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A priori estimates for hypoelliptic differential equations in a half-space

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A PRIORI ESTIMATES FOR HYPOELLIPTIC DIFFERENTIAL EQUATIONS IN A HALF-SPACE

by LEIF ARKERYD

0. Introduction.

Our aim is to show that every distribution solution u of a formally hypoelliptic partial differential equation

$$\mathcal{A}u = f \text{ in } R^n_+ ,$$

satisfying Dirichlet's boundary conditions

$$D_1^j u = 0, j = 0, \dots, l \text{ on } R^{n-1},$$

does belong to C^∞, if f does. In analogy with the elliptic case (cf. Arkeryd [1]), it is natural to try to obtain à priori estimates

$$N_1(u) \leq CN_2(\mathcal{A}u) + N_3(u) \tag{0.1}$$

with suitable norms N_1, N_2, N_3, with in particular N_3 « weaker » than N_1. These estimates are proved in two steps:

1^0. The inequality (0.1) is established for operators with constant coefficients.

2^0. For operators

$$\mathcal{A} = A + \Sigma a_j Q_j ,$$

where A and Q_j have constant coefficients, Q_j is weaker that A and $a_j \in C^\infty$, the inequality (0.1) can be obtained from the constant coefficient case 1^0 if

$$N_2(aQ_j u) \leq C \sup |a| N_1(u) + C' N_3(u).$$

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In Peetre [8] (see also Schechter [9] and Matsuzawa [7])

$$N_2(u) = \left(\int_{R^n_+} | u |^2 \, dx \right)^{1/2}$$

is considered, but then (0.1) is not true for all formally hypoelliptic operators; the second step does not always work. Here we use instead

$$N_2(u) = \inf \left(\int_{R^n} | A_-^{-1} \tilde{u} |^2 \, dx \right)^{1/2},$$

if $A = A_+ \cdot A_-$ is the « canonical » decomposition of A, with inf taken over all $\tilde{u} \in S'(R^n)$, satisfying $\tilde{u} = u$ in R^n_+. In the same way we take

$$N_1(u) = \inf \left(\int_{R^n} | A_+ \tilde{u} |^2 \, dx \right)^{1/2}.$$

Then step 1^0 is immediate (cf. [8], [11]) and the main difficulty is to prove 2^0. This can be done by use of a commutator lemma analogous to Friedrich's lemma, which follows from the basic estimate

$$\left| \frac{\partial A_-}{\partial \xi_\nu} \right| \leq C \, | A_- | \, | \xi' |^{-\varepsilon}, \; \xi \in R^n, \; | \xi' | \geq M.$$

Let us mention that Hörmander [4] has proved a regularity theorem for operators with constant coefficients and general boundary conditions. He does not, however, use à priori estimates, but explicit formulas for the corresponding Green and Poisson kernels.

The plan of the paper is as follows. Section 1 contains some preliminaries concerning the distribution spaces involved. Section 2 contains the proof of the basic estimate of the Friedrich's type mentioned above. In Section 3 and Section 4 the applications to differential equations are given. Since they are rather routine, we have cut down the exposition to a minimum.

1. Spaces $H^+_{B_+, s}$ and $H^+_{B_-^{-1}, s}$.

The Fourier transform of an element f in one of the Schwartz classes S or S' (see [10]) is denoted by Ff, the inverse transform by $\overline{F}f$, $\overline{F}Ff = f$.

We take formally

$$Ff(\xi) = \int_{R^n} e^{-ix\xi} f(x)\, dx$$

and use the notation

$$P(D) f = \bar{F} P F f,$$

where P is a function on R^n. The following functions will often be used;

$$P(\xi) = A(\xi) = \xi_1 + i \left(1 + \sum_2^n \xi_j^2\right)^{1/2},$$

$$P(\xi) = A_1(\xi) = \left(1 + \sum_2^n \xi_j^2\right)^{1/2}.$$

By

$$A = A(D) = A(D_1, D') = \Sigma a_\alpha D^\alpha, \quad D^\alpha = (i)^{-\nu} \frac{\partial}{\partial x_{\alpha_1}} \cdots \frac{\partial}{\partial x_{\alpha_\nu}}$$

