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THE CURVATURE GROUPS OF A SPACE FORM

by IZU VAISMAN

Following an idea, developed in another manner in [7], we shall define the curvature groups of a connection on a principal bundle and we shall study these groups for the case of a Riemannian connection of constant curvature.

1. First, we shall remember some formulas, which can be found in CHERN [2]. Let $B \xrightarrow{p} X$ be a differentiable, principal bundle (X is a C^∞ -manifold of dimension n ; differentiable will always mean C^∞), with group G and let g_{UV} be his transition functions, corresponding to a covering $\{U\}$ of X by coordinate neighbourhoods. Then a connection on B is defined by a collection $\{\theta_U\}$ of 1-forms on U with values in the Lie algebra \mathfrak{g} of G and such that, in $U \cap V$, we have

$$(1) \quad \theta_V = \omega(g_{UV}) + (ad g_{UV}^{-1}) \theta_U.$$

Here ω is the \mathfrak{g} -valued form on G , attaching to each tangent vector of G the corresponding left invariant field.

Let R be a linear representation of G on a finite-dimensional linear space E . A tensor p -form on X , of type R is a collection $T = \{T_U\}$ of p -forms on U with values in E and such that

$$(2) \quad T_U = R(g_{UV}) T_V.$$

For instance, the 2-forms

$$(3) \quad \Theta_U = d\theta_U + \frac{1}{2} [\theta_U, \theta_U]$$

(the notations are those of CHERN [2]), define a tensor form Θ of type ad G , which is the curvature form of the connection θ .

For the tensor forms, there is an operation of covariant differentiation given by

$$(4) \quad DT_U = dT_U + \tilde{K}(\theta_U) \wedge T_U,$$

where \tilde{K} is the representation of \mathfrak{g} associated to R and the following formulas hold

$$(5) \quad D\Theta_U = 0,$$

$$(6) \quad D^2T_U = \tilde{K}(\Theta_U) \wedge T_U.$$

The formula (2) shows that the tensor p -forms of type R on X define a module \mathcal{I}^p over the ring \mathcal{F} of differentiable functions on X and (6) shows that the p -forms $\{D^2T\}$ define an \mathcal{F} -submodule \mathcal{D}^p of \mathcal{I}^p .

Following the classical case of KODAIRA-SPENCER [5], we will define a *tensorial p -jet-form* of type R on X as a pair (T, S) of tensor forms of type R and of degrees p and $p+1$ respectively and will denote by J^p the \mathcal{F} module of these forms.

Consider now the formula [7]

$$(7) \quad D(T, S) = (DT - S, D^2T - DS).$$

It is easy to see, that it defines an operator (homomorphism) $D: J^p \rightarrow J^{p+1}$, whose square is zero and such that the cochain complex $\left(\bigoplus_{p=0}^n J^p, D\right)$ is acyclic.

Let K^p be the submodule of J^p defined by jet-forms (T, S) such that $S \in \mathcal{D}^{p+1}$. Then $K = \left(\bigoplus_{p=1}^n K^p, D\right)$ is a subcomplex of the preceding one, which is no more acyclic.

Let

$$(8) \quad H^i(X, \theta, R) = H^i(K) = \text{Ker } D^i / \text{Im } D^{i-1} \quad (i = 1, 2, \dots, n)$$

be the cohomology groups of the complex K . We shall say that these groups are the *curvature groups* of type R of the connection θ .

Suppose now θ is a linear connection on X defined by the matrices of 1-forms $\theta_U = (-\omega_i^j)$ and with curvature forms $\Theta_U = (\Omega_i^j)$ and let the tensor forms of type R employed be the usual vector forms on X [3, 7] denoted for instance by $\lambda(\lambda^i)$ where λ^i are scalar p -forms on U . The other

notations remain as above. Then we have the formulas

$$(9) \quad \Omega_i^j = -d\omega_i^j + \omega_i^k \wedge \omega_k^j,$$

$$(10) \quad D\lambda^i = d\lambda^i + \omega_j^i \wedge \lambda^j,$$

$$(11) \quad D^2\lambda^i = -\Omega_j^i \wedge \lambda^j.$$

The corresponding curvature groups will be noted by $H^i(X, \theta)$ and called the *curvature groups of the linear connection* θ . Part of them were called in [7] the cohomology groups of θ and some of their properties were established.

