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THE GENERALIZED WEIERSTRASS-TYPE INTEGRAL $\int f(\zeta, \varphi)$

by GARTH WARNER

Introduction.

In a recent paper [5] Cesari showed [5, p. 111] that the Weierstrass-type integrals $\int f(\mathcal{V}, \varphi)$ relative to a mapping (variety) $\mathcal{V}: A \longrightarrow E^m$ and a set function φ which is quasi-additive and of bounded variation with respect to a mesh function δ can be obtained by a standard process of limit as $\delta(D) \longrightarrow 0$. Indeed, Cesari showed under general assumptions on V and f that the set function $\Phi = f(\mathcal{V}, \varphi)$ is again quasi-additive and of bounded variation with respect to the same mesh function δ, and hence $\int f(\mathcal{V}, \varphi)$ can be defined as a Burkill-Cesari integral $\int f(\mathcal{V}, \varphi) = \int \Phi$. In a second paper [6] Cesari proved, under a convenient system of axioms, that $\int f(\mathcal{V}, \varphi)$ can be represented as a Lebesgue-Stieltjes integral, i.e. $\int f(\mathcal{V}, \varphi) = (A)^{*} \int f(\mathcal{V}, \beta)\, d\mu$ in a certain measure space $(A, \mathcal{B}, \mu)$.

The main purpose of this paper is to continue the axiomatic approach of Cesari's and to present in an abstract and general setting some of the main properties of the generalized Weierstrass-type integrals $\int f(\mathcal{V}, \varphi)$. In an earlier paper [12], we carried out this program for the general Burkill-Cesari integral. For the sake of continuity, we adopt the notations and conventions introduced there.

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In more detail, in this paper generalized Weierstrass-type integrals $\int f(\zeta, \varphi)$ where now both ζ and φ are set functions, and φ is quasi-additive and of bounded variation will be investigated. The function f is assumed to satisfy convenient hypotheses of continuity but f is not assumed to be real valued. In fact f and ζ take on values in general locally convex topological vector spaces while the range of φ is assumed to be a uniformly convex Banach space. In Section 1, we show that the set function $\Phi = f(\zeta, \varphi)$ is again quasi-additive and of bounded variation with respect to the same directed set $(T, >> (\varphi))$, and hence $\int f(\zeta, \varphi) = \int \Phi$ is again a Burkill-Cesari integral. Thus this result represents an extension of a main theorem [5, p. 111] of Cesari's to the general setting. In Section 2, we discuss the question of the invariance of $\int f(\zeta, \varphi)$ with respect to change of the generating set functions (ζ, φ). The results obtained extend those of Stoddart [8]. In Section 3 we consider the question of approximation of the integral $\int f(\zeta, \varphi)$. That is, using a convenient notion of convergence of a net of pairs (ζ_a, φ_a) to a pair (ζ, φ), we give conditions which enable us to conclude that $\int f(\zeta_a, \varphi_a) \longrightarrow \int f(\zeta, \varphi)$. The results obtained extend those proved by Cesari in surface area theory, and show, therefore, that also these results hold in the present axiomatic treatment. In Section 4, we prove an abstract Fubini theorem for the integral $\int f(\zeta, \varphi)$ which generalizes a result found by Nishiura [7]. In Section 5, the difficult question of the semi-continuity of « regular » integrals is taken up. Under the assumption of regularity on the integrand f together with a suitable notion of convergence, an abstract semi-continuity theorem for the integral $\int f(\zeta, \varphi)$ is given. Here, of course, f is real valued, while it is assumed that ζ, φ take on values in a Hilbert space H. It should be pointed out, that no representation theorem for the integral $\int f(\zeta, \varphi)$ as a Lebesgue-Stieltjes integral in a measure space (A, B, μ) is used; the semi-continuity is proved directly for the integral $\int f(\zeta, \varphi)$ as a Burkill-Cesari integral. As corollaries, one obtains a classical theorem due to Tonelli as well as the more recent result of Turner's [9].

The results given in this paper are contained in the author's doctoral dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the University of Michigan, 1966. The author would like to thank Professor L. Cesari, under whose direction the dissertation was written, for his kind advice and encouragement.

1. Existence of the integral $\int f(\zeta, \varphi)$. Let F be a uniformly convex Banach space. Let E and G be Hausdorff locally convex topological vector spaces with topologies described by the collections of semi-norms $\{P_U : U \in \mathrm{U}\}$ and $\{R_W : W \in \mathcal{W}\}$ respectively. S^1 will denote the unit sphere in F, whereas for $U \in \mathcal{U}$, e εE, we shall put $E_U = \{e : P_U(e) = 1\}$ with a similar meaning for G_W, $W \in \mathcal{W}$. The following conventions will be adhered to throughout this paper unless the contrary is explicitly stated. (We shall assume the reader is familiar with the concepts and notation of our earlier paper on this subject [12]).

A will denote a given set, or space, and $\{I\}$ will denote a collection of subsets I of A. Φ will denote a collection of interval functions φ where $\varphi : \{I\} \to F$. $\mathfrak{Z}$ will denote a collection of interval functions ζ where $\zeta : \{I\} \to \mathcal{P}(E)$ ($\mathcal{P}(E)$ being as usual the set of all subsets of E; hence ζ is « multiple valued »). We assume there exists a fixed set T such that with each $\varphi \in \Phi$ one may associate a partial ordering $>> \,=\, >>(\varphi)$ of T in such a way that $(T, >>(\varphi))$ is a directed set. With each $t \in T$ there is to be associated a finite collection of disjoint (or non-overlapping) intervals $C_t = [I_1, \dots, I_N]$ and an operator L_t which is a finite sum, that is, $L_t = \sum_{I \in C_t}$. The action of L_t will be prescribed below.

Let us recall that a Banach space F is said to be uniformly convex if for each ε with $0 < \varepsilon \leq 2$ there is some $\delta = \delta(\varepsilon) > 0$ such that $\|x - y\| \geq \varepsilon$ implies $\left\| \frac{1}{2}(x + y) \right\| < 1 - \delta/2$ whenever $\|x\| = \|y\| = 1$. The following estimate will prove to be useful

(1.1) LEMMA. Let F be a uniformly convex Banach space. Let ε, $0 < \varepsilon \leq 2$, be given. Let $\{\varphi_i : i = 1, \dots, n\}$ and $\{\varphi'_j : j = 1, \dots, m\}$ be two sets of vectors in F. Define $\alpha_i = \varphi_i / \|\varphi_i\|$ for $\varphi_i \neq 0$, any unit vector otherwise, and similarly α'_j. Let J be a mapping from $\{1, \dots, n\}$ into the subsets of $\{1, \dots, m\}$. Denote by Σ^i a sum of terms over j for which $j \in J(i)$, by Σ^i_+ a sum of terms over j for which $j \in J(i)$ and $\|\alpha_i - \alpha'_j\| \geq \varepsilon$, and by Σ^i_- a sum of terms over j for which $j \in J(i)$ and $\|\alpha_i - \alpha'_j\| < \varepsilon$. Then there exists a positive number $K = K(\varepsilon)$ such that

$$\varepsilon^2/2K^2 \, \Sigma_i \Sigma^i_+ \|\varphi'_j\| \leq \Sigma_i \|\varphi_i - \Sigma^i \varphi'_j\| + \Sigma_i \big| \|\varphi_i\| - \Sigma^i \|\varphi'_j\| \big| .$$

Proof. Let i, $1 \leq i \leq n$, be fixed. Invoking the Hahn-Banach theorem, let z be an element of F^* such that $z(\varphi_i) = \|\varphi_i\|$ and $\|z\| = 1$. Since F is uniformly convex, there exists $\delta = \delta(\varepsilon)$ such that in particular $\|\alpha_i - \alpha'_j\| \geq \varepsilon$ implies $\left\| \frac{1}{2}(\alpha_i + \alpha'_j) \right\| < 1 - \delta/2$. Pick $K > 0$ so that $\varepsilon/2K < \delta/2$.

Note that if $\| x \| \leq 1 - \delta/2$, then $| z(x) | < 1 - \varepsilon/2K$. Thus, if $\| \alpha_i - \alpha'_j \| \geq \varepsilon$, then

$$2z\left(\frac{1}{2}(\alpha_i + \alpha'_j)\right) = 2z\left(\frac{1}{2}(\varphi_i/\| \varphi_i \| + \varphi'_j/\| \varphi'_j \|)\right)$$

$$< 2[1 - \varepsilon/2K] = 2 - \varepsilon/K.$$

Therefore,

$$z(\varphi_i/\| \varphi_i \| - \varphi'_j/\| \varphi'_j \|) = 1 - z(\varphi'_j/\| \varphi'_j \|) > 1 + \varepsilon/K - 1 = \varepsilon/K.$$

Squaring both sides and remembering $\| z \| = 1$ gives

$$\varepsilon^2 K^{-2} \leq 2 - 2z(\varphi'_j/\| \varphi'_j \|).$$

Hence

$$\varepsilon^2 K^{-2}/2 \, \| \varphi'_j \| \leq \| \varphi'_j \| - z(\varphi'_j).$$

For any j whatsoever, it is true that $0 \leq \| \varphi'_j \| - z(\varphi'_j)$. Thus we may write

$$\varepsilon^2 K^{-2}/2 \, \Sigma^i_+ \| \varphi'_j \| \leq \Sigma^i (\| \varphi'_j \| - z(\varphi'_j))$$

$$= \Sigma^i \| \varphi'_j \| - \| \varphi_i \| + z(\varphi_i - \Sigma^i \varphi'_j)$$

$$\leq | \Sigma^i \| \varphi'_j \| - \| \varphi_i \| | + \| \varphi_i - \Sigma^i \varphi'_j \| .$$

Summing on i yields the result.

Note that the number K so determined depends only on ε and not on the vectors φ_i, φ'_j. Simple examples show that the lemma fails if the uniform convexity requirement on F is dropped.

Fix a pair $(\zeta, \varphi) \in \mathcal{Z} \times \Phi$. Let D be a subset of E with $\cup \zeta(I) \subset D$. Let $f: D \times F \to G$ be a mapping such that:

(f_1) For each $W \in \mathcal{W}$ there exists a positive real number M_W such that $R_W(f(p, q))) < M_W$ for all pairs $(p, q) \in D \times S^1$.

(f_2) Given $W \in \mathcal{W}$ and $\varepsilon > 0$, there exists $U^* = U^*(\varepsilon, W)$ and $\xi = \xi(\varepsilon, W) > 0$ such that $R_W(f(p_1, q_1) - f(p_2, q_2)) < \varepsilon$ for $P_{U^*}(p_1 - p_2) < \xi$ and $\| q_1 - q_2 \| < \xi$ where $p_1, p_2 \in D$; $q_1, q_2 \in S^1$.

(f_3) $f(p, aq) = af(p, q)$ for all $a \geq 0, p \in D, q \in F$.

By hypothesis (f) we shall mean $(f_1), (f_2)$, and (f_3).

A choice function c on $\{ \zeta(I) : I \in \{I\} \}$ is a function c such that $c(\zeta(I)) \in \zeta(I)$ for each $I \in \{I\}$. Let $\mathcal{C}$ denote the set of all such choice functions c. We shall assume the set function ζ is subject to a condition (ζ).

(ζ) For each $\varepsilon > 0$ and each $U \in \mathcal{U}$, there exists $t' = t'(\varepsilon, U) \in T$ such that for every $t_0 >> t'$ there exists $t'' = t''(\varepsilon, U, t_0)$ such that if $t >> t''$, if $c, c' \in C, J \subset I, I \in C_{t_0}, J \in C_t$, then

$$P_U(c\zeta(I) - c'\zeta(J)) < \varepsilon \text{ and } P_U(c\zeta(J) - c'\zeta(J)) < \varepsilon.$$

We point out that in condition (ζ) the ordering « >> » being used is $>>(\varphi)$. For arbitrary $c \in \mathcal{C}$ consider the set function

$$\Phi_c(I) = f(c\zeta(I), \varphi(I)), I \in \}I\{.$$

If the limit

$$\lim_T \sum_{I \in C_t} \Phi_c(I) = \lim_T \sum_{I \in C_t} f(c\zeta(I), \Phi(I))$$

exists, we denote this limit by $\int f(c\zeta, \varphi)$. Needless to say, the ordering of T used in taking the limit is $>>(\varphi)$. The main existence theorem for the integral is

(1.ii) THEOREM. Let $\varphi : \}I\{ \rightarrow F$. Suppose φ is quasi-additive and of bounded variation, that is, $V = V(\|\varphi\|) < +\infty$. If conditions (ζ) and (f) hold, then $\Phi_c(I) = f(c\zeta(I), \varphi(I))$ is quasi-additive and of bounded W-variation for each $W \in \mathcal{W}$. Therefore, if G is complete, the $B - C$ integral of Φ_c exists; that is,

$$\int \Phi_c = \int f(c\zeta, \varphi) = \lim \sum_{I \in C_t} f(c\zeta(I), \varphi(I)).$$

Furthermore, for arbitrary $c_1, c_2 \in \mathcal{C}$ we have

$$\int f(c_1\zeta, \varphi) = \int f(c_2\zeta, \varphi),$$

that is, the value of the integral is independent of the choice function c.

