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Exceptional boundary points for the nondivergence equation which are regular for the Laplace equation - and vice-versa

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EXCEPTIONAL BOUNDARY POINTS FOR THE NONDIVERGENCE EQUATION WHICH ARE REGULAR FOR THE LAPLACE EQUATION - AND VICE-VERSA

by KEITH MILLER (*) (Berkeley)

In [9] Littman, Stampacchia, and Weinberger investigated the notion of regular boundary points for Dirichlet's problem with respect to the uniformly elliptic equation in divergence form

$$Mu = \sum_{i,j=1}^{n} (a_{ij}(x)\, u_{x_i})_{x_j} = 0 \tag{1}$$

when the coefficients are only supposed to be measurable. The coefficient matrix is assumed symmetric and the uniform ellipticity condition may be stated

$$\text{eigenvalues of } (a_{ij}(x)) \in [\alpha, 1], \tag{2}$$

where α is the ellipticity constant, $0 < \alpha < 1$. They proved that the regular points for equation (1) are the same as those for Laplace's equation.

In this paper we prove that the analogous result is false for the uniformly elliptic equation in nondivergence form

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x)\, u_{x_i x_j} = 0 \tag{3}$$

with the same class of coefficients. In fact, it is false for every $\alpha < 1$ when $n = 3$.

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Our method involves consideration of spines of varying degrees of sharpness (see Figure 1) such as the exponentially sharp spine of Lebesgue [4, p. 303]. In Theorem 5 of Section 5 we show that for $n = 3$ and for every $\alpha < 1$ there exists a certain nondivergence equation for which the tip of every algebraic spine is exceptional, although such points are regular for Laplace's equation. The coefficients for this equation, incidentally, are C^∞ in the interior and become discontinuous only on the axis of the exterior spine. In Theorem 6 we take care of the « vice-versa » portion of the title; we show that for $n = 3$ and for every $\alpha < 1$ there exists a certain nondivergence equation for which the tip of every spine (no matter how sharp) is regular, although the tip of the exponential spine of Lebesgue is exceptional for Laplace's equation. Again the coefficients are C^∞ except on the axis of the exterior spine.

Only in the case $n = 3$ do we have examples of « both ways nonequivalence » for α arbitrarily close to 1. However, in Section 3 we will show that for every $n \geq 2$, if $\alpha < \frac{1}{n-1}$, then an isolated boundary point is regular for a certain nondivergence equation, although it is of course exceptional for Laplace's equation. The coefficients involved are discontinuous only at the isolated point and analytic elsewhere in R^n.

There has been great interest in the question of regular boundary points for elliptic equations. Several authors, even before the paper of Littman, Stampacchia, and Weinberger, had established the result that a boundary point is regular for the equations (1) and (3) if and only if it is regular for the Laplace equation; all of these proofs however use smoothness of the coefficients in an essential way. For example, R. M. Hervé [6] proved this result for the nondivergence equation (3) (also with lower order terms included) provided that the coefficients are locally Lipschitz continuous. For other references to previous results with smooth coefficients and to classical results for Laplace's equation see the introduction to [9].

On the positive side for the nondivergence equation (3), the present author has recently shown [10] that every boundary point with an exterior cone (no matter how small the aperature) is regular for (3) (also with lower order terms included). Notice that boundary points with only slightly worse geometry, i. e. with exterior algebraic spines (no matter how blunt), are exceptional for certain nondivergence equations, $n \geq 3$, as is shown in the present paper. In Section 5 we mention several other results which are immediately explained by our construction, for example the results for Laplace's equation that exponential spines are exceptional when $n = 3$ (Lebesgue's example) and that algebraic spines are exceptional when $n > 3$ (which could be computed from Wiener's criterion).

Our construction of a nonsolvable Dirichlet problem in Section 5 follows the method of Lebesgue's classical example and requires that a bounded discontinuity of a solution at a boundary point be removeable. This is always possible only if $\alpha \geq \frac{1}{n-1}$; otherwise examples show that it may be false. When $\alpha \geq \frac{1}{n-1}$, we show in Theorem 2*a* that a singularity at an isolated point or a boundary point is removeable provided that a certain growth condition is satisfied at the point. Pucci [13, p. 157] has previously derived equivalent theorems; Gilbarg and Serrin [5] have used similar techniques, but are concerned with equations with continuous coefficients. When $\alpha < \frac{1}{n-1}$, we show in Theorem 2*b* that an isolated singularity is removeable provided that a certain order Hölder continuity holds at the point. This result has no relevance to our main topic, but it follows immediately and it appears to be new.

1. Preliminaries.

Let $x = (x_1, \dots, x_n)$ denote a real vector in R^n, $n \geq 2$, and u a real valued function defined (and usually of class C^2) in some bounded open set Ω of interest. We let $\mathcal{L}_\alpha$ denote the class of operators L of form (2), (3). We constantly use the weak maximum principle: that is, if $u \in C^0(\bar{\Omega}) \cap C^2(\Omega)$, $Lu \geq 0$ in Ω, $u \leq 0$ on $\partial\Omega$, then $u \leq 0$ on $\bar{\Omega}$.