we denote a hypoelliptic differential operator with constant coefficients and write

$$(1.1) \qquad A(\xi) = A(\xi_1, \xi') = a \prod_1^{m_+} (\xi_1 - \varrho_j^+(\xi')) \prod_1^{m_-} (\xi_1 - \varrho_j^-(\xi')) = a A_+ A_-$$

with $a = a_{1 \dots 1}$. Here m_+ is the number of roots ϱ_j^+ with positive and m_- the number of roots ϱ_j^- with negative imaginary part. We require, that A satisfies the root condition, i. e. that m_+ and m_- are independent of ξ' for $|\xi'| \geq M$. It is no restriction to take $a = 1$. We set

$$(1.2) \qquad B_{(\pm)} = \begin{cases} A_{(\pm)}(\xi) & \text{if } |\xi'| \geq M_1 \\ (\xi_1 (\mp) i)^{m_{(\pm)}} & \text{if } |\xi'| < M_1, \end{cases}$$

where the value of $M_1 \geq M$ will be defined in Section 2. The following norms are used;

$$\| u \| = \left(\int_{R^n} | u(x) |^2 dx \right)^{1/2}, \| u \|_P = \| P(D) u \|, \| u \|_P^+ = \inf \| \tilde{u} \|_P,$$

where inf is taken over all $\tilde{u} \in S'$, whose restrictions to

$$R_+^n = \{x ; x_1 > 0\}$$

are equal to u, and such that

$$P(D)\tilde{u} \in L^2 .$$

The notation $\tilde{u}$ is used below in this sense. Particular norms of this type are

$$\| u \|^{+}_{B_+, s} = \| u \|^{+}_{\Lambda_1^s B_+} ,$$

$$\| u \|^{+}_{B_-^{-1}, s} = \| u \|^{+}_{\Lambda_1^s B_-^{-1}} ,$$

$$\| u \|^{+}_{r, s} = \| u \|^{+}_{\Lambda^r \cdot \Lambda_1^s} .$$

The corresponding spaces are denoted by

$$H^{+}_{B_+, s} , \; H^{+}_{B_-^{-1}, s} \text{ and } H^{+}_{r, s} .$$

The space corresponding to $\| \cdot \|_P$ is denoted by H_P. Paley-Wiener's theorem gives

$$\| u \|^{+}_{B_+, s} \approx \left(\int \Lambda_1^{2s} (\| F_{x'} u (\cdot , \xi') \|^{+}_{B_+(\cdot , \xi')})^2 \, d\xi' \right)^{1/2} , \tag{1.3}$$

$$\| u \|^{+}_{B_-^{-1}, s} \approx \left(\int \Lambda_1^{2s} (\| F_{x'} u (\cdot , \xi') \|^{+}_{B_-^{-1}(\cdot , \xi')})^2 \, d\xi' \right)^{1/2} . \tag{1.4}$$

The local spaces (cf. 2.5 in [4])

$$(H^{+}_{B_+, s})^{\text{loc}} , \; (H^{+}_{B_-^{-1}, s})^{\text{loc}} \text{ and } (H^{+}_{r, s})^{\text{loc}}$$

correspond to the above spaces. About $H^{+}_{0,s}$ we need the following fact, which goes back to Hörmander and Lions [6].

LEMMA 1.1. Let $c \in C_0^\infty (\overline{R^n_+})$. Then

$$\| cv \|^{+}_{0, s} \leq \sup | c | \, \| v \|^{+}_{0, s} + K_s \| v \|^{+}_{0, s-1}$$

for all $v \in H^{+}_{0, s}$, and with the constant K_s independent of v.

Next we state some lemmas in $H_{\Lambda_1^s B_+}$.

Lemma 1.2. C_0^∞ is dense in $H_{\Lambda_1^s B_+}$.

Proof: We prove in Section 2, that (2.4)

$$| B_-(\xi) - B_-(\xi + \eta) | \leq C (1 + |\eta|^2)^{k/2} | B_-(\xi) |,$$

for all $\xi, \eta \in R^n$. Here and below constants are written C and K, sometimes with index. As the same inequality holds for B_+, it follows that

$$| B_+(\xi + \eta) | \leq C' (1 + |\eta|)^{k/2} | B_+(\xi) |,$$

and consequently

$$| \Lambda_1^s(\xi' + \eta') B_+(\xi + \eta) | \leq C' (1 + |\eta|)^{|s| + k/2} | \Lambda_1^s(\xi') B_+(\xi) | .$$

But from this inequality follows that C_0^∞ is dense in $H_{\Lambda_1^s B_+}$. See [5] Remark p. 36 and Theorem 2.2.1).

