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FRACTIONAL POWERS OF ELLIPTIC DIFFERENTIAL OPERATORS (*)

by TAMAR BURAK

1. Introduction and notations.

We investigate in this work fractional powers of elliptic differential operators on a compact C^∞ n dimensional manifold X , without a boundary. Let A be an elliptic differential operator of order m with coefficients in $C^\infty(X)$. We define fractional powers of A under the assumptions that the range of the symbol $\sigma_m(A)$ of A is disjoint to a ray in the complex plane and that zero is a regular point or a pole of the first order of the resolvent of $\bar{A}$, the closure of A in $H_0(X)$. We show that $\bar{A}^s$, restricted to $C^\infty(X)$, is an elliptic pseudodifferential operator of order ms .

Our proofs are based on the possibility of expanding $(q^m - \bar{A})^{-k}$, $k=1, 2, \dots$, considered as a family of operators depending on a parameter q in the angle $\theta_1 \leq \arg q \leq \theta_2$ into an asymptotic sum of canonical families which are defined in the same angle. This sort of expansion does not carry over to the more general case of $(q^m - \bar{A})^{-k}$ with A an elliptic pseudodifferential operator. In particular we obtain as a consequence of the validity of such an expansion a pointwise asymptotic expansion in an angle for the diagonal values of the kernels of $(\lambda - \bar{A})^{-(n/m)-1}$ with the aid of the powers $|\lambda|^{-[n/m]-1+(n-j)/m}$ of $|\lambda|$.

Recently R. T. Seeley (11) has proved that also if A is an elliptic invertible pseudodifferential operator with the range of $\sigma_m(A)$ disjoint to a ray A^s is an elliptic pseudodifferential operator of order ms . Furthermore

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he has investigated the kernels $K(xys)$ of A^s with $s < -n/m$ and proved that $K(xxs)$ has an extension to a meromorphic function of s with simple poles at $(j - n)/m$, $j = 0, 1, 2, \dots$ and $j \neq n$.

In case A is restricted to be a differential operator it is convenient to investigate properties of $K(xys)$ using the above mentioned expansion of $[\lambda - A]^{-[n/m]-1}$. In particular we obtain that the poles of $K(xxs)$ are situated at $(j - n)/m$, $j = 0, 1, 2, \dots$ and $(j - n)/m$ different from a non negative integer. The results thus obtained generalize results in Minakshisundaram and Pleijel (9) where in order to obtain the distribution of the eigenvalues of a Laplacian of a Riemannian manifold the series $\sum \lambda_n^s \Phi_n(x) \overline{\Phi_n(y)}$ is investigated. Here λ_n and Φ_n are the eigenvalues and the eigenfunctions of the Laplacian. In this case the poles of $\sum \lambda_n^s \Phi_n(x) \overline{\Phi_n(x)}$ are situated at $\frac{1}{2}n + j$, $j = 0, 1, 2, \dots$ when n is odd and at $\frac{1}{2}n$, $\frac{1}{2}n + 1, \dots, 2, 1$ when n is even.

We add that the above methods can be adjusted to investigate interior properties of A^s or of resolvents of realizations of A where A is an elliptic differential operator in a bounded domain in R^n . The results on A^s thus obtained generalize results in Kotaké-Narasimhan (8). It is shown there that if A has a positive self adjoint realization A^s has a very regular kernel.

In an appendix we mention results concerning the existence of rays of minimal growth of the resolvent of A , with A an elliptic pseudodifferential operator on X and concerning the completeness of the generalized eigenfunctions of A .

We refer to Kohn-Nirenberg (7) and to R. S. Palais (10) for the definitions and basic properties of pseudodifferential operators on R^n and on a manifold respectively.

We mention also that the results in section 2 here are proved with the aid of a technique similar to that of Kohn-Nirenberg (7) and R. S. Palais (10) and we do not repeat the proofs here. Main details of the proofs are given in (4).

We introduce now the following notations:

Let R^n be the n dimensional Euclidean space. Let xy be the scalar product in R^n . For a multi index $\alpha = (\alpha_1, \dots, \alpha_n)$ let $|\alpha| = \alpha_1 + \dots + \alpha_n$, $\alpha! = \alpha_1! \dots \alpha_n!$ and $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$ when $\xi \in R^n$. We put $D_j = -i \frac{\partial}{\partial x_j}$, $D = (D_1 \dots D_n)$ and $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$. Similarly $\partial_j = \frac{\partial}{\partial \xi_j}$, $\partial = (\partial_1 \dots \partial_n)$ and $\partial^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$.

As usual let S be the space of C^∞ complex functions defined in R^n

which together with all their derivatives die down faster than any power of $|x|$ at infinity. In S we denote $\tilde{u}(\xi) = (2\pi)^{-n/2} \int e^{-ix\xi} u(x) dx$.

For $u \in S$ and real s we denote by $\|u\|_s$ the s norm of u given by

$$1.1 \quad \|u\|_s^2 = \int (1 + |\xi|^2)^s |\tilde{u}(\xi)|^2 d\xi.$$

H_s is then the completion of S in the s norm. For every real s and complex q we introduce in S an s, q norm which is equivalent to the s norm and is given by

$$1.2 \quad \|u\|_{s,q}^2 = \int (1 + |\xi|^2 + |q|^2)^s |\tilde{u}(\xi)|^2 d\xi.$$

Let $A(q)$ be a family of linear operators from S to S defined in a truncated angle $\Delta_M = \{q, \theta_1 \leq \arg q \leq \theta_2, |q| \geq M\}$, with $\theta_1 \leq \theta_2$ and $M > 0$. Let σ be a real number. We say that $A(q)$ is of order σ in Δ_M if for every s there is a C such that if $q \in \Delta_M$ and $u \in S$

$$1.3 \quad \|A(q)u\|_{s,q} < C \|u\|_{s+\sigma,q}.$$

A family as above is of order $-\infty$ in Δ_M if it is of order σ for every σ . It is clear that if $A(q)$ is of order $-\infty$ in Δ_M , for every real s $\|A(q)u\|_s$ dies down as $q \rightarrow \infty$ in Δ_M faster than any power of $|q|$. If $M = 0$ we put $\Delta_M = \Delta$.

Let X be an n dimensional compact C^∞ manifold without a boundary. We fix on X a finite complete set $\chi_j, j = 1, 2, \dots, K$, of coordinates and denote by O_j the open set where χ_j is defined. Let $\Phi_j, j = 1 \dots K$, be a partition of unity on X subordinate to $\{O_j\}$.

For $\Phi \in C^\infty(X)$ we denote by M_Φ the transformation from $C^\infty(X)$ to $C^\infty(X)$ given by $M_\Phi u = \Phi \cdot u$. Let χ be a set of coordinates defined in $C^\infty(0)$. We denote by χ^* the transformation from $C^\infty \chi(0)$ to $C^\infty(0)$ given by $(\chi^* u)(x) = u(\chi(x))$ and we denote by χ_* the transformation from $C^\infty(0)$ to $C^\infty \chi(0)$ given by $(\chi_* u)(x) = u(\chi^{-1}(x))$.