Given an $L \in \mathcal{L}_\alpha$ and a continuous function φ on $\partial\Omega$, the Dirichlet problem is to find a solution u of

$$
\begin{aligned}
Lu &= 0 \quad \text{in} \quad \Omega, \\
u &= \varphi \quad \text{on} \quad \partial\Omega, \qquad (4) \\
u &\in C^0(\bar{\Omega}) \cap \mathcal{H}(\Omega),
\end{aligned}
$$

where $\mathcal{H}(\Omega)$ is some function space with generalized second derivatives such that the maximum principle still holds. Certainly with only measureable coefficients we cannot expect a $C^2(\Omega)$ solution. It is not yet clear what should be our choice of $\mathcal{H}(\Omega)$, although recent results of Aleksandrov [1] and Pucci [14] indicate that the Sobolev space $\mathcal{H}(\Omega) = \mathcal{H}^{2,n}(\Omega)$ is a logical choice.

DEFINITION. *A barrier for L at the point $x^0 \in \partial\Omega$ is* a function w, defined in some relative neighborhood $N = U \cap \Omega$ (U an open neighborhood of x^0), which satisfies

$$w \in \mathcal{H}(N) \cap C^0(\bar{N}),$$

$$w(x^0) = 0,\ w > 0 \text{ on } \bar{N} - \{x^0\},$$

$$Lw \leq 0 \text{ in } N.$$

DEFINITION. x^0 *is a regular boundary point for* L (or for the equation $Lu = 0$) if there exists a barrier for L at x^0.

Otherwise x^0 is an *exceptional point for* L.

In the case of Laplace's equation, solvability of the Dirichlet problem for arbitrary continuous boundary data is equivalent to existence of a barrier at every boundary point. Unfortunately we cannot at present prove the same equivalence for the general nondivergence equation. After all, with the present state of the theory we cannot prove solvability of (4) even on the sphere if the coefficients are not smooth. If we were to find an example (4) with no solution we might have reason to suspect that the difficulty is caused by the nonsmooth coefficients in the interior and not by the pathology of the boundary. However, if the coefficients of L are Hölder continuous on compact subsets of Ω, then we can prove that the equivalence holds, as stated precisely in the following theorem. For this reason we restrict our example to L with smooth (in fact C^∞) coefficients in the interior.

THEOREM 1. *Suppose the coefficients of* L *are Hölder continuous on each compact subset of* Ω. *Any solution of* (4) *must then be in* $C^2(\Omega)$. *There exists a solution of* (4) *for every continuous boundary function* φ *if and only if there exists a barrier for* L *at every boundary point.*

Proof. The proof of the first assertion is trivial. Let u be a solution of (4) and consider any subsphere S with $\bar{S} \subset \Omega$. By the well known existence theory for equations with Hölder continuous coefficients [4, p. 339] there exists a C^2 solution u_1 on S which agrees with u on ∂S. We have required that the maximum principle also hold for the class $\mathcal{H}$; hence, $u = u_1$ on S. See [10, p. 98] for proof of the second assertion.

We now define certain types of spines, in order of increasing sharpness. Let Ω be the body of revolution obtained by rotating (for $n > 2$) the shaded area shown in Figure 1 about the x_n axis. We may assume that $\partial\Omega$ is C^∞ except at 0, the tip of the intruding spine. Let (r, θ) be polar coordinates with the positive x_n axis as polar axis. That is, $r(x) = |x|$ and $\theta(x) = \arccos(x_n / |x|)$. We say that $\partial\Omega$ contains a *conical spine* if there exists a positive constant k such that $(\pi - \theta) \approx k$ as $x \to 0$ on $\partial\Omega$;

$\partial\Omega$ contains an *algebraic spine* if there exist positive constants k and μ such that $(\pi - \theta) \approx kr^{\mu}$ as $x \longrightarrow 0$ on $\partial\Omega$; $\partial\Omega$ contains an *exponential spine* if there exist positive constants k, C, and λ such that $(\pi - \theta) \approx k \exp(-Cr^{-\lambda})$

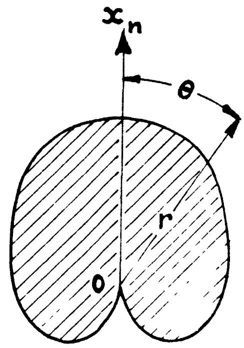

Fig. 1

as $x \longrightarrow 0$ on $\partial\Omega$. As a degenerate case we say that $\partial\Omega$ contains a *line segment spine* if $(\pi - \theta) = 0$ for all x sufficiently near 0 on $\partial\Omega$, for example, $\Omega = \{x : 0 < |x| < 1$ and x is not on the negative x_n axis$\}$.

Finally, we have the case of an *isolated boundary point*, for example, $\Omega = \{x : 0 < |x| < 1\}$. This $\partial\Omega$ may be considered to have a completely degenerate spine; we have made the body of the spine so thin that we have wiped it out completely, leaving only its tip, 0.

2. Removeable singularities.

In this section we very simply and precisely characterize the growth condition, in dependence upon α, which is necessary to insure that a singularity at an interior or boundary point be removeable. What we actually show is that the maximum principle continues to hold as if the singularity were not there. It therefore might be more precise to call such a result an « extended maximum principle », in keeping with the terminology of Gilbarg and Serrin [5], rather than a theorem on « removeable singularities ».

