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**On one inequality in weighted L_p spaces connected with the
problem of existence of traces on hyperplanes**

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$$Zu(x) = u(x) \quad \forall x \in \Pi.$$
$$u_\rho(x_1, \dots, x_N) = u(x).$$
$$Zu(x) = (Zu)_{\rho'}(x_1, \dots, x_{N-1}) = u_{\rho}(x_1, \dots, x_{N-1}, 0).$$
$$B = \begin{pmatrix} B_1^1, B_2^1, \dots, B_N^1 \\ B_1^2, B_2^2, \dots, B_N^2 \\ \dots \dots \dots \dots \dots \dots \\ B_1^N, B_2^N, \dots, B_N^N \end{pmatrix}$$

$$\det B \neq 0$$

The Editorial Committee deeply regrets the author's untimely death.

The space of all infinitely differentiable functions in R^N with compact support will be denoted by $\mathcal{D}(R^N)$.

We use this notation: If $\xi = (\xi_1, \dots, \xi_n)$, $x = (x_1, \dots, x_n)$ then

$$\langle x, \xi \rangle = \sum_{i=1}^n \xi_i x_i.$$

If $\xi = (\xi_1, \dots, \xi_N)$, put $\bar{\xi} = (\xi_1, \dots, \xi_{N-1})$, $\xi = (\bar{\xi}, \xi_N)$ and similary $x = (\bar{x}, x_N)$. If A is a matrix then the inverse and transpose of A we denote by A^{-1} and A' resp., ξA is product of the vector ξ and the matrix A .

Let $u(x) = u(x_1, \dots, x_N)$ be a function of N variables, $u \in \mathcal{D}(R^N)$. Put

$$\mathcal{F}u(\xi) = \int_{R^N} e^{-i \langle \xi, x \rangle} u(x) dx$$

the Fourier transform of u . If u is a function of points in R^N then put

$$\mathcal{F}_{\mathcal{L}} u(\xi) = \mathcal{F}u_{\mathcal{L}}(\xi)$$

and similary for functions of $N - 1$ variables.

Let $u \in \mathcal{D}(R^N)$. Then by lemma 2.3 one has

$$\mathcal{F}_{\mathcal{L}'} Zu(\bar{\xi}) = \frac{1}{2\pi} |\det B|^{-1} \int_{-\infty}^{\infty} \mathcal{F}_{\mathcal{L}_0} u(\xi B^{-1'}) d\xi_N.$$

From properties of $\mathcal{F}_{\mathcal{L}_0} u$ we can deduce properties of $\mathcal{F}_{\mathcal{L}'} Zu$. So, in this paper we will study properties of the operator T given by

$$Tf(\bar{\xi}) = \frac{1}{2\pi} |\det B|^{-1} \int_{-\infty}^{\infty} f(\xi B^{-1'}) d\xi_N,$$

that is of $T = \mathcal{F}_{\mathcal{L}'} Z \mathcal{F}_{\mathcal{L}_0}$.

Properties of T are dependent on the position of hyperplane Π . Suppose

$$(1.1) \quad \Pi = \left\{ x = \sum_{i=1}^N x_i B^i, \sum_{i=1}^N x_i a_i = 0 \right\}$$

The inequality (1.5) can be rewritten in the form

$$|Z^* u|_{H_q^{(0)}(R^{N-1})} \leq C |u|_{H^{(\mathcal{K})}_q},$$

where $Z^* = \mathcal{F}_{\mathcal{L}'}^{-1} T \mathcal{F}_{\mathcal{L}_0}$.

For $u \in \mathcal{D}(R^N) \cap H_q^{(\mathcal{K})}(R^N)$ one has $Z^* u = Zu$. So $Z^* u$ can be treated as trace of u on Π .

Validity of (1.5) depends on the mutual position of the set

$${}_q\mathcal{K} = \{X \in R^N, q^{-1}X \in \mathcal{K}\}, \quad 1/q + 1/p = 1$$

and the $(N - r - 1)$ -dimensional simplex $\mathcal{O}$ given by the coordinate vectors

$$I_1 = (\underbrace{0, \dots, 0}_r, 1, 0, \dots, 0, 0)$$

$$I_2 = (\underbrace{0, \dots, 0}_r, 0, 1, \dots, 0, 0)$$

$$\dots \dots \dots$$

$$I_{N-r} = (0, \dots, 0, 0, 0, \dots, 0, 1).$$

Necessary and sufficient conditions for (1.5) are described in theorem 4.7 and theorem 6.7 (see remark 6.8). It must be ${}_q\mathcal{K} \cap \mathcal{O} \neq \emptyset$ and the set ${}_q\mathcal{K}$ must be in a certain sense «well distributed» with respect to $\mathcal{O}$.

In the following we also use this notation: if $x = (x_1, \dots, x_N) \in R^N$ then

$$x' = (x_1, \dots, x_r) \in R^r, x'' = (x_{r+1}, \dots, x_N) \in R^s, x = (x', x''), s = N - r.$$

Here the number r is given by (1.2).

2. Dual traces.

2.1 LEMMA. Let $u \in \mathcal{D}(R^N)$. Then

$$(2.1) \quad \mathcal{F}_{\mathcal{L}} u(\xi) = |\det B|^{-1} \mathcal{F}_{\mathcal{L}_0} u(\xi B^{-1}).$$

Proof. Using the substitutions $xB = y$ and $u_{\mathcal{L}}(x) = u_{\mathcal{L}_0}(xB)$ one obtains

$$\mathcal{F}_{\mathcal{L}} u(\xi) = |\det B|^{-1} \int_{R^N} e^{-i \langle y, \xi B^{-1'} \rangle} u_{\mathcal{L}_0}(y) dy = |\det B|^{-1} \mathcal{F}_{\mathcal{L}_0}(\xi B^{-1}).$$

By a similar argument one obtains

2.2 LEMMA. Let $u \in \mathcal{D}(R^N)$. Put $v(x) = u(x + x_0)$. Then

$$(2.2) \quad \mathcal{F}_{\mathcal{L}} v(\xi) = e^{i \langle x_0, \xi \rangle} \mathcal{F}_{\mathcal{L}} u(\xi).$$

2.3 LEMMA. Let $u \in \mathcal{D}(R^N)$. Let us denote $v = Zu$ on Π , the trace of u on the hyperplane Π . Then

$$(2.3) \quad \mathcal{F}_{\mathcal{L}'} v(\bar{\xi}) = \frac{1}{2\pi} |\det B|^{-1} \int_{-\infty}^{\infty} \mathcal{F}_{\mathcal{L}_0} u(\xi B^{-1'}) d\xi_N.$$

Proof: It is known

$$\mathcal{F}_{\mathcal{L}'} v(\bar{\xi}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathcal{F}_{\mathcal{L}} u(\xi) d\xi_N.$$

Using the lemma 2.1 one obtains (2.3).

If we put $u = \mathcal{F}_{\mathcal{L}_0}^{-1} f$ in (2.3) we have

$$Tf = \mathcal{F}_{\mathcal{L}'} Z \mathcal{F}_{\mathcal{L}_0}^{-1} f(\bar{\xi}) = \frac{1}{2\pi} |\det B|^{-1} \int_{-\infty}^{\infty} f(\xi B^{-1'}) d\xi_N.$$

2.4 DEFINITION. Let f be a measurable function in R^N such that the (Lebesgue) integral

$$\int_{-\infty}^{\infty} f(\xi B^{-1'}) d\xi_N$$

exists for a. e. $\bar{\xi} = (\xi_1, \dots, \xi_{N-1}) \in R^{N-1}$. Then the function $g = Tf$ of $N - 1$ variables given by

$$(2.4) \quad g(\bar{\xi}) = \frac{1}{2\pi} |\det B|^{-1} \int_{-\infty}^{\infty} f(\xi B^{-1'}) d\xi_N$$

is said to be the dual trace (in the basis $\mathcal{L}'$) of the (dual) function f on the hyperplane Π .

2.5 THEOREM. The dual trace of the function f is independent on the basis $\mathcal{L}$ in this sense: if $\mathcal{L}_* = \{B_*^1, \dots, B_*^N\}$ is another basis which fulfils our conditions,

$$(2.5) \quad B_* = \begin{pmatrix} B_*^1 \\ \vdots \\ B_*^N \end{pmatrix},$$

$$\begin{pmatrix} B_*^1 \\ \vdots \\ B_*^{N-1} \end{pmatrix} = C \begin{pmatrix} B^1 \\ \vdots \\ B^{N-1} \end{pmatrix},$$

where C is a regular $(N-1) \times (N-1)$ -matrix and

$$g^*(\bar{\xi}) = \frac{1}{2\pi} |\det B_*|^{-1} \int_{-\infty}^{\infty} f(\xi B_*^{-1'}) d\xi_N$$

then

$$g(\bar{\xi}) = |\det C| g^*(\bar{\xi} C').$$

In other words: if g is the Fourier transform $\mathcal{F}_{\mathcal{L}'} v$ of some function v defined on Π then $g^* = \mathcal{F}_{\mathcal{L}'_*} v$.

