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A REMARK ON PLANCHEREL'S THEOREM FOR BANACH SPACE VALUED FUNCTIONS

STEPHEN VÁGI

1. Introduction.

Convolution operators acting on Banach space valued functions have turned out to be a useful instrument in proving L^p inequalities, and have found a number of applications in classical analyisis ([1], [9], [10]). Omitting what is not relevant to the present purpose one can summarize this theory, which runs parallel to that of the scalar case, as follows:

A) One has to prove that one's operator is bounded on L^{p_0} for some particular $p_0, p_0 > 1$.

B) This information is then used to show that the operator is of weak type (1,1).

C) Finally, the interpolation theorem of Marcinkiewicz allows to conclude that it is bounded on all L^p's, $1 < p \leq p_0$. This note is concerned with Step *A*.

In the classical case, i. e., when dealing with scalar functions this step is carried out by taking $p_0 = 2$ and using Plancherel's theorem. This is usually quite easy and seems to be the only effective procedure to handle this step.

The same approach succeeds also in the applications which the vector method has found so far. This is so because in every case the Banach spaces which contain the values of the L^2 functions one is working with turn out to be Hillbert spaces, and Plancherel's theorem does hold for Hilbert space valued functions. Now it is remarkable that Steps *B* and *C*, which embody the difficult real variable aspects of the theory and which account for its power, in no way depend on the fact that these spaces

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are Hilbert spaces. To sum up: Extending the method from scalar to vector functions does not effect B and C, however, Step A remains easy only if Plancherel's theorem is available for vector valued L^2 functions. The preceding observations are intended to motivate the interest in asking whether that theorem — in a sense to be made precise below — does hold for quadratically integrable functions with values in a general Banach space, i. e., one which is not linearly homeomorphic to a Hilbert space.

The purpose of this note is to provide a partial answer to the above question. It always turns out to be negative if the Fourier transform is required to be an isometry of L^2; without this restriction it is negative for a class of Banach spaces which includes many of those currently used in classical analysis. Whether the answer is always negative, is not known.

2. Definitions, Notations, and Statement of Results.

E will be a complex Banach space with norm $\| \ \|$, and dual E'. For $1 \leq p < \infty$ $L^p(R, E)$ will denote the Banach space of E-valued, strongly mesurable functions defined on the real line R for which

$$|f|_p = \left(\int_R \|f(x)\|^p \, dx \right)^{1/p} < \infty.$$

Integration in the preceding formula and everywhere else in this paper is with respect to Lebesgue measure. General references on analysis in Banach spaces are the treatises [3] and [4]. For $f \in L^1(R, E)$ the Fourier transform of f is defined by the usual integral formula

$$(\mathcal{F}f)(\xi) = \widehat{f}(\xi) = \int_R e^{-2\pi i \xi x} f(x) \, dx.$$

Note that $\mathcal{F}f$ is a bounded continuous function which vanishes at infinity. Also, note that $L^1(R, E) \cap L^2(R, E)$ is dense in $L^2(R, E)$ Plancherel's theorem in this context is the statement that for $f \in L^1(R, E) \cap L^2(R, E)$ $\mathcal{F}f$ belongs to $L^2(R, E)$ and that $\mathcal{F}$ can be extended to a bounded linear map of $L^2(R, E)$ into itself. If this is the case $\mathcal{F}$ will be said to the extendable (for E). It will follow easily that if $\mathcal{F}$ is extendable, it is a linear homeomorphism of $L^2(R, E)$ onto itself.

The results to be established are the following:

THEOREM 1. *If $\mathcal{F}$ can be extended to an isometry of $L^2(R, E)$ onto itself, then E — with its original norm — is a Hilbert space.*

THEOREM 2. *If $\mathcal{F}$ can be extended to a bounded linear operator of $L^2(R, E)$ into itself and one of the conditions* i, ii, iii *listed below is satisfied, then E is linearly homeomorphic to a Hilbert space.*

i) *E has an unconditional Schauder basis.*

ii) *The dual, or an interated dual of E, has an unconditional Schauder basis.*

iii) *E is the dual, or an iterated dual of a Banach space, which has an unconditional Schauder basis.*

Theorem 2 can be used to prove the non extendability of $\mathcal{F}$ for concrete spaces E. A number of such results are obtained in Section 4.