Consider radially symmetric functions of the form $w(x) = g(r)$. The class $\mathcal{L}_\alpha$ is invariant under translation and rotation of the coordinate axes. At each fixed point $x^0 \neq 0$ we choose coordinate axes with origin at x^0 such that the y_1 axis is in the radial direction and the $y_2, \ldots, y_n$ axes are in the tangential directions. With respect to these axes, $w_{y_1 y_1}(x^0) = g_{rr}$, $w_{y_2 y_2}(x^0) = \ldots = w_{y_n y_n}(x^0) = \frac{g_r}{r}$, and the cross derivatives are all zero. Therefore, the operators $L \in \mathcal{L}_\alpha$, applied to radial functions, are exactly of the form

$$Lw = \alpha_1 g_{rr} + (\alpha_2 + \dots \alpha_n) \frac{g_r}{r} \tag{5}$$

where the functions $\alpha_i(x)$ are measureable and satisfy $\alpha \leq \alpha_i(x) \leq 1$. We consider the particular operator L_0 which at each point has coefficient 1 for the second derivative in the radial direction and coefficient α for the second derivatives in the tangential directions. The equation

$$L_0(w) = g_{rr} + \alpha(n-1)\frac{g}{r} = 0 \tag{6}$$

then has solutions $w(x) = 1$ and

$$w(x) = r^{1-\alpha(n-1)} \quad \left(\text{or } -\log r \text{ if } \alpha = \frac{1}{n-1}\right). \tag{7}$$

Thus for $\alpha \geq \dfrac{1}{n-1}$ the solution w in (7) is positive and tends to $+\infty$ at the origin. Moreover, it is a solution of Pucci's maximizing equation [13],

$$M_\alpha(w) \equiv \max_{L \in \mathcal{L}_\alpha} Lw = 0,$$

as is easily seen from the representation (5), so it is a supersolution (i. e., $Lw \leq 0$) for all $L \in \mathcal{L}_\alpha$. Hence it can be used in the standard fashion to remove singularities which grow more slowly than itself.

THEOREM 2a. *Suppose Ω is a bounded open set and $x^0 \in \bar{\Omega}$ (either $x^0 \in \Omega$ or $x^0 \in \partial\Omega$ is allowed). Suppose $u \in C^0(\bar{\Omega} - \{x^0\}) \cap C^2(\Omega - \{x^0\})$, $u \leq 0$ on $\partial\Omega - \{x^0\}$, $Lu \geq 0$ in $\Omega - \{x^0\}$ where $L \in \mathcal{L}_\alpha$ with $\alpha \geq \dfrac{1}{n-1}$, and $u = o(|x - x^0|^{1-\alpha(n-1)})$ as $x \to x^0$ $\left(\text{or } o(|\log|x - x_0||) \text{ if } \alpha = \dfrac{1}{n-1}\right)$. Then $u \leq 0$ on $\bar{\Omega} - \{x^0\}$.*

2b. *Suppose Ω is a bounded open set, $x^0 \in \Omega$, $u \in C^0(\bar{\Omega}) \cap C^2(\Omega - \{x^0\})$, $u \leq 0$ on $\partial\Omega$, $Lu \geq 0$ in $\Omega - \{x^0\}$ where $L \in \mathcal{L}_\alpha$ with $0 < \alpha < \dfrac{1}{n-1}$, and $|u(x) - u(x^0)| = o(|x - x^0|^{1-\alpha(n-1)})$. Then $u \leq 0$ on $\bar{\Omega}$.*

Proof. As mentioned, the proof of (2a) is standard. We consider the proof of (2b). Suppose, for the sake of contradiction, that u has a positive maximum value M on $\bar{\Omega}$. By the strong maximum principle, this maximum can be assumed only at x^0. Let S be a sphere in Ω centered about x^0, of radius R. Thus $u \leq$ some $N < M$ on ∂S and $u = M$ at x^0. Now, the function $-w(x) = -|x - x_0|^{1-\alpha(n-1)}$ is a solution of the maximizing equation, as is seen from the representation (5). Hence, the function $v(x) = M - (M - N)(|x - x^0|/R)^{1-\alpha(n-1)}$ is a supersolution which is $\geq u$ on the boundary of the deleted sphere $S - \{x^0\}$. Thus $u(x) \leq v(x)$ by the maximum principle, which contradicts the hypothesized growth condition on $|u(x) - u(x^0)|$.

3. The isolated boundary point as a regular point.

When $\alpha < \dfrac{1}{n-1}$ the exponent of the solution w in (7) is positive. Hence w itself is a solution with a non-removeable bounded isolated singularity. Moreover, w is a barrier at the origin for the operator L_0; hence the following result:

THEOREM 3. *The origin is an isolated regular boundary point for the operator L_0 described in* (6) *when $\alpha < \dfrac{1}{n-1}$. This operator has coefficients analytic in R^n minus the origin, as claimed in the introduction.*

Incidentally, w is a solution of Pucci's minimizing equation

$$m_\alpha(w) \equiv \min_{L \in \mathcal{L}_\alpha} Lw = 0\,;$$

hence, within the class of equations $Lu = 0$, $L \in \mathcal{L}_\alpha$, w minimizes the solution (when it exists) of the Dirichlet problem on $\{x : 0 < |x| < 1\}$ for boundary values $\varphi = 0$ at 0 and $\varphi \equiv 1$ on $|x| = 1$.