We now use Lemma 1.2 to approximate elements of $H_{\Lambda_1^s B_+}$ with support in a half-space.

Lemma 1.3. Let $u \in H_{\Lambda_1^s B_+}$, supp $u \subset \overline{R_+^n}$. Then u is the limit in $H_{\Lambda_1^s B_+}$ of a sequence $(u_j)_1^\infty$ of functions

$$u_j \in C_0^\infty(R^n), \text{ supp } u_j \subset R_+^n .$$

Proof. Denote by τ_h translation by h along the x_1-axis. Then

$$\| \tau_h u - u \|_{\Lambda_1^s B_+} = \left(\int | \Lambda_1^s B_+ |^2 | e^{ih\xi_1} - 1 |^2 | Fu |^2 d\xi \right)^{1/2} \leq$$

$$\leq 2 \left(\int_{\complement \Xi} | B_+ \Lambda_1^s Fu |^2 d\xi \right)^{1/2} + \sup_{\Xi} | e^{ih\xi_1} - 1 | \, \| u \|_{\Lambda_1^s B_+} .$$

which can be made arbitrarily small by a suitable choice of Ξ and h. As the statement of the lemma is already established implicitly for $\tau_h u$ by Lemma 1.2, this ends the proof.

Remark. In Lemma 1.3, B_+ can be replaced by B_-^{-1} and $\overline{R_+^n}$ by

$$\overline{R_-^n} = \{x \,;\, x_1 \leq 0\}.$$

By definition, that a function $u \in H^{+}_{B_+, s}$ has the boundary values

$$D_1^j u(0, x') = 0, \qquad j = 0, \ldots, m_+ - 1, \tag{1.5}$$

means, that there is a $\tilde{u} \in H_{A_1^s B_+}$ with

$$\tilde{u} = 0 \text{ for } x_1 < 0, \ \tilde{u} = u \text{ for } x_1 > 0.$$

Finally we need

LEMMA 1.4. A function u satisfying (1.5) is in $H^{+}_{B_+, s}$ if and only if it is in $H^{+}_{B_+, s-1}$ and

$$\frac{u(x_1, x' + h') - u(x_1, x')}{|h'|}$$

is bounded in $H^{+}_{B_+, s-1}$ independently of $h' = (h_2, \ldots, h_n)$

PROOF: The proof is immediate if we notice that with

$$\tilde{u} = u \text{ for } x_1 > 0, \ \tilde{u} = 0 \text{ for } x_1 < 0,$$

there is a characterization of $B_+ \tilde{u}$ in $H_{A_1^s}$ by the same kind of difference quotients.

2. A version of Friedrich's lemma.

The derivation of the à priori inequality mentioned in Section 0, is for $m_- > 0$ based on a commutator lemma analogous to Friedrich's lemma (see e. g. [2]), which is established in this section. The proof depends on a number of lemmas, for which we need the following estimates of hypoelliptic polynomials;

$$|A^{\alpha}(\xi)/A(\xi)| \leq C_{\alpha} |\xi|^{-c|\alpha|} \quad \text{if } \xi \in R^n, \ |\xi'| \geq M,$$

$$\left| \frac{\partial A(\xi)}{\partial \xi_\nu} \middle/ A(\xi) \right| \leq C |\xi|^{-c} \quad \text{if } |\operatorname{Im} \xi_1| \leq C' |\xi'|^c, \ \xi' \in R^{n-1}, \ |\xi'| \geq M, \tag{2.1}$$

$$|\operatorname{Im} \varrho_j(\xi')| > C' |\xi'|^c \quad \text{if } |\xi'| \geq M$$

for some $c > 0$ and with $A^{\alpha}(\xi) = D^{\alpha} A(\xi)$ (see Hörmander [5]).