Proof. We first prove that Φ_c is quasi-additive. Let $\varepsilon, 0 < \varepsilon \leq 1$. and R_W be given. We know from [12, 3. vii] that $\|\varphi\|$ is quasi-additive. Hence there exists $t_1^\# = t_1^\#(\varepsilon)$ such that for every $t_0 >> t_1^\#$ there is also $t_1* = t_1*(\varepsilon, t_0)$ such that $t >> t_1*$ implies

$$\sum_{I \in C_{t_0}} \|\varphi(I) - \Sigma^{(I)}\varphi(J)\| < \varepsilon, \sum_{I \in C_{t_0}} |\|\varphi(I)\| - \Sigma^{(I)}\|\varphi(J)\|| < \varepsilon,$$

and

$$\Sigma'\|\varphi(J)\| < \varepsilon.$$

Moreover, since $\int \| \varphi \|$ exists, there exists $t_* = t_*(\varepsilon)$ such that $t >> t_*$ implies

$$\left| \sum_{I \in C_t} \| \varphi(I) \| - V \right| < \varepsilon.$$

The conditions on f give $M_W > 0$, $U^* \in \mathcal{U}$, $\xi = \xi(\varepsilon, W) > 0$ such that $R_W(f(p, q)) < M_W$ on $D \times S^1$, and $R_W(f(p_1, q_1) - f(p_2, q_2)) < \varepsilon$ for $P_{U^*}(p_1 - p_2) < \xi$ and $\| q_1 - q_2 \| < \xi$ where $p_1, p_2 \in D$, $q_1, q_2 \in S^1$.

Condition (ζ) gives, for this ε and $U^* \in \mathcal{U}$, t' such that for every $t_0 >> t'$ there exists t'' such that if $t >> t''$, if $c, c' \in C$, $J \subset I$, $I \in C_{t_0}$, $J \in C_t$, then

$$P_{U^*}(c\zeta(I) - c'\zeta(J)) < \varepsilon \text{ and } P_{U^*}(c\zeta(J) - c'\zeta(J)) < \varepsilon.$$

Let $K = K(\xi(\varepsilon, W))$ denote the number determined in Lemma (1.i).
Now take $t^\# \in T$ such that $t^\# >> [t'(\xi(\varepsilon, W)), t_1^\#(\varepsilon), t_1^\#(\varepsilon\xi^2(\varepsilon, W)K^{-2})]$ and take any $t_0 >> t^\#$. Let $t_* = t_*(\varepsilon, R_W, t_0) >> [t_1{}_*(\varepsilon\xi^2(\varepsilon, W)K^{-2}, t_0)$, $t_1{}_*(\varepsilon, t_0), t_*(\varepsilon), t''(\xi(\varepsilon, W), t_0)]$. Take any $t >> t_*$, and let $C_t = [J]$, $C_{t_0} = [I]$. Denote by Σ_I a sum over all $I \in C_{t_0}$, by Σ_J a sum over all $J \in C_t$. Denote by $\Sigma_+^{(I)}$ the sum over $J \subset I$ for which $\| \alpha(I) - \alpha(J) \| \geq \xi$, and by $\Sigma_-^{(I)}$ the corresponding sum for which $\| \alpha(I) - \alpha(J) \| < \xi$. Then we have

$$\Sigma_I R_W(f(c\zeta(I), \varphi(I)) - \Sigma^{(I)} f(c\zeta(J), \varphi(J)))$$

$$= \Sigma_I R_W(f(c\zeta(I), \alpha(I)) \| \varphi(I) \| - \Sigma^{(I)} f(c\zeta(I), \alpha(I)) \| \varphi(J) \|$$

$$+ \Sigma^{(I)} f(c\zeta(I), \alpha(I)) \| \varphi(J) \| - \Sigma^{(I)} f(c\zeta(J), \alpha(J)) \| \varphi(J) \|)$$

$$\leq \Sigma_I R_W(f(c\zeta(I), \alpha(I))) \left| \| \varphi(I) \| - \Sigma^{(I)} \| \varphi(J) \| \right|$$

$$+ \Sigma_I(\Sigma_-^{(I)} + \Sigma_+^{(I)}) R_W(f(c\zeta(I), \alpha(I)) - f(c\zeta(J), \alpha(J))) \| \varphi(J) \|$$

$$< M_W \Sigma_I \left| \varphi(I) - \Sigma^{(I)} \| \varphi(J) \| \right| + \varepsilon \Sigma_J \| \varphi(J) \|$$

$$+ 4M_W K^2 \xi^{-2}(\varepsilon, W)[\Sigma_I \| \varphi(I) - \Sigma^{(I)} \varphi(J) \| + \Sigma_I \left| \| \varphi(I) \| - \Sigma^{(I)} \| \varphi(J) \| \right|]$$

$$< (9M_W + V + \varepsilon)\varepsilon.$$

Also

$$\Sigma' R_W(f(c\zeta(J), \varphi(J))) = \Sigma' R_W(f(c\zeta(J), \alpha(J)) \| \varphi(J) \|$$

$$\leq M_W \Sigma' \| \varphi(J) \| < M_W \varepsilon.$$

Since ε is arbitrary, Φ_c is quasi-additive as contended, and, if G is complete, $\int \Phi_c$ exists in view of [12, 3. v].

In order to see that Φ_c is of bounded W-variation, first observe that hypothesis (f) may be applied to $R_W \circ f$ for each $W \in \mathrm{W}$ and consequently $R_W \circ \Phi_c$ is quasi-additive. Moreover for any $t_0 \in T$,

$$\Sigma_I (R_W \circ \Phi_c)(I) = \Sigma_I R_W (f(c\zeta(I), \varphi(I)))$$

$$= \Sigma_I R_W (f(c\zeta(I), \alpha(I)) \| \varphi(I) \|, < M_W \Sigma_I \| \varphi(I) \|.$$

Since $R_W \circ \Phi_c$ is quasi-additive, $\int R_W \circ \Phi_c$ exists; in fact, $\int R_W \circ \Phi_c < M_W V$ an so Φ_c is of bounded W-variation for each $W \in \mathcal{W}$. As a consequence, observe that if the M_W 's are uniformly bounded, that is, if there exists $M > 0$ such that $M_W < M$ for all $W \in \mathcal{W}$, then Φ_c is of bounded variation and $V(\Phi_c) < MV$.

The fact that $\int \Phi_c$ is independent of $c \in \mathcal{C}$ follows by a standard "$\varepsilon/3$" argument and thus the proof of the theorem is complete.

As mentioned before, (1.ii) represents an extension of a result of Cesari's [5, p. 111]; (1.ii) also contains a generalization of Cesari's result due to Stoddart [8, p, 45].

2. Invariance of $\int f(\zeta, \varphi)$. Given pairs of interval functions $(\zeta, \varphi), (\zeta', \varphi')$ one may ask for conditions under which the corresponding integrals $\int f(\zeta, \varphi)$, $\int f(\zeta', \varphi')$ have the same value. This question was discussed by Stoddart [8] who in turn was motivated by Cesari's treatment of the invariance of surface integrals under Fréchet equivalence. The theorems obtained represent extensions to our setting of similar results established in [8] for S_2-type systems and interval functions (ζ, φ) with finite dimensional range. We shall employ freely and without explicit mention the notations and conventions introduced in the preceding section.

We consider a system $A, \{I\}, (T, >>)$ with ζ a map $\zeta: \{I\} \to \mathcal{P}(E)$, $D \subset E$ and $D \supset \cup \zeta(I)$, while $\varphi: \{I\} \to F$; and then a second system A', $\{I'\}, (T', >>')$, with ζ' a map $\zeta': \{I'\} \to \mathcal{P}(E), D \subset E$ and $D \supset \cup \zeta'(I')$, while $\varphi': \{I'\} \to F$. Let $\mathcal{C} = \{c'\}$ denote the set of all choice functions c on $\{\zeta(I): I \in \{I\}\}$, $\mathcal{C}' = \{c'\}$ denote the set of all choice functions c' on $\{\zeta'(I'): I' \in \{I'\}\}$. Let $f: D \times F \to G$ be a map satisfying condition (f).

We already know in view of Theorem (1.ii) sufficient conditions on (ζ, φ) and (ζ', φ') to ensure the existence of the integrals $\int f(\zeta, \varphi)$, $\int f(\zeta', \varphi')$ as points in G.

We shall say that the pair (ζ, φ) is (ϱ-*related* to the pair (ζ', φ') (cf. [8, p. 58]) if for each pair $(t_*, t'_*) \in T \times T'$, each $\varepsilon > 0$, aud each $U \in \mathrm{U}$, there exists a map $m = m(\varepsilon, U, t_*, t'_*) : A \to S'$ and a pair $(t, t') \in T \times T'$ with $t = t(\varepsilon, U, t_*, t'_*) >> t_*$, $t' = t'(\varepsilon, U, t_*, t'_*) >>' t'_*$ such that

(1) For $mI \subset I'$, $I \in C_t$, $I' \in C_{t'}$, and all $c \in \mathcal{C}$, $c' \in \mathcal{C}'$ we have $P_U(c'\zeta'(I') - c\zeta(I)) < \varepsilon$.

(2) (i) $\sum_{I' \in C_{t'}} \| \varphi'(I') - \Sigma^{(I')} \varphi(I) \| < \varepsilon$,

(ii) $\sum_{I' \in C_{t'}} \big| \| \varphi'(I') \| - \Sigma^{(I')} \| \varphi(I) \| \big| < \varepsilon$,

(iii) $\Sigma^* \| \varphi(I) \| < \varepsilon$,

where $\Sigma^{(I')}$ denotes a sum over all $I \in C_t$ such that $mI \subset I'$, and Σ^* denotes a sum over all $I \in C_t$ such that mI is contained in no $I' \in C_{t'}$.

(2.i) THEOREM. Let $f: D \times F \to G$ be a map satisfying condition (f). Suppose (ζ, φ), (ζ', φ') are such that $\int f(\zeta, \varphi)$, $\int f(\zeta', \varphi')$ exist as elements of G, and $V = \limsup \sum_{I \in C_t} \| \varphi(I) \| < +\infty$. Then if (ζ, φ) is ϱ-related to (ζ', φ'), we have

$$\int f(\zeta, \varphi) = \int f(\zeta', \varphi').$$

Proof. It is being assumed that the value of $\int f(\zeta, \varphi)$ or $\int f(\zeta', \varphi')$ is independent of the choice function $c \in \mathcal{C}$ or $c' \in \mathcal{C}'$. Hence fix $c \in \mathcal{C}$, $c' \in \mathcal{C}'$. Let ε', $0 < \varepsilon' < 1$, and $W \in \mathcal{W}$ be given. The conditions on f give $M_W > 0$, $U^* \in \mathcal{U}$, $\xi = \xi(\varepsilon', W) > 0$ such that $R_W(f(p, q)) < M_W$ on $D \times S^1$ and $R_W(f(p_1, q_1) - f(p_2, q_2)) < \varepsilon'$ for $P_{U^*}(p_1 - p_2) < \xi$, and $\| q_1 - q_2 \| < \xi$ where $p_1, p_2 \in D$; $q_1, q_2 \in S^1$.