4. The special solutions.

We now develop some particular equations which have solutions of the form $u(x) = r^\lambda f(\theta)$, $0 < \lambda < 1$, where u is a C^2 function except on the negative x_n axis, but where f has a certain asymptotic growth as $\theta \to \pi^-$. Let β_0 be a positive constant and let β be a function of θ defined as follows:

$$\tag{8} \begin{aligned} &\beta(\theta) \equiv 1 \quad \text{on} \quad [0, \pi/4], \\ &\beta(\theta) \equiv \beta_0 \quad \text{on} \quad [\pi/2, \pi], \\ &\beta \text{ is monotone and } C^\infty \text{ on } [0, \pi]. \end{aligned}$$

At each fixed point $x^0 \neq 0$ we find it convenient to choose the following Euclidean coordinate system with origin at x^0: let the positive y_1

axis lie in the direction of increasing r, let the positive y_2 axis lie in the direction of increasing θ, and let the $y_3 \dots y_n$ axes be chosen arbitrarily in directions orthogonal to the y_1, y_2 plane. We then define L_1 to be the operator whose value at each x^0 is given by

$$L_1 u = u_{y_1 y_1} + u_{y_2 y_2} + \beta (u_{y_{33}} + \dots + u_{y_{nn}}). \tag{9}$$

Clearly the coefficients of L_1 with respect to the original axes are C^∞ except on the negative x_n axis, as claimed in the introduction. For functions of the form $r^\lambda f(\theta)$, λ any real number, we have

$$L_1 (r^\lambda f(\theta)) = r^{\lambda-2} \{\lambda(\lambda - 1) f + (f_{\theta\theta} + \lambda f) + \beta (n-2)(\lambda f + f_\theta \cot\theta)\}. \tag{10}$$

We normalize, setting $f(0) = 1$. The condition that $r^\lambda f(\theta)$ be a C^2 solution of $L_1 (r^\lambda f(\theta)) = 0$ for $0 \leq \theta < \pi$, $r \neq 0$, is then that f be a solution of the initial value problem

$$f_{\theta\theta} + (n-2)\beta \cot\theta f_\theta + (\lambda^2 + \beta(n-2)\lambda) f = 0, \tag{11a}$$

$$f \in C^2 [0, \pi), \quad f(0) = 1, \quad f_\theta(0) = 0. \tag{11b}$$

The final condition insures that $r^\lambda f(\theta)$ is C^2 even on the positive x_n axis.

For our purposes we will always use

$$\beta_0 = \frac{1}{\alpha}, \ 1, \ \text{or } \alpha. \tag{12}$$

In the first case αL is in $\mathcal{L}_\alpha$; in the second L is the Laplacian, and in the third L is in $\mathcal{L}_\alpha$. Actually, the development (8)-(12) is just a specialization of a general representation for $L(r^\lambda f(\theta))$, $L \in \mathcal{L}_\alpha$, to be found in [10], but we have thought it more enlightening to include the complete description of L_1 here.

Now comes the crucial observation concerning the problable behavior of the solution of (11) as $\theta \to \pi^-$. If we look, as in the method of Frobenious [3, pp. 132-135], for solutions of (11a) of the form $(\pi - \theta)^\mu g(\pi - \theta)$, where μ is a constant and g is regular and non zero at the origin, then the third term in (11a), because it has no factor of $(\pi - \theta)^{-2}$, does not enter into the indicial equation for μ. We obtain roots $\mu_1 = 0$ and $\mu_2 = 1 - (n-2)\beta_0$, independently of λ. (In case $\mu_2 = 0$ we expect a solution of the form $\log(\pi - \theta)\, g(\pi - \theta)$). We therefore suspect that the solution of (11) will have the following asymptotic behavior as $(\pi - \theta) \to 0^+$: negative power growth for $(n-2)\beta_0 > 1$, logarithmic growth for $(n-2)\beta_0 = 1$, and bounded growth for $(n-2)\beta_0 < 1$. The following theorem confirms this expectation. We delay its proof to Section 6.

THEOREM 4.

(a) *For every real* λ *and positive* β_0 *there exists a unique solution (call it* F*) of problem* (11).

(b) *If* $1 \leq (n-2)\beta_0$ *and* $0 < \lambda < 1$ *then the solution* F *has exactly one zero (which occurs in* $(\pi/2, \pi)$*), is monotone decreasing on* $[0, \pi)$*, and is asymptotic to a negative constant times* $(\pi - \theta)^{1-(n-2)\beta_0}$ *(or* $-\log(\pi - \theta)$ *when* $(n-2)\beta_0 = 1$*) as* $\theta \to \pi^-$.

(c) *If* $(n-2)\beta_0 < 1$ *then for all real* λ *the solution* F *is continuous on all* $[0, \pi]$*; moreover, for all sufficiently small* $|\lambda|$*,* F *remains positive on* $[0, \pi]$.

5. Exceptional points and vice-versa.

Lebesque mentions in [8, p. 353] that the tip of every algebraic spine is regular for Laplace's equation when $n = 3$. His proof seems mistaken; he introduced the barrier $\sqrt{x_1^N + x_2^N + x_3}$, perhaps under the mistaken notion that its zero level surface $x_3 = -(x_1^N + x_2^N)$ forms an arbitrarily sharp algebraic spine as $N \to \infty$. Actually one can show that no barrier w for Laplace's equation at the tip of an algebraic spine can be Hölder continuous; for, suppose it were Hölder continuous of order $\lambda, \lambda > 0$. Then let $0 < \lambda_1 < \lambda$. The function of the form $u = r^{\lambda_1} f(\theta)$ which is harmonic on $0 \leq \theta < \pi, f(0) = 1, f_\theta(0) = 0$, has a zero $\theta_1 < \pi$, as is seen in theorem 4*b*. Since the spine is sharper than any cone, we could dominate u by a multiple of w near the vertex on the cone $0 \leq \theta \leq \theta_1$, and conclude that u must be Hölder continuous of order λ, a contradiction.