As usual for every real s and $u \in C^\infty(X)$ we put

$$1.4 \quad \|u\|_s^2 = \sum \|x_{j*} \Phi_j u\|_s^2, R^n$$

and denote by $H_s = H_s(X)$ the completion of $C^\infty(X)$ in the s norm.

For $u \in H_s$ and $V \in H - s$ we put

$$1.5 \quad (u, v) = \sum (\chi_{j*} \Phi_j u, \chi_{j*} \Phi_j v)_{R^n}.$$

with this scalar product H_0 is a Hilbert space.

In $C^\infty(X)$ we introduce the s, q norm by

$$1.6 \quad \|u\|_{s,q}^2 = \sum_j \|\chi_{j*} \Phi_j u\|_{s,q,R^n}^2.$$

Again we say that a family $A(q)$, $q \in \Delta_M$, of linear operators from $C^\infty(X)$ to $C^\infty(X)$ is of order σ in Δ_M if for every s there is a C such that 1.3 holds.

We remind that a pseudodifferential operator of order s , on X , is elliptic if $\sigma_s(A) \neq 0$ on $T^*(x)$. Here $\sigma_s(A)$ is the symbol of A which is a complex valued function defined on $T^*(x)$; the bundle obtained from the cotangent bundle of X by deleting zero from each fiber.

We abbreviate in the following « pseudo differential operator » by P.D.O.

2. Expansible families of pseudodifferential operators.

We consider in this section one parametric families $A(q)$, $q \in \Delta$, of P.D.O. on X that have an asymptotic expansion, in a sense made precise in definition 3 below, into a sum of canonical families defined as follows:

DEFINITION 1. A family $A(q)$, $q \in \Delta$, of linear operators from S to S is a canonical family of real degree σ if

$$2.1 \quad \overline{A(q)} u(\xi) = (2\pi)^{-n/2} \int_{R^n} e^{-ix\xi} a(x\xi q) \psi(\xi q) u(x) dx$$

where $a(x\xi q) = (|\xi|^2 + |q|^2)^{\sigma/2} a_0(x\xi q)$ and $a_0(x\xi q)$ is defined for $x \in R^n$, $\xi \in R^n$ and $q \in \Delta$, that satisfy $|\xi|^2 + |q|^2 \neq 0$, and has the following properties: For every $x \in R^n$ $a_0(x\xi q)$ is positive homogeneous of degree zero in (ξq) , $a_0(x\xi q)$ is infinitely differentiable in $x, \xi, q_1 = \text{Re} \cdot q$ and $q_2 = I_m \cdot q$. There exists $\lim_{X \rightarrow \infty} a_0(x\xi q)$. Let $a_0(\infty \xi q) = \lim_{X \rightarrow \infty} a_0(x\xi q)$ then

$$2.2 \quad (1 + |x|^p) D^\alpha \partial^\beta \frac{\partial^{m_1}}{\partial q_1^{m_1}} \frac{\partial^{m_2}}{\partial q_2^{m_2}} (a_0(x\xi q) - a_0(\infty \xi q)) \rightarrow 0$$

as $|x| \rightarrow \infty$ uniformly in (ξ, q) such that $|\xi|^2 + |q|^2 = 1$.

The function $\psi(\xi q)$ is defined for $\xi \in R^n$ and complex q , is infinitely differentiable in ξ , q_1 and q_2 and satisfies: $0 \leq \psi(\xi q) \leq 1$, $\psi(\xi q) = 0$ when $|\xi|^2 + |q|^2 \leq \frac{1}{2}$ and $\psi(\xi q) = 1$ when $|\xi|^2 + |q|^2 \geq 1$, $a(x\xi q)$ is called the symbol of $A(q)$.

Let $A(q)$, $q \in \Delta$, be a canonical family of degree σ . For every $q \in \Delta$ $A(q)$ is a P.D.O. of true order less than or equal to σ . If $\{re^{i\theta}; r \geq 0\}$ if a fixed ray in Δ the symbol of the operator $A(re^{i\theta})$ is given by

$$\sum \frac{r^n}{n!} \frac{\partial^n}{r^n} a(x\xi re^{i\theta})_{r=0}.$$

DEFINITION 2. A family $A'(q)$, $q \in \Delta$, of linear operators from S to S is canonical of degree σ and of second kind if

$$2.3 \quad A'(q)u(x) = (2\pi)^{-n/2} \int e^{-ix\eta} a(x\eta q) \psi(\eta q) \tilde{u}(\eta) d\eta$$

with $a(x\eta q)$ and $\psi(\eta q)$ as in definition 3.1.

We have

LEMMA 1. Let $A(q)$, $q \in \Delta$, be a canonical family of degree σ . $A(q)$ is of order σ in Δ in the sense of 1.3.

We define now

DEFINITION 3. A family $A(q)$, $q \in \Delta$, of linear operators from S to S is called expansible in Δ , if there exists a sequence $A_j(q)$, $q \in \Delta$, $j = 1, 2, \dots$ of canonical families of respective degrees r_j with the following properties: r_j is monotonically decreasing to $-\infty$ and for every n the order of $A(q) - \sum_{j=1}^n A_j(q)$ in Δ is less than r_n .

Let $a(x\xi q)$ be the symbol of $A_j(q)$. The formal sum $\sum a_j(x\xi q)$ is called a symbol of $A(q)$. $A(q)$ is called of degree r_1 , $a_1(x\xi q)$ is the top order symbol of $A(q)$ and we put $\sigma_{r_1}[A(q)](x\xi q) = a_1(x\xi q)$. Lemma 2 below as well as lemma 1 above are proved in analogy to proofs in Kohn-Nirenberg (7). Details of the proofs are given in (4).

LEMMA 2. a) Let $A(q)$, $q \in \Delta$, and $B(q)$, $q \in \Delta$, be canonical families of respective degree s and σ . Let $a(x\xi q)$ be the symbol of $A(q)$ and let $b(x\xi q)$ be the symbol of $B(q)$. The family $A(q)B(q)$, $q \in \Delta$, is expansible of degree

$\sigma + s$. A symbol of $A(q) \cdot B(q)$ is given by

$$2.4 \quad \sum_{r=0}^{\infty} \sum_{|\alpha|=r} \frac{1}{\alpha!} (-D)^{\alpha} a(x\xi q) \partial^{\alpha} b(x\xi q).$$

b) Let $A'(q)$, $q \in \Delta$, be the canonical family of second kind with symbol $a(x\xi q)$. $A'(q)$ is expansible in Δ and a symbol of $A'(q)$ is given by

$$2.5 \quad \sum_{r=0}^{\infty} \sum_{|\alpha|=r} (-1)^r \frac{1}{\alpha!} D^{\alpha} \partial^{\alpha} a(x\xi q).$$

The following theorem ensures the uniqueness of the expansion of an expansible family:

THEOREM 1. Let $A(q)$, $q \in \Delta$, be a canonical family of degree σ with symbol $a(x\xi q)$. If the order of $A(q)$ in Δ is less than σ $a(x\xi q) \equiv 0$.