Proof. Put

$$\mathcal{B} = \begin{pmatrix} B^1 \\ \vdots \\ B^{N-1} \end{pmatrix}, \quad \mathcal{B}_* = \begin{pmatrix} B_*^1 \\ \vdots \\ B_*^{N-1} \end{pmatrix}.$$

Then there is a vector $d = (\bar{d}, d_N)$; $d_N \neq 0$ such that for

$$\mathcal{C} = \left(\frac{C}{\bar{d}} \middle| \frac{0}{d_N} \right),$$

one has

$$(2.6) \quad B_* = \mathcal{C} B,$$

$$B^{-1'} = \mathcal{C}' B_*^{-1'}$$

$$|\det B_*| = |\det C| |d_N| |\det B|$$

$$(\bar{\xi}, \xi_N) \mathcal{C}' = \left(\bar{\xi} C', \sum_{i=1}^N d_i \xi_i \right).$$

By (2.6) we have

$$\begin{aligned}
 g(\bar{\xi}) &= \frac{1}{2\pi} |\det B|^{-1} \int_{-\infty}^{\infty} f(\xi B^{-1'}) d\xi_N = \\
 &= (\text{using the substitution } \xi_N = d_N^{-1} \tau) = \\
 &= \frac{1}{2\pi} |d_N|^{-1} |\det B|^{-1} \int_{-\infty}^{\infty} f((\bar{\xi}, \tau/d_N) B^{-1'}) d\tau = \\
 &= \frac{1}{2\pi} |\det C| |\det B_*|^{-1} \int_{-\infty}^{\infty} f((\bar{\xi}, \tau/d_N) C' B_*^{-1'}) d\tau = \\
 &= |\det C| \frac{1}{2\pi} |\det B_*|^{-1} \int_{-\infty}^{\infty} f\left(\left(\bar{\xi} C', \sum_{i=1}^{N-1} d_i \xi_i + \tau\right) B_*^{-1'}\right) d\tau = \\
 &= \left(\text{using the substitution } \sigma = \sum_{i=1}^{N-1} d_i \xi_i + \tau\right) = \\
 &= |\det C| g^*(\bar{\xi} C').
 \end{aligned}$$

This completes the proof of the first part of theorem 2.5.

Let, now, $g = \mathcal{F}_{\mathcal{Q}'} v$. Then by lemma 2.1 one has

$$(2.7) \quad \mathcal{F}_{\mathcal{Q}'_*} v(\bar{\xi}) = |\det D|^{-1} \mathcal{F}_{\mathcal{Q}'} v(\bar{\xi} D^{-1'}),$$

where D is the matrix of coordinates of vectors $B_*^1, \dots, B_*^{N-1}$ in the basis $\mathcal{Q}'$.

We have $\mathcal{B}_* = C\mathcal{B}$ and so $D = C$. By (2.7) we have

$$g(\bar{\xi}) = \mathcal{F}_{\mathcal{Q}'} v(\bar{\xi}) = |\det C| \mathcal{F}_{\mathcal{Q}'_*} v(\bar{\xi} C') = |\det C| g^*(\bar{\xi} C')$$

and so

$$g^* = \tilde{\mathcal{F}}_{\mathcal{Q}'_*} v.$$

This completes the proof.

3. Conditions for continuity of the operator T .

3.1 THEOREM. The operator T is continuous from $\mathcal{M}_p^{(k)}$ into L_p ($p > 1$) iff for $1/q = 1 - 1/p$ one has

$$(3.1) \quad |T| = \frac{1}{2\pi} |\det B|^{-1/q} \left(\sup_{\xi \in R^{N-1}} \int_{-\infty}^{\infty} [\mathcal{P}_\kappa(\xi B^{-1'})]^{-q} d\xi \right)^{1/q} < \infty.$$

Proof. Let $g = Tf$, $\tilde{h}(\xi) = h(\bar{\xi})$, $u = \mathcal{P}_\kappa f$. Then

$$\begin{aligned} |g|_{L_p} &= \sup_{|h|_{L_q}=1} \int_{R^{N-1}} \tilde{h}(\bar{\xi}) g(\bar{\xi}) d\bar{\xi} = \\ &= \sup_{|h|_{L_q}=1} \frac{1}{2\pi} |\det B|^{-1} \int_{R^N} \tilde{h}(\xi) f(\xi B^{-1'}) d\xi \\ &= \sup_{|h|_{L_q}=1} \frac{1}{2\pi} |\det B|^{-1} \int_{R^N} \tilde{h}(\xi) \frac{u(\xi B^{-1'})}{\mathcal{P}_\kappa(\xi B^{-1'})} d\xi \end{aligned}$$

and

$$\begin{aligned} |T| &= \sup_{|f|_{\mathcal{M}_p^{(k)}}=1} |Tf|_{L_p} = \\ &= \frac{1}{2\pi} |\det B|^{-1} \sup_{|u|_{L_p}=1} \sup_{|h|_{L_q}=1} \int_{R^N} \tilde{h}(\xi) \frac{u(\xi B^{-1'})}{\mathcal{P}_\kappa(\xi B^{-1'})} d\xi = \\ &= (\text{using the substitution } \xi = \eta B') = \\ &= \frac{1}{2\pi} \sup_{|h|_{L_q}=1} \sup_{|u|_{L_p}=1} \int_{R^N} \frac{\tilde{h}(\eta B')}{\mathcal{P}_\kappa(\eta)} u(\eta) d\eta = \frac{1}{2\pi} \sup_{|h|_{L_q}=1} \left| \frac{h(\tilde{\eta} B')}{\mathcal{P}_\kappa(\eta)} \right|_{L_q} = \\ &= \frac{1}{2\pi} \sup_{|h|_{L_q}=1} \left(\int_{R^N} \left[\frac{\tilde{h}(\eta B')}{\mathcal{P}_\kappa(\eta)} \right]^q d\eta \right)^{1/q} = (\text{using the substitution } \eta = \xi B^{-1'}) = \\ &= \frac{1}{2\pi} \sup_{|h|_{L_q}=1} |\det B|^{-1/q} \left(\int_{R^{N-1}} |h(\bar{\xi})|^q \left(\int_{-\infty}^{\infty} |\mathcal{P}_\kappa(\xi B^{-1'})|^{-q} d\xi_N \right) d\bar{\xi} \right)^{1/q} = \\ &= \frac{1}{2\pi} |\det B|^{-1/q} \left(\sup_{\xi \in R^{N-1}} \int_{-\infty}^{\infty} [\mathcal{P}_\kappa(\xi B^{-1'})]^{-q} d\xi_N \right)^{1/q}. \end{aligned}$$

This completes the proof.

The main aim of this paper is to find necessary and sufficient conditions for K and B for (3.1) to hold, that is to estimate the integral

$$(3.2) \quad \int_{-\infty}^{\infty} [\mathcal{P}_{\mathcal{K}}(\xi B^{-1'})]^{-q} d\xi_N.$$

Without loss of generality we can suppose $q = 1$.

In the following, it is denoted by $I'' = (1, \dots, 1)$, $I'_k = (0, \dots, 0, 1, 0, \dots, 0)$, where 1 is on the place k , $I = (0, I'')$, $I_k = (0, I'_k)$.

3.2 LEMMA. Let

$$(3.3) \quad B = \begin{bmatrix} 1, 0, \dots, \dots, & 0 \\ 0, 1, \dots, \dots, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, \dots, 0, 1, 0, \dots, & 0 \\ 0, \dots, 0, 1, 0, \dots, & a_{r+1} \\ 0, \dots, 0, 1, 0, \dots, & a_{r+2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, \dots, \dots, 0, 1, & a_{N-1} \\ 0, \dots, \dots, \dots, 0, 1 \end{bmatrix} \left. \vphantom{\begin{bmatrix} 1, 0, \dots, \dots, & 0 \\ 0, 1, \dots, \dots, & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, \dots, 0, 1, 0, \dots, & 0 \\ 0, \dots, 0, 1, 0, \dots, & a_{r+1} \\ 0, \dots, 0, 1, 0, \dots, & a_{r+2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0, \dots, \dots, 0, 1, & a_{N-1} \\ 0, \dots, \dots, \dots, 0, 1 \end{bmatrix}} \right\} s$$

where $a_{r+1} \neq 0, \dots, a_{N-1} \neq 0$. Then

$$(3.4) \quad \sup_{\xi \in \mathbb{R}^{N-1}} \int_{-\infty}^{\infty} [\mathcal{P}_{\mathcal{K}}(\xi B^{-1'})]^{-1} d\xi_N$$

is finite iff it is finite the number

$$(3.5) \quad \gamma(\mathcal{K}, s) = \sup_{\xi \in \mathbb{R}^N} \int_{-\infty}^{\infty} [\max_{A \in \mathcal{K}} |\xi'|^{A'} |\xi'' - \tau I''|^{A''}]^{-1} d\tau.$$

Proof. One has

$$\xi B^{-1'} = (\xi_1, \dots, \xi_r, \xi_{r+1} + a_{r+1} \xi_N, \dots, \xi_{N-1} + a_{N-1} \xi_N, \xi_N).$$

Using the substitutions

$$\xi_N = \eta_N - \tau, \quad a_i \xi_N + \xi_i = \eta_i \quad (i = r+1, \dots, N-1)$$

one has

$$(3.4) \quad \sup_{\eta \in R^N} \int_{-\infty}^{\infty} [\max_{A \in \mathcal{K}} |\eta'|^{A'} |a_{r+1}|^{Ar+1} \dots |a_{N-1}|^{AN-1} |\eta'' - \tau I''|^{A''}]^{-1} d\tau$$

$$= [\max_{A \in \mathcal{K}} |a_{r+1}|^{Ar+1} \dots |a_{N-1}|^{AN-1}]^{-1} \sup_{\xi \in R^N} \int_{-\infty}^{\infty} [\mathcal{P}_{\mathcal{K}}((\xi', \xi'' - \tau I''))]^{-1} d\tau$$

and the proof is finished.