However, Lebesgue's result still holds. It appears as a problem in Kellog [7, p. 334]. The author has checked it out using Weiner's criterion. One can underestimate the capacity of a cylinder of length h and radius $\varrho < \frac{1}{2}$ by a constant times $h/|\log \varrho|$. One then can underestimate the capacity γ_i of the ith segment of the exterior spine (see Kellog's notation) in terms of the capacity of an enclosed cylinder, thus establishing that the tip of any algebraic spine is regular.

We now establish that the tip of every algebraic spine is exceptional for the operator L_1 when we let $\beta_0 = \frac{1}{\alpha}$, with α any constant satisfying $\frac{1}{2} \leq \alpha < 1$, and $n = 3$. Then for every $\lambda, 0 < \lambda < 1$, the solution $u(x) = r^\lambda F(\theta)$ described in Theorem 4*b* is asymptotic to $-cr^\lambda (\pi - \theta)^{1-\frac{1}{\alpha}}$, c a positive constant. Let Ω be a domain whose boundary contains the algebraic spine $(\pi - \theta) \approx r^{\frac{\lambda\alpha}{1-\alpha}}$. Thus $u(x) \to -c$ as $x \to 0$ on $\partial\Omega$; but $u(x) \to 0$ as $x \to 0$ on straight line rays. Hence u restricted to $\overline{\Omega}$ has continuous boundary values and u assumes these boundary values continuously except for a bounded discontinuity at 0. However, we can show that no solution v of the same equation can assume these boundary values continuously; for if so than $u - v$ would satisfy $L_1(u - v) = 0$ in Ω, $u - v \in C^2(\Omega) \cap C^0(\overline{\Omega} - \{0\})$, and $(u - v) = 0$ on $\partial\Omega - \{0\}$ with at most a bounded discontinuity at 0. Since L_1 has ellipticity constant $\alpha \geq \frac{1}{2}$, we may infer by Theorem 2a that this bounded singularity is removeable, and $(u - v) \equiv 0$ on $\overline{\Omega} - \{0\}$. Therefore, the Dirichlet problem for these boundary values on $\partial\Omega$ is nonsolvable. All points of $\partial\Omega$ other than 0 are regular (there exists an exterior sphere at each of these points). This, with Theorem 1, implies that 0 is not a regular point for this domain and this equation.

Of course 0 is also an exceptional point for this equation for any domain with a sharper spine. Letting $\lambda \to 0^+$ we see that the tip of any algebraic spine, no matter how blunt, is also exceptional.

THEOREM 5. *Let $n = 3$ and $\frac{1}{2} \leq \alpha < 1$. Let L_1 be the operator defined by (9) with $\beta_0 = \frac{1}{\alpha}$. Then L_1 has C^∞ coefficients on R^n — {the closed negative x_3 axis}, $\alpha L_1 \in \mathcal{L}_\alpha$, and 0 is an exceptional boundary point for the equation $L_1 u = 0$ on any domain Ω whose boundary contains an algebraic spine with axis on the negative x_3 axis. However, 0 is a regular boundary point for Laplace's equation on such domains.*

Similarly, using Theorem 4b, we see that the tip of every exponential spine is exceptional for Laplace's equation when $n = 3$.

If we instead let $\beta_0 = \alpha$, $0 < \alpha < 1$, then Theorem 4c tells us that the solution F of (11) stays positive on $[0, \pi]$ for all sufficiently small λ. Thus $r^\lambda F(\theta)$ is a barrier at the origin for L_1 when $\partial\Omega$ contains a line segment spine. Actually, a much more straighforward construction of such a barrier is given in Lemma 7; the barrier there constructed is then used in Lemma 8 to prove Theorem 4c.

THEOREM 6. *Let $n = 3$, $0 < \alpha < 1$, and let L_1 be the operator defined by (9) with $\beta_0 = \alpha$. Then $L_1 \in \mathcal{L}_\alpha$ and 0 is a regular boundary point for L_1 on the domain Ω = {the unit ball minus the closed negative x_3 axis}. However, 0 is an exceptional boundary point for Laplace's equation on this domain.*

The two theorems above were stated only in the case $n = 3$ because only in this dimension do we obtain examples of « both ways nonequivalence of regular boundary points » for α arbitrarily close to 1. However, the asymptotic growth of F revealed in Theorem 4, depending as it does only on $(n-2)\beta_0$, also gives information for higher dimensions. When $n \geq 4$, $\frac{1}{n-1} < \alpha \leq 1$, and $\beta_0 = \frac{1}{\alpha}$, then the tip of every algebraic spine is exceptional for the operator L_1. (Since this includes the case $\alpha = 1$, for which L_1 equals the Laplacian, we have here no example of nonequivalence) When $n \geq 4$, $\alpha < \frac{1}{n-2}$, and $\beta_0 = \alpha$ then the tip of the line segment spine is regular for L_1.