3. Proofs.

Before proving the theorems a few simple properties of the Fourier transform have to be established. Denote by $\mathcal{S}$ the E-valued infinitely often differentiable functions on R which decrease rapidly at infinity ; i. e., such that $(1 + |x|^k) \| f^{(l)}(x) \| \longrightarrow 0$ for $|x| \longrightarrow \infty$, and for all non negative intergers k, l. Exactly as in the scalar case, one shows that $\mathcal{S}$ is dense in $L^p(R, E)$; $1 \leq \leq p < \infty$, and that $\mathcal{F}$ maps $\mathcal{S}$ bijectively onto itself. Define $\overline{\mathcal{F}}f$ for $f \in \in L^1(R, E)$ by $(\overline{\mathcal{F}}f)(\xi) = \mathcal{F}f(-\xi)$. Again it follows exactly as in the scalar case that $\overline{\mathcal{F}}$ restricted to $\mathcal{S}$ is the inverse of $\mathcal{F}$ restricted to $\mathcal{S}$. If $\mathcal{F}$ can be extended to a bounded linear map of $L^2(R, E)$ into itself, then so can $\overline{\mathcal{F}}$, and the two extensions, also denoted by $\mathcal{F}$ and $\overline{\mathcal{F}}$, are inverses of one another; i. e., $\mathcal{F}$ is a linear homeomorphism of $L^2(R, E)$ onto itself. The norms of $\mathcal{F}$ and $\overline{\mathcal{F}}$ are equal and cannot be less than one. The latter fact one sees by considering $f = ug$ with $u \in E$ and g a scalar L^2 function. If the norm of $\mathcal{F}$ is one, then $\mathcal{F}$ is an isometry. It is clear that $\mathcal{F}$ is extendable if E is linearly homeomorphic to a Hilbert space.

The proofs of both Theorem 1 and Theorem 2 are hinged on a lemma which will be proved next.

LEMMA 1. *Let $\mathcal{F}$ be extendable for E. Let k be a positive integer, $n_1, n_2, \dots, n_k$ distinct integers, and $u_1, u_2, \dots, u_k$ (not necessarily distinct) elements of E. Then*

$$c^{-2} \left\| \sum_{j=1}^{k} u_j \right\|^2 \leq \int \left\| \sum_{j=1}^{k} e^{-2\pi i n_j \theta} u_j \right\|^2 d\theta \leq c^2 \left\| \sum_{j=1}^{k} u_j \right\|^2 \tag{1}$$

where c denotes the norm of $\mathcal{F}$.

PROOF: Let N be a positive integer. Define a function f_N by

$$f_N(x) = \sum_{j=1}^{k} u_j \chi(x - N n_j)$$

where χ is the characteristic function of the open interval $(0, 1)$. f_N is a simple function, its Fourier transform $\widehat{f_N}$ is

$$\widehat{f_N}(\xi) = \widehat{\chi}(\xi) \sum_{j=1}^{k} e^{-2\pi i n_j N \xi} u_j,$$

and $\widehat{f_N} \in L^2(R, E)$. The extendability of $\mathcal{F}$ is not needed to show this: $\widehat{\chi} \in L^2$, and its factor in the above formula is bounded. The L^2 norm of $\widehat{f_N}$ is given by

$$|\widehat{f_N}|_2^2 = \int_R \left\| \sum_{j=1}^{k} e^{-2\pi i n_j N\xi} u_j \right\|^2 |\widehat{\chi}(\xi)|^2 \, d\xi. \tag{2}$$

Consider now the function g defined by

$$g(\xi) = \left\| \sum_{j=1}^{k} e^{-2\pi i n_j \xi} u_j \right\|^2.$$

g is periodic of period one, and Lipschitz continuous. Consequently, it has a Fourier series which converges to it everywhere, and the convergence is absolute and uniform ([11] Vol. I). Denote its complex Fourier coefficients by c_n, $-\infty < n < \infty$. By the Lebesgue dominated convergence theorem (2) becomes

$$|\widehat{f_N}|_2^2 = \sum_{n=-\infty}^{\infty} \int_R e^{2\pi i N n \xi} c_n \, |\widehat{\chi}(\xi)|^2 \, d\xi. \tag{3}$$

The Ríemann-Lebesgue lemma for Fourier integrals shows that for N approaching infinity all the terms in the above series which have $n \neq 0$, tend to zero. Since each of the integrals in this series is less than or equal to one in absolute value, and since $\Sigma |c_n|$ converges, it follows that (3) tends to c_0. However

$$c_0 = \int_0^1 g(\theta)\, d\theta = \int_0^1 \left\| \sum_{j=1}^{k} e^{-2\pi i n_j \theta} u_j \right\|^2 d\theta. \tag{4}$$

The lemma now follows from this by observing that

$$|f_N|_2^2 = \sum_{j=1}^{k} \| u_j \|^2$$

is independent of N, and that $\mathcal{F}$ is extendable.