There exists $t_1 \in T$ such that $t >> t_1$ implies

$$R_W\left(\sum_{I \in C_t} f(c\,\zeta(I), \varphi(I)) - \int f(\zeta, \varphi) \right) < \varepsilon'.$$

There exists $t_1' \in T'$ such that $t' >>' t_1'$ implies

$$R_W\left(\sum_{I \in O_{t'}} f(c'\zeta'(I'),\ \varphi'(I')) - \int f(\zeta', \varphi')\right) < \varepsilon'.$$

There exists $t_2 \in T$ such that $t >> t_2$ implies

$$\sum_{I \in O_t} \| \varphi(I) \| < V + \varepsilon'.$$

Let $K = K(\xi(\varepsilon', W))$ denote the number determined in Lemma (1.i). Let $\varepsilon = \text{Min}(\varepsilon', \xi(\varepsilon', W), \varepsilon' \xi^2(\varepsilon', W) K^{-2})$ and let $t_* >> [t_1, t_2]$, $t_*' >>' t_1'$. Use ε, and the pair (t_*, t_*') in relation ϱ above to get a map $m: A \to A'$ and a pair $(t, t') \in T \times T'$ with $t >> t_*$, $t' >>' t_*'$ such that (1), (2) are verified. We now have

$$\begin{aligned}
&R_W\Big(\sum_{I' \in O_{t'}} f(c'\zeta'(I'),\ \varphi'(I')) - \sum_{I \in O_t} f(c\zeta(I),\ \varphi(I))\Big)\\
&\le \sum_{I' \in O_{t'}} R_W\big(f(c'\zeta'(I'),\ \varphi'(I'))\,[\,\|\varphi'(I')\| - \Sigma^{(I')} \|\varphi(I)\|\,]\\
&+ \Sigma^{(I')} [f(c'\zeta'(I'),\ \alpha'(I')) - f(c\zeta(I),\ \alpha(I))]\,\|\varphi(I)\|\big)\\
&+ \Sigma^* R_W(f(c\zeta(I),\ \alpha(I))\,\|\varphi(I)\|\\
&< M_W \sum_{I' \in O_{t'}} |\,\|\varphi'(I')\| - \Sigma^{(I')} \|\varphi(I)\|\,| + \varepsilon' \sum_{I \in O_t} \|\varphi(I)\|\\
&+ 4M_W K^2 \xi^{-2}(\varepsilon') \sum_{I' \in O_{t'}} (\,\|\varphi'(I') - \Sigma^{(I')} \varphi(I)\| +\\
&+ |\,\|\varphi'(I')\| - \Sigma^{(I')} \|\varphi(I)\|\,|) + M_W \sum_{I \in O_t}{}^{*} \|\varphi(I)\|\\
&< (10\, M_W + V + \varepsilon')\,\varepsilon'.
\end{aligned}$$

So

$$R_W\left(\int f(\zeta', \varphi') - \int f(\zeta, \varphi)\right) < (10\, M_W + V + 2 + \varepsilon')\,\varepsilon'.$$

Since ε' is arbitrary, $W \in \mathcal{W}$ is arbitrary, and G is Hausdorff, we conclude that $\int f(\zeta', \varphi') = \int f(\zeta, \varphi)$ as asserted.

By way of applications, we point out that it is not very hard to prove that with respect to suitable interval functions ζ, φ and sets A, $\{I\}$, T that

Fréchet equivalent parametric curves stand in relation ϱ if the map m is requested to be a homeomorphism. The same is true for Fréchet equivalent parametric surfaces although the details are more involved here. We refer the reader to [8] for the proofs of these assertions.

One may also discuss « rotational properties » of integrals $\int f(\zeta, \varphi)$. This problem arises classically when the interval functions (ζ, φ) are generated from a variety in E^m, that is, a mapping $S: A \to E^m$. Any orthogonal transformation $L: E^m \to E^m$ will give a second variety $S' = LS$ in E^m, with will generate corresponding (ζ', φ') and often $>> (\varphi')$ (cf. [10, p. 923] and [8, pp. 76-80]). We shall give below a generalization of a result of Stoddart [8, p. 77] to our setting.

Let $N: F \to F$ be a linear, bijective distance preserving map of F onto itself. Let $M: E \to E$ be a unimorphy, that is, a linear, bijective map of E onto itself such that for each $U \in \mathcal{U}$, all $e_1, e_2 \in E$ we have $P_U(Me_1, Me_2) = P_U(e_1, e_2)$. Relative to a system A, $\}I\{$, T, we shall consider interval functions ζ, ζ' from $\}I\{$ to $\mathcal{P}(E)$, interval functions φ, φ' from $\}I\{$ to F, and orderings of T, $>> = >> (\varphi)$, $>>' = >>' (\varphi')$. Let $D \subset E$ be such that $D \supset \cup \zeta(I)$, $D \supset \cup M^{-1} \zeta'(I)$. Let $f: D \times F \to G$ be a map satisfying condition (f). We shall say that the pair (ζ, φ) is ϱ-(M, N) *related* to the pair (ζ', φ') if for each pair $(t_*, t'_*) \in (T, >>) \times (T, >>') = T \times T'$, each $\varepsilon > 0$, and each $U \in \mathcal{U}$ there exists a pair $(t, t') \in T \times T'$, with $t = t(\varepsilon, U, t_*, t'_*) >> t_*$, $t' = t'(\varepsilon, U, t_*, t'_*) >>' t'_*$ such that

(1) For $I \in C_t$, $I' \in C_{t'}$ and all $c \in \mathcal{C}$, $c' \in \mathcal{C}'$ we have $P_U(c' \zeta'(I') - Mc\,\zeta(I)) < \varepsilon$.

(2) (i) $$\sum_{I' \in C_{t'}} \| \varphi'(I') - N \Sigma^{(I')} \varphi(I) \| < \varepsilon,$$

(ii) $$\sum_{I' \in C_{t'}} \big| \| \varphi'(I') \| - \Sigma^{(I')} \| \varphi(I) \| \big| < \varepsilon,$$

(iii) $$\Sigma' \| \varphi(I) \| < \varepsilon,$$

where $\Sigma^{(I')}$ denotes a sum over all $I \in C_t$ such that $I \subset I'$, and Σ' denotes a sum over all $I \in C_t$ such that I is contained in no $I' \in C_{t'}$.

(2.ii) THEOREM. Let $f: D \times F \to G$ be a map satisfying condition (f). Define $g(p, q) = f(M^{-1} p, N^{-1} q)$. Suppose (ζ, φ), (ζ', φ') are such that $\int f(\zeta, \varphi)$, $\int g(\zeta', \varphi')$ exist as elements of G, and

$$V = \limsup \sum_{I \in C_t} \| \varphi(I) \| < +\infty.$$

Then, if (ζ, φ) is $\varrho - (M, N)$ related to (ζ', φ') we have

$$\int f(\zeta, \varphi) = \int g(\zeta', \varphi').$$

Proof. With m the identity map, observe that (ζ, φ) is ϱ-related to $(M^{-1}\zeta', N^{-1}\varphi')$. Hence by (2.i),

$$\int f(\zeta, \varphi) = \int f(M^{-1}\zeta', N^{-1}\varphi') = \int g(\zeta', \varphi').$$

3. **Approximation of the integral $\int f(\zeta, \varphi)$.** It is the purpose of this section to formulate in abstract terms a theorem of approximation for the integral $\int f(\zeta, \varphi)$. The discussion is motivated by Cesari's treatment in [2, pp. 24-30] of a theorem in the same spirit for surface integrals. We begin with a preliminary estimate.

(3.i) Lemma. Let $\{\varphi_i : i = 1, \dots, n\}$, $\{\varphi_i' : i = 1, \dots, n\}$ be two sets of vectors in F, F a Hilbert space. Define $\alpha_i = \varphi_i / \| \varphi_i \|$ for $\varphi_i \neq 0$, any unit vector otherwise and similarly α_i'. Suppose $\Sigma_1^n \| \varphi_i \| < M$. Then

$$\Sigma_1^n \| \varphi_i \| \, \| \alpha_i - \alpha_i' \|^2 < 2 [M - \Sigma_1^n \varphi_i \cdot \varphi_i' / \| \varphi_i' \|].$$

Proof. We denote the inner product of two vectors in F with a dot «$\cdot$». Observe that

$$1 - \alpha_i \cdot \alpha_i' = \frac{1}{2} [\| \alpha_i \|^2 - 2\alpha_i \cdot \alpha_i' + \| \alpha_i' \|^2] = \frac{1}{2} \| \alpha_i - \alpha_i' \|^2.$$

Hence

$$\| \varphi_i \| \, \| \alpha_i - \alpha_i' \|^2 = 2 [\| \varphi_i \| - \varphi_i \cdot \varphi_i' / \| \varphi_i' \|].$$

and so summing over i gives

$$\Sigma_1^n \| \varphi_i \| \, \| \alpha_i - \alpha_i' \|^2 = 2 [\Sigma_1^n \| \varphi_i \| - \Sigma_1^n \varphi_i \cdot \varphi_i' / \| \varphi_i' \|]$$

$$< 2 [M - \Sigma_1^n \varphi_i \cdot \varphi_i' / \| \varphi_i' \|]$$

which completes the proof.

It is known that every Hilbert space is uniformly convex. We shall specialize in this section and assume the space F is a Hilbert space. However the other conventions enumerated earlier are in force.

Let $\}(\zeta_\beta, \varphi_\beta): \beta \in (\mathcal{L}, \succ)\{$ be a net in $\mathfrak{X} \times \Phi$, let $>\!\!>_\beta$ denote the ordering of T induced by φ_β, let $\mathcal{C}_\beta$ denote the set of choice functions c_β on the set $\}\zeta_\beta(I): I \in \}I\{\{$, and let $D \subset E$ be such that $\bigcup_{\substack{\beta \in \mathcal{L} \\ I \in \}I\{}} \zeta_\beta(I) \subset D$. Let $f: D \times F \to G$ be a map satisfying condition (f). Suppose that the integrals $\mathcal{J} = \int f(\zeta, \varphi)$, $\mathcal{J}_\beta = \int f(\zeta_\beta, \varphi_\beta)$, $(\beta \in \mathcal{L})$ exist — conditions sufficient to ensure this were given in (1.ii). Then for the purposes of this section, we shall mean by the term *the net* $(\zeta_\beta, \varphi_\beta)$ *f-converges to* (ζ, φ) the following: For each $t_* \in T$, for each $\varepsilon > 0$, for each $U \in \mathcal{U}$, and for each $W \in \mathcal{W}$, there exists $\beta_0 = \beta_0(t_*, \varepsilon, U, W) \in \mathcal{L}$ and $t_0 = t_0(t_*, \varepsilon, U, W) \in T$ such that

(f_1) $t_0 >\!\!>_\beta t_*$ for all $\beta \succ \beta_0$;

(f_2) For all $\beta \succ \beta_0$, for all $c \in \mathcal{C}$, $c_\beta \in \mathcal{C}_\beta$, for all $I \in C_{t_0}$, we have

$$P_U(c\zeta(I) - c_\beta \zeta_\beta(I));$$

(f_3) For all $\beta \succ \beta_0$, for all $c \in \mathcal{C}$, $c_\beta \in \mathcal{C}_\beta$,

(i) $$R_W\Big(\sum_{I \in C_{t_0}} f(c_\beta \zeta_\beta(I), \varphi_\beta(I)) - \mathcal{J}_\beta\Big) < \varepsilon$$

(ii) $$R_W\Big(\sum_{I \in C_{t_0}} f(c\zeta(I), \varphi(I)) - \mathcal{J}\Big) < \varepsilon;$$

(f_4) For all $\beta \succ \beta_0$

$$\sum_{I \in C_{t_0}} \| \varphi_\beta(I) - \varphi(I) \| < \varepsilon.$$

We remark that condition (f_2) may be interpreted as «uniform convergence» of the interval functions ζ_β to ζ, while condition (f_3) says roughly that the integrals $\mathcal{J}_\beta$ are obtained from the approximating sums uniformly with respect to T for «large» β.

(3.ii) THEOREM. Let $f: D \times F \to G$ be a map satisfying condition (f). Suppose (ζ, φ), $(\zeta_\beta, \varphi_\beta)$, $\beta \in \mathcal{L}$ are such that $\int f(\zeta, \varphi)$, $\int f(\zeta_\beta, \varphi_\beta)$ exist and $(\zeta_\beta, \varphi_\beta)$ f-converges to (ζ, φ). If φ and each φ_β is of bounded variation V and V_β respectively, then $\lim \mathcal{J}_\beta = \mathcal{J}$ provided $\lim V_\beta = V$.

Proof. It is being assumed that the value of $\mathcal{J}$ and $\mathcal{J}_\beta$ is independent of the choice function $c \in \mathcal{C}$ or $c_\beta \in \mathcal{C}_\beta$. Hence fix $c \in \mathcal{C}$, $c_\beta \in \mathcal{C}_\beta$ and let

$t_* \in T$ be arbitrary. Let ε', $0 < \varepsilon' < 1$, and $W \in \mathcal{W}$ be given. The conditions on f give $M_W > 0$, $U^* \in \mathcal{U}$, $\xi = \xi(\varepsilon', W) > 0$ such that $R_W(f(p,q)) < M_W$ on $D \times S^1$, $R_W(f(p_1, q_1) - f(p_2, q_2)) < \varepsilon'$ for $P_{U^*}(p_1 - p_2) < \xi$ and $\| q_1 - q_2 \| < \xi$ where $p_1, p_2 \in D$; $q_1, q_2 \in S^1$.

