saturate, for $\langle i(A_1), A_2 \rangle$ of $i(\mathcal{N})$. It is clear that the sets of \mathcal{N}_2 are weakly bounded for (D) .

§ 1.4. If λ is another linear form on A we now define :

λ absolutely continuous with respect to μ

to mean :

λ is $\sigma_{\mathcal{N}}$ -continuous on the sets of \mathcal{M} ,

(where $\sigma_{\mathcal{N}}$ is defined by the duality (D_0)).

It follows that $\lambda(Z) = 0$:

$$\zeta \in Z \quad \mu(\zeta\eta) = 0 \quad \forall \eta \in A$$

$$\mu(\zeta N) = 0 \quad \forall N \in \mathcal{N}.$$

Since $\zeta \in M$ for some $M \in \mathcal{M}$ ($\mathcal{F} \subset \mathcal{M}$), the $\sigma_{\mathcal{N}}$ -continuity of λ on M shows that $\lambda(\zeta) = 0$.

λ is thus well-defined on A/Z . We shall denote by $\dot{\lambda}$ the functional so defined, and shall prove that it is $\sigma_{\mathcal{N}_2}$ -continuous on the sets of \mathcal{M}_1 . For this it suffices to establish continuity for the coarser topology $\sigma_{i(\mathcal{N})}$.

Since $\lambda = \dot{\lambda} \circ i$ and i is the canonical map of M onto M/Z , it suffices ([B₁] §. 3.4) to prove that $\sigma_{i(\mathcal{N})}$ is the quotient by Z of $\sigma_{\mathcal{N}}$. This is immediate from consideration of the neighbourhoods for the two topologies since a $\sigma_{\mathcal{N}}$ -neighbourhood $\{\xi: \sup_{\eta \in N} |\mu(\xi\eta)| < \varepsilon\}$ is saturated for the equivalence relation defined by Z ([B₁] II § 3.4).

§ 1.5 We now apply Grothendieck's completion theorem to establish a Lebesgue-Nikodym theorem.

LEMMA. $\sigma_{\mathcal{M}}$ is coarser than the p_0 -topology,

$\sigma_{\mathcal{M}_1}$ is coarser than the p -topology.

PROOF. Since μ is positive

$$|\mu(\xi\eta)|^2 \leq \mu(\xi\xi^*) \mu(\eta^*\eta^{**}) \leq (p_0(\xi))^2 \|\mu\| \|\eta\|^2.$$

Therefore

$$p(\xi) < 1 \implies \sup \{|\mu(\xi\eta)|: \|\eta\| \leq 1\} \leq \|\mu\|^{\frac{1}{2}}$$

so that a $\sigma_{\mathcal{M}}$ neighbourhood defined by $M = U$ contains a p_0 neighbourhood, i. e. $\sigma_{\mathcal{M}} \subset p$ -topology. The second assertion is now immediate.

THEOREM 1. If $\zeta \in (A, \sigma_{\mathcal{M}})^\wedge$

$$A \rightarrow C$$

$$\xi \rightarrow \dot{\mu}(\dot{\xi}\zeta)$$

defines a linear form A which is absolutely continuous with respect to μ .

PROOF. $\dot{\mu}(\dot{\xi}\cdot)$ is $\sigma_{\mathcal{M}_1}$ -continuous on A_2^\wedge for all $\xi \in A_1$,

$(\{\dot{\xi}\} \in \mathcal{M}_1)$, and so extends (uniquely) to $(A_2, \sigma_{\mathcal{M}_1})^\wedge \supset (A_2, \sigma_{\mathcal{M}})^\wedge$.

This extension defines $\dot{\mu}(\dot{\xi}\zeta)$.

$\zeta\mu: \xi \rightarrow \dot{\mu}(\dot{\xi}\zeta)$ is clearly linear on A . To prove that it is absolute continuous with respect to μ , let $M \in \mathcal{M}$, $\varepsilon > 0$. If $\zeta = \{P\}$ (a $\sigma_{\mathcal{M}}$ minimal

Cauchy filter on A_2 ([B₁] II § 3.7)), there exists P such that

$$\sup \{ |\dot{\mu}(\dot{\xi}(\zeta - P))| : \xi \in M \} < \varepsilon/2.$$

Choose $\pi \in P$. $\{\pi\} \in \mathcal{C}$, and so $|\mu(\xi\pi)| < \varepsilon/2$ implies that

$$|\zeta\mu(\xi)| = |\dot{\mu}(\dot{\xi}\zeta)| < \varepsilon \quad \text{for } \xi \in M,$$

i. e., $\zeta\mu$ is $\sigma_{\mathcal{C}}$ -continuous on M .

THEOREM 2. If λ is absolutely continuous with respect to μ there exists $\zeta \in (A, \sigma_{\mathcal{M}})^\wedge$ such that $\lambda = \zeta\mu$.

