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SCUOLA NORMALE SUPERIORE DI PISA
Classe di Scienze

E. BOMBIERI
H. DAVENPORT

Some inequalities involving trigonometrical polynomials

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SOME INEQUALITIES INVOLVING TRIGONOMETRICAL POLYNOMIALS

E. BOMBIERI and H. DAVENPORT

1. Introduction.

Let N be a positive integer and let $a_{M+1}, \dots, a_{M+N}$ be any real or complex numbers. Define

$$S(x) = \sum_{n=M+1}^{M+N} a_n e^{2\pi i n x}. \tag{1}$$

Let $x_1, \dots, x_R$ be any real numbers which satisfy

$$\| x_r - x_s \| \geq \delta \text{ when } r \neq s, \tag{2}$$

where $\| \theta \|$ denotes the difference between θ and the nearest integer, taken positively, and $0 < \delta \leq \frac{1}{2}$.

In a recent paper [1]) we proved that

$$\sum_{r=1}^{R} | S(x_r) |^2 \leq (N^{1/2} + \delta^{-1/2})^2 \sum_{n=M+1}^{M+N} | a_n |^2, \tag{3}$$

and that

$$\sum_{r=1}^{R} | S(x_r) |^2 \leq 2 \max (N, \delta^{-1}) \sum_{n=M+1}^{M+N} | a_n |^2 ;$$

the latter represents an improvement on the former if N and δ^{-1} are of about the same size.

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(1) « On the large sieve method », *Abhandlungen aus Zahlentheorie und Analysis zur Erinnerung an Edmund Landau*, Deutscher Verlag der Wissenschaften, Berlin 1968.

In the present paper we investigate more deeply the two cases in which $N\delta$ is either large or small. The factor on the right of (3) is a little greater than N in the first case, and a little greater than δ^{-1} in the second case. Our object is to determine the order of magnitude of the term that must be added to N or δ^{-1}, as the case may be, to ensure the validity of the inequality.

For the case $N\delta$ large, we prove:

Theorem 1. *If* $N\delta \geq 1$ *then*

$$\sum_{r=1}^{R} |S(x_r)|^2 < N(1 + 5(N\delta)^{-1}) \sum_{n=M+1}^{M+N} |a_n|^2 . \tag{4}$$

On the other hand, if c *is a constant less than* 1 *there exist sums* $S(x)$ *with* δ *arbitrarily small and* $N\delta$ *arbitrarily large for which*

$$\sum_{r=1}^{R} |S(x_r)|^2 > N(1 + c(N\delta)^{-1}) \sum_{n=M+1}^{M+N} |a_n|^2 . \tag{5}$$

There are two new features in the proof of (4), as compared with the proof of (3) in our previous paper. The first of these is a maximization argument (Lemma 1), which has the effect of allowing us to limit ourselves to sums $S(x)$ in which the numbers $|a_n|$ have a measure of approximate equality. The second feature is the use of the function $\Phi_1(t)$, defined in (13), in place of the simpler function $\Phi(t)$ of our previous paper.

For the case $N\delta$ small, we prove:

Theorem 2. *If* $N\delta \leq \frac{1}{4}$, *then*

$$\sum_{r=1}^{R} |S(x_r)|^2 < \delta^{-1}(1 + 270 N^3\delta^3) \sum_{n=M+1}^{M+N} |a_n|^2 . \tag{6}$$

On the other hand there exist sums $S(x)$ *with* $N\delta$ *arbitrarily small for which*

$$\sum_{r=1}^{R} |S(x_r)|^2 > \delta^{-1}\left(1 + \frac{1}{12} N^3\delta^3\right) \sum_{n=M+1}^{M+N} |a_n|^2 . \tag{7}$$

This case represents a problem which is entirely different from that of the first case; it has much in common with the problem of approximating to a Riemann integral by a finite sum. The arguments used in the proof of (6) are quite delicate.

The present paper is self-contained, except for the general inequality (27), due to Davenport and Halberstam, which is needed for the proof of (4).

2. The maximization argument.

Let δ and $x_1, \dots, x_R$ be fixed. For each positive integer N, we define $G(N)$ by

$$N + G(N) = \max \sum_{r=1}^{R} | S(x_r) |^2, \tag{8}$$

where

$$S(x) = \sum_{n=1}^{N} a_n e^{2\pi i n x} \tag{9}$$

and the maximum is taken over all complex numbers $a_1, \dots, a_N$ satisfying

$$\sum_{n=1}^{N} | a_n |^2 = 1. \tag{10}$$

The maximum is obviously attained, and we call a set of coefficients a_n for which it is attained a *maximal set.*

