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ADALBERTO ORSATTI

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A CLASS OF RINGS  
WHICH ARE THE ENDOMORPHISM RINGS  
OF SOME TORSION-FREE ABELIAN GROUPS

ADALBERTO ORSATTI (\*)

**Introduction.**

A well known result of A. L. S. Corner ([2], Theorem A) states that every countable, reduced, torsion-free ring  $A$  is isomorphic with the endomorphism ring  $E(G)$  of some countable, reduced, torsion-free group  $G$ . (A ring  $A$  is called reduced and torsion-free if such is its additive group).

In this paper we establish a similar result for a wider class of rings, precisely for the class  $\mathcal{A}$  consisting of *locally countable, reduced, torsion-free* rings.

We say that a torsion-free ring  $A$  is locally countable if for every prime number  $p$  not dividing  $A$  (i. e.  $pA \neq A$ ) the ring  $A/p^\infty A$  is countable, where  $p^\infty A$  is the intersection of the ideals  $p^n A$  for every natural number  $n$ . (Observe that this definition involves only the additive structure of  $A$ ).

The rings of class  $\mathcal{A}$  are characterized as follows (see Proposition 1). A ring  $A$  belongs to  $\mathcal{A}$  if and only if  $A$  is isomorphic with a pure subring of a direct product  $\prod_p R_p$ ,  $p \in P^*$ , where  $P^*$  is any given set of distinct prime numbers and  $R_p$  a countable, reduced, torsion-free  $Z_p$ -algebra. ( $Z_p$  = ring of rationals whose denominators are prime to  $p$ ).

The following generalization of Theorem A is proved :

**THEOREM A\*.** *Let  $A$  be a locally countable, reduced, torsion-free ring. Then there exists a locally countable, reduced, torsion-free group  $G$ , of the same cardinal as  $A$ , whose endomorphism ring  $E(G)$  is isomorphic with  $A$ .*

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The proof of this theorem relies on Corner's methods and ideas, but does not make use of Theorem A: in fact it is obtained by modifying the proof of Theorem A by means of some localization and globalization techniques as exposed in [5].

It is clear from the above characterization that every ring of class  $\mathcal{A}$  is of cardinal  $\leq 2^{\aleph_0}$ , so that one might wonder if every such ring belongs to the class of the endomorphism rings of *countable*, reduced, torsion-free groups. This last class of rings has been characterized by Corner himself in Theorem 1.1 of [3]. The answer is negative, as proved in Proposition 2, where we construct a ring  $A \in \mathcal{A}$  with the following properties:  $|A| = 2^{\aleph_0}$  and if  $A = E(G)$ , with  $G$  a reduced torsion-free group, then  $|G| \geq 2^{\aleph_0}$ .

It is still an open question if the rings of class  $\mathcal{A}$  satisfy the hypotheses of Theorem 2.2 of [3], which generalizes Theorem 1.1.

### 1. Preliminaries.

All groups considered in this paper are abelian and additively written; rings are associative and with an identity, modules are unitary. If  $f: B \rightarrow A$  is a ring homomorphism,  $f$  always maps the identity of  $B$  into the identity of  $A$ : if  $B$  is a subring of  $A$ ,  $B$  contains the identity of  $A$ . We will regard sometimes the ring possessing only the zero element as a ring with an identity.

Let  $\{H_i\}$  be an indexed family of groups (rings). We denote by  $\prod_i H_i$  the direct product (= cartesian product) of the  $H_i$  and, for every  $x \in \prod_i H_i$ , by  $x_i$  the  $i$ -component of  $x$  ( $x_i \in H_i$ ).  $\sum_i H_i$  will be the subgroup (ideal) of  $\prod_i H_i$  consisting of those elements whose components are almost all zero.

We often attribute to a ring some properties of its additive group; for instance we say that the ring  $A$  is reduced, torsion-free, etc...; or that a subring of  $A$  is pure in  $A$ . By a subgroup of  $A$  we mean a subgroup of the additive group of  $A$ .

Let  $N$  be the set of positive integers and  $P$  the set of prime numbers ( $P \subset N$ ). For every group (ring)  $H$  and for every  $p \in P$ ,  $p^\infty H$  will denote the intersection of all subgroups (ideals)  $p^n H$ ,  $n \in N$ .

