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A THEOREM ON APPROXIMATE DIRECTIONAL DERIVATIVES

by A. M. BRUCKNER [1] and M. ROSENFELD

1. Introduction.

Let f be a measurable real valued function on the Euclidean n space R^n. According to a theorem of Stepanoff [2] (See also Saks [1, p. 300]) f is approximately differentiable a.e. if and only if f possesses approximate partial derivatives a.e.. Stepanoff also gives an example to show that the corresponding statement is false if one deletes the word « approximate » from both the hypothesis and the conclusion. An intermediate property is that of having, at almost every point, directional derivatives in almost every direction. We first show, by example that this property is strictly intermediate if we are dealing with ordinary derivatives. We then study the situation for approximate derivatives ending with a theorem which states that if a measurable function has approximate partials a.e., then it has a.e. directional derivatives in a.e. direction.

For simplicity of notation, we restrict our attention to the case $n = 2$. The higher dimensional cases offer no new difficulties except notational.

2. Preliminaries.

In this section we state the definitions and indicate the notation we shall use in the sequel.

Let M be a measurable set in the Euclidean plane R^2, let $p = (x_0, y_0)$ be a point of R^2, and let θ be an angle (given in radians). If the weak, strong and θ-directional densities of M at p exist, we shall denote them by $\omega(p, M)$, $\sigma(p, M)$ and $d_\theta(p, M)$ respectively. These densities can be given

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by the formulae which follow, χ_M indicating the characteristic function of the set M.

$$d_\theta(p, M) = \lim_{h \downarrow 0} \frac{1}{h} \int_0^h \chi_M(x_0 + t\cos\theta,\ y_0 + t\sin\theta)\, dt$$

$$\omega(p, M) = \lim_{h \downarrow 0} \frac{1}{\pi h^2} \int_{\theta=0}^{2\pi} \int_{r=0}^{h} \chi_M\, r\, dr\, d\theta$$

$$\sigma(p, M) = \lim_{\substack{h \to 0 \\ k \to 0}} \frac{1}{hk} \int_{x_0}^{x_0+h} \int_{y_0}^{y_0+k} \chi_M(x, y)\, dx\, dy.$$

For simplicity of notation we shall write $|A|$ for the Lebesgue measure of a measurable set A. Whether it is one, two or three dimensional measure we are considering will be clear from the context.

3. Results.

We begin with two examples which show that, for ordinary derivatives, the existence for almost all points, of directional derivatives in almost all directions, is intermediate to the existence almost everywhere of partial derivatives and the existence almost everywhere of a total differential.

Example 1. Let F be a residual subset of the real line such that $|F| = 0$. Let $M = F \times F$ and let $f = \chi_M$. For every point $p = (x, y) \in R^2$ such that $x \in \sim F$ and $y \in \sim F$, f has (vanishing) partial derivatives at (x, y). The set of all such points p has full measure. Thus, the partial derivatives of f exist a.e. in R^2.

On the other hand, we show that at *no* point does f have directional derivatives in any direction different from that of the coordinate axes. To show this, we show that any line segment L not parallel to a coordinate axis contains points of M as well as points of $\sim M$. That L contains points of $\sim M$ follows from the fact that $\sim F$ is dense on the real line. Now suppose L is parametrized by the equations $x(t) = at + c$, $y(t) = bt + d$ where a and b are non-zero real numbers and t ranges over the interval $[0, 1]$. We wish to show that there exists t_1, $0 \leq t_1 \leq 1$ such that $at_1 + c \in F$ and $bt_1 + d \in F$. This is equivalent to showing that the sets $\frac{1}{a}(F - c)$ and $\frac{1}{b}(F - d)$ have a point in common in the interval $[0, 1]$. But each of these sets is residual on the real line, so the same is true of their intersection. Thus f does not possess a derivative along L at any point of L.

EXAMPLE 2. Let M be any dense denumerable subset of R^2 possessing no more than two points on any line and let $f = \chi_M$. It is clear that f possesses directional derivatives in all directions at every point not in M but that f possesses a total differential nowhere.

The main result of this article is Theorem 2, below, a result which depends heavily on Theorem 1, which is a sort of Lebesgue density theorem for directional densities.