LEMMA 1. *Suppose N is such that $G(N) > 0$ and*

$$G(N') \leq G(N) \quad \text{for} \quad N' \leq N. \tag{11}$$

Then, for a maximal set of coefficients, we have

$$\sum_{M+1}^{M+H} | a_n |^2 = \frac{H + \theta G}{N + G} \sum_{1}^{N} | a_n |^2, \tag{12}$$

where $|\theta| \leq 1$ and $G = G(N)$; and this holds for all M, H satisfying

$$0 \leq M < M + H \leq N.$$

Proof. Write $e(\theta) = e^{2\pi i\theta}$, and

$$a_n = A_n e(\alpha_n),$$

where $A_n \geq 0$ and α_n is real. For a maximal set of coefficients we have a stationary value of $\Sigma | S(x_r) |^2$ subject to $\Sigma A_n^2 = 1$. Hence we must have

$$\frac{\partial}{\partial A_m} \sum_{r=1}^{R} | S(x_r) |^2 - 2\lambda A_m = 0 \qquad (m = 1, \dots, N)$$

for some λ. Writing $|S|^2 = S\bar{S}$, and noting that

$$\frac{\partial S(x)}{\partial A_m} = e(mx + \alpha_m),$$

we see that the conditions are

$$\sum_{r=1}^{R} e(mx_r + \alpha_m)\overline{S(x_r)} + \sum_{r=1}^{R} e(-mx_r - \alpha_m) S(x_r) = 2\lambda A_m .$$

If this is multiplied by A_m and summed for $m = 1, \dots, N$, it gives

$$2\sum_{r=1}^{R} |S(x_r)|^2 = 2\lambda \sum_{m=1}^{N} A_m^2 = 2\lambda,$$

whence

$$\lambda = N + G.$$

If however the sum is restricted to $m = M + 1, \dots, M + H$, we get

$$\sum_{r=1}^{R} S_H(x_r)\overline{S(x_r)} + \sum_{r=1}^{H} \overline{S_H(x_r)} S(x_r) = 2\lambda \sum_{m=M+1}^{M+H} A_m^2 ,$$

where

$$S_H(x) = \sum_{m=M+1}^{M+H} a_m e(mx).$$

Hence

$$\lambda \sum_{m=M+1}^{M+H} |a_m|^2 \leq \sum_{r=1}^{R} |S_H(x_r)|\,|S(x_r)| \leq \left(\sum_{r=1}^{R} |S_H(x_r)|^2\right)^{1/2} \left(\sum_{r=1}^{R} |S(x_r)|^2\right)^{1/2} .$$

Now

$$\sum_{r=1}^{R} |S(x_r)|^2 = \lambda = N + G .$$

Also, by the definition of $G(H)$ and the hypothesis (11), we have

$$\sum_{r=1}^{R} |S_H(x_r)|^2 \leq (H + G(H)) \sum_{m=M+1}^{M+H} |a_m|^2 \leq (H + G) \sum_{m=M+1}^{M+H} |a_m|^2 .$$

Hence

$$(N + G)^{1/2} \sum_{m=M+1}^{M+H} |a_m|^2 \leq \left((H + G) \sum_{m=M+1}^{M+H} |a_m|^2\right)^{1/2} ,$$

that is,

$$\sum_{m=M+1}^{M+H} |a_m|^2 \leq \frac{H+G}{N+G}.$$

This is on the hypothesis that $\sum_1^N |a_n|^2 = 1$. Plainly that hypothesis can be omitted if we modify the inequality so that it reads

$$\sum_{m=M+1}^{M+H} |a_m|^2 \leq \frac{H+G}{N+G} \sum_{n=1}^{N} |a_n|^2.$$

We can obtain a complementary inequality by applying this to the sums over

$$0 < m \leq M \quad \text{and} \quad M+H < m \leq N,$$

and subtracting from the complete sum. We obtain

$$\sum_{m=M+1}^{M+H} |a_m|^2 \geq \left(1 - \frac{M+G}{N+G} - \frac{N-M-H+G}{N+G}\right) \sum_{n=1}^{N} |a_n|^2 = \frac{H-G}{N+G} \sum_{n=1}^{N} |a_n|^2.$$

The two inequalities together prove (12).

3. A particular Fourier series.

Lemma 2. *Let*

(13) $$\Phi_1(t) = \frac{\sin \pi\lambda t}{t} \cdot \frac{\sin \pi t}{\pi t} \quad \textit{for } 0 < t < 1.$$

Suppose that $\lambda > |\alpha| + 1$. *Then*

(14) $$\int_0^1 \{\Phi_1(t)\}^2 \, dt < \frac{1}{2} \pi^2 \lambda,$$

(15) $$\int_0^1 \Phi_1(t) \cos \pi\alpha t \, dt > \frac{\pi}{2} - \frac{1}{2\pi^2} \left(\frac{1}{(\lambda+\alpha)^2 - 1} + \frac{1}{(\lambda-\alpha)^2 - 1}\right).$$

Proof. (14) is almost immediate, for since $\sin \pi t < \pi t$ we have

$$\int_0^1 \{\Phi_1(t)\}^2\, dt < \int_0^\infty \left(\frac{\sin \pi\lambda t}{t}\right)^2 dt = \frac{1}{2}\pi^2\lambda.$$