Let $\varepsilon = \mathrm{Min}(\varepsilon', \xi(\varepsilon', W), \varepsilon' \xi^2(\varepsilon', W))$, t_* as above, and take $W \in \mathcal{W}$ with corresponding $U^* \in \mathcal{U}$ in the definition of f-convergence above. Hence, we are provided with β_0 and t_0 such that conditions (f_1) through (f_4) above are fulfilled. It may be assumed without loss of generality that β_0 is chosen so that $\beta > \beta_0$ implies $| V - V_\beta | < \varepsilon$ and t_0 is chosen so that $t >> t_0$ implies

$$V - \varepsilon < \sum_{I \in C_t} \| \varphi(I) \| < V + \varepsilon.$$

Let $\beta > \beta_0$. We shall write Σ_I for any sum over all $I \in C_{t_0}$, Σ_I^+ for any sum over all $I \in C_{t_0}$ such that $\| \alpha(I) - \alpha_\beta(I) \| \geq \xi$, Σ_I^- for any sum over all $I \in C_{t_0}$ such that $\| \alpha(I) - \alpha_\beta(I) \| < \xi$. Then we have

$$\begin{aligned}
&R_W(\Sigma_I f(c\zeta(I), \varphi(I)) - \Sigma_I f(c_\beta \zeta_\beta(I), \varphi_\beta(I))) \\
&\quad \leq \Sigma_I R_W(f(c_\beta \zeta_\beta(I), \alpha_\beta(I))) \,|\, \| \varphi(I) \| - \| \varphi_\beta(I) \| \,| \\
&\quad + \Sigma_I R_W(f(c\zeta(I), \alpha(I)) - f(c_\beta \zeta_\beta(I), \alpha_\beta(I))) \| \varphi(I) \| \\
&\quad < M_W \Sigma_I |\, \| \varphi(I) \| - \| \varphi_\beta(I) \| \,| \\
&\quad + \Sigma_I^- R_W(f(c\zeta(I), \alpha(I)) - f(c_\beta \zeta_\beta(I), \alpha_\beta(I))) \| \varphi(I) \| \\
&\quad + \Sigma_I^+ R_W(f(c\zeta(I), \alpha(I)) - f(c_\beta \zeta_\beta(I), \alpha_\beta(I))) \| \varphi(I) \| \\
&\quad < M_W \varepsilon' + \varepsilon'(V + \varepsilon') + 2M_W \Sigma_I^+ \| \varphi(I) \|.
\end{aligned}$$

It remains to study the last sum in the line above. From (3.i) we have

$$\begin{aligned}
&\Sigma_I^+ \| \varphi(I) \| \| \alpha(I) - \alpha_\beta(I) \|^2 \\
&\quad \leq \Sigma_I \| \varphi(I) \| \| \alpha(I) - \alpha_\beta(I) \|^2 \\
&\quad < 2 [V - \Sigma_I \varphi(I) \cdot \varphi_\beta(I) / \| \varphi_\beta(I) \|] + 2\varepsilon \\
&\quad = 2 [V - \Sigma_I \varphi_\beta(I) / \| \varphi_\beta(I) \| \cdot (\varphi_\beta(I) - \Delta_\beta(I)) + 2\varepsilon \\
&\quad < 2 | V - V_\beta | + 2 | V_\beta - \Sigma_I \| \varphi_\beta(I) \| | + 2 \Sigma_I \| \Delta_\beta(I) \| + 2\varepsilon
\end{aligned}$$

where we have written $\Delta_\beta(I) = \varphi_\beta(I) - \varphi(I)$. Observe that

$$\|\varphi_\beta(I)\| = \|(\varphi_\beta(I) - \varphi(I)) + \varphi(I)\| \geq \|\varphi(I)\| - \|\varphi_\beta(I) - \varphi(I)\|.$$

Hence

$$-\Sigma_I \|\varphi_\beta(I)\| \leq \Sigma_I \|\varphi_\beta(I) - \varphi(I)\| - \Sigma_I \|\varphi(I)\| < 2\varepsilon - V$$

and so

$$V - \Sigma_I \|\varphi_\beta(I)\| < 2\varepsilon.$$

Similarly

$$\Sigma_I \|\varphi_\beta(I)\| - V < 2\varepsilon.$$

We conclude that

$$|V - \Sigma_I \|\varphi_\beta(I)\|| < 2\varepsilon.$$

Then

$$|V_\beta - \Sigma_I \|\varphi_\beta(I)\|| \leq |V - V_\beta| + |V - \Sigma_I \|\varphi_\beta(I)\|| < 3\varepsilon,$$

and so

$$\Sigma_I^+ \|\varphi(I)\| \|\alpha(I) - \alpha_\beta(I)\|^2 < 2\varepsilon + 6\varepsilon + 2\varepsilon + 2\varepsilon = 12\varepsilon.$$

Consequently

$$\xi^{-2} \Sigma_I^+ \|\varphi(I)\| \xi^2 \leq \xi^{-2} \Sigma_I^+ \|\varphi(I)\| \|\alpha(I) - \alpha_\beta(I)\|^2 < 12\varepsilon\, \xi^{-2} < 12\varepsilon'.$$

Thereby

$$R_W(\Sigma_I f(c\,\zeta(I), \varphi(I)) - \Sigma_I f(c_\beta \zeta_\beta(I), \varphi_\beta(I)))$$

$$< M_W \varepsilon' + \varepsilon'(V + \varepsilon') + 24 M_W \varepsilon' = \varepsilon'(25 M_W + V + \varepsilon').$$

Writing

$$R_W(\mathcal{J} - \mathcal{J}_\beta)$$

$$\leq R_W(\mathcal{J} - \Sigma_I f(c\,\zeta(I), \varphi(I))) + R_W(\mathcal{J}_\beta - \Sigma_I f(c_\beta \zeta_\beta(I), \varphi_\beta(I)))$$

$$+ R_W(\Sigma_I f(c\,\zeta(I), \varphi(I)) - \Sigma_I f(c_\beta \zeta_\beta(I), \varphi_\beta(I)))$$

$$< \varepsilon'(25 M_W + V + 2 + \varepsilon'),$$

we deduce, since ε' is arbitrary, $W \in \mathcal{W}$ is arbitrary, and G is Hausdorff, that $\lim \mathcal{J}_\beta = \mathcal{J}$ as asserted. This completes the proof of the theorem.

Observe that we may take as a particular case the special map $f: D \times F \rightarrow F$ defined by $f: (p, q) \rightarrow q$ then, under the other conditions, infer that $\lim \int \varphi_\beta = \int \varphi$.

Let us now sketch briefly an illustration of Theorem (3.ii). Let $S = (S, A)$, $S_n = (S_n, A)$. $n = 1, 2, \ldots$, be continuous mappings of finite Lebesgue area $L(S, A)$, $L(S_n, A)$ respectively. Let $[S]$, $[S_n]$ denote the graphs of S, S_n respectively and suppose $[S], [S_n] \subset D \subset E^3$. Let $f(p,q)$, $p = (p_1, p_2, p_3) \in D$, $q = (q_1, q_2, q_3) \in E^3$ be a real valued function defined on $D \times E^3$ such that $f(p, aq) = af(p,q)$ for all $a \geq 0$, $p \in D$, $q \in E^3$, let $U = \{q \in E^3, \| q \| = 1\}$ be the unit sphere in E^3 and suppose f is bounded and uniformly continuous on $D \times U$. Consider the following proposition (P) [4, p. 571]:

(P) Let f be as above. Let $S = (S, A)$, $S_n = (S_n, A)$, $n = 1, 2, \ldots$, be continuous mappings of finite Lebesgue area. Suppose $S_n \to S$ uniformly, $L(S_n, A) \to L(S, A)$ as $n \to \infty$. Then we have also $\lim_{n \to \infty} (S_n) \int f = (S) \int f$.

Now actually this proposition follows from (3.ii) For given a closed simple polygonal region $I \subset A$, we define $\zeta(I) = \bigcup_{W \in I} S(w)$, $\zeta_n(I) = \bigcup_{W \in I} S_n(w)$, $n = 1, 2, \ldots$. Take for $\varphi(I)$, $\varphi_n(I)$ the interval functions defined in [5, p. 106]· recall that φ, φ_n are quasi-additive whith respect to certain mesh functions δ, δ_n relative to finite systems D of nonoverlapping polygonal regions I. Hence the integrals $\int f(\zeta, \varphi)$, $\int f(\zeta_n, \varphi_n)$ exist. Since the variations of φ, φ_n are nothing more than the Lebesgue areas of the surfaces (S, A), (S_n, A) and since by assumption $\lim_n L(S_n, A) = L(S, A)$, we need only verify that (ζ_n, φ_n) f-converges to (ζ, φ). The f-convergence follows since, under the conditions of proposition (P), for arbitrary $\varepsilon > 0$ Cesari has shown ([1, p. 1385]; [3]) that one may find a finite system $D_0 = [I] = [I_1, \ldots, I_M]$ of non-overlapping polygonal regions $I \in \{I\}$ and an integer N_1 such that

$(f_1)_P$ $\delta(D_0) < \varepsilon$, $\delta_n(D_0) < \varepsilon$ for all $n > N_1$;

$(f_4)_P$ For all $n > N_1$

$$\sum_{I \in D_0} \| \varphi_n(I) - \varphi(I) \| < \varepsilon.$$

Then, in view of the uniform convergence of $\{S_n\}$ to S, and simple but technical considerations of surface area theory, one may produce an integer $N_0 > N_1$ such that in addition

$(f_2)_P$ For all $n > N_0$, for all $c \in \mathcal{C}$, $c_n \in \mathcal{C}_n$ for all $I \in D_0$ we have

$$\| c\, \zeta(I) - c_n\, \zeta_n(I) \| < \varepsilon;$$

$(f_3)_P$ For all $n > N_0$, for all $c \in \mathcal{C}$, $c_n \in \mathcal{C}_n$,

(i) $$\sum_{I \in D_0} | f(c\, \zeta(I), \varphi(I)) - \mathcal{I} | < \varepsilon$$

(ii) $$\sum_{I \in D_0} | f(c_n\, \zeta_n(I), \varphi_n(I)) - \mathcal{I}_n | < \varepsilon.$$

These facts are worked out in detail in [2, pp. 27-28]. Hence (ζ_n, φ_n) if f-convergent to (ζ, φ) and so proposition (P) follows from Theorem (3.ii).

We remark that similar considerations may be applied to parametric curves.

4. A Fubini theorem for $\int f(\zeta, \varphi)$. It is the purpose of this section to supplement the results of Section 4 in [12] by formulating a Fubini type theorem for the integrals $\int f(\zeta, \varphi)$. The results obtained represent extensions of those in [7].