Take $k, 1 \leq k \leq s$ and put

$$\begin{aligned} \tilde{\xi}_k &= (|\xi_1|, \dots, |\xi_{r+k-1}|, |\xi_{r+k+1}|, \dots, |\xi_N|) \\ \xi_k(\tau) &= (|\xi_1|, \dots, |\xi_{r+k-1}|, \tau, |\xi_{r+k+1}|, \dots, |\xi_N|) \\ \eta^k(\tau) &= (|\eta_1|, \dots, |\eta_r|, |\eta_{r+1} - \eta_k|, \dots, |\eta_{r+k-1} - \eta_k|, \tau, \\ &\quad |\eta_{r+k+1} - \eta_k|, \dots, |\eta_N - \eta_k|) \end{aligned}$$

for $\xi = (\xi_1, \dots, \xi_N)$, $\eta = (\eta_1, \dots, \eta_N)$.

Further put $(i_1 < i_2 < \dots < i_\ell)$

$$\begin{aligned} P_{i_1, \dots, i_\ell} \xi &= (\xi_1, \dots, \xi_{i_1-1}, \xi_{i_1+1}, \dots, \xi_{i_2-1}, \xi_{i_2+1}, \dots, \xi_{i_\ell-1}, \\ &\quad \xi_{i_\ell+1}, \dots, \xi_N, \xi_{i_1} + \dots + \xi_{i_\ell}). \end{aligned}$$

3.3 LEMMA. Let $1 \leq k \leq s$. Then

$$(3.6) \quad \sup_{\eta \in R^N} \int_0^{\frac{1}{2} \min_{r < i \neq j \leq N} |\eta_i - \eta_j|} [\mathcal{P}_{\mathcal{K}}(\eta^k(\tau))]^{-1} d\tau$$

is finite iff

$$(3.7) \quad \gamma(\mathcal{K}, s, k) = \sup_{\tilde{\xi}_k \in R^{N-1}} \int_0^{\min_{\substack{r < i \leq N \\ i \neq k+r}} |\tilde{\xi}_i|} [\mathcal{P}_{\mathcal{K}}(\xi_k(\tau))]^{-1} d\tau < \infty.$$

Proof. To fix the ideas put $k = 1$. Obviously, (3.6) is less or equal to (3.7). Let $\tilde{\xi}_1$ be such that

$$\gamma(\mathcal{K}, s, 1) < 2 \int_0^{\min_{r+1 \leq i \leq N} |\xi_i|} [\mathcal{P}_{\mathcal{K}}(\xi_1(\tau))]^{-1} d\tau.$$

Without loss of generality, changing arrangement of indices, one can suppose

$$0 < |\xi_{r+2}| < \dots < |\xi_N|.$$

Put

$$\eta_i = \xi_i \quad (i = 1, \dots, r).$$

$$\eta_{r+1} = 0$$

$$\eta_{r+i} = \sum_{j=r+2}^{r+i} |\xi_j| \quad (i = 2, \dots, s).$$

Then

$$\min_{\substack{i \neq j \\ r < i, j \leq N}} |\eta_i - \eta_j| = \min_{1+r < i \leq N} |\xi_i|$$

and for $i = r+1, r < i \leq N$ one has

$$(3.8) \quad |\xi_i| \leq |\eta_i - \eta_{r+1}| = |\eta_i| \leq s |\xi_i|$$

it is

$$\gamma(\mathcal{K}, s, 1) \leq 2 \int_0^{\min_{r < i \neq j \leq N} |\eta_i - \eta_j|} [\mathcal{P}_{\mathcal{K}}(\xi_1(\tau))]^{-1} d\tau.$$

Using (3.8) and the substitution $\tau \rightarrow \frac{1}{2} \tau$ one has

$$\gamma(\mathcal{K}, s, 1) \leq C \int_0^{\frac{1}{2} \min_{r < i \neq j \leq N} |\eta_i - \eta_j|} [\mathcal{P}_{\mathcal{K}}(\eta^k(\tau))]^{-1} d\tau.$$

Now, we use this procedure for a set of $\tilde{\xi}_1$ of positive measure and finish the proof.

3.4 LEMMA. $\gamma(\mathcal{K}, s) < \infty$ iff $\gamma(\mathcal{K}, s, i) < \infty$ ($i = 1, 2, \dots, s$) and

$$\gamma(P_{i_1 i_2} \mathcal{K}, s-1) < \infty \quad (r+1 \leq i_1 < i_2 \leq N).$$

Proof. Put

$$\delta = \frac{1}{2} \min_{r < i \neq j \leq N} |\eta_i - \eta_j|$$

$$\mathcal{I}_k^+ = \left\langle \eta_{r+k}, \eta_{r+k} + \frac{\delta}{2} \right\rangle$$

$$\mathcal{I}_k^- = \left\langle \eta_{r+k} - \frac{\delta}{2}, \eta_{r+k} \right\rangle \quad (k = 1, \dots, s)$$

$$\mathcal{I}_0 = (-\infty, \infty) - \bigcup_{k=1}^s \mathcal{I}_k^+ \cup \mathcal{I}_k^-.$$

Let $\delta = \frac{1}{2} |\eta_{i_1+r} - \eta_{i_2+r}|$, $r < i_1 < i_2 \leq N$. Then for $\tau \in \mathcal{I}_0$ we have

$$\begin{aligned} |\tau - \eta_{i_1+r}| &\leq |\tau - \eta_{i_2+r}| + |\eta_{i_1+r} - \eta_{i_2+r}| = \\ &= |\tau - \eta_{i_2+r}| + 2\delta \leq 5 |\tau - \eta_{i_2+r}| \end{aligned}$$

and so

$$(3.9) \quad \frac{1}{5} |\tau - \eta_{i_2+r}| \leq |\tau - \eta_{i_1+r}| \leq 5 |\tau - \eta_{i_2+r}|.$$

Let $\delta \neq 0$. Then for $\tau \in \mathcal{I}_j^+ \cup \mathcal{I}_j^-$, $j \neq k$ we have

$$\begin{aligned} ||\tau - \eta_{k+r}| - |\eta_{j+r} - \eta_{k+r}|| &\leq |\tau - \eta_{j+r}| \leq \\ &\leq \frac{\delta}{2} \leq \frac{1}{4} |\eta_{j+r} - \eta_{k+r}| \end{aligned}$$

and so

$$(3.10) \quad \frac{3}{4} |\eta_{j+r} - \eta_{k+r}| \leq |\tau - \eta_{k+r}| \leq \frac{5}{4} |\eta_{j+r} - \eta_{k+r}|.$$

Now

$$\int_{-\infty}^{\infty} [\max_{A \in \mathcal{K}} |\eta'|^{A'} |\eta'' - \tau I''|^{A''}]^{-1} d\tau = \int_{\mathcal{I}_0} + \sum_{k=1}^s \left(\int_{\mathcal{I}_k^+} + \int_{\mathcal{I}_k^-} \right).$$

$\int_{-\infty}^{\infty}$ is a bounded function of η iff any of these integrals $\int_{\mathcal{I}_0}$, $\int_{\mathcal{I}_k^+}$, $\int_{\mathcal{I}_k^-}$ is a bounded

function of η . In the integral $\int_{\mathcal{I}_k^+}$ and $\int_{\mathcal{I}_k^-}$ we can use (for $\delta \neq 0$) (3.10) and write $|\eta_{j+r} - \eta_{k+r}|$ instead of $|\tau - \eta_{k+r}|$. Using the substitution $\eta_{k+r} - \tau \rightarrow \tau$ and lemma 3.3 we finally obtain that $\int_{\mathcal{I}_k^+}$ and $\int_{\mathcal{I}_k^-}$ are bounded iff $\gamma(\mathcal{K}, s, k) < \infty$.

For $i_1 \neq i_2$, $r < i_1 < i_2 \leq N$ put

$$\mathcal{M}_{i_1, i_2} = \{\eta, 0 \neq \min_{r < i_1 \neq j \leq N} |\eta_i - \eta_j| = |\eta_{i_1} - \eta_{i_2}|\}.$$

Then $\bigcup_{i_1 \neq i_2} \mathcal{M}_{i_1, i_2} = R^N - \mathcal{M}$, where $\mathcal{M}$ is a set of measure zero.

Let $\eta \in \mathcal{M}_{i_1, i_2}$. Then, using (3.9), we can write $|\tau - \eta_{i_1}|$ instead of $|\tau - \eta_{i_2}|$ in the integral $\int_{\mathcal{I}_0}$. So $\int_{\mathcal{I}_0} \leq C\gamma(P_{i_1, i_2}, \mathcal{K}, s - 1)$.

On the other hand if

$$\sup_{\eta \in R^N} \int_{\mathcal{I}_0} [\max_{A \in \mathcal{K}} |\eta'|^{A'} |\eta'' - \tau I''|^{A''}]^{-1} d\tau < \infty,$$

then taking $\eta_{i_1} \rightarrow \eta_{i_2}$ we have

$$\gamma(P_{i_1, i_2}, \mathcal{K}, s - 1) < \infty.$$

4. Sufficient conditions for $\gamma(\mathcal{K}, s, i) < \infty$.

4.1 LEMMA. Let $\mathcal{K} \in \mathfrak{p}$ and $A^{(0)} \in \mathcal{K}$. Then for $\xi_i \neq 0$ we have

$$|\xi|^{A^{(0)}} \leq \max_{A \in \mathcal{K}^e} |\xi|^A.$$

Proof. $A^{(0)} = \sum_{A \in \mathcal{K}^e} \lambda_A A$, $\lambda_A \geq 0$, $\sum_{A \in \mathcal{K}^e} \lambda_A = 1$ and so

$$\begin{aligned} |\xi|^{A^{(0)}} &= \prod_{A \in \mathcal{K}^e} (|\xi|^A)^{\lambda_A} \leq \prod_{A \in \mathcal{K}^e} (\max_{A \in \mathcal{K}^e} |\xi|^A)^{\lambda_A} = \\ &= (\max_{A \in \mathcal{K}^e} |\xi|^A)^{\sum \lambda_A} = \max_{A \in \mathcal{K}^e} |\xi|^A. \end{aligned}$$

4.2 LEMMA. If $\mathcal{K}_1 \subset \mathcal{K} \in \mathfrak{p}$ then

$$(4.1) \quad \mathcal{P}_{\mathcal{K}_1}(\xi) = \max_{A \in \mathcal{K}_1} |\xi|^A \leq \max_{A \in \mathcal{K}} |\xi|^A = \max_{A \in \mathcal{K}^e} |\xi|^A = \mathcal{P}_{\mathcal{K}}(\xi).$$

Proof. We use lemma 4.1.