PROOF OF THEOREM 1. Let $u, v \in E$ In Lemma 1 set $k = 3, u_1 = u$ $u_2 = u_3 = v$; $n_1 = 0$, $n_2 = 1$, $n_3 = -1$. Since $\mathcal{F}$ is assumed to be an isometry (1) becomes

$$\int_0^1 \| u + 2v \cos 2\pi\theta \|^2 \, d\theta = \| u \|^2 + 2 \| v \|^2 .$$

Replacing v by $-v$ in this formula, adding the two identities thus obtained and finally defining a function ψ of the real variable λ by

(5) $$\psi(\lambda) = \| u + 2\lambda v \|^2 + \| u - 2\lambda v \|^2 - 2 \| u \|^2$$

one obtains

(6) $$\int_0^1 \psi(\cos 2\pi\theta) \, d\theta = 4 \| v \|^2.$$

ψ is readily seen to be a convex, non-negative, and even function of λ. An easy computation shows that the integrand of (6) is symmetric with respect to the point $\theta = 1/4$. This, the evenness, and the periodicity of the integrand allow to rewrite (6) in the form

(7) $$\int_0^{1/4} \psi(\cos 2\pi\theta) \, d\theta = \| v \|^2.$$

Let t be non-negative; introducing tv instead of v in (7), one finds that ψ satisfies the integral equation

(8) $$\int_0^{1/4} \psi(t \cos 2\pi\theta) \, d\theta = \| v \|^2 \, t^2$$

Computation shows that $\psi_0(\lambda) = 8 \| v \|^2 \lambda^2$ is a solution of (8). By the change of variable $t \cos 2\pi\theta = \sqrt{s}$ (8) transforms into the equivalent equation

$$(9) \qquad \int_0^{t^2} \varphi(s) \frac{ds}{\sqrt{t^2 - s}} = 4\pi \| v \|^2 t^2$$

where $\varphi(s) = \psi(\sqrt{s})/\sqrt{s}$. Note that ψ, ψ_0, and their transforms φ and φ_0 are locally integrable on $[0, \infty)$. To show that φ_0 is the only solution of (9) in this class of functions and, hence, ψ_0 the only solution of (8), recall the following fact ([12], p. 323): If f and g are locally integrable on $[0, \infty)$ and g is not identically zero, then

$$\int_0^x f(t)\, g(x - t)\, dt = 0$$

implies that $f(x) = 0$ at almost every x in $[0, \infty)$. Since the kernel of (9) is locally integrable, it now follows from the above theorem and the linearity of (9) that φ_0 is its only solution. Introducing this information into (5), one obtains

$$(10) \qquad \| u + 2\lambda v \|^2 + \| u - 2\lambda v \|^2 = 2 \| u \|^2 + 8 \| v \|^2 \lambda^2.$$

Setting $\lambda = \frac{1}{2}$ in (10) one finally has

$$\| u + v \|^2 + \| u - v \|^2 = 2 \| u \|^2 + 2 \| v \|^2,$$

i.e., the norm of E satisfies the parallelogram identity; in other words, it is an inner product norm. Theorem 1 is proved.

Some further preparation is needed before the proof of Theorem 2 can be taken up. A Schauder basis of a Banach space E is an unconditional basis if the series giving the expansion of an arbitrary element of the space is unconditionally convergent, i.e., if rearrangement does not affect its convergence or sum. It is known [5] that a sequence $\{e_n\}_{n=1}$ of elements of E which span a dense subspace is an unconditional basis of E if and only if there exists $M > 0$ such that for any finite subsets A, B of the integers such that $B \subset A$, and any choice of scalars k_n, one has

$$(11) \qquad \| \sum_{n \in B} k_n e_n \| \leq M \| \sum_{n \in A} k_n e_n \|$$

The following lemma will also be needed.