In what follows « i » will run over 1, 2. Let E_i be Hausdorff locally convex topological vector spaces with associated families of semi-norms given by $\}P_{U_i} : U_i \in \mathcal{U}_i\{$. Let F_1, F_2 be Hilbert spaces as in [7] and let $F = F_1 \otimes F_2$ be their tensor product — thus for $q_1 \in F_1$, $q_2 \in F_2$, $q_1 \otimes q_2 \in F_1 \otimes F_2$, and one has $\| q_1 \otimes q_2 \| = \| q_1 \|_1 \| q_2 \|_2$. Let S_1^α, S_2^α denote the spheres of radius α in F_1, F_2 respectively. Let A_1 and A_2 be sets with corresponding collections of intervals $\}I_1\{$, $\}I_2\{$. Let $(L_{t_i} : t_i \in (T_i, >>_i))$ be as usual. Let ζ_i, φ_i be interval functions where $\zeta_i : \}I_i\{ \rightarrow \mathcal{P}(E_i)$ and $\varphi_i : \}I_i\{ \rightarrow F_i$. Let $\mathcal{C}_i$ denote the set of all choice functions c_i on the collection $\}\zeta_i(I_i) : I_i \in \}I_i\{\{$. Let $D_i \supset \cup \zeta_i(I_i)$. Let $(T, >>)$ be the directed set $T_1 \times T_2$ (« $\times$ » meas Cartesian product) where « $>>$ » is the usual product ordering induced by $>>_1 (\varphi_1)$ and $>>_2 (\varphi_2)$. We then define $A = A_1 \times A_2$, $I = \}I_1\{ \times \}I_2\{$, $C_t = C_{t_1} \times C_{t_2}$, $t = (t_1, t_2) \in T$. The space $E = E_1 \times E_2$ is a Hausdorff locally convex topological vector space if we equip it with the topology determined by the semi-norms $P_{(U_1, U_2)}$ defined by $P_{(U_1, U_2)}(e_1, e_2) = P_{U_1}(e_1) + P_{U_2}(e_2)$ for $(e_1, e_2) \in E_1 \times E_2$. Define the map $\zeta : \}I\{ \rightarrow \mathcal{P}(E_1 \times E_2)$ by stipulating $\zeta(I) = \zeta_1(I_1) \times \zeta_2(I_2)$ for $I = I_1 \times I_2$. Suppose the maps ζ_i satisfy condition (ζ_i). Then we claim that the product map ζ satisfies condition (ζ) with respect to the collection $P_{(U_1, U_2)}$ and the set $(T, >>)$. Indeed, let $\varepsilon > 0$ and $P_{(U_1, U_2)}$ be given. Since ζ_i satisfies condition (ζ_i), there exists $t_i' = t_i'(\varepsilon/2, U_i) \in (T_i, >>_i)$ such that for every $t_{0i} >>_i t_i'$ there exists $t_i'' = t_i''(\varepsilon/2, U_i, t_{0i})$ such that if $t_i >>_i t_i''$, if $c_i, c_i' \in \mathcal{C}_i$, $J_i \subset I_i$, $I_i \in C_{t_{0i}}$, $J_i \in C_{t_i}$ then

$$P_{U_i}(c_i \zeta_i(I_i) - c_i' \zeta_i(J_i)) < \varepsilon/2, \quad P_{U_i}(c_i \zeta_i(J_i) - c_i' \zeta_i(J_i)) < \varepsilon/2.$$

Let $\mathcal{C}$ denote the set of choice functions c on the collection $\}\zeta(I) : I \in \}I\{\{$ — thus each $c \in \mathcal{C}$ determines $c_1 \in \mathcal{C}_1$, $c_2 \in \mathcal{C}_2$ such that $c\,\zeta(I) = (c_1 \zeta_1(I_1), c_2 \zeta_2(I_2))$ for all $I \in \}I\{$. Let $t' = (t_1', t_2')$ and let $t_0 >> t'$. Now $t_0 = (t_{01}, t_{02})$ suitable $t_{0i} \in T_i$ and $t_{0i} >>_i t_i$. Defining $t'' = (t_1'', t_2'') = (t_1''(\varepsilon/2, U_1, t_{01}), t_2''(\varepsilon/2, U_2, t_{02}))$, take any $t = (t_1, t_2) >> t''$. Then if $c, c' \in \mathcal{C}$, $J \subset I$, $I \in C_{t_0}$,

$J \in C_t$, we see that

$$P_{(U_1,\, U_2)}(c\,\zeta(I) - c'\,\zeta(J)) =$$

$$= P_{(U_1,\, U_2)}((c_1\,\zeta_1(I_1) - c_1'\,\zeta_1(J_1), c_2\,\zeta_2(I_2) - c_2'\,\zeta_2(J_2)))$$

$$P_{U_1}(c_1\,\zeta_1(I_1) - c_1'\,\zeta_1(J_1)) + P_{U_2}(c_2\,\zeta_2(I_2) - c_2'\,\zeta_2(J_2)) < \varepsilon.$$

Likewise $P_{(U_1,\, U_2)}(c\,\zeta(J) - c'\,\zeta(J)) < \varepsilon$. Hence the map ζ satisfies condition (ζ) as claimed. Next we define $\varphi = \varphi_1 \otimes \varphi_2 : \{I\} \rightarrow F_1 \otimes F_2$ by the relation $\varphi(I) = \varphi_1(I_1) \otimes \varphi_2(I_2)$ and for $t = (t_1, t_2) \in T$, we let $L_t(\varphi) = \sum_{I \in C_t} \varphi(I)$; observe $L_t(\varphi) = L_{t_1}(\varphi_1) \otimes L_{t_2}(\varphi_2)$. Recall [12, 4.ii] which asserted that if the φ_i were quasi-additive and of bounded variation, then the product $\varphi = \varphi_1 \otimes \varphi_2$ was also quasi-additive and of bounded variation. Let $D \subset E_1 \times E_2$ and suppose $D_i \subset D$. Let $f\colon D \times F \rightarrow G$, G a complete Hausdorff locally convex topological vector space whose topology is described by semi-norms $\{R_W : W \in \mathcal{W}\}$. Assuming f satisfies condition (f), we know from (1.ii), with our assumptions on ζ and φ, that $\int f(\zeta, \varphi)$ exists. Our problem will be to relate this « double integral » to certain « iterated integrals » which will be defined below.

In the Hilbert space F, denote the sphere of radius α by S^α. Let $f\colon D \times F \rightarrow G$ and assume f satisfies condition (f). Then for each $W \in \mathcal{W}$ there exists a positive real number M_W such that $R_W(f(p, q)) < M_W$ for all pairs $(p, q) \in D \times S^1$. Let $q^\# \in S^\alpha$. Writing $f(p, q^\#) = f(p, q^\# / \| q^\# \|) \| q^\# \|$, we see that $R_W(f(p, q^\#)) < \alpha\, M_W$ for all pairs $(p, q) \in D \times S^\alpha$. Likewise there exists $P_{(U_1^*, U_2^*)}$ and $\xi = \xi(\varepsilon, W) > 0$ such that $R_W(f(p_1, q_1) - f(p_2, q_2)) < \varepsilon$ for $P_{(U_1^*, U_2^*)}(p_1 - p_2) < \xi$ and $\| q_1 - q_2 \| < \xi$ where $p_1, p_2 \in D$; $q_1, q_2 \in S^1$. If $q_1^\#$, $q_2^\# \in S^\alpha$, then as above we see that $R_W(f(p_1, q_1^\#) - f(p_2, q_2^\#)) < \alpha\varepsilon$ for $P_{(U_1^*, U_2^*)}(p_1 - p_2) < \xi$ and $\| q_1^\# - q_2^\# \| < \alpha\,\varepsilon$. Keeping these facts in mind, we prove·

(4.i). Lemma. Suppose $f\colon D \times F \rightarrow G$ satisfies condition (f).

(a) Let (p_1, q_1) be a fixed point in $D_1 \times F_1$. Then $g(p_2, q_2) = f(p, q) = f((p_1, p_2), q_1 \otimes q_2)$ as a function on $D_2 \times F_2$ satisfies condition (g).

(b) Let $\varphi_2 : \{I_2\} \rightarrow F_2$ be quasi-additive and of bounded variation. Then $l(p_1, q_1) = \int f(p_1 \times \zeta_2, q_1 \otimes \varphi_2)$ exists for each $(p_1, q_1) \in D_1 \times F_1$. Moreover l regarded as a function on $D_1 \times F_1$ satisfies condition (l).

Proof. By Σ_{I_2} we mean a sum over all $I_2 \in C_{t_2}$. Statement (a) is quite clear in view of the remarks preceding the Lemma and the fact that $\| q_1 \otimes q_2 \| = \| q_1 \|_1 \| q_2 \|_2$. Let us prove (b). First of all the existence of l is a consequence of (1.ii). Homogeneity of l (that is, condition (l_3)) follows since for $a > 0$,

$$a\, l(p_1, q_1) = a \lim \Sigma_{I_2} f(p_1 \times c_2 \zeta_2(I_2),\ q_1 \otimes \varphi_2(I_2))$$

$$= \lim \Sigma_{I_2} f(p_1 \times c_2 \zeta_2(I_2),\ aq_1 \otimes \varphi_2(I_2)) = l(p_1, aq_1).$$

For each $W \in \mathcal{W}$ there exists a positive real number M_W such that $R_W(f(p, q)) < M_W$ for all pairs $(p, q) \in D \times S^1$.
Hence

$$R_W(f((p_1, p_2), q_1 \otimes q_2)) < M_W \| q_1 \otimes q_2 \| = M_W \| q_1 \|_1 \| q_2 \|_2 .$$

Therefore

$$R_W(l(p_1, q_1)) = R_W(\lim \Sigma_{I_2} f(p_1 \times c_2 \zeta_2(I_2), q_1 \otimes \varphi_2(I_2)))$$

$$\leq \lim \Sigma_{I_2} R_W(f(p_1 \times c_2 \zeta_2(I_2). q_1 \otimes \varphi_2(I_2(I_2)))$$

$$\leq \lim \Sigma_{I_2} M_W \| q_1 \|_1 \| \varphi_2(I_2) \|_2 \leq M_W \| q_1 \|_1 V(\| \varphi_2 \|_2).$$

Hence

$$R_W\, l(p_1, q_1)) \leq M_W V(\| \varphi_2 \|_2)$$

for all pairs $(p_1, q_1) \in D_1 \times S_1^1$ and so (l_1) is satisfied.

Now there exists $P_{(U_1^*, U_2^*)}$ and $\xi = \xi(\varepsilon, W) > 0$ such that $R_W(f(p_1, q_1) - f(p_2, q_2)) < \varepsilon$ for $P_{(U_1^*, U_2^*)}(p_1 - p_2) < \xi$ and $\| q_1 - q_2 \| < \xi$ where $p_1, p_2 \in D$; $q_1, q_2 \in S^1$.

Hence if $(p_1^{(1)}, q_1^{(1)}), (p_1^{(2)}, q_1^{(2)}) \in D_1 \times S_1^1$ and $P_{U_1^*}(p_1^{(1)} - p_1^{(2)}) < \xi$, $\| q_1^{(1)} - q_1^{(2)} \|_1 < \xi$, then

$$R_W(f((p_1^{(1)}, p_2), q_1^{(1)} \otimes q_2) - f((p_1^{(2)}, p_2), q_1^{(2)} \otimes q_2)) \leq \varepsilon \| q_2 \|_2 .$$

Hence

$$R_W(l(p_1^{(1)}, q_1^{(1)}) - l(p_1^{(2)}, q_1^{(2)}))$$

$$\leq \lim \Sigma_{I_2} R_W(f((p_1^{(1)} \times c_2 \zeta_2(I_2), q_1^{(1)} \otimes \varphi_2(I_2))$$

$$- f(p_1^{(2)} \times c_2 \zeta_2(I_2), q_1^{(2)} \otimes \varphi_2(I_2)))$$

$$\leq \lim \Sigma_{I_2} \varepsilon \| \varphi_2(I_2) \|_2 = \varepsilon V(\| \varphi_2 \|_2)$$

and thereby (l_2) is fulfilled. Thus l regarded as a function on $D_1 \times F_1$ satisfies condition (l) as required and this completes the proof of the Lemma.

(4.ii) THEOREM. Let $f: D \times F \to G$ be a map satisfying condition (f). Suppose $\varphi_i: \{I_i\} \to F_i$ is quasi-additive and of bounded variation V_i and that $\zeta_i: \{I_i\} \to \mathcal{P}(E_i)$ satisfies condition (ζ_i). Then, with the other assumptions cited earlier, we have

(a) $(A)\int f(\zeta, \varphi)$ exists;

(b) $l(p_1, q_1) = (A_2)\int f(p_1 \times \zeta_2, q_1 \otimes \varphi_2)$ exists for each $(p_1, q_1) \in D_1 \times F_1$, $k(p_2, q_2) = (A_1)\int f(\zeta_1 \times p_2, \varphi_1 \otimes q_2)$ exists for each $(p_2, q_2) \in D_2 \times F_2$. Moreover l, k satisfy conditions (l), k respectively;

(c) the iterated integrals exist, that is $(A_1)\int (A_2)\int f(\zeta, \varphi)$ and $(A_2)\int (A_1)\int f(\zeta, \varphi)$ exist;

(d) the iterated integrals = the double integral of (a).

Proof. We have commented on (a) above. Statement (b) is Lemma (4.i). Statement (c) follows from (b) and (1, ii); (d) is obvious.

5. Semi-continuity of the integral $\int f(\zeta, \varphi)$. In this section we shall prove, under appropriate assumptions, an abstract semi-continuity theorem for integrals of the form $\int f(\zeta, \varphi)$. Usually such theorems are proved for integrals of the form $(X)\int f(\zeta, \varphi)\, d\mu$, μ being a suitable measure on a measure space X. However, to do this one has to introduce various axioms and obtain a representation theorem for the B-C integral as a Lebesgue integral. Using such a representation of the B-C integral, Stoddart [8] succeded in establishing a very general theorem of semi-continuity which in turn contained many of the classical results. On the other hand, Turner [9, p. 112] proved directly a semi-continuity theorem for the Cesari-Weierstrass surface integral without recourse to any representation theorem. We shall give a theorem in this direction which, although not entirely analogous to Turner's, is in the same spirit.