We shall remark that on X there is a canonical 1-form $\omega(dx^i)$ where x^i are coordinates in U and, if θ is without torsion, $D\omega = 0$. If we note by A^p the module of scalar p -forms on X , one may consider the monomorphism

$$(12) \quad h: A^{p-1} \rightarrow J^p$$

given by

$$h(a) = (a \wedge dx^i, 0) \quad (a \in A^{p-1})$$

and it is easy to find

$$hD = Dh.$$

It follows that there is a homomorphism

$$(13) \quad h^*; H^{p-1}(X, R) \rightarrow H^p(X, \theta),$$

$H^{p-1}(X, R)$ being the real cohomology groups of X .

A vector form of the type $a \wedge dx^i$ will still be noted $a \wedge \omega$. More general, the sign \wedge will always note componentwise exterior product.

If θ is a Riemannian connection, his curvature groups will be called the curvature groups of the respective Riemannian space; it is in particular the case of a Riemannian space of constant curvature, which is our object of study in the next.

2. Let $X = V_n$ be a pseudo-Riemannian space with metric $g(g_{ij})$ and let θ be the corresponding Levi-Civita connection. By the formulas

$$(14) \quad \Omega_i^j = \frac{1}{2} R_{i\,kh}^j dx^k \wedge dx^h, \quad R_{ijkh} = g_{js} R_{i\,skh}^s,$$

we get the curvature tensor of the space. It is known [8] that the space V_n has constant (sectional) curvature k if and only if

$$(15) \quad R_{ijkh} = k (g_{ik} g_{jh} - g_{jh} g_{jk}).$$

Hence

$$(16) \quad \Omega_i^j = k \varepsilon_i \wedge dx^j,$$

where $\varepsilon_i = g_{ik} dx^k$ is a covector ε with Pfaff forms as components; the calculus with such tensorial forms will be employed in the next without other references.

The formula (11) becomes now

$$(17) \quad D^2 \lambda^i = k dx^i \wedge a, \quad a = \varepsilon_i \wedge \lambda^i$$

and the module \mathcal{D}^p is defined by vector p -forms ξ of the type

$$(18) \quad \xi = k \omega \wedge a,$$

a being a scalar $(p-1)$ -form. In fact, for $p \geq 1$, it is easy to see that $a = \varepsilon_i \wedge \nu^i$, hence $\xi = D^2 \nu$.

For $k \neq 0$, the forms (18) may be written as

$$(19) \quad \xi = \omega \wedge b$$

and conversely. But, for $k = 0$, the forms (18) vanish. In that case, we shall also consider the modules \mathcal{D}^p given by (19) and shall get cohomology groups by the scheme (8); they will be called the *special groups* of the corresponding flat space.

It is clear that $\omega \wedge a = 0$ if and only if $a = 0$, because $\deg a \leq n-1$, hence if ξ is of the form (19) the corresponding b is uniquely determined. We find then, that the modules K^p ($p = 1, \dots, n-1$) are isomorphic with the modules L^p defined by *pairs*

$$(20) \quad (\lambda, a),$$

where λ is a vector p -form and a a scalar p -form.

Applying the formula (7), we find that, by the above isomorphism, D induces on L^p an operator, noted again by D , whose square is zero and which is given by the formula

$$(21) \quad D(\lambda, a) = (D\lambda - \omega \wedge a, da + k \varepsilon_i \wedge \lambda^i).$$

(Remember that θ is, at us, without torsion).

It is natural now to extend the definitions of L^p and D for $p = 0$ and $p = n$ and to obtain the cochain complex $(L = \bigoplus_{p=0}^n L^p, D)$. We remark that, for a direct verification of $D^2 = 0$ it is necessary to employ the relations

$$(22) \quad D \varepsilon_i = 0, \quad \varepsilon_i \wedge dx^i = 0,$$

which follow from the properties of g_{ij} to be covariantly constant and symmetrical.

As a conclusion of the above discussion we will consider the groups

$$(23) \quad H^i(V_n, L) = H^i(L) \quad (i = 0, 1, \dots, n),$$

i.e. the cohomology groups of the complex L and, in the next, they will be the curvature groups of the space V_n , or the special groups in the case of a flat space.

We shall now show that the special groups of a flat space can be interpreted as cohomology groups of V_n with coefficients in a sheaf.

Let us begin with the remark that, in an euclidian n -space and in cartesian coordinates x^i , the equation

$$(24) \quad \frac{\partial^2 F}{\partial x^i \partial x^j} = C \delta_{ij} \quad (C = \text{const.})$$

characterises the functions $F(x)$, such that $F(x) = \text{const.}$ is a hypersphere.