In summary, when $n = 3$ Theorems 5 and 6 give examples of « exceptional points and vice-versa » for α arbitrarily close to 1. When $n = 2$, Theorem 3 gives an example of « vice-versa » for α arbitrarily close to 1. When $n \geq 4$, the paragraph above and Theorem 3 give examples of « vice-versa » for α sufficiently distant from 1.

6. Proof of Theorem 4.

LEMMA 1. *For every real λ and positive $(n-2)\beta_0$ there exists a unique solution (call it F) of the initial value problem* (11).

Proof: We need only require that β be a positive and continuous function on $[0, \pi]$. Existence is then a simple case of a theorem of the author's [11] which states that there exists a unique solution of the problem

$$g''(t) = G(t, p(t)\, g, t^{-1} g'), \text{ for } 0 < t < h, \tag{13a}$$

$$g(0) = 1, g'(0) = 0, g \in C^2[0, h), \tag{13b}$$

provided that G is continuous in its three variables, Lipschitz continuous in its second and third variables, and monotone nonincreasing in its third variable, and that the function p is continuous for $t > 0$ and $o(t^{-2})$ as $t \to 0^+$.

LEMMA 2a. *There exists a solution H of* (11*a*) *on* $(0, \pi)$ *satisfying* $H \in C^2$ $(0, \pi]$, $H_\theta(\pi) = 0$, $H(\pi) = 1$.

2*b*. *There exists another solution K of* (11*a*) *on* $(0, \pi)$ *which is asymptotic as $\theta \to \pi^-$ to $(\pi - \theta)^{1-(n-2)\beta_0}$ if $1 - (n-2)\beta_0 > 0$, to $\log(\pi - \theta)$ if $1 - (n-2)\beta_0 = 0$, and to $-(\pi - \theta)^{1-(n-2)\beta_0}$ if $1 - (n-2)\beta_0 < 0$.*

2c. *F, the solution of the initial value problem* (11), *is a linear combination of H and K, $F = aH + bK$.*

Proof: Letting $t = \pi - \theta$, we see that Lemma 2 a also follows immediately from the existence theory for problem (13).

Now consider Lemma 2b. Letting $f(t) = g(t)\, t^{1-(n-2)\beta_0}$ (or $g(t) \log t$ if $1-(n-2)\beta_0 = 0$) we see that f satisfies (11a) iffi g satisfies a certain equation of the form

$$g'' + \left(\frac{2-(n-2)\beta_0 + 0(t^2)}{t}\right) g' + o(t^2)\, g = 0. \tag{14}$$

Thus, provided $(n-2)\beta_0 < 2$, (14) is exactly in the form (13a) and there exists a solution with $g'(0) = 0$, $g(0) = 1$.

The above method suffices for the example used in Theorem 5, for there $n = 3$, $\beta_0 = \frac{1}{\alpha}$, and $\frac{1}{2} < \alpha < 1$. We have included it for its simplicity. However, for the general theory with $(n-2)\beta_0$ an arbitrary positive constant we shall have to go to the method of Frobenius and use the analyticity of equation (11a) at $\theta = \pi$.

We try for solutions of the form

$$f(t) = t^{\mu} \sum_0^{\infty} c_j t^j, \qquad c_0 = 1,$$

and get $\mu_1 = 0$ and $\mu_2 = 1 - (n-2)\beta_0$ as solutions of the indicial equation. We summarize some results found in [3, p. 133].

If μ_2 is not an integer then there exist two independent solutions

$$H(t) = \sum_0^{\infty} b_j t^j,\ b_0 = 1,$$

$$K(t) = t^{\mu_2} \sum_0^{\infty} c_j t^j,\ c_0 = 1.$$

If $\mu_2 = 0$, then there exist two independent solutions, $H(t)$ as above, and

$$K(t) = (\log t)\, H(t) + \sum_0^{\infty} c_j t^j.$$

If μ_2 is an integer $-k$ (necessarily negative since β_0 is positive) then there exist two independent solutions, $H(t)$ as above and

$$K(t) = \sum_{-k}^{\infty} c_j t^j,\ c_{-k} = 1,$$

but where the coefficients c_0, c_1, ... are not necessarily constants but may include a factor of log t. In any case we see that Lemma 2b is valid. Finally, H and K are independent, so every solution of (11a), F included, is a linear combination of the two.

LEMMA 3. *For $\lambda > 0$, the solution F must start out strictly decreasing and continue strictly decreasing so long as F remains positive.*

Proof: The expansion $F_\theta(\theta) = \theta F_{\theta\theta}(0) + o(\theta)$ inserted in (11a) yields $F_{\theta\theta}(0) < 0$. Hence F is initially strictly decreasing. However, were F to cease sirictly decreasing while still positive it would either have an interval of constancy (clearly impossible) or a positive relative minimum within $(0, \pi)$ (also clearly impossible since $\lambda^2 + (n-2)\beta\lambda$ is positive and $F_{\theta\theta} \geq 0$ at a minimum). This completes the proof of the lemma.

To prove Theorem 4b it is sufficient to establish that the coefficient b, where $F = aH + bK$, is positive. However, for certain values of λ (eigenvalues) b is zero. For example, $\lambda = 1$ is an eigenvalue for every function β since the linear function $x_n = r^1 \cos\theta$ is a solution of every elliptic equation under consideration. However, by comparison of F with cos θ we will establish that there are no eigenvalues satisfying $0 < \lambda < 1$.