Let H be a real Hilbert space and let S^1 denote the unit sphere in H. Let $\{e_n\}$ be an orthonormal basis for H. Let H_W denote H equipped with the weak topology. A weak neighborhood W of a point $h_0 \in H$ is determined by specifying a number $\varepsilon > 0$ and n continuous linear functionals $z_1, \dots, z_n$; then $W = W(h_0; z_1, \dots, z_n; \varepsilon) = \{h : h \in H \text{ and } |z_i(h) - z_i(h_0)| < \varepsilon, i = 1, \dots, n\}$. It is known that H and H_W have the same set of continuous linear functionals and if z is such a functional, then there exists a unique vector $a \in H$ such that $z(h) = (h, a) = (a, h)$ for all $h \in H$ ((,) means inner product in H). Moreover, regarding z as an element of H^*, $\| z \| = \| a \|$. Finally, since H is reflexive, spheres in H are weakly compact.

We shall consider a set $\mathcal{S}$ of pairs (ζ, φ) where $\zeta : \{I\} \to \mathcal{P}(H)$, $\varphi : \{I\} \to H$ (so, in our previous notation, we are taking $E = H$, $F = H$). We shall assume that the elements of S are « preserved » under isometrys. Precisely, let $m : H \to H$ be a bounded, linear, bijective map such that if $h_1, h_2, \in H$, then $\| mh_1 - mh_2 \| = \| h_1 - h_2 \|$. Given an arbitrary $(\zeta, \varphi) \in \mathcal{S}$, let $\zeta' = m\zeta$ and suppose that there is associated with ζ' a map $\varphi' : \{I\} \to H$ such that $(\zeta', \varphi') \in \mathcal{S}$. We then assume

$$(m)\, \varphi' = m\, \varphi.$$

Let f: $D \times H \to R$, R the real numbers, $D \subset H$ to be specified presently. Conditions (f) and (ζ) are assumed to be in force here for each $(\zeta, \varphi) \in \mathcal{S}$. If we let $D \supset \bigcup_{\mathcal{S}} \bigcup_{I} \zeta(I)$, then condition (f) now reads: (f_1) f is bounded on $D \times S^1$, (f_2) f is uniformly continuous on $D \times S^1$; (f_3) $f(p, aq) = af(p, q)$ for all $a \geq 0$, $p \in D$, $q \in S^1$. For a given pair $(\zeta, \varphi) \in \mathrm{S}$, it is desirable to be able to speak of $\mathcal{J}((\zeta, \varphi), I) = (I) \int f(\zeta, \varphi) = \lim_{(T_I, >>_I(\varphi))} \sum_{J \in C_{t_I}} f(c\, \zeta(J), \varphi(J))$ for each $I \in \{I\}$ where $(T_I, >>_I(\varphi))$ is a certain directed set that is, we want to consider $\mathcal{J}$ as a map, $\mathcal{J}$: $\mathcal{S} \times \{I\} \to R$. For this purpose, it is clearly enough to have at hand a method for extending the definition of the integral of f over A to a corresponding integral of f over each $I \in \}I\{$. In applications this situation always exists. Abstractly such an extension may be realized by a generalization of the method given by Cesari in [6]. In a future paper, we shall give the details of this procedure, but for our purposes here we shall proceed axiomatically and make the following assumptions (which are consquences of the general extension process):

(P_1) With each pair $(\zeta, \varphi) \in \mathcal{S}$ and with each $I \in \}I\{$ there is associated a directed set $(T_I(\zeta, \varphi), >>_I(\zeta, \varphi)) = (T_I, >>_I(\varphi))$ and a collection of intervals $\}J\{ \subset I, J \in \}I\{$. For given $I \in \}I\{$, we assume the set T_I and the collection $\}J\{$ are fixed and independent of the pair (ζ, φ) (although $>>^I(\varphi)$ will in general depent upon the pair (ζ, φ)). With each $t_I \in T_I$ there is associated a finite collection C_{t_I} of intervals $J \in \}J\{$ and an operator $L_{t_I} = \sum_{J \in C_{t_I}}$.

It is assumed that for each $(\zeta, \varphi) \in \mathcal{S}$, for each $I \in \}I\{$,

$$\text{(I)} \qquad \int f(\zeta, \varphi) = \lim_{(T_I, >>_I(\varphi))} \sum_{J \in C_{t_I}} f(c\, \zeta(J), \varphi(J))$$

exists and is independent of the choice function c.

(P_2) We shall suppose that if $f(p,q) \geq 0$ for all $(p,q) \in D \times S^1$, then for arbitrary $t \in T$, we have

$$\mathcal{J}((\zeta,\varphi),A) \geq \sum_{I \in C_t} \mathcal{J}((\zeta,\varphi),I),$$

(ζ,φ) being an arbitrary element of $\mathcal{S}$.

We shall assume that given $(\zeta,\varphi) \in \mathcal{S}$, $I \in \{I\}$, and any isometry m, then the orderings $>>_I(m\,\varphi)$ and $>>_I(\varphi)$ are comparable.

It will prove to be useful to consider « special neighborhood » systems of elements $s_0 = (\zeta_0, \varphi_0)$ in $\mathcal{S}$. Thus, for arbitrary $s_0 \in \mathcal{S}$, it will be said that a net $(s_\alpha, \alpha \in \mathcal{A}, \succ))$ in S is *M-convergent* to $s_0 \in \mathcal{S}$ if the following conditions are satisfiied:

(M_1) For each $\varepsilon > 0$ and $t* \in T$, there exists $\alpha_0 = \alpha_0(\varepsilon, t*)$ and $t** = t**(t*, \varepsilon)$ such that for all $\alpha \succ \alpha_0$:

(*a*) $t** >> (\varphi_\alpha)\, t*$, $t** >> (\varphi_0)\, t*$;

(*b*) For all $c_0 \in \mathcal{C}_0$, $c_\alpha \in \mathcal{C}_\alpha$, for all $I \in C_{t**}$

$\|c_0\,\zeta_0(I) - c_\alpha\,\zeta_\alpha(I)\| < \varepsilon$ and $\sup_{J \subset I} \|c_0\,\zeta_0(I) - c_\alpha\,\zeta_\alpha(j)\| < \varepsilon$.

(M_2) For each $\varepsilon > 0$, for each $I \in \{I\}$, there exists $\alpha_0^\# = \alpha_0^\#(\varepsilon, I)$ such that for each $\alpha \succ \alpha_0^\#$ there exists $t_I^\# = t_I^\#(\alpha,\varepsilon)$ such that for every $t_I^{\#\#} >> (\varphi_\alpha)\, t_I^\#$ we may find a subsystem $C'_{t_I^{\#\#}} \subset C_{t_I^{\#\#}}$ satisfying

$$|([\varphi_0(I) - \Sigma' \varphi_\alpha(J)], e_1)| < \varepsilon,$$

Σ' a sum over all $J \in C'_{t_I^{\#\#}}$, e_1 an element of the orthonormal basis $\{e_n\}$.

We remark that the second condition of M-convergence is related to a lemma of McShane used in surface area theory (see [10, p. 923]). Furthermore, the use of a special type of convergence is not new; thus the « V-convergence » of [8, p. 89] is in the same spirit as M-convergence although quite different in content. Finally we shall assume that isometrys preserve M-convergence, that is, if $m: H \to H$ is an isometry and s_α is M-convergent to s_0 then ms_α is M-convergent to ms_0.

Assuming axioms (P_1), (P_2), we are now in a position to prove a general theorem concerning the lower semi-continuity of integrals $\int f(\zeta,\varphi)$.

(5.1) THEOREM. Consider a pair (ζ_0, φ_0) satisfying the following conditions:

(1) There exists $\varrho > 0$ such that the set $\mathcal{N} = \{p : \|p - \cup\,\zeta_0(I)\| < \varrho\}$ has the property that $f(p,q) \geq 0$ for $p \in \mathcal{N}$, $q \in H$. Assume $D \supset \mathcal{N}$

(2) At each point $(p_0, q_0), p_0 \in \cup \zeta_0(I), \|q_0\| - 1$, for given $\varepsilon > 0$, there exists a $\varrho_0 > 0$, a weakly continuous linear functional $w_0: H \to R$ (hence $w_0(h) = (h, a_0)$ for some $a_0 \in H$), and weak neighborhood $W(p_0; z_1, \ldots \ldots, z_n; 3\varrho_0)$ of p_0 such that for all $p \in W(p_0; z_1, \ldots, z_n; 3\varrho_0)$ we have

(a) $f(p, q) \geq w_0(q)$ for every $q \in H$;

(b) $f(p, q) \leq w_0(q) + \varepsilon/7M \|q\|$ for all $q \in H$ such that $q/\|q\| \in W^*(q_0; z_1^*, \ldots, z_n^*; 3\varrho_0)$, W^* a certain weak neighborhood of q_0. Here M is a constant to be defined below.

(3) Condition (f) is satisfied by f, and $\mathcal{J}((\zeta, \varphi), A)$ exists for each $(\zeta, \varphi) \in \mathcal{S}$.

(4) Each $(\zeta, \varphi) \in \mathcal{S}$ is such that condition (ζ) holds for the map ζ while φ is of bounded variation.

(5) $\cup \zeta_0(I)$ is a weakly compact subset of H.

Then $\int f(\zeta, \varphi)$ is lower semi-continuous at (ζ_0, φ_0) with respect to M-convergence.

Proof. Let $\varepsilon > 0$ be given. Let M be a constant such that

$$M \geq \sup \{f(p, q) : (p, q) \in \mathcal{H} \times S^1\}, M \geq V(\|\varphi_0\|).$$

From (2), for every point $p_0 \in \cup \zeta_0(I)$ and $q_0 \in S^1$, there exists $\varrho_0 > 0$, a weakly continuous linear functional $w_0(q) = (a_0 \cdot q)(q \in H)$, and weak neighborhoods $W(p_0, z_1, \ldots, z_n; 3\varrho_0)$, $W^*(q_0; z_1^*, \ldots, z_m^*; 3\varrho_0)$ of p_0, q_0 respectively such that for all $p \in W$ we have

(5.1) (a) $f(p, q) \geq w_0(q)$ for all $q \in H$;

(b) $f(p, q) \leq w_0(q) \leq w_0(q) + \varepsilon/7M \|q\|$ for all $q \in H$ such that $q/\|q\| \in W^*$.

Since S^1 is weakly compact, invoking (5) we see that $Y = \cup \zeta_0(I) \times S^1$ is compact in $H_w \times H_w$. Hence Y may be covered by a finite number of the open sets

$$W(p_0; z_1, \ldots, z_n; \varrho_0) \times W^*(q_0; z_1^*, \ldots, z_m^*; \varrho_0).$$

Let the centers of such a covering be (p_i, q_i), $i = 1, \ldots, r$, let the «radii» be ϱ_i, let the associated functions be $w_i(q) = (a_i, q)$, and let the neighborhoods be

$$W_i(p_i; z_1^i, \ldots, z_{n_i}^i; \varrho_i), \; W_i^*(q_i; z_1^{*i}, \ldots, z_{m_i}^{*i}; \varrho_i).$$

Now by assumption each w_i is a weakly continuous linear functional on H and hence a fortiori is strongly continuous. We claim that $\|w_i\| = \|a_i\| \leq M$. In fact, we find from (2) — (a) above that $M \geq f(p_i, a_i/\|a_i\|) \geq w_i(a_i/\|a_i\|) = \|a_i\|$ as desired. Let $\varrho < \text{Min}[\varepsilon/7M, \varrho, \varrho_i, \varrho_i/\|z_j^i\|, \varrho_i/z_k^i\| :$

$i = 1, \ldots, r; j = 1, \ldots, n_i; k = 1, \ldots, m_i]$. With σ playing the role of ε in the definition of condition (ζ_0), we may find a $t'' \in T$ so that, among other things, $t >> t''$ yields $(>> = >> (\varphi_0))$

$$\| c_0 \zeta_0 (J) - c_0' \zeta_0 (J) \| < \sigma \text{ for } c_0, c_0' \in \mathcal{C}_0, J \in C_t. \tag{5.2}$$

We chose $\bar{t}$ such that $t_* >> \bar{t}$ implies

$$| \mathcal{I}((\zeta_0, \varphi_0), A) - \sum_{I_* \in C_{t_*}} f(\bar{c}_0 \zeta_0 (I_*), \varphi_0 (I_*)) | < \varepsilon/7. \tag{5.3}$$