LEMMA 2.1. If ξ belongs to the cylinder $|\xi'| \geq M$, then for all ν

$$\left| \frac{\partial A_-(\xi)}{\partial \xi_\nu} \Big/ A_-(\xi) \right| \leq K |\xi'|^{-b}.$$

Here C is independent of ξ and $c^2 > b > 0$.

PROOF. As the coefficients of $A_-(\xi)$ are analytic in $|\xi'| \geq M$ (see [3] p. 289-290), the derivatives $\frac{\partial A_-}{\partial \xi_\nu}$ exist. Cauchy's formula gives

$$\frac{\partial A_-(\xi)}{\partial \xi_\nu} \Big/ A_-(\xi) = \frac{1}{2\pi i} \int_{-\infty + i\varepsilon}^{\infty + i\varepsilon} \frac{\partial}{\partial \xi_\nu} A(\xi_1 - \tau, \xi') \frac{d\tau}{A(\xi_1 - \tau, \xi')\tau}, \min_{1 \leq j \leq m} \operatorname{Im} \varrho_j^+ > \varepsilon > 0.$$

Take q and p such that

$$qc > 1, \frac{1}{q} + \frac{1}{p} = 1.$$

Then by (2.1) and with $\varepsilon = C' |\xi'|^c$ we obtain

$$\left| \frac{\partial A_-(\xi)}{\partial \xi_\nu} \Big/ A_-(\xi) \right| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{C d\sigma}{(|\xi_1 - \sigma| + |\xi'|)^c |\sigma + iC'|\xi'|^c|} \leq$$

$$\leq \frac{C}{2\pi} \left(\int_{-\infty}^{\infty} \frac{d\sigma}{(|\xi_1 - \sigma| + |\xi'|)^{cq}} \right)^{1/q} \cdot \left(\int_{-\infty}^{\infty} \frac{d\sigma}{|\sigma + iC'|\xi'|^c|^p} \right)^{1/p} \leq$$

$$\leq K |\xi'|^{-c/q}.$$

The next lemma compares $A_-(\xi)$ with $A_-(\xi + \eta)$ for small real η. For technical reasons, we only make that comparison in a cylinder

$$|\xi'| \geq M_1 \geq 2M,$$

with M_1 so large that

$$|\xi'| \leq 2 |\xi' + \eta'| \leq 4 |\xi'| \quad \text{if } |\eta| \leq |\xi'|^b.$$

This is the constant M_1 mentioned in formula (1.2).

LEMMA 2.2. Take ξ with $|\xi'| \geq M_1$ and $|\eta| \leq |\xi'|^b$. Then

$$|A_-(\xi+\eta)| \leq K' |A_-(\xi)|, \tag{2.2}$$

$$|A_-(\xi+\eta) - A_-(\xi)| \leq C|\eta||\xi'|^{-b} A_-(\xi), \tag{2.3}$$

wich K' and C independent of ξ and η.

PROOF. We write

$$\log \frac{A_-(\xi+\eta)}{A_-(\xi)} = \int_0^1 \sum_1^n \eta_j A_-^{-1}(\xi+t\eta) \frac{\partial}{\partial \xi_j} A_-(\xi+t\eta)\, dt.$$

The integrand can be estimated by Lemma 2.1. The restrictions on η and M_1 then give

$$\left| \int_0^1 \eta_j A_-^{-1}(\xi+t\eta) \frac{\partial}{\partial \xi_j} A_-(\xi+t\eta)\, dt \right| \leq$$

$$\leq K \int_0^1 |\xi'|^b |\xi' + t\eta'|^{-b}\, dt \leq K 2^b,$$

and so

$$|A_-(\xi+\eta)| \leq |A_-(\xi)| e^{nK2^b} = K' |A_-(\xi)|.$$

The inequality (2.3) follows from

$$|A_-(\xi+\eta) - A_-(\xi)| = \left| \int_0^1 \Sigma\, \eta_j \frac{\partial}{\partial \xi_j} A_-(\xi+t\eta)\, dt \right| \leq$$

$$\leq K|\eta| n \int_0^1 |A_-(\xi+t\eta)| |\xi+t\eta|^{-b}\, dt \leq$$

$$\leq K|\eta| \cdot n K' |A_-(\xi)| 2^b |\xi'|^{-b} = C |\eta| |\xi'|^{-b} |A_-(\xi)|.$$

The estimate that corresponds to (2.3) for $|\eta| > |\xi'|^b$, is much more easily obtained.