We can try then to define hyperspheres in a general pseudo Riemannian space V_n as hypersurfaces which may be given by an equation $F(x) = \text{const.}$ such that

$$(25) \quad F_{|ij} = f(x) g_{ij}.$$

But, by a derivation of (25) we get

$$R_{i \ k j}^s F_{|s} = f_{|k} g_{ij} - f_{|j} g_{ik}$$

and contracting with g^{ih} , then contracting $h = j$ and noting

$$r_k^s = g^{ij} R_{i \ k j}^s$$

we get

$$(26) \quad r_k^s F_{|s} = (n - 1) f_{|k},$$

whence, by a new derivation and by employing Bianchi's identities

$$(27) \quad g^{ij} R_{i \ k m j}^s F_{|s} = 0$$

Hence, in the general case, F is constant on the connected components of V_n and, from (25), $f(x) = 0$. It follows that such hyperspheres as we intended to define do not generally exist. However, for a space of constant curvature, (27) is verified identically and we may hope to find such hyperspheres. (More generally, (27) is verified for any symmetric space V_n).

In this case, by (15) and (26) we get $f_{jh} = -kF_{jh}$, whence $f = -kF + C$ and (25) becomes

$$(28) \quad F_{ij} = (-kF + C)g_{ij};$$

if $k = 0$, we get an equation equivalent to (24). The functions F on V_n , verifying (28), or equivalently (25), will be called S -functions. For a flat space and in cartesian coordinates these functions are locally

$$(29) \quad F = C(\delta_{ij}x^ix^j + a_ix^i + b) \quad (a_i, b, C = \text{const.}).$$

Each S -function F determines a field of contravariant vectors, i.e. a vector 0-form λ —the normal field of the hyperspheres $F = \text{const.}$, which, by (25), is verifying the condition

$$(30) \quad \lambda^i_{;j} = f\delta^i_j.$$

Conversely, if a vector 0-form λ satisfies (30), we get

$$\lambda_{ij} = fg_{ij}$$

which shows that λ_i is the gradient of an S -function F . The vector 0-forms λ , characterized by (30), will be called S -fields.

The S -fields of V_n define an additive abelian group but not an \mathcal{F} -module. It is clear that, for a given S -field λ , the function f in (30) is uniquely determined and is of the form $-kF + C$.

Let us note by S the group of S -fields on V_n . Then, by the above remarks we get a monomorphism

$$(31) \quad i: S \rightarrow L^0$$

given by $i(\lambda) = (\lambda, f)$ and, if we note by \mathcal{S} , \mathcal{L}^p the sheaves of germs associated to S , L^p respectively, we get a sequence of sheaves and homomorphisms

$$(32) \quad 0 \rightarrow \mathcal{S} \xrightarrow{i} \mathcal{L}^0 \xrightarrow{D} \mathcal{L}^1 \xrightarrow{D} \dots \xrightarrow{D} \mathcal{L}^n \xrightarrow{D} 0,$$

which is exact at \mathcal{S} and where the sheaves \mathcal{L}^p are fine. Moreover we have $D^2 = Di = 0$ (because of (30) and $f = -kF + C$).

Suppose now we are in the case of a flat space, hence $k = 0$, and suppose we have an element (λ, a) of L^p which belongs to the kernel of D , i.e.

$$(33) \quad D\lambda = \omega \wedge a \quad da = 0.$$

Then, by the Poincaré lemma we have locally $a = db$. Moreover, it is known that we can choose, locally, cartesian coordinates and get $D\lambda^i = d\lambda^i$, such that the first relation (33) will become

$$d\lambda^i = dx^i \wedge db,$$

whence

$$\lambda^i = d\sigma^i - dx^i \wedge b = D\sigma^i - dx^i \wedge b.$$

Hence (33) implies locally $(\lambda, a) = D(\sigma, b)$ and we conclude that the sequence (32) is exact (the exactity at \mathcal{L}^0 is trivial).

So, we got the interpretation looked for: *the special groups of a paracompact, flat, pseudo-Riemannian space are the cohomology groups of the space with coefficients in the sheaf of germs of the S -fields of the space.*

We still remark here that for the curvature groups of a space V_n the homomorphisms (13) remain valid.

3. In this section we shall suppose that the space V_n is Riemannian (g -positive definite), compact and oriented, i.e. V_n is a compact oriented space form and shall obtain a theory of «harmonic forms» for the curvature groups.