Every group (ring) is a topological group (ring) in the *natural topology* obtained by taking the subgroups (ideals)  $nH$ ,  $n \in N$ , as a basis of neighbourhoods of 0. This topology will be our main tool; for its principal properties see [2]. We recall here some of them. Let  $H$  be a reduced torsion-free group (ring): then  $H$  is Hausdorff in the natural topology. Let  $L$  be a subgroup (subring) of  $H$  and endow  $H$  with the natural topology.

If  $L$  is pure in  $H$ , then the natural topology of  $L$  coincides with the relative topology.  $L$  is dense in  $H$  if and only if the group  $H/L$  is divisible. If  $H$  is divisible by every positive integer prime to some fixed  $p \in P$ , then the natural topology of  $H$  coincides with the  $p$ -adic topology. The (group or ring) homomorphisms are uniformly continuous mappings with respect to the natural topologies of the corresponding structures.

$Z$  will denote the ring of integers,  $Z_p$  ( $p \in P$ ) the ring of rationals whose denominators are prime to  $p$ ,  $\widehat{Z}_p$  the ring of  $p$ -adic integers. Maps are written on the left.

The group theoretical terminology is that of Fuchs's book [4].

Let  $A$  be a reduced torsion-free ring. For every  $p \in P$ , consider the ring  $A_p^* = (A/p^\infty A) \otimes Z_p$  (tensor product of  $Z$ -algebras) and the ring homomorphism  $\varphi_p: A \rightarrow A_p^*$  resultant of the canonical maps  $A \rightarrow A/p^\infty A$  and  $A/p^\infty A \rightarrow A_p^*$ .  $A/p^\infty A$  is torsion-free and without elements ( $\neq 0$ ) of infinite  $p$ -height; then the map  $A/p^\infty A \rightarrow A_p^*$  is injective — so that the kernel of  $\varphi_p$  is  $p^\infty A$  — and  $A_p^*$  is a reduced torsion-free  $Z_p$ -algebra. Define  $A^* = \prod_p A_p^*$ ,  $p \in P$ , and let  $\varphi: A \rightarrow A^*$  be the canonical map given by

$$\varphi(a)_p = \varphi_p(a) \quad (a \in A, p \in P)$$

where  $\varphi(a)_p$  is the  $p$ -component of  $\varphi(a)$ . Then  $A^*$  is a reduced torsion-free ring and  $\varphi$  is a ring homomorphism.

In [5] we defined for every group  $G$  the groups  $G_p^* = (G/p^\infty G) \otimes Z_p$ ,  $G^* = \prod_p G_p^*$ ,  $p \in P$ , and the canonical homomorphisms  $G \rightarrow G_p^*$ ,  $G \rightarrow G^*$ .  $G_p^*$  and  $G^*$  were called respectively the *Hausdorff  $p$  localization* and the *natural pre-completion* of  $G$ . This terminology will be used also for  $A$ .

From the embedding lemma of [5] we obtain the following

**LEMMA 1.** *Let  $A$  be a reduced torsion-free ring. Then the canonical homomorphism  $\varphi: A \rightarrow A^*$  is injective;  $\varphi(A)$  is a pure subring of  $A^*$ ; the group  $A^*/\varphi(A)$  is divisible, i. e.  $\varphi(A)$  is dense in  $A^*$  endowed with the natural topology.*

Denote by  $A^p$  the image of  $\varphi(A)$  under the canonical projection  $A^* \rightarrow A_p^*$ ;  $A^p$  will be called the  *$p$ -projection* of  $A$ .