LEMMA 2.3. If $|\xi'| \geq M_1$, $|\xi' + \eta'| \geq M_1$ and $|\eta| \geq |\xi'|^b$, then

$$|A_-(\xi) - A_-(\xi + \eta)| \leq C |\eta|^d |\xi'|^{-c} |A_-(\xi)|,$$

where d is independent of ξ and η.

PROOF. Under the given restrictions on ξ and η, for some $a > 0$ the following inequalities hold;

$$|\varrho_j(\xi')| \leq |\xi'|^a \leq |\eta|^{a/b}.$$

$$|\varrho_j(\xi' + \eta')| \leq |\xi' + \eta'|^a \leq C_a |\eta|^{a/b},$$

$$|\operatorname{Im} \varrho_j(\xi')| \geq C |\xi'|^c.$$

Hence

$$\left| \frac{\varrho_j(\xi') - \varrho_j(\xi' + \eta')}{\xi_1 - \varrho_j(\xi')} \right| \leq C \frac{(1 + |\eta|)^{a/b}}{|\xi'|^c},$$

which gives the desired estimate, when inserted into

$$\frac{A_-(\xi) - A_-(\xi + \eta)}{A_-(\xi)} = \frac{\prod_1^{m_-} (\xi_1 - \varrho_j(\xi')) - \prod_1^{m_-} (\xi_1 + \eta_1 - \varrho_j(\xi' + \eta'))}{\prod_1^{m_-} (\xi_1 - \varrho_j(\xi'))} =$$

$$= \sum_{j=1}^{m_-} \frac{-\eta_1 - \varrho_j(\xi') + \varrho_j(\xi' + \eta')}{\xi_1 - \varrho_j(\xi')} \prod_{\nu > j} \frac{\xi_1 + \eta_1 - \varrho_\nu(\xi' + \eta')}{\xi_1 - \varrho_\nu(\xi')}.$$

Recalling from (1.2) that

$$B_-(\xi) = \begin{cases} A_-(\xi), & |\xi'| \geq M_1 \\ (\xi_1 + i)^{m_-}, & |\xi'| < M_1, \end{cases}$$

and using Lemmas 2.2 and 2.3, the main step in the proof of our commutator lemma easily follows.

LEMMA 2.4. There are constants k and C independent of $\xi, \eta \in R^n$, such that

$$\left| \frac{B_-(\xi) - B_-(\xi + \eta)}{B_-(\xi)} \right| \leq C \frac{(1 + |\eta|^2)^{k/2}}{(1 + |\xi' + \eta'|^2)^{b/2}}. \tag{2.4}$$

PROOF. The points ξ and $\xi + \eta$ can be situated inside or outside the cylinder $|\xi'| = M_1$. This gives four cases, which are treated separately.

1^0. By Lemmas 2.2 and 2.3, the inequality (2.4) is fulfilled for $|\xi'| \geq M_1$ and $|\xi' + \eta'| \geq M_1$.

2^0. For $|\xi'| \geq M_1$ and $|\xi' + \eta'| < M_1$, write the left-hand side of (2.4) as in the proof of Lemma 2.3.

$$\frac{B_-(\xi) - B_-(\xi + \eta)}{B_-(\xi)} = \sum_{j=1}^{m_-} \frac{-\eta_1 - \varrho_j(\xi') - i}{\xi_1 - \varrho_j(\xi')} \prod_{\nu > j} \frac{\xi_1 + \eta_1 + i}{\xi_1 - \varrho_\nu(\xi')}.$$

Each factor can be estimated by

$$C' \frac{(1 + |\eta|^2)^{k'}}{(1 + |\xi' + \eta'|^2)^{c/2}}$$

for some k' and C', which obviously implies (2.4).

3^0. The case $|\xi'| < M_1$, $|\xi' + \eta'| \geq M_1$ is treated analogously.

4^0. If $|\xi'| < M_1$ and $|\xi' + \eta'| < M_1$ the inequality is well-known.