4.3 LEMMA. If $\mathcal{K}_1 \subset \mathcal{K} \in \mathfrak{p}$ then

$$\gamma(\mathcal{K}_1, s) \geq \gamma(\mathcal{K}, s),$$

$$\gamma(\mathcal{K}_1, s, i) \geq \gamma(\mathcal{K}, s, i) \quad (i = 1, 2, \dots, s).$$

Proof. Lemma is immediate consequence of (4.1).

4.4 LEMMA. Let there be $A^{(0)} \in \mathcal{K} \in \mathfrak{p}$ such that

$$A_i^{(0)} = 0 \ (i = 1, \dots, r), \ 0 \leq A_i^{(0)} \ (i = r + 1, \dots, N), \ A_{k+r}^{(0)} < 1 \text{ and } \sum_{i=r+1}^N A_i^{(0)} = 1.$$

Then $\gamma(\mathcal{K}, s, k) < \infty$.

Proof. For example, let $k = s$. Then

$$\begin{aligned} \gamma(\mathcal{K}, s, s) &\leq \gamma(\{A^{(0)}\}, s, s) = \sup_{\bar{\xi} \in R^{N-1}} \sup_{\tau \in R^{N-1}} \int_0^{\min_{r < i < N} |\xi_i|} |\xi_s(t)|^{-A^{(0)}} d\tau = \\ &= \sup_{\bar{\xi} \in R^{N-1}} \int_0^{\min_{r < i < N} |\xi_i|} |\bar{\xi}|^{-\overline{A^{(0)}}} \tau^{-A_N^{(0)}} d\tau \leq \\ &\leq \sup_{\bar{\xi} \in R^{N-1}} |\bar{\xi}|^{-\overline{A^{(0)}}} (1 - A_N^{(0)})^{-1} \left(\min_{r < i < N} |\xi_i| \right)^{1-A_N^{(0)}} \leq \\ &\leq \sup_{\bar{\xi} \in R^{N-1}} \left(\min_{r < i < N} |\xi_i| \right)^{-\sum_{i=1}^{N-1} A_i^{(0)}} (1 - A_N^{(0)})^{-1} \left(\min_{r < i < N} |\xi_i| \right)^{1-A_N^{(0)}} \leq \frac{1}{1 - A_N^{(0)}} < \infty. \end{aligned}$$

4.5 LEMMA. Let $\mathcal{K}_1 \subset \mathcal{K} \in \mathfrak{p}$ and let $\mathcal{K}_1$ be a segment $\overline{A^{(1)} A^{(2)}}$, where $A^{(1)} = I_k + A^{(0)}$, $A^{(2)} = I_k - A^{(0)}$, $A_{r+k}^{(0)} > 0$. Then

$$\gamma(\mathcal{K}, s, k) < \infty.$$

Proof. Suppose $k = s$. Then

$$\begin{aligned} \frac{1}{2} \gamma(\mathcal{K}, s, s) &\leq \frac{1}{2} \gamma(\mathcal{K}_1, s, s) \leq \\ &\leq \sup_{\bar{\xi} \in R^{N-1}} \int_0^{\min_{r \leq i < N} |\xi_i|} (|\bar{\xi}|^{A^{(0)}} \tau^{A_N^{(0)}} + |\bar{\xi}|^{-A^{(0)}} \tau^{-A_N^{(0)}})^{-1} \tau^{-1} d\tau \leq \\ &\leq \sup_{\bar{\xi} \in R^{N-1}} \int_0^\infty \leq \sup_{\bar{\xi} \in R^{N-1}} \left(|\bar{\xi}|^{A^{(0)}} \int_0^{|\bar{\xi}|^{-A^{(0)}/A_N^{(0)}} \tau^{A_N^{(0)}-1} d\tau + |\bar{\xi}|^{-A^{(0)}} \int_{|\bar{\xi}|^{-A^{(0)}/A_N^{(0)}}^\infty \tau^{-A_N^{(0)}-1} d\tau \right) \leq \\ &\leq \frac{2}{A_N^{(0)}} < \infty. \end{aligned}$$

4.6 DEFINITION. The convex hull of the set $\{I_1, \dots, I_s\}$ is denoted by $\mathcal{O}$. We say that $\mathcal{K}$ regularly penetrates the hyperplane $X_{r+k} = 1$ if there is a segment $\overline{A^{(1)} A^{(2)}}$ such that $I_k \in \overline{A^{(1)} A^{(2)}}$, $A_{r+k}^{(1)} < 1 < A_{r+k}^{(2)}$, $A^{(1)} \in \mathcal{K}$, $A^{(2)} \in \mathcal{K}$.

The mapping

$$P_t = P_{i_1^{(t)}, i_2^{(t)}} P_{i_1^{(t-1)}, i_2^{(t-1)}} \dots P_{i_1^{(1)}, i_2^{(1)}}$$

(that is

$$P_t X = P_{i_1^{(t)}, i_2^{(t)}} (P_{i_1^{(t-1)}, i_2^{(t-1)}} (\dots (P_{i_1^{(1)}, i_2^{(1)}} X) \dots))$$

is said to be the admissible projection of order t ($1 \leq t \leq s-1$) if $r+1 \leq i_1^{(k)} < i_2^{(k)} \leq N-k+1$. P_0 is defined as the identity.

4.7 THEOREM. Let $\mathcal{K} \in \mathfrak{p}$ and

- 1) $\mathcal{K} \cap \mathcal{O} \neq \emptyset$
- 2) if P_t is the admissible projection of order t and

$$P_t \mathcal{K} \cap P_t \mathcal{O} = \{(\underbrace{0, \dots, 0}_r, \underbrace{0, \dots, 0, 1}_k, \underbrace{0, \dots, 0}_{s-t-k})\}$$

then $P_t \mathcal{K}$ regularly penetrates the hyperplane $X_{r+k} = 1$ (in R^{N-t}).

Then $\gamma(\mathcal{K}, s) < \infty$ ⁽⁴⁾.

Proof. We use mathematical induction. We prove that for any admissible projection of order t ($0 \leq t \leq s-1$) is

$$\gamma(P_t \mathcal{K}, s-t) < \infty.$$

First, if $t=s-1$ then $P_t = P_{1,2,\dots,s}$. It is clear that $P_{s-1} \mathcal{S} = \{(0, \dots, 0, 1)\}$ and $P_{s-1} \mathcal{K} \cap P_{s-1} \mathcal{S} \neq \emptyset$. So, $P_{s-1} \mathcal{K} \cap P_{s-1} \mathcal{S} = \{(0, \dots, 0, 1)\}$. By condition 2) and lemma 4.5 we have $\gamma(P_{s-1} \mathcal{K}, 1) = \gamma(P_{s-1}, 1, 1) < \infty$.

Let $\gamma(P_t \mathcal{K}, s-t) < \infty$ for any admissible projection P_t of order $t > t_0$. Suppose P_{t_0} is an admissible projection of order t_0 . Then

$$(4.2) \quad \gamma(P_{i_1, i_2} P_{t_0} \mathcal{K}, s-t_0-1) < \infty$$

for any $r < i_1 < i_2 \leq N-t_0$. On the other hand $P_{i_1, i_2} P_{t_0} \mathcal{K} = P_{i_1, i_2}(P_{t_0} \mathcal{K})$. Using condition 1) we have $P_{t_0} \mathcal{K} \cap P_{t_0} \mathcal{S} \neq \emptyset$. If

$$P_{t_0} \mathcal{K} \cap P_{t_0} \mathcal{S} \neq \{(0, \dots, 0, \underbrace{0, \dots, 0}_r, 1, \underbrace{0, \dots, 0}_{s-t_0-k})\}$$

then using lemma 4.4 one has

$$(4.3) \quad \gamma(P_{t_0} \mathcal{K}, s-t_0, k) < \infty.$$

If

$$P_{t_0} \mathcal{K} \cap P_{t_0} \mathcal{S} = \{(0, \dots, 0, \underbrace{0, \dots, 0}_r, \underbrace{0, \dots, 0}_k, 1, \underbrace{0, \dots, 0}_{s-t_0-k})\},$$

using condition 2) and lemma 4.5, one obtains (4.3). It follows from (4.2), (4.3) and lemma 3.4 that

$$\gamma(P_{t_0} \mathcal{K}, s-t_0) < \infty.$$

So we can conclude (for $t_0 = 0$)

$$\gamma(\mathcal{K}, s) < \infty.$$

and the proof is finished.

(4) For $t=0$ the condition 2) takes the form: $\mathcal{K} \cap \mathcal{S} = \{I_k\} \implies \mathcal{K}$ regularly penetrates the hyperplane $X_{r+k} = 1$.

In the following we will prove that sufficient conditions 1, 2) of theorem 4.7 are also necessary for $\gamma(\mathcal{K}, s) < \infty$.