From the definition of  $\varphi$  we get  $A^p = \varphi_p(A)$ . Since  $A_p^*/\varphi_p(A)$  is a divisible torsion group with trivial  $p$ -primary component ([5], pag. 5) and  $A_p^*$  is torsion-free, we have

**LEMMA 2.** *Let  $A$  be a reduced torsion-free ring. Then  $A^p$  is a  $p$ -pure subring of  $A_p^*$ ; the pure subring (subgroup) of  $A_p^*$  generated by  $A^p$  coincides with  $A_p^*$ ;  $A^p$  is dense in  $A_p^*$  endowed with the natural topology.*

The natural topology of  $A_p^*$  coincides with the  $p$ -adic topology. Let  $\widehat{A}_p^*$  be the natural ( $= p$ -adic) completion of  $A_p^*$ ; then  $\widehat{A}_p^*$  is a reduced torsion-free ring which contains  $A_p^*$  as a pure and dense subring ([2], Lemma 1.4). Extending by continuity the  $Z_p$ -algebra structure of  $A_p^*$ ,  $\widehat{A}_p^*$  becomes a  $\widehat{Z}_p$ -algebra. Moreover  $\widehat{A}_p^*$  is torsion-free over  $\widehat{Z}_p$ , otherwise the additive group of  $\widehat{A}_p^*$  would contain some cyclic  $p$ -group.

$A^*$  is a pure and dense subring of  $\prod_p \widehat{A}_p^*$ ,  $p \in P$ , which is complete in the natural topology, as easily verified. By means of the injection  $\varphi$ , we identify  $A$  with  $\varphi(A)$ : by Lemma 1,  $A$  becomes a pure and dense subring of  $A^*$ . Then the natural completion  $\widehat{A}$  of  $A$  coincides with  $\widehat{A}^*$ , hence with  $\prod_p \widehat{A}_p^*$ . (See [5], P. 5. and Teorema 1). Now the following pure and dense  $p$  inclusions hold:

$$(1) \quad A_p^* \subseteq \widehat{A}_p^*; \quad A \subseteq A^* = \prod_p A_p^* \subseteq \prod_p \widehat{A}_p^* = \widehat{A}.$$

Let  $\widehat{Z} = \prod_p \widehat{Z}_p$ ,  $p \in P$ , be the natural completion of  $Z$  and identify  $Z$  with the (pure and dense) subring of  $\widehat{Z}$  generated by the identity of  $\widehat{Z}$ . Extending by continuity the obvious  $Z$ -algebra structure of  $A$ ,  $\widehat{A}$  becomes a  $\widehat{Z}$ -algebra. The product of an element  $\pi \in \widehat{Z}$  by an element  $a \in \widehat{A}$  is given by the following relations on  $p$ -components:

$$(2) \quad (\pi a)_p = \pi_p a_p \quad (p \in P, \pi_p \in \widehat{Z}_p, a_p \in \widehat{A}_p^*).$$

This is an immediate consequence of the principle of the extension of identities, [1], because (2) holds for  $\pi \in Z$  and  $a \in A$ .

We conclude this section with the following remark.

**LEMMA 3.** *Let  $L$  be a  $p$ -pure subgroup of the reduced torsion-free  $Z_p$ -module  $H$ . Then the group  $L \otimes Z_p$  is canonically isomorphic with the pure subgroup of  $H$  generated by  $L$ .*

**PROOF.** Let  $L_p$  be the pure subgroup of  $H$  generated by  $L$ .  $L_p/L$  is a divisible torsion group with trivial  $p$ -primary component. Then the canonical isomorphism is obtained from the exact sequence  $0 \rightarrow L \rightarrow L_p \rightarrow L_p/L \rightarrow 0$  (where the maps are the natural ones) by tensor multiplication by  $Z_p$ .

**2. The proof of Theorema A\*.**

Let  $A (\neq 0)$  be a ring satisfying the hypotheses of Theorem A\*. Since  $A$  is reduced and torsion-free, the above remarks hold, in particular inclusions (1) are verified.

Let  $P^*$  be the non void subset of  $P$  consisting of the primes  $p$  such that  $pA \neq A$ . If  $p \notin P^*$  we have  $A_p^* = \widehat{A}_p^* = 0$ , hence the  $p$ -component of every element of  $\widehat{A}$  is zero and we may take  $\widehat{A} = \prod_p \widehat{A}_p^*$ ,  $p \in P^*$ . Note that, if  $p \in P^*$ ,  $A_p^*$  is countable.