THEOREM 1. *Let M be a measurable subset of R^2. Then for almost every point $p \in M$, $d_\theta(p, M) = 1$ for almost every θ.*

PROOF. Assume first that M is closed. Let $\mathcal{M} = M \times [0, 2\pi)$. Let $\mathcal{D} = \{(p, \theta) \in \mathcal{M} : d_\theta(p, M) = 1\}$. We begin by showing that $\mathcal{D}$ is measurable. For each pair of positive integers (n, k), let

$$Q_{nk} = \Big\{(x, y, \theta) : |\{r \in [0, b] :$$

$$(x + r\cos\theta,\ y + r\sin\theta) \in M\}| \geq \left(1 - \frac{1}{n}\right) b \text{ whenever } b \leq \frac{1}{k}\Big\}.$$

Then $\mathcal{D} = \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} Q_{nk}$. To show that $\mathcal{D}$ is measurable, it suffices to show that for each n and k, the set Q_{nk} is closed. Thus let $\{(x_j, y_j, \theta_j)\}$ be a sequence of points in Q_{nk} converging to a point (x_0, y_0, θ_0). Let $b \leq \frac{1}{k}$ and, for each $j = 0, 1, 2, \ldots$, let

$$M_j = \{r \in [0, b] : (x_j + r\cos\theta_j,\ y_j + r\sin\theta_j) \in M\}.$$

Since $(x_j, y_j, \theta_j) \in Q_{nk}$ for each $j \geq 1$, $|M_j| \geq \left(1 - \frac{1}{n}\right) b$ We show $|M_0| \geq \left(1 - \frac{1}{n}\right) b$. Now if $r \in M_j$ for infinitely many j, say, $r \in M_{j_m}$, $m = 1, 2, \ldots$ then $(x_0 + r\cos\theta_0,\ y_0 + r\sin\theta_0)$ is in the closure of the set $\bigcup_m \{(x_{j_m} + r\cos\theta_{j_m},\ y_{j_m} + r\sin\theta_{j_m})\}$. Since M is closed, we infer $(x_0 + r\cos\theta_0,\ y_0 + r\sin\theta_0) \in M$, so $r \in M_0$. Therefore $M_0 \supset \bigcap_s \bigcup_{j>s} M_j$ from which it follows that $|M_0| \geq \left(1 - \frac{1}{n}\right) b$. Therefore $(x_0, y_0, \theta_0) \in Q_{nk}$ and Q_{nk} is closed. Therefore $\mathcal{D}$ is measurable.

Now fix θ, $0 \leq \theta < 2\pi$. For almost all $p \in M$, $d_\theta(p, M) = 1$. Thus, for every θ, the θ-section of $\mathcal{M} - \mathcal{D}$ has zero (two dimensional) measure. Since $\mathcal{M}$ and $\mathcal{D}$ are measurable, it follows from Fubini's Theorem that $|\mathcal{M} - \mathcal{D}| = 0$. Employing Fubini's Theorem again, we see that for almost every $p \in M$, the p-section of $\mathcal{M} - \mathcal{D}$ has (one dimensional) measure zero. That is, for almost every p, the point (p, θ) is in $\mathcal{D}$ for almost every θ in $[0, 2\pi)$. This means that for almost every p, $d_\theta(p, M) = 1$ for almost every θ.

It remains to prove the theorem for M an arbitrary measurable set. Let $\{F_k\}$ be a sequence of closed sets contained in M such that $|M \sim \bigcup_{k=1}^{\infty} F_k| = 0$. For each k, there exists a set $Z_k \subset F_k$ such that $|Z_k| = 0$ and if $p \in F_k \sim Z_k$, then for almost every θ, $d_\theta(p, F_k) = 1$. For such a p and θ, $d_\theta(p, M) = 1$. Thus, if $p \notin (M \sim \bigcup_{k=1}^{\infty} F_k) \cup \bigcup_{k=1}^{\infty} Z_k$, $d_\theta(p, M) = 1$ for almost all θ.

This completes the proof of the theorem.

Comparing Theorem 1 with the Lebesgue Density Theorem, we see that if M is measurable, then almost every point of M is both a point of two dimensional density of M and a point of directional density in almost every direction. Neither of these two conditions, however, implies the other at individual points as the following examples show.

EXAMPLE 3. Let $M_1 = \{(x, y) : 0 < y < x^2\}$ and let $M = R^2 \sim M_1$. It is easy to verify that $d_\theta(p, M) = 1$ for all θ while $\sigma(p, M)$ does not exist. In fact, the lower strong density of M at p is zero.

EXAMPLE 4. We give an example of a set M and a point p such that $\sigma(p, M) = 0$ while $d_\theta(p, M)$ does not exist for any θ. If $N = R^2 \sim M$ then $\sigma(p, N) = 1$ while $d_\theta(p, N)$ exists for no θ.