LEMMA 2. *Let A be a finite set, $\{u_\alpha\}_{\alpha\in A}$ a family of elements of E, $\{k_\alpha\}_{\alpha\in A}$ a family of scalars. Then*

$$\| \sum_{\alpha\in A} k_\alpha u_\alpha \| \leq 4 \sup_{\alpha\in A} |k_\alpha| \sup_{B\subset A} \| \sum_{\alpha\in B} u_\alpha \| . \tag{12}$$

This lemma has been announced in a more general context in [8]. For the sake of completeness a proof will be included for the special case used here. If E is the field of complex numbers (11) is easily proved by separating the left hand side into its real and imaginary parts. Let now f be a continuous linear functional on E. The left hand side of (11) is the supremum of

$$|f(\sum_{\alpha\in A} k_\alpha u_\alpha)|$$

taken over all f's whose norm does not exceed one. Using (12) for the scalar case, one has

$$|f(\sum_{\alpha\in A} k_\alpha u_\alpha)| \leq 4 \sup_{\alpha\in A} |k_\alpha| \sup_{C\subset A} | \sum_{\alpha\in C} f(u_\alpha)| . \tag{13}$$

But if $\|f\| \leq 1$, then

$$| \sum_{\alpha\in C} f(u_\alpha)| \leq \| \sum_{\alpha\in C} u_\alpha \| \leq \sup_{B\subset A} \| \sum_{\alpha\in B} u_\alpha \| . \tag{14}$$

(13) and (14) together prove (12).

PROOF OF THEOREM 2 UNDER THE ASSUMPTION i): It is no restriction of generality also to assume that $\|e_n\| = 1$ for all n. Let E_0 be the dense subspace of E spanned by the basis; i.e., E_0 consists of all finite linear combinations of e_n's. Define a linear map $T: E_0 \to l^2$ by $Te_n = \eta_n$ where $\{\eta_n\}_{n=1}^{\infty}$ is the standard orthonormal basis of l^2. T maps E_0 injectively onto a dense subspace of l^2.

Let now $u = \sum_{j=1}^{k} \lambda_j e_j \in E_0$. Apply Lemma 1 to the vectors $\lambda_j e_j, j = 1,2,\ldots,k$, and arbitrary distinct integers $n_1, \ldots, n_k$ to obtain

$$c^{-2} \| Tu \|_{l^2}^2 \leq \int_0^1 \left\| \sum_{j=1}^{k} e^{-2\pi i n_j \theta} \lambda_j e_j \right\|^2 d\theta \leq c^2 \| Tu \|_{l^2}^2 . \tag{15}$$

Let $A = \{1, 2, \ldots, k\}$ and apply Lemma 2 to $\sum_{j=1}^{k} e^{-2\pi i n_j \theta} \lambda_j e_j$ and to $\sum_{j=1}^{k} \lambda_j e_j$ by setting $u_j = \lambda_j e_j$, $k_j = e^{-2\pi i n_j \theta}$ in the first case, and $u_j = e^{-2\pi i\, j\theta} \lambda_j e_j$, $k_k = e^{+2\pi i n_j \theta}$ in the second. By (11)

$$\left\| \sum_{j=1}^{k} e^{-2\pi i n_j \theta} \lambda_j e_j \right\| \leq 4M \| u \| \text{ and } \| u \| \leq 4M \left\| \sum_{j=1}^{k} e^{-2\pi i n_j \theta} \lambda_j e_j \right\| .$$

These two inequalities combined with (15) give

$$(16) \qquad (4cM)^{-2} \| u \|^2 \leq \| Tu \|_{l^2}^2 \leq (4cM)^2 \| u \|^2 .$$

(16) shows that T can be extended to all of E as a bounded linear map, that it has a bounded inverse, and consequently a closed range. Since $T(E_0)$ is dense in l^2, T maps E onto l^2, i.e., T is a linear homeomorphism of E with l^2.

Theorem 2, assuming ii or iii, is a direct consequence of the following

LEMMA 3. *Let F and G be Banach spaces, B a bounded bilinear form on $F \times G$ which is norm determining for both F and G. Then $\mathcal{F}$ is extendable for F if and only if it is extendable for G.*

PROOF OF LEMMA 3. Let $f \in L^2(R, F)$, $g \in L^2(R, G)$. Introduce

$$(f, g) = \int_R B(f, g)\, dx .$$

$(,)$ is a bounded bilinear form on $L^2(R, F) \times L^2(R, G)$, and norm determining for both $L^2(R, F)$ and $L^2(R, G)$, i.e., for $f \in L^2(R, F)$, $g \in L^2(R, G)$

$$| f |_2 = \sup \{ | (f, h) | ; h \in L^2(R, G) \text{ and } | h |_2 \leq 1 \} ,$$

$$| g |_2 = \sup \{ | (k, g) | ; k \in L^2(R, F) \text{ and } | k |_2 \leq 1 \} .$$

This is easily checked by a minimal modification of the argument given in [9] to prove Lemma 8 of that paper, which states essentially the same fact.