The first part of the proof is a localization process: for every  $p \in P^*$  we construct a countable pure subgroup  $G_p$  of  $\widehat{A}_p^*$ ,  $G_p \supset A_p^*$ , whose endomorphism ring is isomorphic with  $A_p^*$ ; in this part we will follow exactly, except for small details, the proof of Theorem A of [2].

For a given  $p \in P^*$ , choose in  $\widehat{A}_p^*$  a maximal family  $\{f_i, i \in I\}$  of elements of  $A_p^*$  linearly independent over  $\widehat{Z}_p$ . Then for every  $v \in A_p^*$  there exist a non negative integer  $n_v$  and elements  $\pi_v^i, i \in I$ , of  $\widehat{Z}_p$  such that, in  $\widehat{A}_p^*$ ,

$$p^{n_v} v = \sum_i \pi_v^i f_i$$

where almost all the  $\pi_v^i$  vanish. If we take always the smallest possible  $n_v$ , then  $v$  uniquely determines the  $\pi_v^i$ , since  $\widehat{A}_p^*$  is torsion-free over  $\widehat{Z}_p$ . Let  $\Pi_p$  be the pure subring of  $\widehat{Z}_p$  generated by these  $\pi_v^i (i \in I, v \in A_p^*)$ . Since  $A_p^*$  is countable, so is  $\Pi_p$ . Moreover we have

**LEMMA 4.** *If in  $\widehat{A}_p^* (p \in P^*)$ :*

$$\sum_{j=1}^n \gamma_j v_j = 0 \tag{n \in N}$$

where the  $\gamma_j$  are elements of  $\widehat{Z}_p$  linearly independent over  $\Pi_p$  and the  $v_j \in A_p^*$ , then the  $v_j$  all vanish.

The proof of this lemma is the same as the one of Lemma 2.1. of [2].

For every  $v \in A_p^*$ , choose two elements  $\alpha_p(v), \beta_p(v) \in \widehat{Z}_p$  such that they all form a family which is algebraically independent over  $\Pi_p$ . This is possible because  $A_p^*$  is countable and  $\widehat{Z}_p$  is of transcendence degree  $2^{\aleph_0}$  over  $\Pi_p$ . Let  $A^p$  be the  $p$ -projection of  $A : A^p \subseteq A_p^*$ . For every  $u \in A^p$  define

the element  $e_p(u) \in \widehat{A}_p^*$  by putting

$$(3) \quad e_p(u) = \alpha_p(u) 1_p + \beta_p(u) u$$

where  $1_p$  is the identity of  $\widehat{A}_p^*$ , and let  $G_p$  be the pure subgroup of  $\widehat{A}_p^*$  generated by  $A^p$  and by the subgroups  $e_p(u) A^p$ ,  $u \in A^p$ , of  $\widehat{A}_p^*$ .  $G_p$  is a countable reduced torsion-free  $Z_p$ -module. Now  $G_p$  contains  $A_p^*$ : in fact  $A_p^*$  is pure in  $\widehat{A}_p^*$  and so, by Lemma 2,  $A_p^*$  is the minimal pure subgroup of  $\widehat{A}_p^*$  containing  $A^p$ . Moreover, for every  $u \in A^p$ ,  $G_p$  contains the pure subgroup  $H(u)$  of  $\widehat{A}_p^*$  generated by  $e_p(u) A^p$  and it is easily verified that  $H(u)$  contains  $e_p(u) A_p^*$ . It is now clear that  $G_p$  coincides with the pure subgroup of  $\widehat{A}_p^*$  generated by  $A_p^*$  and the  $e_p(u) A_p^*$ ,  $u \in A^p$ . It follows that  $G_p A_p^* = G_p$  and so every right multiplication in  $\widehat{A}_p^*$  by an element of  $A_p^*$  induces an endomorphism on  $G_p$ . These multiplications are distinct because  $G_p$  contains the identity of  $A_p^*$ . We now prove that every endomorphism of  $G_p$  is obtained in this way. Let  $\delta$  be an arbitrary endomorphism of  $G_p$ . Since  $G_p$  is pure and dense in  $\widehat{A}_p^*$ , the natural ( $=p$ -adic) completion of  $G_p$  coincides with the additive group of  $\widehat{A}_p^*$ . Consequently  $\delta$  extends to a  $\widehat{Z}_p$ -endomorphism  $\widehat{\delta}$  of  $\widehat{A}_p^*$  ([2], Lemma 1.4). Let  $u$  be an arbitrary element of  $A^p$  and consider  $\delta(e_p(u))$ . We have by (3):

$$(4) \quad \delta(e_p(u)) = \widehat{\delta}(\alpha_p(u) 1_p + \beta_p(u) u) = \alpha_p(u) \delta(1_p) + \beta_p(u) \delta(u).$$