Let $s_k = 1 + \frac{1}{2} + \dots + \frac{1}{k}$ for each integer k. Consider the following sequence of sets:

$$A_1 = \left\{(r \cos\theta, r \sin\theta) : \quad \frac{1}{2} \leq r \leq 1 \quad \text{and} \quad s_1 \leq \theta \leq s_2\right\}$$

$$A_2 = \left\{(r \cos\theta, r \sin\theta) : \quad \frac{1}{2^{n_2}} \leq r \leq \frac{1}{2^{n_2 - 1}} \quad \text{and} \quad s_2 \leq \theta \leq s_3\right\}$$

where n_2 is chosen so large that $|A_2|$ is less than $\frac{1}{2}|A_1|$.

We continue in this manner obtaining a sequence of sets $\{A_k\}$ such that for each k, $A_k = \left\{(r\cos\theta, r\sin\theta) : \frac{1}{2^{n_k}} \leq r \leq \frac{1}{2^{n_k - 1}} \text{ and } s_k \leq \theta \leq s_{k+1}\right\}$ where n_k is chosen so large that $n_k > n_{k-1}$ and $|A_k| < \frac{1}{2}|A_{k-1}|$.

For θ_0 a small positive number define $G_{\theta_0}=\left(\theta_0,\frac{\pi}{2}-\theta_0\right)\cup\left(\frac{\pi}{2}+\theta_0,\pi-\theta_0\right)\cup$ $\cup\left(\pi+\theta_0,\frac{3\pi}{2}-\theta_0\right)\cup\left(\frac{3\pi}{2}+\theta_0,2\pi-\vartheta_0\right)$. Choose a sequence $\{\theta_k\}$ such that $\theta_k\downarrow 0$ and $k\cos\theta_k\sin\theta_k\longrightarrow\infty$. Let $B_k=\{(r\cos\theta, r\sin\theta\}: \theta\in G_{\theta_k}$ and $(r\cos\theta, r\sin\theta)\in A_k\}$, M be the union of the sets B_k and subsets of the coordinate axes with no linear density at $(0,0)$ and let p be the origin.

Then it is easy to see that $d_\theta(p,M)$ does not exist for any θ. We verify that $\sigma(p,M)=0$. Let R be a rectangle with one corner at p and the opposite one at (x,y). By symmetry we may assume (x,y) is in the first quadrant. Let k be the smallest integer in $\{j: B_j\cap R\neq\Phi\}$ and let $n=n_k$. Then $\frac{|R\cap M|}{|R|}\leq 2\frac{|A_k|}{|R|}$. Since R contains a point $(r\cos\theta, r\sin\theta)$ such that $r\geq\frac{1}{2^n}$ and $\theta\in G_{\theta_k}$, $|R|\geq\frac{1}{4^n}\cos\theta_k\sin\theta_k$. Since $|A_k|\leq\frac{1}{k}\,\frac{1}{2^{n-1}}\cdot$ $\cdot\left(\frac{1}{2^{n-1}}-\frac{1}{2^n}\right)=\frac{2}{k}\,\frac{1}{4^n}$ and since $k\longrightarrow\infty$ as the diameter of R tends to zero,

$$0\leq\overline{\lim_{|R|\to 0}}\frac{|R\cap M|}{|R|}\leq\lim_{k\to\infty}2\,\frac{\left(\frac{2}{k}\right)\frac{1}{4^{n_k}}}{\frac{1}{4^{n_k}}\cos\theta_k\sin\theta_k}=\lim_{k\to\infty}\frac{4}{k\cos\theta_k\sin\theta_k}=0.$$

REMARK. Example 3 showed that it is possible to have $d_\theta(p,M)=1$ for all θ without having $\sigma(p,M)=1$. However, the relation $d_\theta(p,M)=1$ for almost all θ does imply the equality $\omega(p,M)=1$. This result follows readily from the Schwarz inequality. In fact,

$$\int_0^h\chi_M(x_0+r\cos\theta, y_0+r\sin\theta)\,r\,dr\leq$$

$$\leq\left[\int_0^h\chi_M(x_0+r\cos\theta, y_0+r\sin\theta)\,dr\int_0^h r^2\,dr\right]^{\frac{1}{2}}\leq$$

$$\leq\frac{h^2}{\sqrt{3}}\left[\frac{1}{h}\int_0^h\chi_M(x_0+r\cos\theta, y_0+r\sin\theta)\,dr\right]^{\frac{1}{2}}.$$