LEMMA 4. *When $0 < \lambda < 1$ no solution f of* (11a) *can have both a zero θ_2 of f and an earlier zero θ_1 of f_θ in $[0, \pi/2]$. (By symmetry the same applies in $[\pi/2, \pi]$ with $\theta_2 < \theta_1$).*

Proof: Notice that f satisfies equation (11a) with coefficient $\lambda^2 + \beta(n-2)\lambda$ for the zero order term, but cosine satisfies the same equation with a larger coefficient $1^2 + \beta(n-2)\,1$. A change of the independent variable puts the equation in self adjoint form and one then introduces the Prüfer substitution. Now at θ_1 the Prüfer phase functions $\omega(\theta)$ and $\bar{\omega}(\theta)$, of the solutions f and cosine respectively, satisfy $\bar{\omega}(\theta_1) \geq \pi/2 = \omega(\theta_1)$. Hence, the proof of the Sturm Comparison Theorem [2, p. 259] establishes that $\bar{\omega}(\theta) > \omega(\theta)$ for all $\theta > \theta_1$; thus, f cannot have a zero (i.e., ω cannot reach π) until strictly after the first zero of cosine.

An alternate proof would involve comparing the solution $r^\lambda f(\theta)$ (notice $L_1(r^\lambda f(\theta)) = 0$) with the super solution $r^\lambda \cos\theta$ (notice $L_1(r^\lambda \cos\theta) < 0$) on the region $\{x : \theta_1 < \theta(x) < \theta_2\}$ after first using the linear solution $r^1 \cos\theta = x_n$ as a « barrier at infinity » to get a Phragmen-Lindelöf theorem (see proof of Lemma 8 for example) for all regions contained in the upper half space $x_n > 0$.

LEMMA 5. *For $0 < \lambda < 1$, the solution F must be strictly decreasing on all $[0, \pi)$.*

Proof: We have seen, Lemma 3, that F continues strictly decreasing so long as it stays positive. Now F cannot have a zero relative minimum in $(0, \pi)$, for then we would have $F' = F = 0$ at an interior point, and the unique solution of this initial value problem would be $F \equiv 0$. Also, F cannot have a negative relative minimum in $(0, \pi)$, for then we would have $\pi/2 < \theta_2 < \theta_1 < \pi$ with $F(\theta_2) = 0$ and $F_\theta(\theta_1) = 0$, which contradicts Lemma 4 applied to the second quadrant.

LEMMA 6. *For $0 < \lambda < 1$, the solution F must be of the form $aH + bK$ with $b > 0$.*

Proof: We show first that $b \neq 0$. Suppose, for the sake of contradiction, that $F = aH$. Certainly $a \neq 0$. If $a > 0$, then by Lemma 5 F has stayed positive and strictly decreasing on $[0, \pi)$; but applying the same Lemma 5 to H in the opposite direction yelds that H must be a strictly decreasing function of $(\pi - \theta)$, a contradiction. If $a < 0$ then the graph of F would cross the axis (in the second quadrant by Lemma 4) and F would have both a zero and a later zero of its derivative (at $\theta = \pi$) in the second quadrant, contradicting Lemma 4.

Finally, since $b \neq 0$, F is strictly decreasing, and $K(\theta) \to -\infty$ as $\theta \to \pi^-$ (or $K_\theta(\theta) \to \infty$ in case $(n-2)\beta_0 < 1$), it is clear that $b > 0$.

This completes the proof of Theorem 4*b*. We continue to the proof of Theorem 4*c* by first constructing a barrier at the origin on the domain $\Omega = R^n -$ $\{$negative x_n axis$\}$ for the operator L_1.

LEMMA 7. *Let $(n-2)\beta_0 < 1$. There exists a function f and numbers $\eta, \mu > 0$ such that*

$$f \in C^2[0, \pi] \cap C^0[0, \pi], f_\theta(0) = 0, f(0) = 1,$$

$$f > 0 \text{ on } [0, \pi], \text{ and} \tag{13}$$

$$L_1(r^\lambda f(\theta)) \leq -\eta r^{\lambda-2} \text{ on } r > 0, 0 \leq \theta < \pi,$$

for all $|\lambda| \leq \mu$.

Proof: Let $\bar\beta$ be a constant satisfying $(n-2)\beta_0 < (n-2)\bar\beta < 1$. Then let

$$f(\theta) = 1 + (\pi - \theta)^{1-(n-2)\bar\beta} \text{ on } [\pi/2, \pi] \tag{14}$$

and extend it to $[0, \pi/2]$ in such a way that

$$
(15) \qquad \begin{aligned}
&f \in C^2[0,\pi), \qquad f_\theta(0) = 0,\\
&f_\theta \leq 0 \text{ on } [0,\pi/2), \text{ and}\\
&f_{\theta\theta} < 0 \text{ on } [0,\pi/2).
\end{aligned}
$$

Normalization later will take care of the $f(0) = 1$ requirement. We denote $l_\lambda f(\theta) \equiv L_1(r^\lambda f(\theta))/r^{\lambda-2}$ and consider the case $\lambda = 0$. On $[\pi/2, \pi)$ we have

$$
\begin{aligned}
l_0(f) &\equiv f_{\theta\theta} + (n-2)\beta \cot\theta f_\theta\\
&= -(n-2)(1-(n-2)\bar\beta)(\pi-\theta)^{-(n-2)\bar\beta-1}(\bar\beta + \beta(\pi-\theta)\cot\theta)
\end{aligned}
$$

which is uniformly negative on $[\pi/2, \pi]$ since $(\pi-\theta)\cot\theta \geq -1$ and $\bar\beta > \beta \equiv \beta_0$ there. Likewise on $[0, \pi/2)$ we have $l_0(f) < 0$ uniformly. Hence there exists an $\eta > 0$ such that

$$
l_0(f) < -2\eta \text{ on } [0,\pi).
$$

But, since this f is bounded and λ multiplies only f in l_λ, see (10), $l_\lambda(f(\theta))$ depends continuously on λ, uniformly on $[0,\pi)$; hence $l_\lambda(f(\theta)) < -\eta$ on $[0,\pi)$ for all $|\lambda| \leq$ some positive μ.