Now  $\delta(e_p(u))$ ,  $\delta(1_p)$ ,  $\delta(u)$  are elements of  $G_p$  and so, by the definition of  $G_p$ , there exist  $m, n \in N$  such that

$$(5) \quad \left\{ \begin{array}{l} m \delta(e_p(u)) = b + \sum_{i=1}^n e_p(u_i) b_i \\ m \delta(1_p) = c + \sum_{i=1}^n e_p(u_i) c_i \\ m \delta(u) = d + \sum_{i=1}^n e_p(u_i) d_i \end{array} \right.$$

where the  $u_i$  are distinct elements of  $A^p$ ,  $u_1 = u$  and  $b, c, d, b_i, c_i, d_i \in A_p^*$ .

Substituting from (5) in (4) we obtain

$$\begin{aligned}
 & b + \sum_{i=1}^n (\alpha_p(u_i) 1_p + \beta_p(u_i) u_i) b_i = \\
 & = \alpha_p(u) \left[ c + \sum_{i=1}^n (\alpha_p(u_i) 1_p + \beta_p(u_i) u_i) c_i \right] + \beta_p(u) \left[ d + \sum_{i=1}^n (\alpha_p(u_i) 1_p + \beta_p(u_i) u_i) d_i \right].
 \end{aligned}$$

As the  $\alpha_p(u_i), \beta_p(u_i)$  are algebraically independent over  $II_p$ , from Lemma 4 we get

$$(6) \quad b_1 = c, \quad ub_1 = d$$

while the other  $b$ 's,  $c$ 's and  $d$ 's all vanish. By the last two of (5) and by (6) we have:

$$m\delta(1_p) = c, \quad m\delta(u) = uc.$$

Since  $A_p^*$  is pure in  $G_p$ , it follows  $\delta(1_p) \in A_p^*$ , and since  $G_p$  is torsion-free,  $\delta(u) = u\delta(1_p)$ . So we see that  $\delta$  coincides on  $A^p$  with the right multiplication by  $\delta(1_p) \in A_p^*$ . Now by Lemma 2  $A^p$  is dense in  $A_p^*$ , and  $A_p^*$  is dense in  $G_p$  for  $G_p$  is pure in  $\widehat{A}_p^*$ . It follows that  $A^p$  is dense in  $G_p$  endowed with the natural topology. Then, by the principle of the extension of identities,  $\delta$  coincides with the right multiplication by  $\delta(1_p)$  on the whole of  $G_p$ .

The first part of the proof is now complete.

The second part consists in constructing a group  $G$  with the required properties by means of the  $G_p, p \in P^*$ , using a local-global argument.

For every  $a \in A$  consider the elements  $\alpha(a), \beta(a) \in \widehat{Z}$  defined as follows:

$$(7) \quad \begin{cases} \alpha(a)_p = \alpha_p(a_p), & \beta(a)_p = \beta_p(a_p) & \text{if } p \in P^* \\ \alpha(a)_p = \beta(a)_p = 0 & & \text{if } p \notin P^*. \end{cases}$$

Note that  $a_p$ , being the  $p$ -component of  $a$ , belongs to  $A^p$ . Define the elements  $e(a) \in \widehat{A}$  by putting

$$e(a) = \alpha(a) 1 + \beta(a) a \quad (a \in A)$$

where  $1$  is the identity of  $\widehat{A}$ . Every element of  $\widehat{A}$  is determined by its  $p$ -components with  $p \in P^*$ ; by (2), (3) and (7) we have, for every  $p \in P^*$  and  $a \in A$ :

$$(8) \quad e(a)_p = \alpha_p(a_p) 1_p + \beta_p(a_p) a_p = e_p(a_p).$$



Let  $G$  be the pure subgroup of  $\widehat{A}$  generated by  $A$  and the  $e(a)A$ ,  $a \in A$ .  $G$  is reduced torsion-free and of the same cardinal as  $A$ . In  $\widehat{A}$  we have  $GA = G$ , so that every right multiplication by an element of  $A$  induces an endomorphism on  $G$ ; these multiplications are distinct because  $G$  contains the identity of  $A$ . In order to complete the proof of the theorem, it suffices to show that every endomorphism of  $G$  is obtained in this way and  $G$  is locally countable.