PROOF: We rely in the proof on the following theorem in S. Agmon (3): Let T be a bounded linear operator in $L_2(\Omega)$, Ω an open set in R^n possessing the cone property. Suppose that the range of T and the range of its adjoint T^* are contained in $H_m(\Omega)$ for some $m > n$. (m not necessarily an integer if $\Omega = R^n$). Then T is an integral operator, $Tf = \int_{\Omega} K(xy) f(y) dy$, $f \in L_2(\Omega)$, with a continuous and bounded kernel $k(xy)$ satisfying

$$2.6 \quad |K(xy)| < \gamma (\|T\|_m + \|T^*\|_m)^{n/m} \|T\|_0^{1-n/m}$$

where γ is a constant depending only on m , n and on the dimension of the cone in the cone property of Ω .

It is easily seen that it is sufficient to prove the theorem for $\sigma < -n$. We suppose first that $a(x\xi q) \geq 0$. Let $\sigma - \delta$ with $\delta > 0$ be the order of $A(q)$ in Δ . Let $A'(q)$ be the canonical family of second kind with symbol $a(x\xi q)$. For every $u, v \in S$ $(A(q)u, v) = (u, A'(q)v)$. As a result also $A'(q)$ is of order $\sigma - \delta$ in Δ . It follows from 1.3 and 1.2 that there exists a constant c such that

$$2.7 \quad \|A(q)u\|_0 < C \|q\|^{-|\sigma|-\delta}, \quad \|A(q)u\|_{|\sigma|+\delta} < C, \quad \|A'(q)u\|_{|\sigma|+\delta} < C.$$

Let $K(xyq)$ be the kernel of $A(q)$. From 2.6 it follows by 2.7 that

$$2.8 \quad |K(xyq)| < C \|q\|^{-|\sigma|-\delta+n}.$$

But from definition 1 of canonical families it follows that

$$2.9 \quad K(x\xi q) = (2\pi)^{-n} |q|^{-|\sigma|+n} \int_{R^n} a\left(x\xi \frac{q}{|q|}\right) d\xi.$$

Combining 2.8 with 2.9 we obtain that if $\{re^{i\theta}; r \geq 0\}$ is a ray in Δ and $q = re^{i\theta}$ 2.9 holds only if $\int_{R^n} a(x\xi e^{i\theta}) d\xi = 0$ for every $x \in R^n$. It follows then from the assumption $a(x\xi q) \geq 0$ that $a(x\xi q) \equiv 0$.

In the general case where we do not assume $a(x\xi q) \geq 0$, let $C(q)$, $q \in \Delta$ be the canonical family of degree 2σ and symbol $|a(x\xi q)|^2$. As $|a(x\xi q)|^2 \geq 0$, to prove that $a(x\xi q) \equiv 0$ it is sufficient to show that the order of $C(q)$ in Δ is less than 2σ . Let $\sigma - \delta$ with $\delta > 0$ be the order of $A(q)$ in Δ . Let $A'(q)$, $q \in \Delta$, be the canonical family of second kind with symbol $\overline{a(x\xi q)}$. Again $A'(q)$ is of order $\sigma - \delta$ in Δ and $A(q)A'(q)$ is of order $2\sigma - 2\delta$ in Δ . By lemma 2, a and b , $A(q)A'(q)$ differs from $C(q)$ by a family of order less than 2σ in Δ . Thus the order of $C(q)$ in Δ is also less than 2σ .

We mention the following properties of canonical families which we need in the following sections:

THEOREM 2. a) Let $A(q)$ $q \in \Delta$, and $B(q)$ $q \in \Delta$, be expansible families of respective degrees σ and s $A(q)B(q)$ is expansible of degree $\sigma + s$ in Δ . The symbol of $A(q)B(q)$ is given by

$$2.10 \quad \sum_{jka} (-D)^a a_j(x\xi q) \partial^a b_k(x\xi q)$$

where $\sum a_j(x\xi q)$ is the symbol of $A(q)$ and $\sum b_j(x\xi q)$ is the symbol of $B(q)$.

b) Let $a_j(x\xi q)$, $q \in \Delta$, $j = 1, 2, \dots$ be a sequence of symbols of canonical families of respective degrees r_j . Suppose r_j is monotonically decreasing to $-\infty$. There exists a family $A(q)$, expansible in Δ , with symbol $\sum a_j(x\xi q)$.

c) Let $A(q)$ be expansible in Δ , there exist families $A^*(q)$ and $A'(q)$ expansible in Δ such that $A(q) - A'(q)$ is of order $-\infty$ in Δ and such that for every $u, v \in S$ $(A'(q)u, v) = (u, A^*(q)v)$. Let $\sum a_j(x\xi q)$ be the symbol of $A(q)$. The symbol of $A^*(q)$ is given by

$$2.11 \quad \sum_{aj} \frac{(-1)^{|a|}}{\alpha!} D^a \partial^a a_j \overline{a_j(x\xi q)}.$$

d) Let $A(q)$ be expansible in Δ . Let Ω_i , $i = 1, 2$, be open sets in R^n with positive distance. For every real s there is a C such that if $u_i \in S$

$i = 1, 2$ and has its support in Ω_i then for every $q \in \Delta$

$$2.12 \quad \|(A(q)u_1, u_2)\| < C \|u_1\|_{s, q} \|u_2\|_{s, q}.$$

e) Let Ω be an open set in R^n . Let χ be a C^∞ diffeomorphism from Ω into R^n . Let $\Phi_i(x) \in C_0^\infty(\Omega)$ $i = 1, 2, \dots$. Let $A(q)$ be an expansible family in Δ of degree s . Let $a(x\xi q)$ be the top order symbol of $A(q)$. Let $B(q)$ be the family defined in Δ by $B(q) = M_{\Phi_1} \chi^* A(q) \chi_* M_{\Phi_2}$. $B(q)$ is expansible in Δ and of degree s . Let $b(x\xi q)$ be the top order symbol of $B(q)$

$$2.13 \quad b(x\xi q) = \Phi_1(x) a(\chi(x), J^{-1}(x)\xi, q) \Phi_2(x)$$

where $J(x)$ is the adjoint of $\frac{d\chi}{dx}$.

Theorem 2 is proved in analogy to proofs in Kohn-Nirenberg (7) and R. S. Palais (10). Details of the proofs are given in (4).

Let X be a compact n dimensional C^∞ manifold without a boundary. We introduce families $A(q)$ $q \in \Delta$ of linear operators from $C^\infty(X)$ to $C^\infty(X)$ which are expansible in Δ in the sense of the following definition :

DEFINITION 4. Let $A(q)$ $q \in \Delta$ be a family of linear operators from $C^\infty(X)$ to $C^\infty(X)$. $A(q)$ is expansible in Δ if it satisfies :

1) Let Φ_i $i = 1, 2$ be functions in $C^\infty(X)$ with disjoint supports. The family $M_{\Phi_1} A(q) M_{\Phi_2}$ is of order $-\infty$ in Δ .