LEMMA 8. *For all λ satisfying $0 < \lambda < \mu$, the solution F of Theorem 4 is $\geq f$ on $[0,\pi]$, where f and μ are defined in Lemma 7.*

Proof. We first establish the following Phragmen-Lindelöf type result for the operator L_1: if Ω is an open set contained in the open cone $T_\pi = \{x : x$ is not on the closed negative x_n axis$\}$, $w \in C^2(\Omega) \cap C^0(\bar\Omega)$, $L_1 w \geq 0$ in Ω, $w \leq 0$ on $\partial\Omega$, and $w(x) = o(r^\mu)$ as $x \to \infty$ in Ω, then $w \leq 0$ on $\bar\Omega$. Our proof uses the positive supersolution $r^\mu f(\theta)$ as a « barrier at infinity ». By the growth condition on w, for every $\varepsilon > 0$ there exists an R sufficiently large that $v_\varepsilon \equiv w - \varepsilon r^\mu f(\theta)$ is ≤ 0 on the boundary of $\Omega_R \equiv \Omega \cap \{|x| < R\}$. Hence $v_\varepsilon \leq 0$ on $\bar\Omega_R$ by the maximum principle for bounded open sets. This, with the arbitrariness of ε, establishes the desired result. More general Phragmen-Lindelöf theorems on cones may be found in [12].

We know that both f and F are continuous on $[0,\pi]$ and equal 1 at 0, that $r^\lambda f(\theta)$ is a supersolution for L_1 on Ω, and that $r^\lambda F(\theta)$ is a solution. Suppose now for the sake of contradiction that there exists $\psi \in (0,\pi]$ where $F(\psi) = kf(\psi)$ with $0 < k < 1$. Application of the above Phragmen-Lindelöf result to the subsolution $w = r^\lambda F(\theta) - r^\lambda kf(\theta)$ on the open cone $\Omega = \{x : x \neq 0$ and $0 \leq \theta(x) < \psi\}$ would then imply that $F(\theta) \leq kf(\theta)$ for $0 \leq \theta \leq \psi$, yielding a contradiction when $\theta = 0$. This completes the proof of the lemma and of Theorem 4c.

REFERENCES

[1] ALEKSANDROV, A. D. *Uniqueness conditions and bounds for the solution of the Dirichlet problem* (in Russian), Vestnich Leningrad. Univ. Ser. Mat. Mech. Astron., 18 (1963) no. 3, pp. 5-29.

[2] BIRKHOFF, G., and ROTA, G., *Ordinary differential equations*, Ginn and Co., Boston (1962), p. 259.

[3] CODDINGTON, E., and LEVINSON, N., *Theory of ordinary differential equations*, McGraw-Hill, New York (1955), see pp. 132-135.

[4] COURANT, R., and HILBERT, D., *Methods of mathematical physics*, Vol. 2, Interscience, New York (1961), see p. 303 and p. 339.

[5] GILBARG, D., and SERRIN, J., *On isolated singularities of solutions of second order elliptic differential equations*, J. d'Analyse Math., Vol. 4 (1956), pp. 309-340.

[6] HERVÉ, R. M., *Recherches axiomatique sur la théorie des fonctions surharmoniques et du potentiel*, Ann. Inst. Fourier, Grenoble 12 (1962), pp. 415-571.

[7] KELLOG, O. D., *Foundations of potential theory*, Dover, New York (1953), see p. 334 and p. 330.

[8] LEBESGUE, H., *Conditions de regularitè, conditions d'irrégularité, conditions d'impossibilitè dans le problème de Dirichlet*, Comptes Rendus, Vol. 178 (1924), pp. 352-354.

[9] LITTMAN, W., STAMPACCHIA, G., and WEINBERGER, H. F., *Regular points for elliptic equations with discontinuous coefficients*, Ann. Scuola Norm. Sup. di Pisa, XVII (1963) pp. 45-79.

[10] MILLER, K., *Barriers on cones for uniformly elliptic operators*, Ann. Mat. Pura Appl., LXXVI (1967), pp. 93-105.

[11] MILLER. K., *Existence theory for certain ordinary differential equations with a monotone singularity*, to appear in Proc. Amer. Math. Soc.

[12] MILLER, K., *Extremal barriers on cones with Phragmen-Lindelöf theorems and other applications*, (to appear).

[13] PUCCI, C., *Operatori ellittici estremanti*, Ann. Mat. Pura Appl., Vol. 72 (1966), pp. 141-170.

[14] PUCCI, C., *Limitazioni per soluzioni di equazioni ellittiche*, Ann. Mat. Pura Appl., Vol. 74 (1966), pp. 15-30.