For this purpose, let us determine the Hausdorff  $p$ -localizations  $G_p^*$ ,  $p \in P$ , and the natural pre-completion  $G^*$  of  $G$ . Since  $G$  is pure in  $\widehat{A}$  we have for every  $p \in P$

$$(9) \quad p^\infty G = (p^\infty \widehat{A}) \cap G.$$

If  $p \notin P^*$ , then  $p^\infty \widehat{A} = \widehat{A}$ , hence  $p^\infty G = G$  and  $G_p^* = 0$ .

Suppose then  $p \in P^*$  and let  $\varepsilon_p$  be the canonical projection of  $\widehat{A}$  onto  $\widehat{A}_p^*$ :  $\varepsilon_p$  maps every element of  $\widehat{A}$  into its  $p$ -component. By (9),  $p^\infty G$  consists of the elements of  $G$  whose  $p$ -component is zero; consequently  $G/p^\infty G$  is canonically isomorphic with  $\varepsilon_p(G)$  which, as easily verified, is  $p$ -pure in  $\widehat{A}_p^*$ . Hence, by Lemma 3, we identify  $G_p^*$  with the pure subgroup of  $\widehat{A}_p^*$  generated by  $\varepsilon_p(G)$ . Next we prove that  $G_p^* = G_p$  ( $p \in P^*$ ), from which it follows that  $G$  is locally countable, because  $G_p$  is countable.  $G_p^*$  contains  $\varepsilon_p(G)$  which, by the definition of  $G$ , contains  $A^p = \varepsilon_p(A)$  and  $e_p(a_p)A^p$  for every  $a \in A$ . But, when  $a$  runs over  $A$ ,  $a_p$  exhausts  $A^p$ ; hence, by the definition of  $G_p$ ,  $G_p^* \supseteq G_p$ . On the other hand a straightforward calculation shows that  $\varepsilon_p(G) \subseteq G_p$ ; this implies  $G_p^* \subseteq G_p$  and so  $G_p^* = G_p$ .

From the above remarks it follows:

$$G^* = \prod_p G_p, p \in P^*; \quad G \subseteq G^* \subset \widehat{A}.$$

Now, let  $\omega$  be an arbitrary endomorphism of  $G$ . By P. 2. of [5]  $\omega$  extends uniquely to an endomorphism  $\omega^*$  of  $G^*$ . (The proof of P. 2. suggests a way for constructing  $\omega^*$ ). Observe that if  $p$  and  $q$  are distinct primes of  $P^*$ ,  $\text{Hom}(G_p, G_q) = 0$  because  $G_p$  is a  $Z_p$ -module,  $G_q$  is a  $Z_q$ -module and both are reduced and torsion-free. Recalling how we obtained the endomorphisms of  $G_p$ , we see that every endomorphism of  $G^* = \prod_p G_p$ ,  $p \in P^*$ , is induced by a right multiplication in  $\widehat{A}$  by an element of  $A^* = \prod_p A_p^*$ ,  $p \in P^*$ . Then  $\omega$  coincides on  $G$  with the right multiplication by  $\omega^*(1) = \omega(1) \in A^* \cap G$ . If

we can prove that  $\omega(1) \in A$ , the conclusion is reached. In fact, we will prove that  $A^* \cap G = A$  <sup>(4)</sup>. It is clear that  $A^* \cap G \supseteq A$ ; conversely, if  $g \in G \cap A^*$ , then  $mg = b + \sum_{i=1}^n e(a_i) b_i$  with  $m, n \in \mathbb{N}$ ,  $b, a_i, b_i \in A$ , and  $\sum_{i=1}^n e(a_i) b_i = mg - b = c \in A^*$ . As for the  $p$ -components, for every  $p \in P^*$ , we obtain by (8)