To prove (15) we start from the relations

$$\int_0^1 \Phi_1(t)\cos \pi\alpha\, t\, dt = \frac{1}{2}\int_0^1 \frac{\sin \pi(\lambda+\alpha)t + \sin \pi(\lambda-\alpha)t}{t}\cdot\frac{\sin \pi t}{\pi t}\, dt$$

$$= \frac{1}{4}\int_0^1 \frac{\cos \pi(\lambda+\alpha-1)t - \cos \pi(\lambda+\alpha+1)t}{\pi t^2}\, dt$$

$$+ \frac{1}{4}\int_0^1 \frac{\cos \pi(\lambda-\alpha-1)t - \cos \pi(\lambda-\alpha+1)t}{\pi t^2}\, dt.$$

For $B > A > 0$ we have

$$\int_0^\infty \frac{\cos At - \cos Bt}{\pi t^2}\, dt = \int_0^\infty \frac{2\sin^2 \frac{1}{2}Bt - 2\sin^2\frac{1}{2}At}{\pi t^2}\, dt$$

$$= \frac{B-A}{\pi}\int_0^\infty \left(\frac{\sin t}{t}\right)^2 dt = \frac{1}{2}(B-A) = \pi$$

if $A = \pi(\lambda+\alpha-1)$ and $B = \pi(\lambda+\alpha+1)$. Hence

$$\int_0^1 \Phi_1(t)\cos\pi\alpha\, t\, dt = \frac{\pi}{2} - \frac{1}{4\pi}\,\mathcal{R}\,(J(\alpha)+J(-\alpha)), \tag{16}$$

where

$$J(\alpha) = \int_1^\infty \left(e^{i\pi(\lambda+\alpha-1)t} - e^{i\pi(\lambda+\alpha+1)t}\right) t^{-2}\, dt.$$

To estimate $J(\alpha)$ we rotate the line of integration through an angle $\frac{\pi}{2}$ in the complex plane, so that it becomes the line $1 + iu$, $u > 0$. The contribution of the quadrant at infinity vanishes.

We get

$$J(\alpha) = \int_0^\infty \left(e^{i\pi(\lambda+\alpha-1)(1+iu)} - e^{i\pi(\lambda+\alpha+1)(1+iu)}\right)(1 + iu)^{-2}\, i\, du$$

$$= -\, i e^{i\pi(\lambda+\alpha)} \int_0^\infty \left(e^{-\pi(\lambda+\alpha-1)u} - e^{-\pi(\lambda+\alpha+1)u}\right)(1 + iu)^{-2}\, du.$$

Hence

$$|J(\alpha)| \leq \int_0^\infty \left(e^{-\pi(\lambda+\alpha-1)u} - e^{-\pi(\lambda+\alpha+1)u}\right) du$$

$$= \frac{1}{\pi}\left(\frac{1}{\lambda + \alpha - 1} - \frac{1}{\lambda + \alpha + 1}\right) = \frac{2}{\pi\,((\lambda + \alpha)^2 - 1)}.$$

Putting this, for α and $-\alpha$, in (16), we obtain (15).

LEMMA 3. *Suppose that K satisfies $K\delta \geq 2$. Let N_0 be any positive integer. There exists a Fourier series*

$$\psi(x) = \sum_{-\infty}^{\infty} b_n\, e(nx), \tag{17}$$

with real coefficients b_n satisfying $b_{-n} = b_n$, such that

$$\psi(x) = 0 \quad \textit{if} \quad \| x \| > \frac{1}{2}\delta, \tag{18}$$

$$\sum_{-\infty}^{\infty} b_n^2 < \frac{1}{2}\pi^2 (N_0 + K)\,\delta^2, \tag{19}$$

and, for $|n| \leq N_0$

$$b_n^{-2} < \frac{4}{\pi^2\delta^2}\left(1 + \frac{1}{7\{(N_0 + K - |n|)^2\delta^2 - 1\}}\right). \tag{20}$$

Proof. We define $\lambda = (N_0 + K)\,\delta$, and we define $\psi(x)$ for $|x| \leq \frac{1}{2}$ by

$$\psi(x) = \begin{cases} \Phi_1(2\delta^{-1}|x|) & \text{if} \quad |x| < \frac{1}{2}\delta, \\ 0 & \text{if} \quad \frac{1}{2}\delta \leq |x| \leq \frac{1}{2}. \end{cases}$$

We define $\psi(x)$ for other real x by periodicity with period 1. Then $\psi(x)$ is an even function of x and satisfies (18). The Fourier coefficients b_n of $\psi(x)$ are given by

$$b_n = \int_{-\delta/2}^{\delta/2} \psi(x)\, e(-nx)\, dx = \delta \int_0^1 \Phi_1(t) \cos \pi\alpha\, t\, dt,$$

where $\alpha = n\delta$.