5. Some geometrical lemmas.

5.1 LEMMA (Helly, cfr. [5], [19], [20]). Let $\mathcal{M}_i (i = 1, \dots, n)$ be convex sets in R^N . Let be for any $j_1, j_2, \dots, j_{N+1}$

$$\bigcap_{i=1}^{N+1} \mathcal{M}_{j_i} \neq \emptyset.$$

Then

$$\bigcap_{i=1}^n \mathcal{M}_i \neq \emptyset.$$

Proof. Obviously, for $N = 1$ the lemma is true. Let the lemma take place in any Euclidean space of dimension $< N$.

First suppose $n = N + 2$. Put $\mathcal{N}_i = \bigcap_{k=1, k \neq i}^{N+2} \mathcal{M}_k \neq \emptyset$. Then there is $X^{(i)} \in \mathcal{N}_i$. The convex hull of $X^{(1)}, \dots, X^{(i-1)}, X^{(i+1)}, \dots, X^{(N+2)}$ is denoted by Δ_i ; the convex hull of $X^{(1)}, \dots, X^{(N+2)}$ is denoted by Δ . Obviously

$$\Delta_i \subset \mathcal{M}_i, \quad X^{(i)} \in \bigcap_{k=1, k \neq i}^{N+2} \Delta_k.$$

If the dimension of Δ is $< N$ then $\bigcap_{i=1}^{N+2} \Delta_i \neq \emptyset$ and so $\bigcap_{i=1}^{N+2} \mathcal{M}_i \neq \emptyset$.

If the dimension of Δ is N then there is at least one Δ_i whose dimension is N . Suppose that this is for Δ_{N+2} . Then

$$X^{(N+2)} = \sum_{i=1}^{N+1} \lambda_i X^{(i)}, \quad \sum_{i=1}^{N+1} \lambda_i = 1.$$

Without loss of generality we can suppose

$$\lambda_1 \geq 0, \dots, \lambda_k \geq 0, \lambda_{k+1} < 0, \dots, \lambda_{N+1} < 0.$$

Put

$$\mu_i = \mu_{N+2} \lambda_i \quad (i = 1, \dots, k)$$

$$\mu_i = -\mu_{N+2} \lambda_i \quad (i = k+1, \dots, N+1)$$

$$\mu_{N+2} = \left(\sum_{i=1}^k \lambda_i \right)^{-1}.$$

Then

$$\mu_i \geq 0, \sum_{i=1}^k \mu_i = \sum_{i=k+1}^{N+2} \mu_i = 1$$

and

$$X^* = \sum_{i=1}^k \mu_i X^{(i)} = \sum_{i=k+1}^{N+2} \mu_i X^{(i)}.$$

So $X^* \in \Delta_i (i=1, 2, \dots, N+2)$ and $X^* \in \bigcap_{i=1}^{N+2} \Delta_i \subset \bigcap_{i=1}^{N+2} \mathcal{M}_i \neq \emptyset$. So, for $n = N+2$, the assertion of the lemma is true.

Now suppose that the lemma takes place for any $n < n_0$. Let $\mathcal{M}_1, \dots, \mathcal{M}_{n_0}$ be convex sets such that for any $j_1, \dots, j_{N+1}$ one has $\bigcap_{i=1}^{N+1} \mathcal{M}_{j_i} \neq \emptyset$. By the first part of the proof we know $\bigcap_{i=1}^{N+2} \mathcal{M}_{j_i} \neq \emptyset \forall j_1, j_2, \dots, j_{N+2}$. Put $\mathcal{M}_i^* = \mathcal{M}_i \cap \mathcal{M}_{n_0} (i=1, \dots, n_0-1)$. The number of convex sets $\mathcal{M}_i^*$ is n_0-1 and intersection of any $N+1$ sets $\mathcal{M}_i^*$ is not empty. So $\bigcap_{i=1}^{n_0-1} \mathcal{M}_i^* = \bigcap_{i=1}^{n_0} \mathcal{M}_i \neq \emptyset$ and the proof is finished.

5.2 LEMMA. Let $f_i (i=1, 2, \dots, n), f$ be linear functionals on R^N such that

$$f_i(X) \leq 0 \forall i=1, \dots, n \implies f(X) \leq 0.$$

Then there are $\lambda_i \geq 0$ such that $f = \sum_{i=1}^n \lambda_i f_i$.

Proof. Let this assertion be true for any Euclidean space of dimension $< N$. Put

$$\begin{aligned} \mathcal{M}_i &= \{X \in R^N; f_i(X) \leq 0\} \\ \mathcal{M} &= \{X \in R^N; f(X) > 0\}. \end{aligned}$$

The sets $\mathcal{M}_i, \mathcal{M}$ are convex and $\bigcap_{i=1}^n \mathcal{M}_i \cap \mathcal{M} = \emptyset$. On the other hand $0 \in \bigcap_{i=1}^n \mathcal{M}_i \neq \emptyset$ and, using lemma 5.1, there are sets $\mathcal{M}_1, \dots, \mathcal{M}_N$ such that $\bigcap_{j=1}^n \mathcal{M}_{j_j} \cap \mathcal{M} = \emptyset$, it is

$$f_{j_j}(X) \leq 0 (j=1, \dots, N) \implies f(X) \leq 0.$$

So, in the following, we can consider only the case $n = N$. Obviously

$$(5.1) \quad f = \sum_{i=1}^N \lambda_i f_i.$$

Put $R = \{X \in R^N; f_i(X) = 0 \forall i=1, \dots, N\}$.

If f_i are linearly dependent then the dimension of R^N/R is $< N$. For $X \in \tilde{X} \in R^N/R$ put $f_i^*(\tilde{X}) = f_i(X)$, $f^*(\tilde{X}) = f(X)$. Then $f_j^*(\tilde{X}) \leq 0 \implies f^*(\tilde{X}) \leq 0$ and there are $\lambda_i \geq 0$ such that $f^* = \sum_{i=1}^N \lambda_i f_i^*$ and so $f = \sum_{i=1}^N \lambda_i f_i$ ($\lambda_i \geq 0$).

If f_i are linearly independent then there are $X^{(1)}, \dots, X^{(N)}$ such that $f_i(X^{(j)}) = -\delta_{ij}$ (it is 0 for $i \neq j$, $= -1$ for $i = j$). Obviously $X^{(1)}, \dots, X^{(N)} \in \mathcal{M}_i$ ($i = 1, \dots, N$) and so $f(X^{(i)}) \leq 0$ ($i = 1, \dots, N$). Using (5.1) we have

$$f(X^{(j)}) = \sum_{i=1}^N \lambda_i f_i(X^{(j)}) = -\lambda_j \leq 0$$

and so $\lambda_j \geq 0$.

In case $\lambda \mathcal{M} = \mathcal{M}$ for $\lambda > 0$ the convex set $\mathcal{M}$ is said to be a cone. So the cones need not be closed.

5.3 LEMMA. Let $\mathcal{M}$ be a cone in R^N , $\mathcal{M} \neq R^N$. Then there is a linear functional $f \neq 0$ such that for $X \in \mathcal{M}$ it is $f(X) \geq 0$ and in any inner point X of $\mathcal{M}$ it is $f(X) > 0$.

Proof. There are $P \neq 0$ on the boundary of $\mathcal{M}$, a constant C and a functional $f \neq 0$ such that for $X \in \mathcal{M}$ one has $f(X) + C \geq 0$ and $f(P) + C = 0$. The points λP ($\lambda > 0$) are boundary points of $\mathcal{M}$ and so

$$(5.2) \quad \lambda f(P) + C = f(\lambda P) + C \geq 0 \quad (\lambda > 0).$$

From (5.2) it follows $f(P) = C = 0$ and so for $X \in \mathcal{M}$ one has $f(X) \geq 0$. The rest of assertion is obvious.

5.4 LEMMA Let $\mathcal{M}, \mathcal{N}$ be cones, $\mathcal{M} \neq R^N \neq \mathcal{N}$ and $\mathcal{M} \cap \mathcal{N} \subset \{0\}$. Then there is a linear functional $f \neq 0$ such that

$$f(X) \geq 0 \text{ for } X \in \mathcal{M}$$

$$f(X) \leq 0 \text{ for } X \in \mathcal{N}$$

and the inequalities are sharp inside of $\mathcal{M}$ and $\mathcal{N}$.

Proof. $\mathcal{M} \cap (-\mathcal{N})$ is a convex cone and so

$$\dim(\mathcal{M} \cap (-\mathcal{N})) = N \implies \mathcal{M} \cap (-\mathcal{N}) = R^N \implies \mathcal{M} = R^N.$$

So under the hypotheses of the lemma

$$\dim (\mathcal{M} \cap -\mathcal{M}) < N, \dim (\mathcal{N} \cap -\mathcal{N}) < N$$

and there is $Z \notin (\mathcal{M} \cap -\mathcal{M}) \cup (\mathcal{N} \cap -\mathcal{N}) \cup \{0\}$.

The convex hull of $\mathcal{M} \cap (-\mathcal{N})$ will be denoted by $\mathcal{K}$ and suppose $Z \in \mathcal{K}$, $-Z \in \mathcal{K}$. Then there are $\lambda, \mu, 0 \leq \lambda \leq 1, 0 \leq \mu \leq 1, X_1 \in \mathcal{M}, Y_1 \in \mathcal{M}, X_2 \in \mathcal{N}, Y_2 \in \mathcal{N}$ such that

$$Z = \lambda X_1 + (1 - \lambda) X_2 = -\mu Y_1 - (1 - \mu) Y_2$$

and so

$$\lambda X_1 + \mu Y_1 + (1 - \lambda) X_2 + (1 - \mu) Y_2 = 0.$$

If either $\lambda = \mu = 0$ or $\lambda = \mu = 1$ then $Z \in \mathcal{M} \cap \mathcal{N}$ and this is in contradiction with the fact that $\mathcal{M} \cap \mathcal{N} \subset \{0\}$. So $\lambda + \mu \neq 0 \neq 2 - (\lambda + \mu)$. Put

$$Z_1 = \frac{\lambda X_1 + \mu Y_1}{\lambda + \mu}, Z_2 = \frac{(1 - \lambda) X_2 + (1 - \mu) Y_2}{2 - (\lambda + \mu)}.$$

Then

$$Z_1 = -\frac{2 - (\lambda + \mu)}{\lambda + \mu} Z_2.$$

On the other hand $Z_1 \in \mathcal{M}, Z_2 \in -\mathcal{N}$ and so $Z_1 \in \mathcal{M} \cap \mathcal{N}$ which is in contradiction with $\mathcal{M} \cap \mathcal{N} \subset \{0\}$ and $Z \neq 0$.