PROOF. λ is $\sigma_{\mathcal{C}}$ -continuous on the sets of \mathcal{M} , by definition, therefore $\dot{\lambda}$ is $\sigma_{i(\mathcal{C})}$ -continuous on the sets of $i(\mathcal{M}) = \mathcal{M}_1$, and so λ is $\sigma_{\mathcal{C}_2}$ -continuous on the sets of \mathcal{M}_1 since $\sigma_{i(\mathcal{C})} \subset \sigma_{\mathcal{C}_2}$. Thus $\dot{\lambda} \in (A_2^\wedge, \sigma_{\mathcal{M}_1})^\wedge$ by Grothendieck's theorem.

Now $\sigma_{\mathcal{M}} \subset p_0$ -topology and so, if $Z_{\mathcal{M}}$ denotes the adherence of 0 in $(A_2, \sigma_{\mathcal{M}})^\wedge$,

$$(A_2, p_0)^\wedge / Z_{\mathcal{M}} = A_2^\wedge / Z_{\mathcal{M}} \subset (A_2, \sigma_{\mathcal{M}})^\wedge$$

by [F₁] Prop. 2.3. It follows that

$$(1) \quad (A_2^\wedge / Z_{\mathcal{M}}, \sigma_{\mathcal{M}_1})^\wedge \subset ((A_2, \sigma_{\mathcal{M}})^\wedge, \sigma_{\mathcal{M}_1})^\wedge.$$

Since $\sigma_{\mathcal{M}_1}$ is clearly the associated separated topology on $(A_2, \sigma_{\mathcal{M}})^\wedge$, we have

$$(2) \quad ((A_2, \sigma_{\mathcal{M}})^\wedge, \sigma_{\mathcal{M}_1})^\wedge = (A_2, \sigma_{\mathcal{M}})^\wedge.$$

Finally, since, denoting by Z_0 the adherence of 0 in $(A_2, p_0)^\wedge$, it is easy to verify that $Z_0 = Z_{\mathcal{M}}$, we have

$$\begin{aligned} A_2^\wedge &= A_2^\wedge / Z_0 \quad \text{by [B}_1\text{] II § 3.8,} \\ &= A_2^\wedge / Z_{\mathcal{M}}, \end{aligned}$$

there follows, from (1) and (2),

$$(A_2^\wedge, \sigma_{\mathcal{M}_1})^\wedge \subset (A_2, \sigma_{\mathcal{M}})^\wedge.$$

This shows that $\dot{\lambda}$, as a linear form on A_2^\wedge , is represented via the duality (D) by $\zeta \in (A_2, \sigma_{\mathcal{M}})^\wedge$. Thus

$$\lambda(\xi) = \dot{\lambda}(\dot{\xi}) = \lim \dot{\mu}(\dot{\xi} P),$$

where $\xi = \{P\}$ is a $\sigma_{\mathcal{M}_1}$ Cauchy filter on A_2 .

$$\lambda(\xi) = \dot{\mu}(\dot{\xi} \lim P)$$

since $\dot{\mu}(\dot{\xi})$ is $\sigma_{\mathcal{M}_1}$ -continuous on A_2 , $(\{\dot{\xi}\} \in \mathcal{M}_1)$, and so

$$\lambda(\xi) = \dot{\mu}(\dot{\xi} \zeta) = (\zeta \mu)(\xi).$$

§ 1.6. We apply our abstract Lebesgue-Nikodym theorem to Radon measures on a compact space.

Let T be a compact Hausdorff space. We take A to be $\mathcal{K}(T)$, the algebra of continuous real-valued functions with compact supports, (which here coincides with $\mathcal{C}(T)$ —continuous functions). Our involution is the identity map, and the norm on A is the uniform norm, $\|\xi\| = \sup_{t \in T} |\xi(t)|$. (A has a unit and is complete for this norm). We take μ to be a positive Radon measure, and so μ is continuous for the norm of A .

If λ is a Radon measure on T which is absolutely continuous with respect to μ in the sense of [B₃] V § 5.5 we have shown [F₂] Prop. 3.1 that it is $\sigma_{\mathcal{K}}$ -continuous on K (in the present context $\mathcal{M} = \mathcal{S}_p$, $\mathcal{K} = \mathcal{S}_2$ and $\mathcal{K}\mathcal{S} = \mathcal{S}$, in the notation of [F₂]). In order to apply our abstract Lebesgue-Nikodym theorem we must show that λ is $\sigma_{\mathcal{S}_2}$ -continuous on \mathcal{M} , the family obtained by saturating $\{U\}$. It clearly suffices to establish $\sigma_{\mathcal{S}_2}$ -continuity on the weak closure of U for the duality (D_0) . However, U is already weakly closed (in \mathcal{K}_1 , for if this were not so there would exist $f \in \mathcal{K}_1 \cap U^c$ ($|f(t_0)| > 1$ for some $t_0 \in T$) which is weakly adherent to U , i.e. given $\varepsilon > 0$ and $\varphi \in \mathcal{K}_2$ there would exist $h \in U$ such that $|\mu((f-h)\varphi)| < \varepsilon$, which is clearly false.