2) Let O be an open set in X . Let χ be a set of local coordinates defined in O . Let Φ_1 and Φ_2 be in $C^\infty(X)$ with supports in O . The family $\chi_* M_{\Phi_1} A(q) M_{\Phi_2} \chi^*$ is expansible in Δ in the sense of definition 3.

If for every O , Φ_i and χ as above $\chi_* M_{\Phi_1} A(q) M_{\Phi_2} \chi^*$ is of degree s , $A(q)$ is called of degree s .

The validity of definition 4 follows from properties (d) and (e) in theorem 2.

Let $A(q)$, $q \in \Delta$, be an expansible family as in definition 4 of degree s . It follows from property (e) in theorem 2 that there exists a function $\sigma_s[A(q)]$ defined in a subset of the cartesian product of the cotangent bundle of X and Δ and such that

$$2.14 \quad \Phi_1(x) \Phi_2(x) \sigma_s[A(q)](\sum \xi_j d\chi_j(x), q) = \sigma_s[\chi_* M_{\Phi_1} A(q) M_{\Phi_2} \chi^*](\chi(x), \xi, q)$$

for every $\sum \xi_j d\chi_j(x)$ in the cotangent bundle of X and $q \in \Delta$ that satisfy $\sum |\xi_j|^2 + |q|^2 \neq 0$. $\sigma_s[A(q)]$ is called the symbol of $A(q)$.

We define also

DEFINITION 5. Let $M > 0$. A family $A(q)$, $q \in \Delta_M$, of linear operators from $C^\infty(X)$ to $C^\infty(X)$ is *expansible* in Δ_M if there exists a family $A'(q)$ *expansible* in Δ such that $A(q) - A'(q)$ is of order $-\infty$ in Δ_M .

3. Expansible families with non vanishing symbols.

The asymptotic expansion of the diagonal values of the kernels
 $(\lambda - A)^{-[n/m]-1}$.

THEOREM 1. Let $A(q)$, $q \in \Delta$, be an *expansible* family of degree s . Let $\sigma_s[A(q)](\Sigma \xi_j d\chi_j(x), q) \neq 0$ when $q \in \Delta$ and $\Sigma \xi_j^2 + |q|^2 \neq 0$. There exists a family $B(q)$ *expansible* of degree $-s$ in Δ such that $A(q)B(q) - I$ and $B(q)A(q) - I$ are of order $-\infty$ in Δ .

$$3.1 \quad \sigma_s[A(q)] \sigma_{-s}[B(q)] = 1.$$

PROOF. Let $A(q)$ be an *expansible* family of P.D.O. in R^n of degree r . Let $a(x\xi q) = \sigma_r[A(q)](x\xi q)$ and let $a(x\xi q) \neq 0$ when $x \in \Omega$, $\xi \in R^n$, $q \in \Delta$ and $|\xi|^2 + |q|^2 \neq 0$. Let θ and Φ be in $C_0^\infty(\Omega)$ such that $\theta\Phi = \Phi$. There exist families $B_1(q)$ and $B_2(q)$, of P.D.O. in R^n , *expansible* in Δ and of degree $-r$ such that $M_\Phi A(q)B_1(q)M_\theta = M_\Phi$ and $M_\Phi B_2(q)A(q)M_\theta = M_\Phi$ are of order $-\infty$ in Δ . In fact let $\psi \in C_0^\infty(\Omega)$ and let $\psi\theta = \theta$. $\psi(x)a(x\xi q)^{-1}$ is a symbol of a canonical family $W(q)$ of degree $-r$ in Δ . It follows from theorem 2 (a) in 2 that $S_1(q) = -A(q)W(q) + M\psi$ and $S_2(q) = -W(q)A(q) + M\psi$ are *expansible* and of negative order in Δ .

Let $S(q)$, $q \in \Delta$, be an *expansible* family of P.D.O. in R^n of negative order $-\varrho$. There exists then an *expansible* family $T(q)$, $q \in \Delta$, such that $(I - S(q))T(q) - I$ and $T(q)(I - S(q)) - I$ are of order $-\infty$ in Δ . In fact $T(q)$ is a family corresponding by theorem 2, (b), 2 to the symbol whose terms of degrees bigger than $-(n+1)\varrho$ coincide with the terms of degrees

bigger than $-(n+1)\varrho$ in the expansion of $\sum_{i=0}^n S(q)^i$.

Let $T_i(q)$ $i = 1, 2$ $q \in \Delta$ be *expansible* families such that $(I - S_1(q))T_1(q) - I$ and $T_2(q)(I - S_2(q)) - I$ are of order $-\infty$ in Δ . Let $B_1(q) = W(q)T_1(q)$ and $B_2(q) = T_2(q)W(q)$. It is easily seen that $B_i(q)$ are families as required and that $\sigma_{-r}[B_i(q)](x\xi q) = \sigma_r[A(q)](x\xi q)^{-1}$ for every x such that $\theta(x) = 1$.

Let $A(q)$ be as in the statement of this theorem. With the aid of a suitable choice of partitions of unity on X and the above local result we obtain families $B_1(q)$ and $B_2(q)$ of P.D.O. on X , *expansible* in Δ and of

degree $-s$ such that $A(q)B_1(q) - I$ and $B_2(q)A(q) - I$ are of order $-\infty$ in Δ , and $\sigma_{-s}[B_i(q)]\sigma_s[A(q)] = 1$, $i = 1, 2$. As $A(q)B_1(q) - I$ and $B_2(q)A(q) - I$ are of order $-\infty$ in Δ also $B_1(q) - B_2(q)$ is of order $-\infty$ there. Thus $B(q) = B_1(q)$ satisfies the theorem.

It follows from the proof of this theorem and from the proof of theorem 2(b) 2 that $B(q)$ can be chosen so that for every $u \in C^\infty(X)$ $B(q)u$ is a continuous function from Δ to $H_0(X)$.

Let A be an elliptic differential operator of order m with coefficients in $C^\infty(X)$. Let the range of $\sigma_m(A)$ be disjoint to $\Gamma = \{\lambda; \lambda = q^m, q \in \Delta\}$. The family $A - q^m$, $q \in \Delta$, satisfies the assumptions of theorem 1 here. It is well known, and it follows also from theorem 1 here, that there exist constants M and C such that $\{\lambda; \lambda = q^m, q \in \Delta_M\} \subset \varrho(\bar{A})$, with $\varrho(\bar{A})$ the resolvent set of $\bar{A}$, and in Δ_M $\|(q^m - \bar{A})^{-1}\| < C|q|^{-m}$. Comparing $(q^m - \bar{A})^{-1}$, $q \in \Delta_M$, with the family $B(q)$ given by theorem 1 we obtain easily

THEOREM 2. Let A be an elliptic differential operator of order m with coefficients in $C^\infty(X)$. Let the range of $\sigma_m(A)$ be disjoint to $\Gamma = \{\lambda; \lambda = q^m, q \in \Delta\}$. Let $M > 0$ be such that $\{\lambda; \lambda = q^m, q \in \Delta_M\} \subset \varrho(\bar{A})$. $(q^m - \bar{A})^{-1}$, restricted to $C^\infty(X)$, is an expansible family in Δ_M of degree $-m$.