So we have proved that either $Z \notin \mathcal{K}$ or $-Z \notin \mathcal{K}$ ($Z \neq 0$) and so $\mathcal{K} \neq R^N$. Using lemma 5.3 we can find a linear functional $f \neq 0$ such that for $X \in \mathcal{K} \supset \supset \mathcal{M} \cap (-\mathcal{N})$

$$f(X) \geq 0$$

and the proof is finished.

5.5 LEMMA. Let $\mathcal{K} \in \mathfrak{p}$ and Π be a hyperplane in R^N . Suppose that there is at most one extremal point of $\mathcal{K}$ in Π . Let $X \in \mathcal{K} \cap \Pi$ and X be not extremal point of $\mathcal{K}$. Then there is $B \in R^N$ such that $X + B \in \mathcal{K} - \Pi, X - B \in \mathcal{K} - \Pi$ (that is the segment $\overline{X + B, X - B}$ penetrates the hyperplane Π).

Proof. Assertion of this lemma is sufficiently obvious.

5.6 LEMMA Let $\mathcal{K} \in \mathfrak{p}$. Let

$$\mathcal{M}(\mathcal{K}) = \{X \in R^N; \langle A, X \rangle \leq 0 \ \forall \ A \in \mathcal{K}\}$$

have empty interior. Then $0 \in \mathcal{K} - \mathcal{K}^e$.

Proof. Let $0 \in (R^N - \mathcal{K}) \cup \mathcal{K}^c$. Then there is a hyperplane Π such that $0 \in \Pi$, $\Pi \cap \mathcal{K} \subset \{0\}$ and $\mathcal{K}$ lies in one of the semispaces R_1, R_2 defined by Π , $R^N = \Pi \cup R_1 \cup R_2$. Suppose $\mathcal{K} \subset R_1 \cup \Pi$. There is a point $S \in R_2$, $S \neq 0$, $\langle S, X \rangle = 0 \forall X \in \Pi$. Then there is certain neighborhood $\mathcal{U}$ of S such that $\mathcal{U} \subset \mathcal{M}(\mathcal{K})$ and this is a contradiction. So $0 \in \mathcal{K} - \mathcal{K}^c$.

5.7 LEMMA. Let $\mathcal{K} \in \mathfrak{p}$ and

$$\mathcal{M}(\mathcal{K}) = \{0\}.$$

Then $\mathcal{K}$ is a neighbourhood of 0.

Proof. By lemma 5.6, $0 \in \mathcal{K}$. If 0 is boundary point of $\mathcal{K}$ then there is a linear functional $f \neq 0$, $f(X) \leq 0 \forall X \in \mathcal{K}$. Let $f(X) = \langle A, X \rangle$. Then, using lemma 5.2, one has

$$A = \sum_{X \in \mathcal{K}^e} \lambda_x X, \quad \lambda_x \geq 0.$$

Obviously $\lambda = \sum_{X \in \mathcal{K}^e} \lambda_x > 0$ and $\lambda^{-1} A \in \mathcal{K}$. Further $f(\lambda^{-1} A) = \langle A, \lambda^{-1} A \rangle = \lambda^{-1} \langle A, A \rangle > 0$ and this is the contradiction.

6. Necessary conditions for $\gamma(\mathcal{K}, s, k) < \infty$.

Put

$$R_+^N = \{X \in R^N, X = (X_1, \dots, X_N); X_i > 0 \forall i\}.$$

For $X \in R_+^N$ put

$$lg X = (lg X_1, \dots, lg X_N) = (lg X', lg X'') = (lg \bar{X}, lg X_N).$$

For $X \in R^N$ put

$$e^x = (e^{x_1}, \dots, e^{x_N}).$$

If $\mathcal{M} \subset R_+^N$ then

$$lg \mathcal{M} = \{X \in R^N, e^X \in \mathcal{M}\}$$

If $\mathcal{M} \subset R^N$ then

$$e^{\mathcal{M}} = \{X \in R_+^N, lg X \in \mathcal{M}\}.$$

If $\mathcal{M} \in (0, \infty)$ put

$$\mu_{lg} \mathcal{M} = \int_{\mathcal{M}} \frac{d\tau}{\tau}.$$

6.1 LEMMA. Let $\mathcal{K} \in \mathfrak{p}$ and

$$\sup_{\xi_k \in R_+^{N-1}} \int_0^\infty [P_{\mathcal{K}}(\xi_k(\tau))]^{-1} d\tau < \infty.$$

Then the interior of the set

$$\mathcal{M}_k(\mathcal{K}) = \{X \in R^N, \langle A - I_k, X \rangle \leq 0 \ \forall \ A \in \mathcal{K}\}$$

is empty.

Proof. Suppose $k = s$, that is $\xi_k(\tau) = (\bar{\xi}, \tau)$. Put for $\bar{\xi} \in R_+^N$

$$m(\bar{\xi}) = \{\tau, \tau \in (0, \infty); |\xi_s(\tau)|^{A-I_s} \leq 1 \ \forall \ A \in \mathcal{K}\}.$$

Then

$$\tau \in m(\bar{\xi}) \iff \frac{P_{\mathcal{K}}(\xi_s(\tau))}{\tau} \leq 1 \iff$$

$$\iff \langle A - I_s, \lg \xi_s(\tau) \rangle \leq 0 \ \forall \ A \in \mathcal{K} \iff \lg \xi_s(\tau) \in \mathcal{M}_s(\mathcal{K}).$$

One has

$$\int_0^\infty \frac{d\tau}{\mathcal{P}_{\mathcal{K}}(\xi_s(\tau))} \geq \int_{m(\bar{\xi})} \frac{\tau}{\mathcal{P}_{\mathcal{K}}(\xi_s(\tau))} \frac{d\tau}{\tau} \geq \mu_{\lg} m(\bar{\xi}).$$

Put $X(T) = \lg(\xi(\tau)) = (\bar{X}, T)$. For $\bar{X} = \lg \bar{\xi}$ one has

$$\inf m(\bar{\xi}) = \inf_{\langle A - I_s, X(T) \rangle \leq 0 \ \forall \ A \in \mathcal{K}} e^T = \inf_{X(T) \in \mathcal{M}_s(\mathcal{K})} e^T,$$

$$\sup m(\bar{\xi}) = \sup_{X(T) \in \mathcal{M}_s(\mathcal{K})} e^T$$

and $m(\bar{\xi})$ is a segment (with ends $\inf m(\bar{\xi})$ and $\sup m(\bar{\xi})$) or it is empty.

On the other hand

$$(6.1) \quad \mu_{\lg} m(\bar{\xi}) \begin{cases} = 0 & \text{if } m(\bar{\xi}) = \emptyset \\ = \lg \sup m(\bar{\xi}) - \lg \inf m(\bar{\xi}) = \\ = \sup_{X(T) \in \mathcal{M}_s(\mathcal{K})} T - \inf_{X(T) \in \mathcal{M}_s(\mathcal{K})} T & \text{if } m(\bar{\xi}) \neq \emptyset. \end{cases}$$

The set $\mathcal{M}_s(\mathcal{K})$ is a cone. If $\mathcal{M}_s(\mathcal{K})$ has at least one inner point then there is a ball $\mathcal{U} \subset \mathcal{M}_s(\mathcal{K})$ such that $\lambda \mathcal{U} \subset \mathcal{M}_s(\mathcal{K})$ ($\lambda > 0$). Then, obviously, using (6.1), one has

$$\infty = \sup_{\bar{\xi} \in R_+^{N-1}} \mu_{lg} m(\bar{\xi}) \leq \sup_{\bar{\xi} \in R_+^{N-1}} \int_0^\infty \frac{d\tau}{\mathcal{P}_{\mathcal{K}}(\xi_s(\tau))}$$

which is the contradiction. So the interior of $\mathcal{M}_s(\mathcal{K})$ must be empty.

6.2 LEMMA. Under the hypotheses of lemma 6.1 is $I_k \in \mathcal{K} - \mathcal{K}^e$. If there is at most one extremal point of $\mathcal{K}$ on the hyperplane $X_{r+k} = 1$ then $\mathcal{K}$ regularly penetrates the hyperplane $X_{r+k} = 1$.

Proof. Using the lemma 5.6 one has $I_k \in \mathcal{K} - \mathcal{K}^e$. Then one can use the lemma 5.5.

6.3 LEMMA. Let $\mathcal{K} \in \mathfrak{p}$ and

$$(6.2) \quad \sup_{\bar{\xi}_k \in R^{N-1}} \int_0^\infty \frac{d\tau}{\mathcal{P}_{\mathcal{K}}(\xi_k(\tau))} < \infty.$$

Then $\mathcal{K}$ regularly penetrates the hyperplane $X_{r+k} = 1$.