When Q is weaker than A, we have

$$\| B_-^{-1} \Lambda_1^s \, a \, Qu \| \leq$$
$$\leq \left\| B_-^{-1} \Lambda_1^s (aB_- - B_- a) \frac{Q}{B} B_+ u \right\| + \left\| \Lambda_1^s \, a \frac{Q}{B} B_+ u \right\|.$$

Because A is hypoelliptic, we have

$$\left| \frac{Q(\xi)}{B(\xi)} \right| \leq C \quad \text{for } \xi \in R^n$$

(see [5] p. 102). Then the first term on the right side can be estimated by Lemma 2.4 as follows;

$$\left\| B_-^{-1} \Lambda_1^s (aB_- - B_- a) \frac{Q}{B} B_+ u \right\| =$$
$$= \left\| B_-^{-1}(\xi) \Lambda_1^s(\xi') \int Fa(\eta) (B_-(\xi - \eta) - B_-(\xi)) \frac{Q(\xi - \eta)}{B(\xi - \eta)} B_+(\xi - \eta) Fu(\xi - \eta) d\eta \right\|$$
$$\leq C \left\| \Lambda_1^s(\xi') \int Fa(\eta) \frac{\Lambda^k(\eta)}{\Lambda_1^b(\xi')} B_+(\xi - \eta) Fu(\xi - \eta) d\eta \right\| \leq$$
$$\leq C \left\| \int Fa(\eta) \Lambda^k(\eta) \Lambda_1^{|s-b|}(\eta) \Lambda_1^{s-b}(\xi' - \eta') B_+(\xi - \eta) Fu(\xi - \eta) d\eta \right\| \leq$$
$$\leq C \int |Fa \Lambda^k \Lambda_1^{|s-b|}| d\eta \, \| \Lambda_1^{s-b} B_+ u \|.$$

This estimate, together with the inequality

$$\| u \|^{+}_{B_+, s-b} \leq \varepsilon \| u \|^{+}_{B_+, s} + C_\varepsilon \| u \|^{+}_{B_+, s-1}$$

of Ehrling-Nirenberg type, gives

THEOREM 2.1. *Let* $a \in C_0^\infty(\overline{R^n_+})$. *Then for* $\varepsilon > 0$,

$$\| (aB_- - B_- a) QB_-^{-1} u \|^{+}_{B_-^{-1}, s} \leq \varepsilon \| u \|^{+}_{B_+, s} + C_\varepsilon \| u \|^{+}_{B_+, s-1},$$

with C_ε *independent of* $u \in H^{+}_{B_+, s}$.

3. A priori inequalities for hypoelliptic operators.

THEOREM 3.1. *Let*

$$\mathcal{A}(x, D) = A(D) + \sum_1^m a_j(x) Q_j(D),$$

where $A(D)$ *is hypoelliptic and* $Q_1(D), \ldots, Q_m(D)$ *are weaker than* $A(D)$. *If* $a_1(x), \ldots, a_m(x) \in C_0^\infty(\overline{R^n_+})$ *and*

$$\sum_j \sup | a_j(x) | < \varepsilon$$

for some sufficiently small $\varepsilon > 0$, *then*

$$\| u \|^{+}_{B_+, s} \leq C (\| \mathcal{A} u \|^{+}_{B_-^{-1}, s} + \| u \|^{+}_{B_+, s-1}) \tag{3.1}$$

for all $u \in H^{+}_{B_+, s}$, *satisfying the boundary conditions* (1.5).

PROOF. We prove the theorem for $m_- > 0$. The modifications in the simpler case $m_- = 0$ are obvious. As Lemma 1.3 shows, it is sufficient to prove the theorem for $u \in C_0^\infty(R^n_+)$. According to a theorem by Peetre ([8], Lemma 4)

$$\| F_{x'} u(\cdot, \xi') \|^{+}_{A_+(\cdot, \xi)} \leq \| A(\cdot, \xi') F_{x'} u(\cdot, \xi') \|^{+}_{A_-^{-1}(\cdot, \xi)} . \tag{3.2}$$

if $|\xi'| \geq M$ and if $u \in C_0^\infty(R_+^n)$. The proof is based on the Paley-Wiener theorem. We multiply (3.2) by Λ_1^s and integrate in ξ' (cf. (1.3), (1.4)) getting

$$\| u \|^+_{B_+, s} \leq \| Au \|^+_{B_-^{-1}, s} + \sqrt{1 + M_1^2}\, \| u \|^+_{B_+, s-1}.$$