Here $\bar{c}_0$ is an arbitrary element of $\mathcal{C}_0$ which will remain fixed throughout the remainder of the proof. Now the point $(\bar{c}_0 \zeta_0 (I_*), \varphi_0 (I_*)/\| \varphi_0 (I_*) \|)$ is in one of the sets of the cover above (if $\| \varphi_0 (I_*) \| = 0$, $f(\bar{c}_0 \zeta_0 (I_*), \varphi_0 (I_*) = 0$ and thus will not effect our calculations). Denote this element of the cover by

$$W_{i(I_*)} (p_{i(I_*)}; z_1^{i(I_*)}, \ldots, z_{n_{i(I_*)}}^{i(I_*)}; \varrho_{i(I_*)}) \times W^*_{i(I_*)} (q_{i(I_*)},$$

$$z_1^{*i(I_*)}, \ldots, z_{m_{i(I_*)}}^{*i(I_*)}, \varrho_{i(I_*)})$$

or more simply by

$$W_{i(I_*)} (\varrho_{i(I_*)}) \times W^*_{i(I_*)} (\varrho_{i(I_*)}).$$

Suppose the net $(s_\alpha, \alpha \in (\mathcal{A}, \succ))$ is M-convergent to s_0. In the definition (M_1) of M-convergence, let σ play the role of ε, and take $t* >> (t'', \bar{t})$. Hence there exists $\alpha_0 \in \mathcal{A}$ and $t** \in T$ such that the relations stated there hold. In particular, $t** >> (\varphi_\alpha)\, t*$ for all $\alpha \succ \alpha_0$ and $t** >> t*$. Thus $t** >> (t'', \bar{t})$ and relations (5.2), (5.3) above are in force. We shall let $t_* = t**$. Let N be the number of elements $I_* \in C_{t_*}$. Associate with each $I_* \in C_{t_*}$ an isometry $m_{i(I_*)}$ which will be defined below. Then for each $I_* \in C_{t_*}$, $m_{i(I_*)} s_\alpha = (m_{i(I_*)} \zeta_\alpha, m_{i(I^*)} \varphi_\alpha) = (\zeta_\alpha', \varphi_\alpha')$ is M-convergent to $m_{i(I_*)} s_0 = (m_{i(I_*)} \zeta_0, m_{i(I_*)} \varphi_0) = (\zeta_0', \varphi_0')$ in view of our general hypotheses. We remember that by assumption the orderings $>>_{I_*} (m_{i(I_*)} \varphi_\alpha)$ and $>>_{I_*} (\varphi_\alpha)$ are comparable. Employing part (M_2) of the definition of M-convergence, use the constant $\sigma^* = \mathrm{Min}(\sigma, \varepsilon/7MN)$ for $m_{i(I_*)} s_0$, and thus obtain N elements $\alpha_0^\# (\sigma^*, I_*)$, one for each $I_* \in C_{t_*}$. Let $\bar{\alpha} \succ \{\alpha_0^\# (\sigma^*, I_*); I_* \in C_{t_*}\}$. Consider any $\alpha \succ \bar{\alpha}$. Then we infer the existence of N elements $t_{I_*}^\# = t_{I_*}^\# (\alpha, \sigma^*)$, one for each $I_* \in C_{t_*}$, such that if $t_{I_*}^{\#\#} >> I_* (\varphi_\alpha)\, t_{I_*}^\#$, then there is a subsystem $C'_{t_{I_*}^{\#\#}} \subset C_{t_{I_*}^{\#\#}}$ satisfying

$$| (\varphi_0' (I_*) - \Sigma' \varphi_\alpha' (J), e_1) | < \varepsilon/7MN. \tag{5.4}$$

Odserve that it may be assumed $t^{\#}_{t_*}$ so selected as to ensure in addition that if $t^{\#\#}_{I_*} >>_{I_*}(\varphi_a)\, t^{\#}_{I_*}$, then

$$(5.5) \qquad | \mathcal{J}((\zeta_a, \varphi_a), I_*) - \Sigma_{I_*} f(\bar{c}_a \zeta_a(J), \varphi_a(J)) | < \varepsilon/7N,$$

$\bar{c}_a$ being a fixed choice function in $\mathcal{C}_a$. Here, in (5.4) and (5.5), Σ' denotes a sum over all $J \in C'_{t^{\#\#}_{I_*}}$, Σ_{I_*} a sum over all $J \in C_{t^{\#\#}_{I_*}}$.

Next we note that for any $c_0 \in \mathcal{C}_0$ and any $I_* \in C_{t_*}$ we have, for arbitrary $z_j^{i(I_*)}\ (j = 1, \dots, n_{i(I_*)})$

$$\begin{aligned} &| z_j^{i(I_*)}(c_0 \zeta_0(I_*) - p_{i(I_*)}) | \\ &\quad \leq | z_j^{i(I_*)}(c_0 \zeta_0) - \bar{c}_0 \zeta_0(I_*) | + | z_j^{i(I_*)}(\bar{c}_0 \zeta_0(I_*) - p_{i(I_*)}) | \\ &\quad \leq \| z_j^{i(I_*)} \| \sigma + \varrho_{i(I_*)} < 2\, \varrho_{i(I_*)} \end{aligned}$$

where we used relation (5.2) in passing from the second to the third line.

Similarly, for arbitrary $c_a \in \mathcal{C}_a$,

$$| z_j^{i(I_*)}(c_a \zeta_a(I_*) - p_{i(I_*)}) | \leq 2\varrho_{i(I_*)}.$$

Hence we may conclude $\zeta_0(I_*) \subset W_{i(I_*)}(3\varrho_{i(I_*)})$, $\zeta_a(I_*) \subset W_{i(I_*)}(3\varrho_{i(I_*)})$, for each $I_* \in C_{t_*}$. Therefore, inequality (5.1)-(*a*) holds for all $q \in H_0(I_*)$ or $p \in \zeta_a(I_*)$, and (5.1)-(*b*) for $\varphi_0(I_*)/\| \varphi_0(I_*) \|$. Then

$$\begin{aligned} &\mathcal{J}((\zeta_a, \varphi_a), A) - \mathcal{J}((\zeta_0, \varphi_0), A) = \\ &\quad = [\mathcal{J}((\zeta_a, \varphi_a, A) - \sum_{I_* \in C_{t_*}} \mathcal{J}((\zeta_a, \varphi_a), I_*)] \\ &\quad + \sum_{I_* \in C_{t_*}} [\mathcal{J}((\zeta_a, \varphi_a), I_*) - \Sigma_{I_*} f(\bar{c}_a \zeta_q(J), \varphi_a(J))] \\ &\quad + \sum_{I_* \in C_{t_*}} [\Sigma_{I_*} f(\bar{c}_a \zeta_a(J), \varphi_a(J)) - \Sigma' f(\bar{c}_a \zeta_a(J), \varphi_a(J)] \\ &\quad + \sum_{I \in C_{t_*}} [\Sigma' f(\bar{c}_a \zeta_a(I), \varphi_a(J)) - (a_{i(I_*)}, \varphi_a(J))] \\ &\quad + \sum_{I_* \in C_{t_*}} [(a_{i(I_*)}, \Sigma' \varphi_a(J) - \varphi_0(I_*))] \\ &\quad + \sum_{I_* \in C_{t_*}} [a_{i(J_*}, \varphi_0(I_*)) - f(\bar{c}_0 \zeta_0(I_*), \varphi_0(I_*))] \\ &\quad + \sum_{I_* \in C_{t_*}} f(\bar{c}_0 \zeta_0(I_*), \varphi_0(I_*)) - \mathcal{J}((\zeta_0, \varphi_0), A)] \\ &\quad = \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4 + \sigma_5 + \sigma_6 + \sigma_7. \end{aligned}$$

We shall now examine the sums $\sigma_i (i = 1, \dots, 7)$. It follows from axiom (P_2) that $\sigma_1 \geq 0$. Relation (5,5) implies $|\sigma_2| < \varepsilon/7$. Since

$$\sup_{\substack{J \subset I \\ c_0 \in \mathcal{C}_0^*,\, c_\alpha \in \mathcal{C}_\alpha}} \| c_0 \zeta_0 (I_*) - c_\alpha \zeta_\alpha (J) \| < \sigma < \varrho .$$

since $C'_{t_{I_*}} \subset C_{t_{I_*}}$, and since $f(p, q) \geq 0$ for $p \in \mathcal{N} = \{ p : \| p - \cup \zeta_0 (I) \| < \varrho \}$, $q \in H$, we see that $f(\overline{c_\alpha} \zeta_\alpha (J), \varphi_\alpha (J)) \geq 0$ and hence $\sigma_3 \geq 0$. Next, given $I_* \in C_{t_*}$ and $J \subset I_*$, an argument used before shows that

$$| z_j^{i(I_*)} (\overline{c_\alpha} \zeta_\alpha (J) - p_{i(I_*)}) | < 3 \varrho_{i(I_*)} .$$

Thus, in virtue of (5.1)-(a), $\sigma_4 \geq 0$. To discuss σ_5, observe that our isometry $m_{i(I_*)}$ may be chosen so that $a_{i(I_*)} / \| a_{i(I_*)} \| = m^*_{i(I_*)} e_1$, $m^*_{i(I_*)}$ denoting the adjoint of $m_{i(I_*)}$. Hence

$$| (a_{i(I_*)}, \Sigma' \varphi_\alpha (J) - \varphi_0 (I_*)) | \leq$$

$$\leq \| a_{i(I_*)} \| \, | (m^*_{i(I_*)} e_1 , \Sigma' \varphi_\alpha (J) - \varphi_0 (I_*)) |$$

$$\leq \| a_{i(I_*)} \| \, | (e_1 , m_{i(I_*)} (\Sigma' \varphi_\alpha (J) - \varphi_0 (I_*)) |$$

$$\leq \| a_{i(I_*)} \| \, | (e_1 , (\Sigma' \varphi'_\alpha (J) - \varphi'_0 (I_*)) |$$

$$< \varepsilon / 7MN \| a_{i(I_*)} \| \text{ (by relation (5.4).)}$$

Consequently

$$| \sigma_5 | < \varepsilon / 7MN \sum_{I_* \in C_{t_*}} \| a_{i(I_*)} \| \leq \varepsilon / 7MN \, MN = \varepsilon / 7,$$

where we recalled $\| a_{i(I_*)} \| = \| w_{i(I_*)} \| \leq M$.

From (5.1)-(b) we deduce

$$f (\overline{c_0} \zeta_0 (I_*), \varphi_0 (I_*)) \leq (a_{i(I_*)}, \varphi_0 (I_*)) + \varepsilon / 7M \| \varphi_0 (I_*) \|.$$

Therefore

$$\sigma_6 \geq \sum_{I_* \in C_{t_*}} - \varepsilon / 7M \| \varphi_0 (I) \| \geq \varepsilon / 7M (- V) \geq - \varepsilon / 7.$$

Relation (5.3) implies $| \sigma_7 | < \varepsilon / 7$.

Thus we see that

$$\mathcal{I}((\zeta_\alpha, \varphi_\alpha), A) - \mathcal{I}((\zeta_0, \varphi_0), A) > -4\,\varepsilon/7 > -\varepsilon$$

for all $\alpha \succ \bar{\alpha}$, and from this it follows that I is lower-semicontinuous at (ζ_0, φ_0) with respect to M-convergence. This completes the proof.

We shall refer to condition (1) in (5.1) by saying that f is « non-negative near s_0, » and condition (2) by saying that f is « s_0-convex. »

We now give two illustrations of the preceding theorem.