It follows from theorem 2 that for every natural k , $(q^m - \bar{A})^{-k}$ restricted to $C^\infty(X)$, is an expansible family of degree $-mk$ in Δ_M . In particular for every real σ there is C such that if $u \in C^\infty(X)$

$$3.2 \quad \|(q^m - \bar{A})^{-K}u\|_{mk+\sigma, q} < C \|U\|_{\sigma, q}.$$

We mention also the following lemma which we need in section 4.

LEMMA 1. Let A satisfy the assumptions of theorem 2. Let zero be a regular point or a pole of first order of the resolvent of $\bar{A}$. Let $\lambda \in \varrho(\bar{A})$ if $\lambda = q^m$, $\lambda \neq 0$ and $q \in \Delta$. Let $B(q)$, $q \in \Delta$, be the family corresponding to $A(q) = q^m - A$ by theorem 1 here. Let $M > 0$. For every real s and ϱ there is a C such that if $q \in \Delta$, $|q| < M$, $q \neq 0$ and $u \in C^\infty(X)$

$$3.3 \quad \|(q^m - \bar{A})^{-1} - B(q)Au\|_s < C \|U\|_{\varrho}$$

with the aid of theorem 2 here and theorem 3.1 in S. Agmon (3) which we already mentioned in the proof of theorem 1.2 we obtain the following result:

THEOREM 3. Let A be an elliptic differential operator of order m with coefficients in $C^\infty(X)$. Let the range of $\sigma_m(A)$ be disjoint to $\Gamma = \{\lambda; \lambda =$

$= q^m, q \in \mathcal{A}$. Let $M > 1$ be such that $\{\lambda; \lambda = q^m, q \in \mathcal{A}_M\} \subset \varrho(\bar{A})$. Let χ be a set of local coordinates defined in $\Omega \subset X$. Let $O \subset \bar{O} \subset \Omega$. Let $\theta \in C^\infty(X)$ with support in Ω such that $\theta = 1$ in O . Let $x' = \chi(x)$. Let $R(x', y', \lambda)$, $\lambda = q^m, q \in \mathcal{A}_M$ be the kernel of $\chi_* M_\theta(\lambda - \bar{A})^{-[n/m]-1} M_\theta \chi^*$. Let $a_j(x', \xi q)$ be the symbol of the canonical family of degree $-m([n/m] + 1) - j$ in the expansion of $\chi_* M_\theta(q^m - \bar{A})^{-[n/m]-1} M_\theta \chi^*$. For every k there is a C_k such that if $x \in O$ and $\lambda = q^m$ with $q = |q| e^{i\theta}$.

$$3.4 \quad \left| R(x', x', \lambda) - (2\pi)^{-n} \sum_{j=0}^k |\lambda|^{-[n/m]-1+(n-j)/m} \int_{R^n} a_j(x', \xi e^{i\theta}) d\xi \right| < C_k |\lambda|^{-[n/m]-1+(n-k-1)/m}$$

$a_j(x', \xi e^{i\theta})$ is determined recursively by the coefficients of A .

PROOF: To justify the statement of the theorem we notice that for every $\lambda \in \varrho(\bar{A})$, $(\lambda - \bar{A})^{-[n/m]-1}$ and $(\lambda - \bar{A})^{-[n/m]-1*}$ are bounded operators from $H_0(X)$ to $H_m([n/m] + 1)(X)$ and as a result $(\lambda - \bar{A})^{-[n/m]-1}$ has a continuous kernel. Let $A_j(q)$ be the canonical family with symbol $a_j(x', \xi q)$. For every $u \in \mathcal{S}$ and $q \in \mathcal{A}$ such that $|q| > 1$ we have

$$3.5 \quad \widehat{A_j(q)u}(\xi) = (2\pi)^{-n/2} \int e^{-ix\xi} a_j(x', \xi q) u(x') dx'.$$

As a result the diagonal values $\alpha_j(x', x', q)$ of the kernel, $\alpha_j(x', y', q)$, of $A_j(q)$ satisfy

$$3.6 \quad \alpha_j(x', x', q) = (2\pi)^{-n} |q|^{-([n/m]+1)m+n-j} \int_{R^n} a_j(x', \xi e^{i\theta}) d\xi.$$

It follows from definition 3.2 of an expansible family of P.D.O. in R^n , and from the above mentioned theorem in S. Agmon (3) that for every k there is a C_k and an $N(k)$ such that for $x \in O$ we have

$$3.7 \quad \left| (R(x', x', \lambda)) - \sum_{j=0}^{N(k)} \alpha_j(x', x', \lambda^{1/m}) \right| < C_k |\lambda|^{-([n/m]+1)+(n-k-1)/m}.$$

Combining 3.6 with 3.7 we obtain 3.4.

It follows from theorem 2(a).2 and from the relation $(q^m - \bar{A})^{-[n/m]-1} \cdot (q^m - A)^{[n/m]+1} = I$ that $a_j(x', \xi e^{i\theta})$, with $x \in O$, are determined recursively by the coefficients of A .

We mention also the following result of theorem 2.

THEOREM 4. Let the assumptions of theorem 3 be satisfied. Let $R(xy\lambda)$ be the kernel of $[\lambda - A]^{-[n/m]-1}$. Let $x \neq y$. For every α, β $\partial_x^\alpha \partial_y^\beta R(xy\lambda)$ tends rapidly to zero as $\lambda \rightarrow \infty$ in Γ , uniformly in $(xy) \in \Omega_1 \times \Omega_2$ where Ω_1 and Ω_2 are disjoint sets in X with positive distance.

PROOF: To justify the statement of the theorem we notice that for every $\lambda \in \varrho(\bar{A})$, $(\lambda - \bar{A})^{-[n/m]-1}$ is a P.D.O. on X and as a result the kernel of $(\lambda - A)^{-[n/m]-1}$ is infinitely differentiable in the complement of the diagonal of $X \times X$. The statement of the theorem follows easily from theorem 2 here, from property 1 in the definition 4.2 of an expansible family, from theorem 13.9 in S. Agmon (2) and from Sobolev's theorems.

4. Fractional powers of elliptic differential operators.

Let A be an elliptic differential operator of order m with coefficients in $C^\infty(X)$. We define fractional powers of A under the assumption that the range of $\sigma_m(A)$ is disjoint to a fixed ray in the complex plane. It follows then that a whole angle, $\{\lambda; \lambda \neq 0, \theta_1 \leq \arg \lambda \leq \theta_2\}$ is disjoint to the range of $\sigma_m(A)$. We assume also that zero is a regular point or a pole of first order of the resolvent of $\bar{A}$. It follows from the discreteness of the spectrum of $\bar{A}$ and from the known behavior of $(\lambda - \bar{A})^{-1}$ at infinity that a subangle, Γ , of the above angle belongs to $\varrho(\bar{A})$ and that there is a C such that if $\lambda \in \Gamma$ $\|\lambda(\lambda - \bar{A})^{-1}\| < C$. For simplicity we have assumed that Γ contains the negative real axis. We have the following theorem:

THEOREM 1: Let A be as above. Let $0 < \alpha < 1$. Let A_α be the operator defined in $C^\infty(X)$ by

$$4.1 \quad A_\alpha u = \frac{\sin \pi \alpha}{\pi} \int_0^\infty \lambda^{\alpha-1} (\lambda + \bar{A})^{-1} \bar{A} u d\lambda.$$

A_α is an elliptic P.D.O of order $m\alpha$. $\sigma_{m\alpha}(A_\alpha) = \sigma_m(A)^\alpha$.