By Parseval's formula,

$$\sum_{-\infty}^{\infty} b_n^2 = \int_{-\delta/2}^{\delta/2} \psi^2(x)\, dx = \delta \int_0^1 \{\Phi_1(t)\}^2\, dt.$$

Hence, by (14) of Lemma 2,

$$\sum_{-\infty}^{\infty} b_n^2 < \frac{1}{2}\pi^2 \lambda\delta = \frac{1}{2}\pi^2 (N_0 + K)\,\delta^2.$$

It remains only to prove (20). For $|n| \leq N_0$, we have

$$\lambda - |\alpha| = (N_0 + K - |n|)\,\delta \geq K\delta \geq 2.$$

By (15) of Lemma 2,

$$b_n > \frac{1}{2}\pi\delta\left(1 - \frac{2}{\pi^3((\lambda - |\alpha|)^2 - 1)}\right).$$

Now

$$\frac{2}{\pi^3\{(\lambda - |\alpha|)^2 - 1\}} < \frac{2}{3\pi^3} < \frac{1}{46},$$

and for $0 < \gamma < \frac{1}{46}$ we have

$$(1 - \gamma)^{-2} < 1 + 2\cdot 1\,\gamma.$$

Hence

$$b_n^{-2} < \frac{4}{\pi^2\delta^2}\left(1 + \frac{4\cdot 2}{\pi^3\{(\lambda - |\alpha|)^2 - 1\}}\right).$$

Since $4\cdot 2\,\pi^{-3} < \frac{1}{7}$ this gives (20), on recalling that

$$\lambda - |\alpha| = (N_0 + K - |n|)\,\delta.$$

4. Proof of the first part of Theorem 1.

We observe first that there is no loss of generality in taking $M = 0$ in the definition of $S(x)$ in (1), for we can reduce the general case to this by putting $n = M + n'$. Thus we can take $S(x)$ to be defined by (9).

For fixed δ and fixed $x_1, \dots, x_R$, let N be the least positive integer for which $G(N)$, defined in § 2, satisfies

$$G(N) > 5\,\delta^{-1}\,; \tag{21}$$

if there is no such integer the desired conclusion holds. For this N, the hypothesis (11) of Lemma 1 is satisfied. For a maximal set of coefficients, (12) holds; and of course we also have

$$\sum_{r=1}^{R} |\,S(x_r)\,|^2 = (N + G) \sum_{1}^{N} |\,a_n\,|^2. \tag{22}$$

Suppose first that N is odd, say $N = 2N_0 + 1$. We define

$$a'_n = a_{n+N_0+1} \quad \text{for} \quad -N_0 \leq n \leq N_0\,, \tag{23}$$

and have

$$|\,S(x)\,| = |\,S_0(x)\,|\,, \tag{24}$$

where

$$S_0(x) = \sum_{n=-N_0}^{N_0} a'_n\, e(nx). \tag{25}$$

By (12),

$$\sum_{-m}^{m} |\,a'_n\,|^2 = \sum_{-m+N_0+1}^{m+N_0+1} |\,a_n\,|^2 \geq \frac{2m + 1 - G}{N + G} \sum_{1}^{N} |\,a_n\,|^2 \tag{26}$$

if $0 \leq m \leq N_0$.

Suppose next that N is even, say $N = 2N_0$.

We define a'_n as above in (23) for $-N_0 \leq n < N_0$, and put $a'_{N_0} = 0$. Then (24) is still valid, with the same definition (25). Also (26) is still valid, provided $0 \leq m < N_0$.

Let $\psi(x)$ be any even function of period 1 satisfying the condition (18) of Lemma 3. It was proved by Davenport and Halberstam [2] that

$$\sum_{r=1}^{R} |\,S(x_r)\,|^2 \leq \left(\sum_{-\infty}^{\infty} b_n^2 \right) \left(\sum_{-N_0}^{N_0} b_n^{-2}\, |\,a_n\,|^2 \right). \tag{27}$$

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[2] « The values of a trigonometrical polynomial at well spaced points », *Mathematika* 13 (1966), 91-96, and 14 (1967), 229-232.

With the particular function $\psi(x)$ of Lemma 3, this gives

$$\sum_{r=1}^{R} |S(x_r)|^2 \leq 2(N_0+K)\left(\sum_{-N_0}^{N_0} |a_n'|^2 \left(1+\frac{1}{7}c_n\right)\right),$$

where

$$c_n = \frac{1}{(N_0+K-|n|)^2\delta^2 - 1}.$$

In view of (22) we can write the result as

$$(N+G)\sum_{1}^{N} |a_n|^2 \leq 2(N_0+K)\left(\sum_{1}^{N} |a_n|^2 + \frac{1}{7}S\right), \tag{28}$$

where

$$S = \sum_{-N_0}^{N_0} c_n |a_n'|^2.$$

Applying partial summation, and using the fact that $c_{-n} = c_n$, we obtain

$$\sum_{-N_0}^{N_0} c_n |a_n'|^2 = c_{N_0}\sum_{-N_0}^{N_0} |a_n'|^2 - \sum_{m=0}^{N_0-1}(c_{m+1}-c_m)\sum_{n=-m}^{m} |a_n'|^2.$$