Proof. Suppose $k = s$. Let us denote $\mathcal{K}_\alpha^e = \mathcal{K}^e - \{X \in R^N, X_N = 1\}$ and by $\mathcal{K}_\alpha$ the convex hull of $\mathcal{K}_\alpha^e$, $\mathcal{K}_\beta^e = \mathcal{K}^e - \mathcal{K}_\alpha^e$, $\mathcal{K}_\beta$ the convex hull of $\mathcal{K}_\beta^e$. Then

$$d = \text{dist}(\mathcal{K}_\alpha^e, \{X \in R^N, X_N = 1\}) > 0.$$

For $\bar{\xi} \in R_+^{N-1}$ put

$$\begin{aligned} m(\bar{\xi}) &= \{\tau \in (0, \infty); |\xi_s(\tau)|^{A-I_s} \leq 1 \forall A \in \mathcal{K}_\alpha^e\} = \\ &= \{\tau \in (0, \infty); |\xi_s(\tau)|^{A-I_s} \leq 1 \forall A \in \mathcal{K}_\alpha\}. \end{aligned}$$

Then

$$\begin{aligned} \int_0^\infty [\mathcal{P}_{\mathcal{K}}(\xi_s(\tau))]^{-1} d\tau &\geq \int_{m(\bar{\xi})}^\infty [\mathcal{P}_{\mathcal{K}}(\xi_s(\tau))]^{-1} \frac{d\tau}{\tau} \geq \\ &\geq \int_{m(\bar{\xi})}^\infty \frac{1}{\max(1, \tau^{-1} \mathcal{P}_{\mathcal{K}_\beta}(\xi_s(\tau)))} \frac{d\tau}{\tau} \geq [\max(1, \mathcal{P}_{\mathcal{K}_\beta}(\xi_s(\tau)))^{-1} \mu_{lg} m(\bar{\xi})]. \quad (2) \end{aligned}$$

$$(2) \quad m(\bar{\xi}) = \{X \in R^N, X_N = 1, A \in \mathcal{K}_\alpha\}.$$

Put $X(T) = l g \xi(\tau)$,

$$\mathcal{M} = \{X \in \mathcal{R}^N, \langle A - I_s, X \rangle \leq 0 \ \forall \ A \in \mathcal{K}_\alpha\}$$

$$\mathcal{N} = \{X \in \mathcal{R}^N, \langle A - I_s, X \rangle \leq 0 \ \forall \ A \in \mathcal{K}_\beta\}$$

$$\mu(\mathcal{M}, \bar{X}) = \mu\{T \in (-\infty, \infty); (\bar{X}, T) \in \mathcal{M}\}.$$

By a similar argument as in the proof of lemma 6.1 one has

$$\mu_{lg} m(\bar{\xi}) = \mu(\mathcal{M}, \bar{X}).$$

Further

$$\mathcal{P}_{\mathcal{K}_\beta \div I_s}(\xi_s(\tau)) = e^{\max_{A \in \mathcal{K}_\beta^e} \langle A - I_s, X \rangle} = e^{\max_{A \in \mathcal{K}_\beta^e} \langle \bar{A}, \bar{X} \rangle}$$

By (6.2) one has

$$(6.3) \quad \sup_{\bar{X} \in \mathbb{R}^{N-1}} \frac{\mu(\mathcal{M}, \bar{X})}{\max_{A \in \mathcal{K}_\beta^e} \langle \bar{A}, \bar{X} \rangle, 0)} < \infty.$$

The interior of $\mathcal{M}$ will be denoted by $\mathcal{M}^0$ and the projection $(\bar{X}, X_N) \rightarrow \bar{X}$ will be denoted by P .

If $\bar{X} \in P\mathcal{N} \cap P\mathcal{M}^0$, $\bar{X} \neq 0$ then for $\lambda > 0$ one has $\lambda \bar{X} \in P\mathcal{N} \cap P\mathcal{M}^0$, $\langle \bar{A}, \lambda \bar{X} \rangle \leq 0 \ \forall \ A \in \mathcal{K}_\beta^e$ and

$$g(\lambda) = \frac{\mu(\mathcal{M}, \lambda \bar{X})}{\max_{A \in \mathcal{K}_\beta^e} \langle \bar{A}, \lambda \bar{X} \rangle, 0)} \geq \mu(\mathcal{M}, \lambda \bar{X}) = \lambda \mu(\mathcal{M}, \bar{X}).$$

From $\bar{X} \in P\mathcal{M}^0$ it follows $\mu(\mathcal{M}, \bar{X}) > 0$ and so

$$\lim_{\lambda \rightarrow \infty} g(\lambda) = \infty.$$

On the other hand

$$f(\bar{X}) = \frac{\mu(\mathcal{M}, \bar{X})}{\max_{A \in \mathcal{K}_\beta^e} \langle \bar{A}, \bar{X} \rangle, 0)}$$

is a continuous function of $\bar{X}$ (because $d > 0$, that is $\mu(\mathcal{M}, \bar{X})$ is a continuous function of $\bar{X}$) and so $\sup_{\bar{X} \in K^{N-1}} f(\bar{X}) = \infty$; this is the contradiction. So one has

$$(6.4) \quad P\mathcal{M} \cap P\mathcal{M}^0 \subset \{0\}.$$

Evidently, $P\mathcal{M}$ is closed and $P\mathcal{M}^0$ is open.

There are three possibilities :

- 1) $P\mathcal{M} = \{0\}$
- 2) $\mathcal{M}^0 = \emptyset$
- 3) $\mathcal{M}^0 \neq \emptyset$ and there is $\bar{X} \in P\mathcal{M}$, $\bar{X} \neq 0$.

Ad 1) $P\mathcal{M} = \{0\}$. Then $\mathcal{K}_\beta$ is a neighbourhood of I_s in the hyperplane $X_N = 1$ and by (6.3) there is at least one point $\bar{X} \neq 0$ such that $\mu(\mathcal{M}, \bar{X}) < \infty$. So, there are $X^{(1)} \in \mathcal{K}$, $X^{(2)} \in \mathcal{K}$ such that $X_N^{(1)} < 1$, $X_N^{(2)} > 1$. Then the convex hull K_γ of $\mathcal{K}_\beta \cup \{X^{(1)}, X^{(2)}\}$ is a neighbourhood of I_s in R^N , $K_\gamma \subset \mathcal{K}$ and the assertion of lemma 6.3 is true.

Ad 2) $\mathcal{M}^0 = \emptyset$. Then by lemma 5.6 $I_s \in \mathcal{K}_a$. There is no extremal point of $\mathcal{K}_a$ in the hyperplane $X_N = 1$. By lemma 5.5 $\mathcal{K}_a$ regularly penetrates the hyperplane $X_N = 1$ and the assertion of lemma 6.3 is true too.

Ad 3) $\mathcal{M}^0 \neq \emptyset$ and there is $\bar{X} \in P\mathcal{M}$, $\bar{X} \neq 0$. Then $\bar{X} \in P\mathcal{M}^0$ and so $P\mathcal{M}^0 \neq R^{N-1}$. $\mathcal{M}^0 \neq \emptyset$ implies $P\mathcal{M}^0 \neq \emptyset$; there is $\bar{Y} \in P\mathcal{M}^0$, $\bar{Y} \neq 0$. So $P\mathcal{M} \neq R^{N-1}$. By lemma 5.4 there is $\bar{Z} \in R^{N-1}$, $\bar{Z} \neq 0$ such that for $X \in P\mathcal{M}^0$ one has $\langle \bar{Z}, \bar{X} \rangle > 0$ and for $\bar{X} \in P\mathcal{M}$ one has $\langle \bar{Z}, \bar{X} \rangle \leq 0$. Then

$$\langle \bar{Z}, 0 \rangle, X \rangle > 0 \quad \text{for } X \in \mathcal{M}^0$$

$$\langle \bar{Z}, 0 \rangle, X \rangle \leq 0 \quad \text{for } X \in \mathcal{M}.$$

By lemma 5.2

$$\bar{Z} = \sum_{X \in \mathcal{K}_\beta^e} \lambda_X \bar{X}, \quad \lambda_X \geq 0$$

and obviously $\lambda = \sum \lambda_X > 0$ and $\lambda^{-1} \bar{Z} \in P\mathcal{K}_\beta$. Therefore $B = I_s + \lambda^{-1} \langle \bar{Z}, 0 \rangle \in \mathcal{K}_\beta$. Further $d > 0$ and so there is $C > 0$ such that

$$(6.4) \quad \mu(\mathcal{M}, \bar{X}) = 0 \quad \text{if } \langle \bar{Z}, \bar{X} \rangle \leq 0$$

$$\mu(\mathcal{M}, \bar{X}) \leq C \langle \bar{Z}, \bar{X} \rangle \quad \text{if } \langle \bar{Z}, \bar{X} \rangle \geq 0.$$

(6.4) implies

$$\begin{aligned}\mu(\mathcal{M}, \bar{X}) e^{-\langle \bar{Z}, \bar{X} \rangle} &= 0 \quad \text{if } \langle \bar{Z}, \bar{X} \rangle \leq 0 \\ \mu(\mathcal{M}, \bar{X}) e^{-\langle \bar{Z}, \bar{X} \rangle} &\leq C \langle \bar{Z}, \bar{X} \rangle e^{-\langle \bar{Z}, \bar{X} \rangle} \quad \text{if } \langle \bar{Z}, \bar{X} \rangle \geq 0.\end{aligned}$$

The convex hull of ${}^c\mathcal{K}_\alpha^e \cup \{B\}$ will be denoted by ${}^c\mathcal{K}_\delta$.
Then

$$(6.5) \quad \int_{m(\bar{\xi})} \frac{d\tau}{\mathcal{P}_{\mathcal{K}_\delta}(\xi_s(\tau))} \leq \int_{m(\bar{\xi})} \frac{1}{|\xi_s(\tau)|^{B-I_s}} \frac{d\tau}{\tau} = \frac{\mu(\mathcal{M}, \bar{X})}{e^{\langle \bar{Z}, \bar{X} \rangle}} \leq C_1 < \infty.$$