$$\sum_{i=1}^n (\alpha_p(a_{ip}) 1_p + \beta_p(a_{ip}) a_{ip}) b_{ip} = c_p.$$

But, as  $a_{ip}, b_{ip}, c_p$  belong to  $A_p^*$ , it follows from Lemma 4 that  $c_p = 0$  for every  $p \in P^*$ . This means  $c = 0$  and so  $mg = b \in A$ . Hence  $g \in A$ , because  $A$  is pure in  $G$ .

### 3. The rings of class $\Delta$ and an example.

Let  $\Delta$  be the class of the rings satisfying the hypotheses of Theorem A\*.  $\Delta$  may be characterized as follows.

**PROPOSITION 1.** *A ring  $A$  belongs to  $\Delta$  if and only if  $A$  is isomorphic with a pure subring of a ring of type  $R = \prod_{p \in P^*} R_p$ , where  $P^*$  is a set of distinct primes and, for every  $p \in P^*$ ,  $R_p$  is a countable reduced torsion-free  $Z_p$ -algebra.*

**PROOF.** The necessity follows immediately from Lemma 1. Let  $A$  be a pure subring of  $R$ :  $A$  is reduced and torsion-free. We have  $A/p^\infty A = A/(p^\infty R) \cap A$  for every  $p \in P$  and  $A \not\subseteq pA$  if and only if  $p \in P^*$ . For every  $p \in P^*$ ,  $A/p^\infty A$  is isomorphic with a  $p$ -pure subring of  $R_p$  and is countable.

Finally we show that the class  $\Delta$  is not contained in the class of the endomorphism rings of countable reduced torsion-free groups; these rings are characterized by Theorem 1.1. of [3].

**PROPOSITION 2.** *There exists in  $\Delta$  a ring  $A$  of cardinal  $2^{\aleph_0}$  such that every reduced torsion-free group, whose endomorphism ring is isomorphic with  $A$ , is of cardinal  $\geq 2^{\aleph_0}$ .*

**PROOF.** For every  $p \in P$ , let  $R_p$  be a countable pure subring of  $\widehat{Z}_p$  of rank  $> 1$ .  $R_p$  properly contains  $Z_p$  as a pure and dense subring. Define

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<sup>(4)</sup> I am indebted to F. Menegazzo for this suggestion.

$R = \prod_p R_p, p \in P$ , and consider the subring  $A$  of  $R$ :

$$A = \{\alpha \in R \mid \alpha_p \in Z_p \text{ for almost all } p\}$$

where, as usual,  $\alpha_p$  is the  $p$ -component of  $\alpha$ . We have the proper inclusions

$$\sum_p R_p \subset A \subset \prod_p R_p \quad (p \in P);$$

it is easily verified that  $A$  is pure in  $R$  so that, by Proposition 1,  $A \in \mathcal{A}$ ;  $A$  is of cardinal  $2^{\aleph_0}$ .

Let  $G$  be reduced torsion-free group such that  $E(G) = A$ . For every  $p \in P$  consider the element  $\varepsilon_p \in A$  such that the  $p$ -component of  $\varepsilon_p$  is the identity  $1_p$  of  $R_p$  whereas the other components all vanish. As  $\varepsilon_p$  is an idempotent element of  $A$ ,  $G$  splits into the direct sum of the endomorphic images  $\varepsilon_p(G)$  and  $(1 - \varepsilon_p)(G)$ , where  $1$  is the identity of  $A$ . Let  $G_p$  be the subgroup of  $G$  consisting of those elements which are divisible by every prime different from  $p$ ;  $G_p$  is a reduced torsion-free  $Z_p$ -module. Since  $\varepsilon_p$  is divisible in  $R$  and hence in  $A$  by every prime different from  $p$ , while  $1 - \varepsilon_p$  is divisible by every power of  $p$ , we have  $\varepsilon_p(G) \subseteq G_p$  and  $(1 - \varepsilon_p)(G) \subseteq p^\infty G$ . On the other hand  $G_p \cap p^\infty G = 0$  because  $G$  is reduced and torsion-free. Hence  $\varepsilon_p(G) = G_p, (1 - \varepsilon_p)(G) = p^\infty G$  and