Therefore

$$\frac{1}{\pi h^2}\int_{\theta=0}^{2\pi}\int_{r=0}^{h}\chi_M(x_0+r\cos\theta,\, y_0+r\sin\theta)\, r\, dr\, d_\theta \leq$$

$$\leq \int_{\theta=0}^{2\pi}\frac{1}{\sqrt{3\pi}}\left[\frac{1}{h}\int_0^h \chi_M(x_0+r\cos\theta,\, y_0+r\sin\theta)\, dr\right]^{\frac{1}{2}} d\theta.$$

Now, if $d_\theta(p, M) = 0$ for almost all θ, then the expression in brackets approaches the value 0 almost everywhere as $h \to 0$. It follows from the Lebesgue bounded convergence theorem that, as $h \to 0$, the last mentioned double integral approaches 0 with h. We have shown that if p is a point of directional dispersion for M for almost all θ, it is also a point of weak dispersion for M. By considering complements, the desired result follows.

Example 1 above showed us that the existence almost everywhere of partial derivatives does not imply the existence almost everywhere of any of the directional derivatives. For approximàte differentiation, the situation is different.

Theorem 2. *Let f be measurable on R^2. If the approximate partial derivatives exist almost everywhere, then for almost all $p \in R^2$, the approximate directional derivatives exist for almost all directions.*

Proof. Let M be the set of points for which it is not the case that f possesses approximate directional derivatives in almost all directions. We show $|M| = 0$. Suppose, then, that M has positive outer measure ε. According to a theorem of Whitney [3], there exists a continuously differentiable function g which agrees with f except on a set S of measure less than ε. The set $M \sim S$ has positive outer measure. By Theorem 1, for almost every point in $\sim S$, $d_\theta(p, \sim S) = 1$ for almost every θ. Thus there is such a point $p \in M \sim S$. Now g has a directional derivative for every direction θ at p. This directional derivative is the approximate directional derivative of f at p for any direction θ for which $d_\theta(p, \sim S) = 1$; that is, for almost every θ. But this means $p \in \sim M$, a contradiction.

Remarks: It is not difficult to verify that the converse to Theorem 2 is valid. Thus, the following four conditions are equivalent for a function f measurable in R^2.

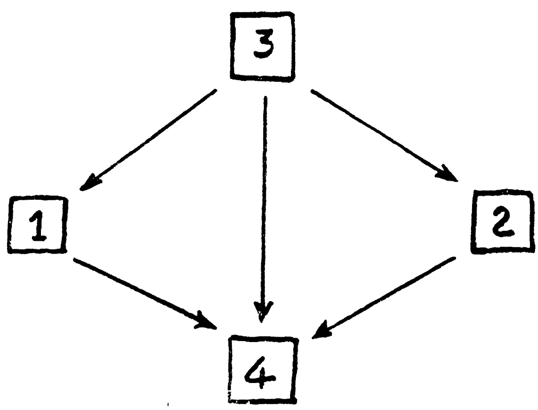
1. f possesses approximate partial derivatives a. e.

2. For almost all points, f possesses approximate directional derivatives in almost all directions.

3. (Stepanoff) f has an approximate differential a. e.

4. (Whitney) For every $\varepsilon > 0$, there exists a set M whose complement has measure less than ε and a continuously differentiable function g such that f and g agree on M.

Furthermore, if we drop the term « approximate » wherever it appears, the only valid implications among the four modified conditions are indicated in the chart below



The only counterexample remaining is $2 \not\longrightarrow 1$.

EXAMPLE 5. Let F be a residual subset of the real numbers R^1 such that $|F| = 0$. Let $S = F \times R^1$. Then $|S| = 0$. We shall construct a subset M of S with the properties that M intersects every horizontal line L in a set dense in L, while M intersects every other line in at most two points. The characteristic function of M has the desired properties. Let $S(y: r, s) = \{(x, y): r < x < s\}$ where y, r and s are real numbers. This family has the cardinality of the continuum. Let Ω be the first ordinal equivalent to the continuum and let the family $\{S(y: r, s)\}$ be well ordered by $\Omega: S_1, S_2, \ldots$ Let $p_1 \in S_1 \cap S$. Assuming we have for all $\alpha < \beta$, $p_\alpha \in S_\alpha \cap S$ such that no three of the points in the family $\{p_\alpha: \alpha < \beta\}$ are collinear (except possibly collinear on a horizontal line), choose $p_\beta \in S_\beta \cap S$ such that this property holds now for the family $\{p_\alpha: \alpha \leq \beta\}$. That this is possible follows from the fact that $S_\beta \cap S$ has cardinality of the continuum while the number of lines already chosen has cardinality less than that of the continuum. The set $M = \{p_\alpha: \alpha < \Omega\}$ has the desired properties.

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