It follows that

$$\| u \|^+_{B_+, s} \leq \left\| \left(A + \sum_1^m a_j Q_j\right) u \right\|^+_{B_-^{-1}, s} + \tag{3.3}$$

$$+ \sum_1^m \| a_j Q_j u \|^+_{B_-^{-1}, s} + C \| u \|^+_{B_+, s-1}.$$

But

$$\| a_j Q_j u \|^+_{B_-^{-1}, s} \leq \| B_- a_j Q_j B_-^{-1} u \|^+_{B_-^{-1}, s} +$$

$$+ \| (a_j B_- - B_- a_j) Q_j B_-^{-1} u \|^+_{B_-^{-1}, s}.$$

The last term can be estimated by use of Theorem 2.1, and in view of Lemma 1.1, we can estimate the first term in the following way;

$$\| B_- a_j Q_j B_-^{-1} u \|^+_{B_-^{-1}, s} \leq \| a_j Q_j B_-^{-1} u \|^+_{0, s} \leq$$

$$\sup | a_j |\, \| Q_j B_-^{-1} u \|^+_{0, s} + K \| Q_j B_-^{-1} u \|^+_{0, s-1} \leq$$

$$\leq C_j (\sup | a_j |\, \| u \|^+_{B_+, s} + K \| u \|^+_{B_+, s-1}).$$

Here C_j is independent of u and a_j. We have now proved that

$$\| a_j Q_j u \|^+_{B_-^{-1}, s} \leq (C_j \sup | a_j | + \varepsilon) \| u \|^+_{B_+, s} + C \| u \|^+_{B_+, s-1},$$

which together with (3.3) gives the desired estimate (3.1), if we assume for instance, that $m\varepsilon + \sum_{j=1}^m C_j \sup | a_j | < 1/2$.

4. Regularity.

In this section

$$\mathcal{A} = A + \Sigma a_j Q_j$$

is formally hypoelliptic. Before the main regularity theorem we formulate a result on regularity in the x'-directions.

THEOREM 4.1. *Let* $u \in H^+_{B_+, r}$ *for some* r *and let* u *satisfy* (1.5). *Define* $\mathscr{A}(x, D)$ *as in Theorem* 3.1 *Then*

$$u \in H^+_{B_+, s} \text{ if } \mathscr{A}u \in H^+_{\underline{B}-1, s}.$$

PROOF. It is always possible to choose r, so that $r = s - \nu$ for some integer ν. If $r \leq s - 1$ then the quotient

$$\frac{u(x_1, x' + h) - u(x_1, x')}{|h|}$$

is bounded in $H^+_{B_+, r}$ by Theorem 3.1. Then by Lemma 1.4, $u \in H^+_{B_+, r+1}$. By iteration, this proves the theorem.

THEOREM 4.2. *Let* $u \in D'(\overline{\dot{R}^n_+})$ *and satisfy* (1.5) *Then*

$$u \in (\dot{H}^+_{B_+, s})^{\mathrm{loc}} \quad \text{if} \quad \mathscr{A}\, u \in (H^+_{\underline{B}-1, s})^{\mathrm{loc}}.$$

PROOF. The theorem means that $\psi\, u \in H^+_{B_+, s}$ if $\psi \in C_0^\infty(\overline{R^n_+})$. It is no restriction to take all Q_j hypoelliptic and ψ with « small » support. For each such function ψ, we take another Φ of the same type with $\Phi = 1$ in a neighbourhood of supp ψ. We first show that $\Phi\, u \in H^+_{B_+, r}$ for some r, when supp Φ is small enough for $\mathscr{A}$ to fulfil the conditions of Theorem 3.1 in some open set $\omega \supset$ supp Φ. From the fact that

$$|B_-| \leq K \Lambda^{m_-} \Lambda_1^{m_0}$$

for some m_0, it follows

$$\mathscr{A}u \in (H^+_{-m_-, s - m_0})^{\mathrm{loc}}$$

if

$$\mathscr{A}u \in (H^+_{\underline{B}-1, s})^{\mathrm{loc}}.$$

As the Q_j's are hypoelliptic, there is a $d > 0$ such that for large ξ'

$$|Q_j^\alpha / Q_j| \leq |\xi|^{-d|\alpha|}, \quad |A^\alpha / A| \leq |\xi|^{-d|\alpha|}.$$