Since $c_{m+1} - c_m > 0$ for $m \geq 0$, we can apply the inequality (26) in the inner sum on the right. This gives

$$S \leq \left\{c_{N_0} - \sum_{m=0}^{N_0-1}(c_{m+1}-c_m)\frac{2m+1-G}{N+G}\right\}\sum_{1}^{N} |a_n|^2$$

$$= \frac{1}{N+G}\left\{\sum_{-N_0}^{N_0} c_m + (2c_{N_0}-c_0)G + (N-2N_0-1)c_{N_0}\right\}\sum_{1}^{N} |a_n|^2.$$

Since $2N_0+1 \geq N$, the last term in the bracket can be omitted.

Substitution in (28) gives

$$(N+G)^2 \leq 2(N_0+K)\left(N+G+\frac{1}{7}\Delta\right),$$

where

$$\Delta = \sum_{-N_0}^{N_0} c_m + (2c_{N_0}-c_0)G. \tag{29}$$

By the inequality of the arithmetic and geometric means, we have

$$N+G \leq N_0+K+\frac{1}{2}N+\frac{1}{2}G+\frac{1}{14}\Delta,$$

and since $N_0 \leq \frac{1}{2} N$ this implies that

$$G \leq 2 \cdot K + \frac{1}{7} \Delta. \tag{30}$$

Since the function c_n, for a continuous variable n, has a positive second derivative, we have

$$\sum_{-N_0}^{N_0} c_n < \int_{-N_0-\frac{1}{2}}^{N_0+\frac{1}{2}} c_x\,dx = 2\int_0^{N_0+\frac{1}{2}} \frac{dx}{(N_0+K-x)^2\,\delta^2-1}$$

$$< 2\,\delta^{-1}\int_{(K-\frac{1}{2})\delta}^{\infty} \frac{du}{u^2-1} = \delta^{-1}\log\frac{\left(K-\frac{1}{2}\right)\delta+1}{\left(K-\frac{1}{2}\right)\delta-1}.$$

We now take $K = 2\delta^{-1}$, and have

$$\sum_{-N_0}^{N_0} c_n < \delta^{-1}\log\frac{3-\frac{1}{2}\delta}{1-\frac{1}{2}\delta} < \delta^{-1}\log\frac{11}{3} < 1\cdot 3\,\delta^{-1}.$$

Also

$$2c_{N_0} - c_0 < 2c_{N_0} = \frac{2}{K^2\,\delta^2-1} = \frac{2}{3}.$$

Hence

$$\Delta < 1\cdot 3\,\delta^{-1} + \frac{2}{3}\,G.$$

From (30) we now obtain

$$G < 4\delta^{-1} + \frac{1}{7}\left(1\cdot 3\,\delta^{-1} + \frac{2}{3}\,G\right) < 4\cdot 2\,\delta^{-1} + \frac{1}{10}\,G,$$

which gives a contradiction to (21). This contradiction proves the first part of Theorem 1.

We have not used the hypothesis that $N\delta \geq 1$, but the result (4) becomes of little value if $N\delta < 1$.

5. Proof of the second part of Theorem 1.

We give a simple example, with δ arbitrarily small and $N\delta$ arbitrarily large, for which

$$\sum_{r=1}^{R} | S(x_r) |^2 = (N + \delta^{-1} - 1) \sum | a_n |^2. \tag{31}$$

This suffices for (5), since $\delta^{-1} - 1 > c\delta^{-1}$ if δ is sufficiently small.

Let h and L be arbitrarily large positive integers, and take

$$S(x) = \sum_{\nu=-L}^{L} e^{2\pi i (2h+1)\nu x}.$$

For this sum, regarded as a case of (1), we have

$$N = 2(2h+1)L + 1, \qquad \sum | a_n |^2 = 2L + 1.$$

Take $\delta = 1/(2h+1)$, and take the points x_r to be

$$0, \ \pm \frac{1}{2h+1}, \ \pm \frac{2}{2h+1}, \dots, \pm \frac{h}{2h+1}.$$

At each of these points we have $S(x) = 2L + 1$, and therefore

$$\sum_r | S(x_r) |^2 = (2h+1)(2L+1)^2.$$

Also

$$(N + \delta^{-1} - 1) \sum | a_n |^2 = (2(2h+1)L + 1 + 2h + 1 - 1)(2L+1)$$
$$= (2h+1)(2L+1)^2.$$

This proves (31).