Finally, we observe that the space $(A, \sigma_{\mathcal{M}})^\wedge$ of Theorems 1 and 2 is here $L(\mu)$, by [F₁] Prop. 3.4, since for compact T $L_{loc}(\mu) = L(\mu)$ and $\mathcal{K}\mathcal{S} = \mathcal{S}$. Now [F₁] Prop. 2.7 and Prop. 3.4 ensure that $\zeta\mu$ as here defined for $\zeta \in L(\mu)$ is none other than the product $\xi\mu$ of integration theory. Our Lebesgue-Nikodym theorem now asserts that λ is absolutely continuous with respect to μ if and only if $\lambda = \zeta\mu$.

§ 2.1 We now consider an extension of Theorems 1 and 2 which will enable us to establish the Lebesgue-Nikodym theorem for Radon measures on a locally-compact space. The extension has very little in the way of novel features now that the more restricted Theorems 1 and 2 have been established, but in the interests of ease of presentation it seemed worthwhile to establish the simpler results first.

We consider a $*$ -algebra A which is the inductive limit, as a topological vector space ([B₂] II § 4), of a directed family $\{A_j\}_J$ of normed $*$ -subalgebras A_j with norms $\|\cdot\|_j$. Let μ be a linear form on A having the properties $F_{1,2,3}$ of § 1.2. As in § 1.2 μ defines a prehilbert structure on A , with semi-norm p_0 and associated norm p , and $A^\wedge = (A, p_0)^\wedge$ with norm p is a Hilbert space. If $U_j = \{\xi \in A_j : \|\xi\|_j \leq 1\}$ and $V = \{\xi \in A : p_0(\xi) \leq 1\}$, since

$$(p_0(\xi))^2 = \mu(\xi\xi^*) \leq \|\mu\|_j \|\xi\xi^*\|_j \leq \|\mu\|_j \|\xi\|_j^2$$

if $\xi \in A_j$ (where $\|\mu\|_j$ is the norm of the restriction of μ to A_j), we have

$$U_j \subset m_j V \quad \text{for some } m_j > 0.$$

It follows that if, as before, i is the canonical map of A into the separated completion A^\wedge , then the weak closures in A^\wedge of the sets $i(U_j)$ are compact for $\sigma(A^\wedge, i(A))$.

We define, as before, a duality (D_0) on $A_1 \times A_2$ and now take as \mathcal{M} the saturate for (D_0) of $\{U_j\}_J$ considered as a family of subsets of A_1 . \mathcal{N} will be the same family considered as a family of subsets of A_2 . \mathcal{M}_1 and \mathcal{N}_2 , families in $i(A_1)$ and A_2 , respectively, are defined as before. We define absolute continuity of another linear form λ with respect to μ as in § 1, and Theorems 1 and 2 can then be established in this new context, the only change on the argument that is needed is an obvious modification of the proof of the lemma preceding Theorem 1.

§ 2.2 In applying the theorems to Radon measures on a locally compact space T we take the A_j to be the $*$ -normed algebras $\mathcal{K}(K_j)$, where $\{K_j\}_J$ are the compact sets of T . A Radon measure μ on T has the properties $F_{1,2,3}$. If λ is absolutely continuous with respect to μ (in the sense of [B₃] V § 5.5) we have seen ([F₂] Prop. 3.1) that it is $\sigma_{\mathcal{K}_2 \mathcal{S}_2}$ -continuous on the sets of $\mathcal{K}_1 \mathcal{S}_1$, in the notation of [F₂], so that to show that λ is absolutely continuous with respect to μ in the present sense it suffices to observe i) that each U_j is contained in a set of $\mathcal{K}_1 \mathcal{S}_1$ of the form fS_0 where $f \in \mathcal{K}$ has the value 1 on K_j and $S_0 = \{\varphi \in \mathcal{K} : |\varphi| \leq 1\}$, and ii) that $\mathcal{K}_2 \mathcal{S}_2 \subset \mathcal{N}$ since

each gS is contained in a set gcS_0 , where c is a constant, and $gcS_0 \subset dU_j$ for some constant d and support $(g) \subset K_j$.

Finally, in interpreting the conclusions of our abstract Lebesgue-Nikodym theorem in the present context we note that $\sigma_{\mathcal{M}} = \sigma_{\mathcal{K}_1 \mathcal{S}_1}$: i) above, shows that $\sigma_{\mathcal{M}} \subset \sigma_{\mathcal{K}_1 \mathcal{S}_1}$, and ii) shows that $\sigma_{\mathcal{K}_1 \mathcal{S}_1} \subset \sigma_{\mathcal{M}}$. Thus $(\mathcal{K}, \sigma_{\mathcal{M}})$ is isomorphic, as a topological vector space, with $L_{loc}(\mu)$.

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