PROOF: It follows from the assumption on A that, in an angle Δ that, contains the positive axis, $A + q^m$ satisfies the assumptions of theorem 1.3. Let $B(q)$, $q \in \Delta$, be a family expansible in Δ such that $(A + q^m)B(q) - I$ and $B(q)(A + q^m) - I$ are of order $-\infty$ in Δ , chosen so that for

every $u \in C^\infty(X)$ $B(q)u$ is a continuous function from Δ to $H_0(X)$. It follows from theorem 2.3 and lemma 1.3 that the operator S defined in $C^\infty(X)$ by

$$4.2 \quad Su = m \int_0^\infty q^{m(\alpha-1)} q^{m-1} ((\bar{A} + q^m)^{-1} - B(q)) A u dq$$

is of order $-\infty$ in Δ . Thus it is sufficient to show that the operator B_α defined in $C^\infty(X)$ by

$$4.3 \quad B_\alpha u = \frac{\sin \pi \alpha}{\pi} m \int_0^\infty q^{m(\alpha-1)} q^{m-1} B(q) u dq,$$

is an elliptic P.D.O. of order $m(\alpha-1)$ with $\sigma_{m(\alpha-1)}(B_\alpha) \cdot \sigma_m(A)^{1-\alpha} = 1$. It follows easily from the definition of an expansible family of P.D.O on X that to prove this result it is sufficient to check that if $A(q)$ is a canonical family of P.D.O. in R^n , defined in Δ , of degree $\sigma < -m$ and with symbol $a(x\xi q)$ the operator P defined in S by

$$4.4 \quad Pu = m \int_0^\infty q^{-1+m\alpha} A(q) u dq$$

differs by an operator of order $-\infty$ from the canonical operator with symbol $m \int_0^\infty q^{-1+m\alpha} a(x\xi q) dq$. Let $A'(q)$, $q \in \Delta$, be the family of linear operators from S to S given by

$$4.5 \quad \overline{A'(q)} u(\xi) = 2\pi^{-2/n} \int e^{-ix\xi} \zeta(\xi) a(x\xi q) dx$$

where as in the definition of a canonical P.D.O. in R^n , $\zeta(\xi) \in C^\infty(R^n)$ $0 \leq \zeta(\xi) \leq 1$, $\zeta(\xi) = 0$ when $|\xi| \leq \frac{1}{2}$ and $\zeta(\xi) = 1$ when $|\xi| \geq 1$. Let P' be defined in S by

$$4.6 \quad P' u = m \int_0^\infty q^{-1+m\alpha} A'(q) u dq.$$

It follows from definition 1.2 of canonical families, from 4.4, 4.5 and 4.6 $P - P'$ is of order $-\infty$. Let $u, v \in S$ then

$$4.7 \quad (P' u, v) = m \int_0^\infty q^{-1+m\alpha} (A'(q) u, v) dq.$$

It follows from Parseval's equality that

$$4.8 \quad \overline{(P' \cdot u, \tilde{v})} = m \int_0^\infty q^{-1+m\alpha} \overline{(A'(q)u, \tilde{v})} dq.$$

It follows from Fubini's theorem with the aid of 4.5 that

$$4.9 \quad \overline{(P' u, \tilde{v})} = \int (2\pi)^{-n/2} \left(\int_{\mathbb{R}^n} e^{-ix\xi} \zeta(\xi) \left(m \int_0^\infty q^{-1+m\alpha} a(x\xi q) dq \right) u(x) dx \right) \tilde{v}(\xi) d\xi$$

and as a result P' coincides with the canonical P.D.O. determined by the symbol $m \int_0^\infty q^{-1+m\alpha} a(x\xi q) dq$.

Let $\bar{A}_\alpha$ be the closure of A_α in $H_0(X)$. It follows from the ellipticity of A_α that the domain of definition of $\bar{A}_\alpha$ is $H_{m\alpha}(X)$.

As $\bar{A}$ is a closed operator, densely defined in $H_0(X)$, such that $\varrho(\bar{A})$ contains a whole angle $\theta_1 \leq \arg \lambda \leq \theta_2$, except possibly the origin, and an estimate $\|\lambda(\lambda - \bar{A})^{-1}\| < C$ holds in this angle, a definition for the power $\bar{A}^\alpha$ is given by Kato (6). $\bar{A}_\alpha$ coincides with $\bar{A}^\alpha$. In fact the closed operator $\bar{A}^\alpha$ is defined in Kato (6) indirectly by

$$4.10 \quad (\lambda + \bar{A}^\alpha)^{-1} = \frac{\sin \pi \alpha}{\pi} \int_0^\infty \frac{\mu^\alpha}{\lambda^2 + 2\lambda \mu^2 \cos \pi \alpha + \mu^{2\alpha}} (\mu + \bar{A})^{-1} d\mu.$$

which holds for $\lambda > 0$ (and for $\lambda \geq 0$ if $0 \in \varrho(\bar{A})$).

In our case where zero is at most a pole of the resolvent of $\bar{A}$ it follows easily from 4.10 that $D(\bar{A}) \subset D(\bar{A}^\alpha)$ and in $D(\bar{A})$

$$4.11 \quad \bar{A}^\alpha u = \frac{\sin \pi \alpha}{\pi} \int_0^\infty \lambda^{\alpha-1} (\lambda + \bar{A})^{-1} \bar{A} u d\lambda.$$

So that $\bar{A}^\alpha$ is closed extension of A_α with $\{\lambda; \lambda > 0\} \subset \varrho(\bar{A}^\alpha)$. But it follows from the ellipticity of A_α , from $\sigma_{m\alpha}(A_\alpha) = \sigma_m(A)^\alpha$ and from the remarks in the appendix 5 that if $\lambda > 0$ is sufficiently large also $\lambda \in \varrho(\bar{A}_\alpha)$ so that $\bar{A}_\alpha = \bar{A}^\alpha$.

As usual we put for $s = n + \alpha$ with n natural and $0 < \alpha < 1$ $\bar{A}^s = \bar{A}^n \bar{A}^\alpha$. Again it follows from the ellipticity of A and A_α that the domain of defi-

nition of $\bar{A}^{-s}$ is $H_{ms}(X)$ and $\bar{A}^{-s}$ is the closure in $H_0(x)$ of an elliptic P.D.O. of order ms . Also if $0 \in \varrho(\bar{A})$ and as usual $\bar{A}^{-s}$ with $s > 0$ is defined by $\bar{A}^{-s} = (\bar{A}^s)^{-1}$, in $C^\infty(X)$, $\bar{A}^{-s}$ is an elliptic P.D.O. of order $-ms$.