6. Lemmas for Theorem 2.

As observed at the beginning of § 4, we can take $M = 0$ in (1), so that $S(x)$ is defined by (9). We suppose that

$$N\delta \leq \frac{1}{4}. \tag{32}$$

Define $\Phi(z)$ by

$$\Phi(z) = \sum_{m=1}^{N} \sum_{n=1}^{N} a_m \bar{a}_n \left(1 - \frac{\pi\delta(m-n)}{\sin \pi\delta(m-n)}\right) e((m-n)z). \tag{33}$$

We note that in every term $|\delta(m-n)| \leq \delta N \leq \frac{1}{4}$.

Lemma 4. *We have*

$$\delta\, |S(x)|^2 = \int_{x-\delta/2}^{x+\delta/2} (|S(z)|^2 - \Phi(z))\, dz. \tag{34}$$

Proof. Since

$$|S(z)|^2 = \sum_{m=1}^{N} \sum_{n=1}^{N} a_m \bar{a}_n\, e((m-n)z),$$

we have

$$|S(z)|^2 - \Phi(z) = \sum_{m=1}^{N} \sum_{n=1}^{N} a_m \bar{a}_n \frac{\pi\delta(m-n)}{\sin \pi\delta(m-n)} e((m-n)z).$$

The result now follows from

$$\int_{-\delta/2}^{\delta/2} e((m-n)z)\, dz = \frac{\sin \pi\delta(m-n)}{\pi(m-n)}.$$

Lemma 5. *Let* $F(z) = |S(z)|^2$. *Then*

$$\Phi(z) = \sum_{k=1}^{\infty} (-1)^{k-1} c_k\, \delta^{2k} F^{(2k)}(z), \tag{35}$$

where the c_k *are positive constants, and* $c_1 = \frac{1}{24}$, *and, for* $0 < \theta < 2\pi$,

$$\sum_{1}^{\infty} c_k\, \theta^{2k} = \frac{\frac{1}{2}\theta}{\sin \frac{1}{2}\theta} - 1. \tag{36}$$

Proof. For any θ with $|\theta| < 2\pi$, we have

$$\frac{\frac{1}{2}\theta}{\sin \frac{1}{2}\theta} = \prod_{n=1}^{\infty} \left\{1 - \left(\frac{\theta}{2n\pi}\right)^2\right\}^{-1} = 1 + \sum_{k=1}^{\infty} c_k\, \theta^{2k},$$

where the c_k are positive constants. Also $c_1 = \frac{1}{24}$ from $\sum_1^\infty n^{-2} = \frac{\pi^2}{6}$. Putting $\theta = 2\pi\,\delta\,(m-n)$ and substituting in (33), we obtain

$$\Phi(z) = -\sum_{m=1}^{N} \sum_{n=1}^{N} a_m \bar{a}_n \sum_{k=1}^{\infty} c_k \,(2\pi\,\delta\,(m-n))^{2k}\, e\,((m-n)\,z).$$

Now

$$F^{(2k)}(z) = (2\pi\, i)^{2k} \sum_{m=1}^{N} \sum_{n=1}^{N} a_m \bar{a}_n\,(m-n)^{2k}\, e\,((m-n)\,z).$$

These two equations yield (35).

Lemma 6. *For any positive integer k,*

$$\int_0^1 |\, F^{(k)}(z)\,|\, dz \leq (4\pi\, N)^k \sum_{n=1}^{N} |\, a_n\,|^2. \tag{37}$$

Proof. Let $T(z) = \overline{S(z)}$, so that $F(z) = S(z)\,T(z)$. By Leibniz's formula,

$$F^{(k)}(z) = \sum_{l=0}^{k} \binom{k}{l} S^{(l)}(z)\, T^{(k-l)}(z).$$

By Cauchy's inequality,

$$\int_0^1 |\, S^{(l)}(z)\, T^{(k-l)}(z)\,|\, dz \leq \left\{ \int_0^1 |\, S^{(l)}(z)\,|^2\, dz \right\}^{1/2} \left\{ \int_0^1 |\, T^{(k-l)}(z)\,|^2\, dz \right\}^{1/2}.$$

Now

$$S^{(l)}(z) = (2\pi\, i)^l \sum_{n=1}^{N} a_n\, n^l\, e\,(nz),$$

whence

$$\int_0^1 |\, S^{(l)}(z)\,|^2\, dx = (2\pi)^{2l} \sum_{n=1}^{N} |\, a_n\,|^2\, n^{2l}.$$

There is a similar result for the integral containing T. Finally

$$\int_0^1 |\, F^{(k)}(z)\,|\, dz \leq \sum_{l=0}^{k} \binom{k}{l} \left\{ (2\pi)^{2k} N^{2k} \left(\sum_{n=1}^{N} |\, a_n\,|^2 \right)^2 \right\}^{1/2} = (4\pi\, N)^k \sum_{n=1}^{N} |\, a_n\,|^2.$$

LEMMA 7. *Let E denote the set of real numbers, considered modulo 1, for which*

$$\Phi(z) - F(z) > 0. \tag{38}$$

Then

$$\delta \Sigma \mid S(x_r) \mid^2 \leq \sum_1^N \mid a_n \mid^2 + \int_E (\Phi(z) - F(z))\, dz. \tag{39}$$