$\mathcal{F}$ can be defined on $L^1(R, F)$ and $L^1(R, G)$. For $f \in \mathcal{S}(R, F)$ and $g \in \mathcal{S}(R, G)$ it is readily verified that

$$(17) \qquad (\mathcal{F}f, g) = (f, \mathcal{F}g).$$

Now if $\mathcal{F}$ is extendable for F it has a bounded transpose with respect to (,). By (17) this transpose coincides on the dense subspace $\mathcal{S}$ of $L^2(R, G)$ with $\mathcal{F}$, i.e., $\mathcal{F}$ defined on $\mathcal{S}$ is extendable to $L^2(R, G)$. Since F and G enter the statement of Lemma 3 symmetrically, this concludes the proof of that lemma.

To prove Theorem 2 assuming, say condition ii, let $E^{(n)}$ be the iterated dual of E which has an unconditional basis. If $\mathcal{F}$ is extendable for E, it is extendable for E' by Lemma 3, B being the bilinear form giving the duality betwen E and E'. In n steps one extends $\mathcal{F}$ to $E^{(n)}$, this space then satisfies condition i of Theorem 2 and is, therefore, linearly homeomorphic to a Hilbert space; this implies that E itself is linearly homeomorphic to a Hilbert space, too. It is clear now, how to prove Theorem 2 assuming condition iii.

4. Resuls on Non-Extendability.

In this section, the non-extendability of $\mathcal{F}$ for a number of specific Banach spaces will be shown. These examples will illustrate sufficiently how other cases might be dealt with. The method will consist in using Theorem 2 in combination with the following simple

LEMMA 4. *Let E and F be Banach spaces. Let $\mathcal{F}$ be non-extendable for F; then it is non-extendable for E if either of the following two conditions holds.*

i) *E contains a closed subspace which is linearly homeomorphic to F.*

ii) *E can be mapped linearly and continuously onto F.*

The statement is obvious in Case i. In Case ii, it follows by duality from Theorem 2 and Part i of Lemma 4.

a) Sequence spaces. Theorem 2 implies that if E has an unconditional basis and is not linearly homeomorphic to a Hilbert space, then $\mathcal{F}$ is not extendable. Let c, c_0 and l^p, $1 \leq p \leq \infty$ denote the familiar sequence spaces. Except for l^∞ all there spaces have obvious unconditional bases. Now l^p, $p \neq 2$ is not linearly homeorphic to a Hilbert space ([2], p. 120) So by the above remark, $\mathcal{F}$ is not extendable for l^p, $1 \leq p < \infty$, $p \neq 2$. Since $l^\infty = (l^1)'$ and $(c_0)' = l^1$, non extendability follows for l^∞ and c_0 from Theorem 2. Finally, c contains c_0 as a direct summand.

b) L^p spaces. Let (S, Σ, μ) be a measure space, then $\mathcal{F}$ is non-extendable for $L^p(S, \Sigma, \mu)$, $1 \leq p < \infty$ unless this space is finite dimensional or $p = 2$. It is easy to see that L^p is finite dimensional if and only if the collection of measurable sets of positve finite measure consists of finitely many atoms, each of finite measure. Excluding this trivial case, one has to deal with a measure space which contains infinitely many measurable sets of positive finite measure. One can always find in such a space a sequence $\{A_n\}$ of disjoint measurable sets which have positive finite measure. This is clear if the space contains infinitely many atoms of finite measure. If not, let B_0 be a measurable set of finite positive measure which is not a finite union of atoms; such a set exists by assumption. B_0 contains a proper subset of positive measure B_1 such that $0 < \mu(B_1) < \mu(B_0)$. Let $A_1 = B_0 - B_1$; $\mu(A_1) > 0$. Now proceed inductively. If χ_i denotes the characteristic function of A_i, then $(\mu(A_i))^{-1/p}\chi_i$ has L^p norm one, and the closed subspace of L^p spanned by these functions is linearly isometric to the sequence space l^p. Lemma 4 and a) now imply that $\mathcal{F}$ is non-extendable. If (S, Σ, μ) is totally sigma-finite, then L^∞ is the dual of L^1, hence $\mathcal{F}$ is non-extendable to L^∞.