We investigate now properties of the kernels $\bar{A}^s$, with $s < -n/m$, or more generally properties of the kernels of the operators K_s , $s < -n/m$, given by

$$4.12 \quad K_s = \frac{\sin \pi s}{\pi} \int_1^\infty \lambda^s \left((\lambda + \bar{A})^{-1} - \frac{E_0}{\lambda} \right) d\lambda + \frac{1}{2\pi i} \int_{|\lambda|=1} \lambda^s \left((\lambda - \bar{A})^{-1} - \frac{E_0}{\lambda} \right) d\lambda,$$

where E_0 is the spectral projection on the null space of $\bar{A}$.

In case $0 \in \varrho(\bar{A})$ $K_s = \bar{A}^s$. K_s satisfies the following lemma:

LEMMA 1. Let $s < -n/m$ and let K_s be given by 4.12 K_s has a continuous kernel $K(xys)$.

PROOF: In case $m \leq n$ we consider the following expression for K_s :

$$4.13 \quad K_s = \frac{\sin \pi s}{\pi} \frac{[n/m]!}{(s+1) \dots (s+[n/m])} \int_1^\infty \lambda^{s+[n/m]} (\lambda + \bar{A})^{-[n/m]-1} d\lambda + \frac{\sin \pi s}{\pi} \frac{E_0}{s} \\ + \frac{1}{2\pi i} \frac{[n/m]!}{(s+1) \dots (s+[n/m])} \int_{|\lambda|=1} \lambda^{s+[n/m]} \left((\lambda - \bar{A})^{-[n/m]-1} - \frac{E_0}{\lambda^{[n/m]+1}} \right) d\lambda.$$

From 3.2 it follows

$$4.14 \quad \|(\lambda + \bar{A})^{-[n/m]-1}\|_{[n/m]+1} \leq C; \|(\lambda + \bar{A})^{-[n/m]-1}\|_0 \leq C |\lambda|^{-[n/m]-1}$$

$$4.15 \quad \|(\lambda + \bar{A})^{-[n/m]-1*}\|_{[n/m]+1} \leq C.$$

4.15 is obtained by an application of 3.2 to $(\lambda + \bar{A})^{-[n/m]-1*}$ which coincides with $(\lambda + \bar{A}^*)^{-[n/m]-1}$ where A^* is the formal adjoint of A in $C^\infty(X)$. It follows from 4.14, 4.15 and 2.6 that the continuous kernel $S(xy\lambda)$ of $(\lambda + \bar{A})^{-[n/m]-1}$ satisfies

$$4.16 \quad |S(xy\lambda)| \leq C |\lambda|^{-[n/m]-1+n/m}.$$

Also $S(xy\lambda)$ is a continuous function of $(xy\lambda)$ for $(xy) \in XxX$ and $\lambda \geq 1$.

As a result if $s < -n/m$ K_s has a continuous kernel given by

$$4.17 \quad \frac{\sin \pi s}{\pi} \frac{[n/m]!}{(s+1) \dots (s+[n/m])} \int_1^\infty \lambda^{s+[n/m]} s(xy\lambda) d\lambda + \frac{\sin \pi s}{\pi} \frac{E_0(xy)}{s} +$$

$$\frac{1}{2\pi i} \frac{[n/m]!}{(s+1) \dots (s+[n/m])} \int_{|\lambda|=1} \lambda^{s+[n/m]} \left(R(xy\lambda) - \frac{E_0(xy)}{\lambda^{[n/m]+1}} \right) d\lambda.$$

Here $R(xy\lambda)$ is the kernel of $(\lambda - A)^{-(n/m)-1}$ and $E_0(xy)$ is the C^∞ kernel of E_0 .

If $m > n$ we replace $\frac{[n/m]!}{(s+1) \dots (s+[n/m])}$ in 4.17 by 1.

THEOREM 2. Let $s < -n/m$. Let $x \neq y$. $K(xys)$ has an extension to an entire analytic function of s which vanishes at the positive integers and also at zero if $0 \in \rho(\bar{A})$. The extension $K(xys)$ thus obtained belongs to C^∞ in the complement of the diagonal of XX .

PROOF. Let θ and ψ be functions in $C^\infty(X)$ with disjoint supports. Let $S(xy\lambda)$ be the infinitely differentiable kernel of $M_\theta(\lambda + \bar{A})^{-1} M_\psi$. It follows from theorem 4.3 that for every $\alpha\beta \partial_x^\alpha \partial_y^\beta S(xy\lambda)$ tends rapidly to zero as $\lambda \rightarrow \infty$, uniformly in (xy) . Also $\partial_x^\alpha \partial_y^\beta S(xy\lambda)$ is a continuous function of $(xy\lambda)$. As a result the operator

$$4.18 \quad \frac{\sin \pi s}{\pi} \int_1^\infty \lambda^s M_\theta(\lambda + \bar{A})^{-1} M_\psi d\lambda$$

with $s < n/m$ has the infinitely differentiable kernel given by

$$4.19 \quad L(xys) = \frac{\sin \pi s}{\pi} \int_1^\infty \lambda^s S(xy\lambda) d\lambda$$

and for every (xy) $L(xys)$ has an extension to an entire analytic function of s which vanishes at the integers. The extension $L(xys)$ thus obtained is in $C^\infty(XX)$. Similarly the operator

$$4.20 \quad \frac{1}{2\pi i} \int_{|\lambda|=1} \lambda^s M_\theta \left((\lambda - \bar{A})^{-1} - \frac{E_0}{\lambda} \right) M_\psi d\lambda$$

with $s < -n/m$ has the infinitely differentiable kernel given by

$$4.21 \quad M(xys) = \frac{1}{2\pi i} \int_{|\lambda|=1} \lambda^s (R(xy\lambda) - \theta(x) E_0(xy) \psi(y)) d\lambda.$$

Here $R(xy\lambda)$ is the kernel of $M_\theta(\lambda - \bar{A})^{-1} M_\psi$ and $E_0(xy)$ is the kernel of E_0 . For every $(xy)M(xys)$ has an extension to an entire analytic function of s which vanishes at the non negative integers as a result of the regularity of $R(xy\lambda) - \frac{\theta(x) E_0(xy) \psi(y)}{\lambda}$ in the closed unit circle. The extension $M(xys)$ thus obtained is in $C^\infty(X \times X)$.

To complete the proof we notice that if $E_0 \neq 0$ and $s < -n/m$

$$4.22 \quad \frac{\sin \pi s}{\pi} M_\theta \int_1^\infty \lambda^{s-1} E_0 d_\lambda M_\psi$$

has the infinitely differentiable kernel

$$4.23 \quad \frac{\sin \pi s}{\pi s} \theta(x) E_0(xy) \psi(y).$$

For $x = y$ we have the following result:

THEOREM 3: Let $K(xys)$ be the kernel of K_s with $s < -n/m$. every $x \in X$ $K(xxs)$ has an extension to a meromorphic function of s with simple poles at the points $(j - n/m)$, $j = 0, 1, 2, \dots$, that differ from the naturals and from zero.