Take

$$\Phi_0 \in C_0^\infty(R^n_+)$$

with $\Phi_0 = 1$ in a neighbourhood of supp Φ and supp $\Phi_0 \subset \omega$. As $u \in D'$, we have

$$\Phi_0 u \in H^+_{\sigma \tau}$$

for some integer σ and real τ. If $\sigma < m_+$ we construct a sequence of $C_0^\infty(\overline{R^n_+})$-functions

$$\Phi_0, \ \Phi_1, \dots, \Phi_\mu = \Phi, \ \mu = m_+ - \sigma$$

with $\Phi_{j-1} = 1$ in a neighbourhood of supp Φ_j. Let $m_1 = m_+ + m_-$ be the order of the derivative D_1 in $\mathscr{A}$ and m' the total order. As

$$(D^\alpha \Phi_1)(A^\alpha + \Sigma a_j Q_j^\alpha) \Phi_0 u \in H^+_{\sigma - m_1 + 1, \tau - m' + 1} \text{ when } \alpha \neq 0$$

and

$$\Phi_1 (A + \Sigma a_j Q_j) u \in H^+_{-m_-, s - m_0},$$

Leibniz' formula shows that

$$\Phi_1 (A + \Sigma a_j Q_j) u + \sum_{|\alpha| \neq 0} D_\alpha \Phi_1 (A^\alpha + \Sigma a_j Q_j^\alpha) \Phi_0 u =$$

$$= (A + \Sigma a_j Q_j) \Phi_1 u \in H^+_{\sigma - m_1 + 1, \min(\tau - m' + 1, s - m_0)}.$$

Then by partial regularity (see e. g. [5]), for some τ'

$$\Phi_1 u \in H^+_{\sigma+1, \tau'}$$

and so, by iteration, for some r'

$$\Phi u \in H^+_{m_+, r'}.$$

For some r this will give

$$\Phi u \in H^+_{B_+, r}. \tag{4.1}$$

Take r so that with $q = c/b$

$$\nu = \frac{(s - r) q}{d}$$

is an integer. Let $(\psi_j)_0^\nu$ be a sequence analogous to $(\Phi_j)_0^\mu$, and with

$$\psi_0 = \Phi, \ \psi_\nu = \psi.$$

The terms in

$$(A + \Sigma a_j Q_j) \psi_1 u = \psi_1 (A + \Sigma a_j Q_j) u + \sum_{|\alpha| \neq 0} D^\alpha \psi_1 (A^\alpha + a_j Q_j^\alpha) \psi_0 u$$

can be estimated as in the proof of Theorem 2.1 and Theorem 3.1. With our choice of d this gives

$$\| a_j D^\alpha \psi_1 Q_j^\alpha \psi_0 u \|^+_{\underline{B}^{-1}, r+d/q} \leq K \| \psi_0 u \|^+_{B_+, r},$$

$$\| D^\alpha \psi_1 A^\alpha \psi_0 u \|^+_{\underline{B}^{-1}, r+d/q} \leq K \| \psi_0 u \|^+_{B_+, r},$$

and so

$$(A + \Sigma a_j Q_j) \psi_1 u \in H^+_{\underline{B}^{-1}, r+d/q}.$$

Then by Theorem 4.1

$$\psi_1 u \in H^+_{B_+, r+d/q}.$$

Repeating this ν times gives

$$\Phi u \in H^+_{B_+, s},$$

and so

$$u \in (H^+_{B_+, s})^{\text{loc}}.$$

CORR. 4.1. If $\mathcal{A} u \in C^\infty (\overline{R^n_+})$ and $u \in \mathcal{D}' (\overline{R^n_+})$ satisfies (1.5), then $u \in C^\infty (\overline{R^n_+})$.

PROOF. This follows by partial regularity from Theorem 4.2.

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