$$(10) \quad G = G_p \oplus p^\infty G.$$

We now show that the endomorphism ring  $E(G_p)$  of  $G_p$  is isomorphic with  $R_p$ . By the direct decomposition (10), every endomorphism  $\beta$  of  $G_p$  extends to an endomorphism  $\bar{\beta}$  of  $G$  such that  $\bar{\beta}(G) \subseteq G_p, \bar{\beta}(p^\infty G) = 0$ . Since  $\varepsilon_p$  induces the identity on  $G_p$ , we have  $\bar{\beta} = \varepsilon_p \bar{\beta} \in \varepsilon_p A$ . Conversely, every element of  $\varepsilon_p A$  induces an endomorphism on  $G_p$  and vanishes on  $p^\infty G$ . It follows that  $E(G_p)$  is isomorphic with the ring  $\varepsilon_p A$ , hence with  $R_p$ . Every non trivial endomorphism of  $G_p$  is injective: in fact, since  $G_p$  is in a natural way an  $R_p$ -module,  $\widehat{G}_p$  is a module over  $\widehat{R}_p = \widehat{Z}_p$ ; since  $G_p$  is a torsion-free group,  $\widehat{G}_p$  is torsion-free over  $\widehat{Z}_p$ ; then  $G_p$  is torsion-free over  $R_p$ .

It is clear that, for every  $p \in P$ ,  $G_p$  coincides with the Hausdorff  $p$ -localization  $G_p^*$  of  $G$  and  $\varepsilon_p$  coincides with the canonical homomorphism  $G \rightarrow G_p^*$ . By means of the  $\varepsilon_p, p \in P$ , we construct the canonical homomorphism  $\varepsilon$  of  $G$  in its natural pre-completion  $G^* = \prod_p G_p$  and identify  $G$  with the pure and dense subgroup  $\varepsilon(G)$  of  $G^*$ . Since, if  $p$  and  $q$  are distinct primes,  $\text{Hom}(G_p, G_q) = 0$ , the endomorphism ring of  $\prod_p G_p$  is  $\prod_p R_p$ . As  $A \subset \prod_p R_p$ ,

every endomorphism of  $G$  extends to an endomorphism of  $\prod_p G_p$ . Consequently the effect of  $\alpha \in A$  on  $g \in G$  is described by the following formulae on the  $p$ -components :

$$(11) \quad \alpha (g)_p = \alpha_p (g_p) \quad (p \in P).$$

It is easily verified that  $G$  contains  $\sum_p G_p$ . Moreover this inclusion is proper since the endomorphism ring of  $\sum_p G_p$  is  $\prod_p R_p$ , while  $E(G) = A$  which is not isomorphic with  $\prod_p R_p$ . Then we can find an element  $\bar{g} \in G$  and an infinite subset  $\bar{P}$  of  $P$  such that  $\bar{g}_p \neq 0$  for every  $p \in \bar{P}$ . Let  $\bar{A}$  be the ideal of  $A$  consisting of all  $\alpha \in A$  such that  $\alpha_p = 0$  if  $p \notin \bar{P}$ .  $|\bar{A}| = 2^{\aleph_0}$  because  $A \supset \prod_p Z_p, p \in \bar{P}$ . Consider the additive homomorphism  $\gamma: \bar{A} \rightarrow G$  mapping  $\alpha \in \bar{A}$  into  $\alpha(\bar{g}) \in G$ . If  $\alpha \in \bar{A}, \alpha \neq 0$ , there exists  $p \in \bar{P}$  such that  $\alpha_p \neq 0$ ; since every non trivial endomorphism of  $G_p$  is injective  $\alpha_p(\bar{g}_p) \neq 0$ ; by (11) this implies  $\alpha(\bar{g}) \neq 0$ , i. e.  $\gamma$  is injective. Hence  $|G| \geq 2^{\aleph_0}$ .

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*Seminario Matematico  
Università di Padova, Italia*