EXAMPLE 1. Let the set A be a finite closed interval, say $[0, 1]$, of the line; let $\{I\}$ be the class of all intervals $I = [a, b] \subset A$, and $T = \mathcal{D}$ the class of all finite subdivisions $t = D = [I_j : j = 1, \dots, N]$ of $[0, 1]$. Consider the collection $\Omega = \Omega(L)$ of all continuous parametric rectifiable curves C: $x = x(w)$, $0 \leq w \leq 1$, $x = (x_1, \dots, x_k)$ with range in some set $L \subset E^k$. Then with each such x we may associate an interval function $\zeta_x(I) = \zeta(I) = \bigcup_{W \in I} x(w)$. Each ζ_x satisfies condition (ζ_x). Moreover put $\varphi_x(I) = \varphi(I) = (\varphi_1, \dots, \varphi_k)$, $\varphi_r(I) = x_r(b) - x_r(a)$, $r = 1, \dots, k$, for every $I = [a, b] \subset A$. Our collection $\mathcal{S}$ will then be the collection of all pairs (ζ_x, φ_x). Given a pair (ζ_x, φ_x), the set $T = \mathcal{D}$ is directed by the mesh function $\delta_x(D) = h_x(D) + \max |I_j|$, where $h_x(D) = \max_{I \in D}$ (oscillation of x on I), if we require that $D_1 = t_1 >> (\varphi_x)\, t_2 = D_2$ provided $\delta_x(D_1) \leq \delta_x(D_2)$. Given x, an orthogonal transformation $m: E^k \longrightarrow E^k$ determines a new map mx and corresponding ζ_{mx} in the obvious way. Moreover, it is known that $\varphi_{mx} = m\,\varphi_x$ (see [9, p. 21]), hence condition (m) is satisfied by the collection $\mathcal{S}$. Suppose $\{x_n\}$ is a sequence of continuous parametric rectifiable curves which converge uniformly to a continuous parametric rectifiable curve x_0, that is, $\sup_{W \in A} \| x_n(w) - x(w) \| \longrightarrow 0$ as $n \longrightarrow \infty$. Then we claim that the sequence (ζ_n, φ_n) is M-convergent to (ζ_0, φ_0). In fact, let $\varepsilon > 0$ and $I = [a, b] \subset A$ be given. Let us first verify (M_1) — (a) Evidently, for this purpose. it suffices to show that there exists an integer N_0 and a system $t{**} = D{**} \in \mathcal{D}$ such that for all $n \geq N_0$ we have $\delta_n(D{**}) < \varepsilon$ and $\delta_0(D{**}) < \varepsilon$. Since x_n converges uniformly to x_0, these functions $(x_n, n = 0, 1, \dots)$ are equicontinuous and so there is a rational number of the form $q^{-1} < \varepsilon/4$ such that $\| x_n(w') - x_n(w'') \| < \varepsilon/4$ if $| w' - w'' | < q^{-1}$. Subdivide $[0, 1]$ into q equal parts, and let $t{**} = D{**}$ denote the subdivision so obtained. Notice that for all n, $h_n(D{**}) < \varepsilon/4$. Choose N_0 such that $n \geq N_0$ implies $\| x_n(w) - x_0(w) \| < \varepsilon/4$ for all $w \in [0, 1]$. We then have

(a) $\delta_n(D{**}) = \operatorname{Max} | I_j | + h_n(D{**}) < \varepsilon/4 + \varepsilon/4 < \varepsilon$ for all $n = 0, 1, 2, \dots$.

(b) Fix $I \in D^{**} = C_{t_{**}}$, and let $n \geq N_0$. We consider $\| c_0 \zeta_0 (I) - c_n \zeta_n (I) \|$. Let $c_0 \zeta_0 (I) = x_0 (w')$, $c_n \zeta_n (I) = x_n (w'')$ where w', $w'' \in I$. Thus

$$\| c_0 \zeta_0 (I) - c_n \zeta_n (I) \|$$

$$\leq \| x_0 (w') - x_0(w'') \| + \| x_0 (w'') - x_n (w'') \| < \varepsilon.$$

The fact that $\sup_{J \subset I} \| c_0 \zeta_0 (I) - c_n \zeta_n (J) \| < \varepsilon$ is clear. Hence (M_1) is verified. Next, we check condition (M_2). In order to do this we first have to specify, for each $I \in \{I\}$ in accordance with (P_1), the set T_I and intervals $\{J\} \subset I$. So, if $I = [a, b] \subset [0, 1]$, we let T_I be the set of all finite subdivisions of $[a, b]$ and $\{J\} = \{[c, d] : [c, d] \subset I\}$. We can then define, in an analogous way to the case $A = [0, 1]$, the mesh functions δ_I and the orderings $>>_I (\varphi)$ of T_I. Having done this, we now observe, since one interval is as good as the next, that it is sufficient to check (M_2) for $I = [0, 1]$ only. But, noticing that for any subdivision $D = [I_j : j = 1, \ldots, N]$ of $[0, 1]$ whatsoever we have in general $\varphi ([0, 1]) = x (1) - x (0) = \Sigma_j \varphi (I_j)$, one easily sees that (M_2) is trivially true. Hence we may conclude: Uniform convergence of x_n to x_0 implies M-convergence of (ζ_n, φ_n) to (ζ_0, φ_0). Moreover, since uniform convegence is preserved under isometrys m, we infer that $(m\zeta_n, m\varphi_n)$ is M-convergent to $(m\zeta_0, m\varphi_0)$.

Now let $f : L \times E^k \to R$ be a parametric integrand satisfying the usual hypotheses of continuity (that is, condition (f)). For each $I \in \{I\}$, and pair $(\zeta, \varphi) \in \mathcal{S}$, it is evident that $(I) \int f (\zeta, \varphi)$ is meaningful (each φ is quasi-additive, see [5, p. 103]) and conditions (P_1), (P_2) above are satisfied. The integral which arises here then is nothing more than the classical Weierstrass integral over a rectifiable continuous curve. Observe that $\cup \zeta_0 (I) =$ range x_0 is a compact subset of E^k. Thus, if the conditions (1), (2) of Theorem (5.i) on f are satisfied, we are able to state a semi-continuity theorem in the calculus of variations due to Tonelli.

Theorem (Tonelli). Let $f : L \times E^k \to R$ satisfy condition (f). Let Ω be the class of all continuous rectifiable parametric curves $x : [0, 1] \to L \subset E^k$. If $x_0 \in \Omega$ is such that f is non-negative near $s_0 = (\zeta_0, \varphi_0)$ and is s_0-convex, then the integral $\mathcal{J} ((\zeta, \varphi), A)$ is lower semi-continuous at x_0 in Ω with the uniform topology.

EXAMPLE 2. We wish to deduce here a theorem proved by Turner [9, p. 112] on the semi-continuity of the Cesari-Weierstrass surface integral. The reader is referred to [12, Section 5] for the notations and definitions to be employed here. Let the admissible set A be the square. Let

$\}I\{$ be the class of all closed simple polygonal regions $I \subset A$, and $T = \mathcal{D}$ the class of all finite systems $D = [I] = [I_1, \dots, I_N]$ of non-overlapping regions $I \in \}I\{$. Let $\Omega = \Omega(L)$ be the class of all continuous BV mappings $x : A \longrightarrow L \subset E^3$ (in other words, continuous surfaces S in L with finite Lebesgue area). Then with each such x we may associate an interval function $\zeta(I) = \zeta_x(I) = \bigcup_{w \in I} (x(w))$. Moreover, we put

$$\varphi(I) = \varphi_x(I) = (\varphi_1(I), \varphi_2(I), \varphi_3(I)), \varphi_r(I) = v(x_r, I), r = 1,$$

2, 3 where $v(x_r, I)$ is the relative variation (signed area) of the plane mapping (x_r, I). It is known that φ is quasi-additive with respect to $\mathcal{D}$ and $\delta(D)$ [12, Section 5]. Our collection $\mathcal{S}$ will then be the collection of all pairs (ζ_x, φ_x). The set $T = \mathcal{D}$ is directed in the usual fashion for S_2-type systems. Given x, an orthogonal transformation $m : E^3 \longrightarrow E^3$ determines a new map mx and corresponding ζ_{mx} in the obvious way. The fact that $\varphi_{mx} = m\varphi_x$ is well-known [4]; hence condition (m) is satisfied by the collection $\mathcal{S}$. Suppose $\}x_n\{$ is a sequence of continuous parametric surfaces of finite Lebesgue area which converge uniformly to a continuous parametric surface x_0 of finite Lebesgue area, that is $\sup_{w \in A} \| x_n(w) - x_0(w) \| \longrightarrow 0$. Then we claim that the sequence $\}(\zeta_n, \varphi_n)\{$ is M-convergent to (ζ_0, φ_0). Let us first verify (M_1)-(a). It suffices to show that there exists an integer N_0 and a system $t_{**} = D_{**} \in \mathcal{D}$ such that for all $n \geq N_0$ we have $\delta_n(D_{**}) < \varepsilon$ and $\delta_0(D_{**}) < \varepsilon$. This fact, however, is a consequence of a lemma proved by L. Cesari [1. p. 1385]. Likewise. (M_1)-(b) is immediate (for details see [2, pp. 27-28]). Hence (M_1) holds. It remains to verify (M_2). In order to do this, we first have to specify for each $I \in \}I\{$, in accordance with (P_1), the set T_1 and intervals $\}J\{ \subset I$. So, given $I \in \}I\{$, we take for T_1 the set of all simple closed polygonal regions J such that $J \subset I$. We can then define the mesh functions δ_I and the orderings $>>_I(\varphi)$ of T_I in the usual way. Let $\varepsilon > 0$ be given along with $I \in \}I\{$. Let $\}e_1, e_2, e_3\{$ be the usual orthonormal basis of Euclidean space E_3. Then $(\varphi_0, e_1) = \varphi_{01}$ where φ_{01} is the signed area function of the plane mapping $r_1 x_0 = x_{01} : A \longrightarrow E^2$, r_1 denoting the projection of E^3 onto the yz plane. According to a lemma to be found in [11, p. 198], there is a number $\nu > 0$ with the following property:

For every continuous BV plane mapping $y : A \longrightarrow E^2$ with $\| y(w) - x_{01}(w) \| < \nu$ for all $w \in A$, there is an $\eta = \eta(\varepsilon, y) > 0$ such that every finite system $D \in \mathcal{D}$ with $\delta_y(D) < \eta$ has a subsystem $D' \subset D$ satisying

$$| \varphi_{01}(A) - \Sigma_{D'} \varphi_y(J) | < \varepsilon$$

where $\Sigma_{D'}$ denotes a sum over all $J \in D'$. The lemma is equally applicable if we take $A = I$, $I \in \}I\{$.

Hence choose N_0 so that $\sup_{w \in I} \| x_n(w) - x_0(w) \| < \nu_I$ for all $n \geq N_0$ (thus N_0 plays the role of $\alpha_0^{\#}$ in condition (M_2)). Then take any $n \geq N_0$; certainly $\sup_{w \in A} \| x_n(w) - x_0(w) \| < \nu_I$, and thus

$$\| x_{n1}(w) - x_{01}(w) \| = \| r_1 x_n(w) - r_1 x_0(w) \|$$

$$\leq \| x_n(w) - x_0(w) \| < \nu_I$$

on I too. Hence there exists $\eta_I = \eta_I(\varepsilon, x_n) > 0$ such that every $t_I^{\#\#} = D_I^{\#\#}$ with $\delta_{x_{n1}}(D_I^{\#\#}) < \eta_I$ has a subsystem $D_I'^{\#\#} \subset D_I^{\#\#}$ satisfying $| \varphi_{01}(I) - \Sigma' \varphi_{n1}(J) | < \varepsilon$ that is $| (\varphi_0(I) - \Sigma' \varphi_n(J)), e_1 | < \varepsilon$ where Σ' denotes a sum over all $J \in D_I'^{\#\#}$. So select $t_I^{\#} = D_I^{\#}$ with $\delta_{x_n}(D_I^{\#}) < \eta_I$. If $t_I^{\#\#} = D_I^{\#\#} >> (\varphi_n)\, t_I^{\#} = D_I^{\#}$, that is, if $\delta_{x_n}(D_I^{\#\#}) \leq \delta_{x_n}(D_I^{\#}) < \eta_I$, then $\delta_{x_{n1}}(D_I^{\#\#}) \leq \delta_{x_n}(D_I^{\#\#}) < \eta_I$ from a known property of the mesh δ (see [4, p. 331]). Therefore condition (M_2) follows. We conclude: Uniform convergence of $\}x_n\{$ to x_0 implies M-convergence of $\}(\zeta_n, \varphi_n)\{$ to (ζ_0, φ_0). Since uniform convergence is preserved under isometrys m, we infer that $\}(m\zeta_n, m\varphi_n)\{$ is M-convergent to $(m\zeta_0, m\varphi_0)$.

Let $f: L \times E^3 \rightarrow R$ be a parametric integrand with the usual properties. The conditions (P_1), (P_2) above are satisfied here, the integral (I) $\int f(\zeta, \varphi)$ being the Cesari-Weierstrass integral over a continuous BV surface. Finally, $\cup \zeta_0(I) =$ range x_0 is a compact subset of E^3. We can now deduce from Theorem (5.i) the Turner theorem.

Theorem (Turner). Let $f: L \times E^3 \rightarrow R$ satisfy condition (f). Let $\Omega = \Omega(L)$ be the class of all continuous surfaces of finite Lebesgue area $x: A \rightarrow L \subset E^3$. If $x_0 \in \Omega$ is such that f is non negative near $s_0 = (\zeta_0, \varphi_0)$ and is s_0 convex, then the integral $\mathcal{I}((\zeta, \varphi), A)$ is lower semi-continuous at x_0 in Ω with the uniform topology.

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