Proof. By Lemma 4 the left hand side is

$$\sum_r \int_{x_r - \delta/2}^{x_r + \delta/2} \left\{ \mid S(z) \mid^2 - \Phi(z) \right\} dz,$$

and by (2) the intervals of integration are disjoint. We have

$$\int_0^1 \mid S(z) \mid^2 dz = \sum_1^N \mid a_n \mid^2, \qquad \int_0^1 \Phi(z)\, dz = 0,$$

the latter being a consequence of the definition of $\Phi(z)$ in (33), since the terms with $m = n$ in the double sum have coefficients 0. It follows that

$$\delta \sum_r \mid S(x_r) \mid^2 = \sum_1^N \mid a_n \mid^2 + \int_{E'} \{\Phi(z) - F(z)\}\, dz,$$

where $F(z) = \mid S(z) \mid^2$ as before, and where E' denotes the complement of the set of intervals $(x_r - \delta/2,\ x_r + \delta/2)$. The integral in the last expression can only be increased if we replace E' by E, since E comprises all z for which the integrand is positive.

7. Proof of the first part of Theorem 2.

If we represent numbers z by points on the circumference of a circle of perimeter 1, the set E of Lemma 7 consists of a finite number of open intervals. Each interval has length less than δ, for by Lemma 4 it is impossible for (38) to hold throughout any interval of length δ.

We divide the intervals of E into two types. The first type are those for which $F'(z)$ does not vanish in the interval, and we denote the union of these by E_1. The second type are those for which $F'(z)$ vanishes at some point of the (open) interval, and we denote the union of these by E_2.

Let I be one of the intervals of E_1. Since $\Phi(z) - F(z) = 0$ at the end points of I, we have

$$\int_I \{\Phi(z) - F(z)\}\, dz = -\int_I (z-\gamma)\{\Phi'(z) - F'(z)\}\, dz$$

for any real number γ. Since $F'(z)$ is of constant sign, we can choose γ in I so that

$$\int_I (z-\gamma) F'(z)\, dz = 0.$$

We now have

$$\left| \int_I \{\Phi(z) - F(z)\}\, dz \right| = \left| \int_I (z-\gamma)\, \Phi'(z)\, dz \right| \leq \delta \int_I |\Phi'(z)|\, dz.$$

Hence

$$\int_{E_1} \{\Phi(z) - F(z)\}\, dz \leq \delta \int_0^1 |\Phi'(z)|\, dz.$$

By lemmas 5 and 6, the right hand side is

$$\leq \delta \sum_{k=1}^{\infty} c_k\, \delta^{2k} \int_0^1 |F^{(2k+1)}(z)|\, dz$$

$$\leq \left\{ \sum_{k=1}^{\infty} c_k\, (4\pi N\delta)^{2k+1} \right\} \left\{ \sum_1^N |a_n|^2 \right\}. \tag{40}$$

We now turn to the set E_2. Since $F(z) \geq 0$ always, we have

$$\int_{E_2} \{\Phi(z) - F(z)\}\, dz \leq \int_{E_2} \Phi(z)\, dz.$$

By Lemmas 5 and 6, this is

$$\leq \frac{\delta^2}{24} \int_{E_2} F''(z)\, dz + \sum_{k=2}^{\infty} c_k\, \delta^{2k} \int_{E_2} |F^{(2k)}(z)|\, dz$$

$$\leq \frac{\delta^2}{24} \int_{E_2} F''(z)\, dz + \left\{ \sum_{k=2}^{\infty} c_k\, (4\pi N\delta)^{2k} \right\} \sum_1^N |a_n|^2. \tag{41}$$

It remains to estimate the integral over E_2 of $F''(z)$. Suppose first that E_2 consists of only one or two intervals; as noted earlier, each has length less than δ. Thus in this case

$$\int_{E_2} F''(z)\, dz \leq 2\delta \max |F''(z)|.$$

Now

$$F''(z) = -(2\pi)^2 \sum_{m=1}^{N} \sum_{n=1}^{N} a_m \bar{a}_n (m-n)^2 e((m-n)z),$$

and therefore

$$|F''(z)| \leq (2\pi)^2 N^2 \left(\sum_{n=1}^{N} |a_n|\right)^2 \leq (2\pi)^2 N^3 \sum_{1}^{N} |a_n|^2.$$

Hence, in the present case,

$$\int_{E_2} F''(z)\, dz \leq 8\pi^2 N^3 \delta \sum_{1}^{N} |a_n|^2. \tag{42}$$

Now suppose that E_2 consists of at least 3 intervals, say $(\alpha_1, \beta_1), \dots$ $\dots, (\alpha_s, \beta_s)$.