The set $m(\bar{\xi})$ is either empty or a segment. Put

$$\begin{aligned}m_n &= \max\left(0, \inf m(\bar{\xi}) - \frac{1}{n}\right) \\ M_n &= \sup m(\bar{\xi}) + \frac{1}{n}.\end{aligned}$$

Then

$$\int_{(0, \infty) - m(\bar{\xi})} \frac{d\tau}{\mathcal{P}_{\mathcal{K}_\delta}(\xi_s(\tau))} = \lim_{n \rightarrow \infty} \int_{(0, \infty) - (m_n, M_n)} \frac{d\tau}{\mathcal{P}_{\mathcal{K}_\delta}(\xi_s(\tau))}.$$

Let $m_n \neq 0$. Then there is $A^{(1)} \in \mathcal{K}_\alpha^e$ such that $|\xi_s(\tau)|^{A^{(1)} - I_s} \geq 1$ for $\tau \in (0, m_n)$. That is $A_N^{(1)} < 1$ and $m_n \leq |\bar{\xi}|^{\overline{A^{(1)}}/(1-A_N^{(1)})}$. Then

$$(6.6) \quad \int_0^{m_n} \frac{d\tau}{\mathcal{P}_{\mathcal{K}_\delta}(\xi_s(\tau))} \leq \int_0^{|\bar{\xi}|^{\overline{A^{(1)}}/(1-A_N^{(1)})}} \frac{d\tau}{|\bar{\xi}|^{A^{(1)}} \tau^{A_N^{(1)}}} = \frac{1}{1 - A_N^{(1)}}.$$

Let $M_n \neq \infty$. Then there is $A^{(2)} \in \mathcal{K}_\alpha^e$ such that $|\xi_s(\tau)|^{A^{(2)} - I_s} \geq 1$ for $\tau \in (M_n, \infty)$, that is $A_N^{(2)} > 1$ and $M_n \geq |\bar{\xi}|^{\overline{A^{(2)}}/(1-A_N^{(2)})}$. Then

$$(6.7) \quad \int_{M_n}^\infty \frac{d\tau}{\mathcal{P}_{\mathcal{K}_\delta}(\xi_s(\tau))} \leq \int_{|\bar{\xi}|^{\overline{A^{(1)}}/(1-A_N^{(2)})}}^\infty \frac{d\tau}{|\bar{\xi}|^{A^{(2)}} \tau^{A_N^{(2)}}} = \frac{1}{A_N^{(2)} - 1}.$$

Using (6.5), (6.6), (6.7) one has

$$\sup_{\substack{\xi \in R^{N-1} \\ +}} \int_0^\infty [\mathcal{P}_{\lambda_\delta}(\xi_s(\tau))]^{-1} d\tau < \infty.$$

By lemma 6.2 $\mathcal{K}_\delta$ regularly penetrates the hyperplane $X_N = 1$, $\mathcal{K}_\delta \subset \mathcal{K}$ and the proof is finished.

Let $\mathcal{K} \in \mathfrak{p}$. The convex hull of the set $\mathcal{K} \cup \bigcup_{\substack{i \neq k \\ i=1, 2, \dots, s}} (2I_k - I_i)$ will be denoted by $\mathcal{K}^{(k)}$.

6.4 LEMMA

$$(6.8) \quad \sup_{\substack{\xi_k \in R^{N-1} \\ +}} \int_0^\infty \frac{d\tau}{\mathcal{P}_{\mathcal{K}^{(k)}}(\xi_k(\tau))} \leq \gamma(\mathcal{K}, s, k) + 1.$$

Proof. Let $\min_{\substack{r < i \leq N \\ i \neq k+r}} |\xi_i| = |\xi_j|$ ($j \neq k+r$). Then

$$\int_{\substack{\min |\xi_i| \\ r < i \leq N \\ i \neq k+r}}^\infty \frac{d\tau}{\mathcal{P}_{\mathcal{K}^{(k)}}(\xi_k(\tau))} \leq \int_{|\xi_j|}^\infty \frac{d\tau}{\tau^2 |\xi_j|^{-1}} = 1$$

and

$$\int_0^{\min |\xi_i|} \frac{d\tau}{\mathcal{P}_{\mathcal{K}^{(k)}}(\xi_k(\tau))} \leq \int_0^{\min |\xi_i|} \frac{d\tau}{\mathcal{P}_{\mathcal{K}}(\xi_k(\tau))} = \gamma(\mathcal{K}, s, k).$$

6.5 LEMMA. Let $\mathcal{K} \in \mathfrak{p}$ and $\gamma(\mathcal{K}, s, k) < \infty$ for $k = 1, 2, \dots, s$. Then $\mathcal{K} \cap \mathcal{O} \neq \emptyset$.

Proof. By lemmas 6.4, 6.2 $I_k \in \mathcal{K}^{(k)}$. Put

$$\pi_k = \left\{ X = \sum_{i=1}^s \lambda_i I_i, \lambda_k \geq 0, \sum_{i=1}^s \lambda_i = 1 \right\}.$$

Then $\mathcal{O} = \bigcap_{i=1}^s \pi_k$. Put $\mathcal{K}^* = \mathcal{K} \cap \bigcup_{i=1}^s \pi_k$. Then $I_k \in \mathcal{K}^{(k)}$ implies $\bigcap_{i \neq k} \pi_i \cap \mathcal{K}^* \neq \emptyset$.

By lemma 5.1 $\bigcap_{i=1}^s \pi_k \cap \mathcal{K}^* \neq \emptyset$ and so $\mathcal{K} \cap \mathcal{O} \neq \emptyset$.

6.6 LEMMA. Let $\mathcal{K} \in \mathfrak{p}$, $\mathcal{K} \cap \mathcal{S} = \{I_k\}$ and

$$\sup_{\xi_k \in R_+^{N-1}} \int_0^\infty \frac{d\tau}{\mathcal{P}_{\mathcal{K}^{(k)}}(\xi_k(\tau))} < \infty.$$

Then $\mathcal{K}$ regularly penetrates the hyperplane $X_{r+k} = 1$.

Proof. The convex hull of $\bigcup_{i \neq k} \{2I_k - I_i\}$ will be denoted by $\mathcal{S}^{(k)}$. By lemma 6.3 there is B such that $I_k + B \in \mathcal{K}^{(k)}$, $I_k - B \in \mathcal{K}^{(k)}$, that is there are λ, μ , $0 \leq \lambda \leq 1$, $0 \leq \mu \leq 1$; $X^{(1)}, X^{(2)} \in \mathcal{S}^{(k)}$; $Y^{(1)}, Y^{(2)} \in \mathcal{K}$ such that

$$I_k + B = \lambda X^{(1)} + (1 - \lambda) Y^{(1)}$$

$$I_k - B = \mu X^{(2)} + (1 - \mu) Y^{(2)}.$$

Obviously $\lambda + \mu \neq 2$. Suppose $\lambda + \mu \neq 0$. Then

$$2I_k = (\lambda + \mu) X^{(3)} + (2 - \lambda - \mu) Y^{(3)}$$

where

$$X^{(3)} = \frac{\lambda X^{(1)} + \mu X^{(2)}}{\lambda + \mu} \in \mathcal{S}^{(k)}, \quad Y^{(3)} = \frac{(1 - \lambda) Y^{(1)} + (1 - \mu) Y^{(2)}}{2 - \lambda - \mu} \in \mathcal{K}.$$

Put

$$\alpha = (2 - \lambda - \mu)/(\lambda + \mu) > 0.$$

Then

$$2I_k - X^{(3)} = \alpha Y^{(3)} + (1 - \alpha) I_k$$

and

$$Y^{(3)} = I_k + \alpha^{-1} (I_k - X^{(3)}).$$

Obviously $2I_k - X^{(3)} \in \mathcal{S}$, $2I_k - X^{(3)} \neq I_k$.

Let $\alpha \leq 1$. Then $I_k \in \mathcal{K}$, $Y^{(3)} \in \mathcal{K}$ implies

$$2I_k - X^{(3)} = (1 - \alpha) I_k + \alpha Y^{(3)} \in \mathcal{K}$$

and so $I_k \neq 2I_k - X^{(3)} \in \mathcal{K} \cap \mathcal{S}$ which is the contradiction.

Let $\alpha^{-1} < 1$. Then $I_k \in \mathcal{S}$, $2I_k - X^{(3)} \in \mathcal{S}$ and so

$$Y^{(3)} = (1 - \alpha^{-1}) I_k + \alpha^{-1} (2I_k - X^{(3)}) \in \mathcal{S}.$$

Therefore

$$I_k \neq Y^{(3)} \in \mathcal{K} \cap \mathcal{O}$$

and it is a contradiction too.

Assumption $\lambda + \mu \neq 0$ is false and so $\lambda = \mu = 0$ and $I_k + B \in \mathcal{K}$, $I_k - B \in \mathcal{K}$.

6.7 THEOREM. Let $\mathcal{K} \in \mathfrak{p}$. Then the conditions 1), 2) of theorem 4.7 are sufficient and necessary for $\gamma(\mathcal{K}, s) < \infty$.

Proof. The assertion is an immediate consequence of theorem 4.7, lemma 6.5, lemma 6.4, lemma 6.6 and lemma 3.4.

6.8 REMARK. Validity of the inequality (1.5) is equivalent to $\gamma(q\mathcal{K}, s) < \infty$ where $1/q = 1 - 1/p$, $1 < p < \infty$ (see theorem 3.1 and lemma 3.2).

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