First we remember [2, 6] that, in this case, we have a global scalar product of scalar p -forms given by

$$(34) \quad \langle a, b \rangle = \int_{V_n} a \wedge * b$$

and the pairs of adjoint operators $d, \delta; e(a), i(a)$, the last two being respectively the exterior and interior products; they are linked by

$$(35) \quad \delta = (-1)^{np+n+1} * d *, \quad i(a) = (-1)^{n(p+1)} * e(a) *$$

when acting on p forms and $\deg a = 1$. Then, we have the Laplace operator

$$(36) \quad \Delta = d\delta + \delta d,$$

which is self-adjoint and commutes with $d, \delta, *$.

Let $\lambda(\lambda^i)$ be a vector p -form. Then, it is easy to see that the forms $*\lambda^i$ define a vector $(n-p)$ -form $*\lambda$, $e(a)\lambda^i$ define a vector form $e(a)\lambda$ and $i(a)\lambda^i$ a vector form $i(a)\lambda$.

For the vector p -forms $\lambda(\lambda^i)$, $\mu(\mu^i)$ we have again a global scalar product given by [5]

$$(37) \quad \langle \lambda, \mu \rangle = \int_{V_n} g_{ij} \lambda^i \wedge * \mu^j,$$

which is clearly positive definite and commutative and, with respect to it, the operators $e(a)$ and $i(a)$ are again adjoints.

Further we have

$$\begin{aligned} 0 &= \int_{V_n} d(g_{ij} \lambda^i \wedge * \mu^j) = \int_{V_n} D(g_{ij} \lambda^i \wedge * \mu^j) = \int_{V_n} g_{ij} D\lambda^i \wedge * \mu^j + \\ &\quad + (-1)^{p-1} \int_{V_n} g_{ij} \lambda^i \wedge **^{-1} D * \mu^j, \end{aligned}$$

whence the operator

$$(38) \quad \bar{D} = (-1)^p **^{-1} D * = (-1)^{n\mu+n+1} * D *$$

is adjoint to D and satisfies the relation

$$(39) \quad \langle D\lambda, \mu \rangle = \langle \lambda, \bar{D}\mu \rangle.$$

We get then the following expressions for D and \bar{D}

$$(40) \quad D\lambda^i = d\lambda^i + e(\omega_j^i) \lambda^j, \quad \bar{D}\lambda^i = \delta\lambda^i - i(\omega_j^i) \lambda^j.$$

Let now (λ, a) and (μ, b) be elements of L^p . Then, the formula

$$(41) \quad \langle (\lambda, a), (\mu, b) \rangle = \langle \lambda, \mu \rangle + \langle a, b \rangle$$

defines a positive definite and commutative scalar product on L^p and we wish to find the adjoint of the operator D on pairs.

We remark, first, that $e(dx^i) = e(\omega)$ and $e(\varepsilon_i) = e(\varepsilon)$ may be applied componentwise to tensor forms, this operation leading again to tensor forms with a contravariant, respectively covariant, index more. This implies that $i(\omega)$

and $i(\varepsilon)$ may also be applied to tensor forms. A straightforward calculation gives

$$(42) \quad \begin{aligned} \langle e(\omega) a, \mu \rangle &= \langle a, i(\varepsilon_j) \mu^j \rangle, \\ \langle e(\varepsilon_i) \lambda^i, b \rangle &= \langle \lambda, i(\omega) b \rangle. \end{aligned}$$

(Of course the degrees of the forms are taken in the necessary manner).

The following calculation holds now

$$\begin{aligned} \langle D(\lambda, a), (\mu, b) \rangle &= \langle D\lambda, \mu \rangle - \langle e(\omega) a, \mu \rangle + \langle da, b \rangle + \\ &+ k \langle e(\varepsilon_i) \lambda^i, b \rangle = \langle \lambda, \bar{D}\mu \rangle - \langle a, i(\varepsilon_j) \mu^j \rangle = \langle a, \delta b \rangle + \\ &+ k \langle \lambda, i(\omega) b \rangle = \langle (\lambda, a), \bar{D}(\mu, b) \rangle, \end{aligned}$$

where

$$(43) \quad \bar{D}(\mu, b) = (\bar{D}\mu + ki(\omega) b, \delta b - i(\varepsilon_j) \mu^j)$$

is the adjoint operator of D .

Further we shall consider according to the known scheme the self adjoint Laplace operator on L^p

$$(44) \quad \square = D\bar{D} + \bar{D}D,$$

and the pairs in $Ker \square$ will be called *harmonic pairs*.