Let χ be a set of local coordinates defined in $\Omega \subset X$. Let $x' = \chi(x)$. Let 0 be an open set such that $0 \subset \bar{0} \subset \Omega$. Let θ be in $C^\infty(X)$ with support in Ω and let $\theta(x) = 1$ in 0 . Let $K'(x'x's)$ be the kernel of $\chi_* M_\theta K_s M_\theta \chi^*$. Let $a_j(x'\xi q)$ be the symbol of the canonical family of degree $-m([n/m] + 1) - j$ in the expansion of $\chi_* M_\theta (q^m + \bar{A})^{-(n/m)-1} M_\theta \chi^*$. Let i be a natural number

$$4.24 \quad K'(x'x'i) = - \frac{(2\pi)^{-n} [n/m]!}{(i+1) \dots (i+[n/m])} \int a_{mi+n}(x'\xi 1) d\xi.$$

If $E_0 = 0$ 4.24 holds also for $i = 0$.

If $E_0 \neq 0$, let $E'_0(x'y')$ be the kernel of $\chi_* M_\theta E_0 M_\theta \chi^*$, then

$$4.25 \quad K'(x'x'0) = - (2\pi)^{-n} \int a_n(x'\xi 1) d\xi + E'_0(x'x').$$

The residues of $(K'(x'x's))$ are as follows:

Let i be a negative integer with $i \geq -n/m$. The residue of $K'(x'x's)$ at $s = i$ is given by

$$4.26 \quad - (2\pi)^{-n} \frac{[n/m]!}{\pi \prod_{\substack{j=1 \\ j \neq -i}}^{[n/m]} (j+i)} \int_{R^n} a_{mi+n}(x' \xi 1) d\xi.$$

Let $(j-n)/m$, $j = 0, 1, 2, \dots$, be different from an integer. The residues of $K'(x'x's)$ at $s = (j-n)/m$ are given by

$$4.27 \quad - (2\pi)^{-n} \frac{\sin(\pi(j-n)/m)}{\pi} \frac{[n/m]!}{\left(\frac{j-n}{m} + 1\right) \dots \left(\frac{j-n}{m} + [n/m]\right)} \int a_j(x' \xi 1) d\xi.$$

PROOF: We consider the expression 4.13 for K_s . Let $s < -n/m$. It is easily seen that $\frac{1}{2\pi i} \frac{[n/m]!}{(s+1) \dots (s+[n/m])} \int_{|\lambda|=1} \lambda^{s+[n/m]} \left((\lambda - \bar{A})^{-[n/m]-1} - \frac{E_0}{\lambda^{[n/m]+1}} \right) d\lambda$ has a continuous kernel $M(xys)$ and for every (xy) $M(xys)$ has an extension to an entire analytic function of s which vanishes at the bigger from $-n/m$ integers. Thus to prove the theorem it is sufficient to show that if $N(xys)$ with $s < -n/m$ is the kernel of

$$4.28 \quad N_s = \frac{\sin \pi s}{\pi} \frac{[n/m]!}{(s+1) \dots (s+[n/m])} \int_1^\infty \lambda^{s+[n/m]} (\lambda + \bar{A})^{-[n/m]-1} d\lambda$$

$N(xxs)$ has an extension to a meromorphic function of x with simple poles at $(j-n)/m$, $j = 0, 1, 2, \dots$, with $(j-n)/m$ different from the non negative integers, and to determine $N(xxi)$ and the residues of $N(xxs)$.

An application of theorem 3.3 to $(\lambda + A)^{-[n/m]-1}$, with $\lambda \geq 1$, yields the asymptotic expansion

$$4.29 \quad (2\pi)^{-n} \sum_{j=0}^{\infty} \lambda^{-[n/m]-1+(n-j)/m} \int_{R^n} a_j(x' \xi 1) d\xi$$

for the diagonal values of the kernel of $\chi_* M_\theta(q^m + \bar{A})^{-[n/m]-1} M_\theta \chi^*$ where $a_j(x' \xi 1)$ are as in the statement of the theorem.

Let $N'(x'x's)$ be the kernel of $\chi_* M_\theta N_s M_\theta \chi^*$. The location of the poles of $N'(x'x's)$ follows from 4.28, 4.29 and from

$$4.30 \quad \int_1^\infty \lambda^{s+[n/m]} \lambda^{-[n/m]-1+(n-j)/m} d\lambda = - \frac{1}{s - \frac{j-n}{m}}.$$

The values of $K'(x'x'i)$ and the residues of $K'(x'x's)$ are given by 4.24-4.27 as a result of 4.28, 4.29, 4.30 and the relation of $N'(x'x's)$ to $K'(x'x's)$.

5. Appendix.

We add here some remarks concerning the existence of rays of minimal growth of the resolvent of an elliptic P. D. O. on X and concerning the completeness of the generalized eigenfunctions of A .

We prove in (4) the following theorems which we do not prove here:

THEOREM 1: Let A be an elliptic P. D. O. of positive order s . Let the range of $\sigma_s(A)$ be disjoint to the angle $\Gamma = \{\lambda; \theta_1 \leq \arg \lambda \leq \theta_2\}$ with $\theta_1 \leq \theta_2$. Let $\bar{A}$ be the closure of A in $H_0(X)$. There exist constants M and C such that if $|\lambda| > M$ and $\lambda \in \Gamma$, $\lambda \in \rho(\bar{A})$ and $\|\lambda(\lambda - \bar{A})^{-1}\| < C$.

We have also the following:

THEOREM 2: Let A be an elliptic P. D. O. of positive order s . Let $\lambda_0 \in \rho(\bar{A})$. For every $\varepsilon > 0$ $(\lambda_0 - \bar{A})^{-1} \in C_{n/s+\varepsilon}$.

The proof of this theorem is based on $(\lambda_0 - \bar{A})^{-1}$, restricted to $C^\infty(X)$, being an elliptic P. D. O. of order $-m$ and on the following lemma in S. Agmon (1):

Let T be a compact operator in $H_0^\sharp$ which carries $H_0^\sharp$ into $H_0^\sharp$ with $\sigma > 0$. Let $\{\lambda_j\}$ be the sequence of eigenvalues of T each repeated according to the multiplicity. For every $\varepsilon > 0$ $\sum |\lambda_j|^{n/\sigma+\varepsilon} < \infty$.

From theorems 2 and 3 it follows with the aid of (XI.9.31) in Dunford-Schwartz (5):

THEOREM 3. Let A be an elliptic P. D. O. of positive order s . Let the range of $\sigma_s(A)$ be contained in an angle $\Gamma: \{\lambda; \theta_1 < \arg \lambda < \theta_2\}$ with $\theta_2 - \theta_1$ less than $\pi s/n$. The generalized eigenfunctions of $\bar{A}$ are complete in $H_0(X)$.

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