c) Let K be a compact Hausdorff space, and $C(K)$ the space of continuous complex valued functions on K. Let $C(K)'$ be the dual of this space, i.e., the set of all finite signed Baire measures on K. $\mathcal{F}$ is non-extendable for $C(K)'$ and hence for $C(K)$. To see this let μ be a non-zero element of $C(K)'$, $\mu \geq 0$. $L^1(\mu)$ is a closed subspace of $C(K)'$, $\mathcal{F}$ is non extendable for $L'(\mu)$ by b). Hence, by Lemma 4, it is non-extendable to $C(K)'$.

d) Let $C^k(R^n)$ be the space of bounded continuous functions on Euclidean n-space which have bounded continuous derivatives up to and including those of order k. $C^k(R^n)$ is a Banach space with the norm

$$\|f\| = \|f\|_\infty + \sum_{|\alpha|=1}^{k} \|D^\alpha f\|_\infty$$

where D^α is the, by now, standard multi-index notation for derivatives. Restriction to a line maps $C^k(R^n)$ continuously onto $C^k(R)$. Restriction to the interval $I = [0,1]$ maps $C^k(R)$ continuously into $C^k(I)$. It is easily checked that this map is also onto. It is known ([0], p. 184) that $C^k(I)$ is linearly homeomorphic to $C(I)$ (The norm of $C^k(I)$ used in [0] is different from the one above, however, they are easily seen to be equivalent). It follows by Lemma 4 and c) that $\mathcal{F}$ is not extendable for $C^k(R^n)$.

e) To conclude, consider the following case: Let H be a complex infinite dimensional Hilbert space and let $B(H)$ denote the algebra of bounded linear operators on H. $\mathcal{F}$ is not extendable for $B(H)$. To see this, let $A \in B(H)$ be self-adjoint and such that the norm closure of the subalgebra generated by A is infinite dimensional. (It suffices to take an A whose spectrum is not a finite set.) From spectral theory one knows that this subalgebra is isometrically isomorphic to the algebra of continuous complex valued functions on a compact Hausdorff space ([6], p. 95), i.e., the Banach space $B(H)$ contains a closed subspace for which $\mathcal{F}$ is not extendable. One can go slightly further: if $C(H)$ denotes the closed ideal of $B(H)$ consisting of the compact linear operators on H, and if H is separable, it follows that $\mathcal{F}$ is not extendable for $C(H)$ because ([7], p. 208) $B(H)$ is linearly isometric to the second dual of $C(H)$.

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REFERENCES

[0] S. Banach, *Théorie des opérations linéaires*, Warsaw 1932.

[1] A. Benedek, A. P. Calderón and R. Panzone, *Convolution operators on Banach space valued functions*, Proc. Nat. Acad. Sci., U.S.A. 48, No. 3 (1962), 356-365.

[2] M. M. Day, *Normed linear spaces*, Springer Verlag, Berlin, 1958.

[3] N. Dunford and J. Schwartz, *Linear operators*, Part I, Interscience, New York, 1958.

[4] E. Hille, *Functional analysis and semigroups*, A. M. S. Colloquium Publication, XXXI, New York, 1948.

[5] R C. James, *Bases and reflexivity in Banach spaces*, Ann. of Math. (2), 52 (1950), 518-527.

[6] L. Loomis, *An introduction to abstract harmonic analysis*, Van Nostrand, New York, 1953.

[7] M. A. Neumark, *Operatorenalgebren im Hilbertschen Raum*, in: Sowjetische Arbeiten Zur. Funktionalanalysis, Verlag Kultur und Fortschritt, Berlin, 1954.

[8] J. R. Retherford, *On Čebyšev subspaces and unconditional bases in Banach spaces*, Bull. A. M. S., 73 (1967), 238-241.

[9] J. Schawartz, *A remark on inequalities of Calderón-Zygmuud type for vector valued functious*, Comm. on Pure and Appl. Math., XIV (1961), 785-799.

[10] S. Vági, *On multipliers and singular integrals in L^p spaces of vector valued functions*, Ph. D. Thesis, U. of Chicago 1965.

[11] A. Zygmund, *Trigonometric series*, 2nd. Ed., Cambridge Univ. Press, New Vork, 1959.

[12] E. C. Titchmarsh, *Introduction to the theory of Fourier integrals*, 2nd. Ed., Oxford Univ. Press, 1948.