We get

$$(45) \quad \begin{aligned} \square(\lambda, a) &= (\square \lambda + kDi(\omega) a - \bar{D}e(\omega) a - e(\omega) \delta a + ki(\omega) da + \\ &+ e(\omega) i(\varepsilon_j) \lambda^j + k^2 i(\omega) e(\varepsilon_j) \lambda^j, \Delta a - di(\varepsilon_j) \lambda^j + \\ &+ k\delta e(\varepsilon_j) \lambda^j + ke(\varepsilon_i) \bar{D} \lambda^i - i(\varepsilon_i) D \lambda^i + k^2 e(\varepsilon_j) i(dx^j) a + i(\varepsilon_j) e(dx^j) a), \end{aligned}$$

where $\square \lambda$ is given by (44) applied to vector-forms.

The formulas (40) and (45) show that \square is a strongly elliptic operator as considered for instance in [1]. Hence, by classical theorems [1] we have the decomposition in direct sum

$$(46) \quad L^p = Ker \square \oplus Im \square,$$

which gives the decomposition

$$(47) \quad L^p = Ker \square \oplus Im D \oplus Im \bar{D}$$

the three terms being mutually orthogonal. Moreover, we have

$$(48) \quad \text{Ker } \square = \text{Ker } D \cap \text{Ker } \bar{D}$$

and it is a real linear space of a finite dimension.

If we note by \square^p the action of \square on L^p we see from (47) and (8) that there are isomorphisms

$$(49) \quad H^p(V_n, L) \simeq \text{Ker } \square^p \quad (p = 0, 1, \dots, n),$$

hence

$$\dim H^p(V_n, L) = l_p(V_n)$$

are finite, non negative, integers. They will be called the *curvature numbers* of V_n . The sum

$$(50) \quad \chi(V_n, L) = \sum_{p=0}^n (-1)^p l_p$$

will be the *curvature characteristic* of V_n .

The problem of effective calculation of the numbers l_p seems to be difficult. We shall remark that this is the problem of finding the number of independent solutions of the equation

$$(51) \quad \square(\lambda, a) = 0,$$

or of the equivalent system

$$(52) \quad D\lambda - \omega \wedge a = 0, \quad da + k\varepsilon_i \wedge \lambda^i = 0, \quad \bar{D}\lambda + ki(\omega)a = 0, \\ \delta a - i(\varepsilon_j)\lambda^j = 0.$$

Another equation equivalent to (51), is

$$(53) \quad \langle \square(\lambda, a), (\lambda, a) \rangle = 0,$$

which, after some calculations are done, gives

$$(54) \quad \langle \square \lambda, \lambda \rangle + \langle i(\varepsilon_j)\lambda^j, i(\varepsilon_j)\lambda^j \rangle + k^2 \langle e(\varepsilon_j)\lambda^j, \\ e(\varepsilon_j)\lambda^j \rangle + \langle \Delta a, a \rangle + (pk^2 + n - p) \langle a, a \rangle + 2 \langle a, k \delta e(\varepsilon_j)\lambda^j + \\ + ke(\varepsilon_j)\bar{D}\lambda^j - di(\varepsilon_j)\lambda^j - i(\varepsilon_j)D\lambda^j \rangle = 0.$$

From (52), we get that if $k = 0$ and $p = n$, a pair $(0, a)$ is harmonic if and only if a is so; hence $l_n \geq 1$. In the same case $p = n$, even if $k \neq 0$,

there are no non vanishing harmonic pairs of the form $(\lambda, 0)$. In the general case, $\lambda = 0$ for a pair implies $a = 0$. We still remark that the last term in (54) does not allow us to apply the methods used for instance in [4].

Finally, we make another remark which holds good for the general case of a pseudo-Riemannian manifold of constant curvature.

In [3] it is given a relation of *F-Relatedness* between the vector forms of two manifolds linked by an application $F: X \rightarrow Y$. From the definition given there it follows that, if F is a covering mapping, to every vector p -form on Y there corresponds a form on X which is F -related to the given one. Now, if X, Y are pseudo-Riemannian spaces of constant curvature and F a covering, locally isomorphic mapping, it is obvious that we get, in the indicated manner, monomorphisms

$$(56) \quad H^i(Y, L) \rightarrow H^i(X, L).$$

If X, Y are Riemannian, compact, orientable spaces, this means

$$(57) \quad l_n(Y) \leq l_n(X).$$

The same relations for Betti numbers are classical [4].

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