By the definition of E_2, each interval (α_j, β_j) contains some ξ_j for which

$$F'(\xi_j) = 0.$$

By Rolle's theorem, there exists some η_j between ξ_j and ξ_{j+1} for which

$$F''(\eta_j) = 0.$$

We make the obvious convention that $\xi_{s+1} = \xi_1$, and so on.

We have

$$\int_{\alpha_j}^{\beta_j} F''(z)\, dz = \int_{\alpha_j}^{\beta_j} \left\{\int_{\eta_{j-1}}^{z} F'''(y)\, dy\right\} dz.$$

Both η_{j-1} and z are contained in the interval (ξ_{j-1}, ξ_{j+1}). Hence, since $\beta_j - \alpha_j < \delta$,

$$\int_{\alpha_j}^{\beta_j} F''(z)\, dz \leq \delta \int_{\xi_{j-1}}^{\xi_{j+1}} |F'''(y)|\, dy.$$

The intervals (ξ_{j-1}, ξ_{j+1}) cover the whole interval of length 1 twice. Hence

$$\int_{E_2} F''(z)\,dz \leq 2\delta \int_0^1 |F'''(y)|\,dy \leq 2\delta\,(4\pi N)^3 \sum_1^N |a_n|^2 \tag{43}$$

by Lemma 6. Comparing this with (42), we see that (43) is valid in both the two cases.

It follows from (41) that

$$\int_{E_2} \{\Phi(z) - F(z)\}\,dz \leq \left\{\frac{(4\pi N\delta)^3}{12} + \sum_{k=2}^{\infty} c_k\,(4\pi N\delta)^{2k}\right\} \sum_1^N |a_n|^2 .$$

Adding to this the estimate in (40) for the integral over E_1, and substituting in (39), we obtain

$$\sum_r |S(x_r)|^2 \leq \delta^{-1}(1 + U) \sum_1^N |a_n| ,$$

where

$$U = \sum_{k=1}^{\infty} c_k\,(4\pi N\delta)^{2k+1} + \frac{(4\pi N\delta)^3}{12} + \sum_{k=2}^{\infty} c_k\,(4\pi N\delta)^{2k}.$$

Since $c_1 = \dfrac{1}{24}$, this implies

$$U \leq \frac{1}{8}(4\pi N\delta)^3 + 2 \sum_{k=2}^{\infty} c_k\,(4\pi N\delta)^{2k} .$$

By (36) with $\theta = \pi$, we have

$$\sum_{k=2}^{\infty} c_k\,\pi^{2k} = \frac{\pi}{2} - 1 - \frac{1}{24}\pi^2 < 0.16.$$

Hence, for $N\delta \leq \dfrac{1}{4}$,

$$U \leq \frac{1}{8}(4\pi N\delta)^3 + 2\,(4N\delta)^4\,(0.16) < 270\,(N\delta)^3.$$

This proves (6).

8. Proof of the second part of Theorem 2.

We take $N = 2$, and

$$S(x) = 1 + e^{2\pi i x},$$

so that $|S(x)|^2 = 4\cos^2 \pi x$. We have

$$\sum_{n=1}^{N} |a_n|^2 = 2.$$

We take the points x_r at

$$0, \quad \pm\delta, \quad \pm 2\delta, \ldots, \pm m\delta$$

where

$$(2m + 1 + \zeta)\,\delta = 1,$$

and $0 < \zeta < 1$. Note that the gap (mod 1) between $m\delta$ and $-m\delta$ is $1 - 2m\delta > \delta$.

We have

$$\sum_r |S(x_r)|^2 = 4 \sum_{r=-m}^{m} \cos^2 \pi r \delta = 2 \sum_{r=-m}^{m} (1 + \cos 2\pi r \delta)$$

$$= 2\left(2m + 1 + \frac{\sin(2m+1)\pi\delta}{\sin \pi\delta}\right).$$

Now $(2m + 1)\,\delta = 1 - \zeta\,\delta$, so $\sin(2m+1)\pi\delta = \sin \pi\,\zeta\,\delta$.

Hence

$$\sum_r |S(x_r)|^2 = 2\left(2m + 1 + \zeta + \left(\frac{\sin \pi\,\zeta\,\delta}{\sin \pi\,\delta} - \zeta\right)\right) = 2\delta^{-1}(1 + V),$$

where

$$V = \delta\left(\frac{\sin \pi\,\zeta\,\delta}{\sin \pi\,\delta} - \zeta\right).$$

For fixed ζ, as $\delta \to 0$, we have

$$V \sim \frac{1}{6}\pi^2 \zeta (1 - \zeta^2)\,\delta^3,$$

and on taking $\zeta = 1/\sqrt{3}$ we get

$$V \sim \frac{\pi^2}{9\sqrt{3}}\,\delta^3 = \frac{\pi^2}{72\sqrt{3}}\,N^3 \delta^3.$$

Since $\dfrac{\pi^2}{72\sqrt{3}} > \dfrac{1}{12}$, this example satisfies (7).