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THE UNBOUNDED GROWTH OF SOLUTIONS OF LINEAR PARABOLIC DIFFERENTIAL EQUATIONS (*)

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Introduction.

The object of this paper is to study the behavior for large t of positive solutions $u(x, t)$ of the general second order linear parabolic inequality

$$(0) \quad Lu + cu - u_t \leq 0,$$

where

$$Lu \equiv a_{ij} u_{ij} + b_i u_i.$$

Here the summation convention is used, and subscripts on u denote derivatives: $u_t = \frac{\partial u}{\partial t}$, $u_i = \frac{\partial u}{\partial x_i}$, etc. The coefficients are functions of $x = (x_1, x_2, \dots, x_n)$ and t . The parabolicity is assumed to be uniform in t but not necessarily in x .

When $c \leq 0$ and the above inequality is reversed, a variety of theorems (see, for example [3-5]) are available concerning the limiting behavior as $t \rightarrow \infty$ of solutions defined in $E^n \times (0, \infty)$. An example from [4] is the following. Suppose $\Sigma(a_{ii} + b_i x_i) > \alpha > 0$ for all x and t , and suppose $u(x, t)$ satisfies $Lu - u_t = 0$ for $t > 0$. If $\lim_{|x| \rightarrow \infty} u(x, 0) = 0$, then $\lim_{t \rightarrow \infty} u(x, t) = 0$ uniformly in x .

In the present paper, on the other hand, we require that $c > 0$ for certain values of its arguments, and investigate under what conditions positive solutions will approach ∞ as $t \rightarrow \infty$. The results show that the behavior of positive solutions as $t \rightarrow \infty$ is intimately related to the possible

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behavior as $|x| \rightarrow \infty$ of positive solutions $V(x)$ of the corresponding elliptic inequality (with $c = 0$).

More specifically, the results in a large part framed in terms of barriers and antibarriers of L . These concepts were used in [6] and elsewhere; they are positive functions $V(x)$ satisfying $LV \leq 0$ for large $|x|$ and for each t . Barriers approach 0 and antibarriers approach ∞ as $|x| \rightarrow \infty$. It is known from [6] that if an operator has a barrier, it cannot have an antibarrier. Explicit conditions on the coefficients are given in [6] (see our corollaries to Theorems 1 and 4) which insure that one or the other of these functions exists. In the case of the Laplace operator, a barrier exists when $n \geq 3$, and an antibarrier when $n \leq 2$. If an antibarrier exists, $c \geq 0$, and $c \not\equiv 0$, then positive solutions of the parabolic problem treated here tend to infinity exponentially (Theorem 1). If a barrier exists, this is not always true (Theorem 4), but is true in any case if $c \geq 0$ and c is large enough for x in some domain (Theorem 2). Theorem 3 shows that exponential growth is possible even when $c(x, t) \rightarrow -\infty$ as $|x| \rightarrow \infty$ provided that c is large enough for x in a fixed domain and a growth condition is placed on the coefficients. Furthermore an explicit lower bound for $u(x, t)$ can be obtained in this case.

A regular solution of (0) will be taken to mean a function continuous for $t \geq 0$ whose second spacial derivatives and first time derivatives are continuous for $t > 0$, and which satisfies (0) for $t > 0$.

The following functions will be used extensively in the argument:

$$A(x, t) = a_{ij}(x, t) x_i x_j / |x|^2$$

$$B(x, t) = a_{ii}(x, t) + b_i(x, t) x_i.$$

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1. Main Theorems.

THEOREM 1. *Assume there exist positive continuous functions $\kappa(x)$ and $M(x)$ such that for all $\xi_1, \xi_2, \dots, \xi_n$,*

$$(1) \quad a_{ij}(x, t) \xi_i \xi_j \geq \kappa(x) \sum_{i=1}^n \xi_i^2;$$

$$(2) \quad |a_{ij}(x, t)|, |b_i(x, t)| \leq M(x).$$

Assume there exists an antibarrier $V(x)$ defined in a neighborhood of infinity such that $LV \leq 0$ for each t , and $V(x) \rightarrow \infty$ as $|x| \rightarrow \infty$.

Let $c(x, t)$ be a function satisfying $c(x, t) \geq 0$ and $c(x, t) \geq c_1$ for $|x| < \delta$ where c_1 and δ are some positive numbers. Let $u(x, t) \geq 0$ be a regular solution of $Lu + cu - u_t \leq 0$ in $E^n \times (0, \infty)$. If $u(x, 0) \not\equiv 0$ then

$$\lim_{t \rightarrow \infty} u(x, t) = \infty;$$

in fact, there is a function $\psi(x) > 0$ and a number $\gamma > 0$ such that

$$(3) \quad u(x, t) \geq \psi(x) e^{\gamma t}$$

for $t \geq 1$.

REMARK 1. It will be clear from the proof that in place of (2) one need merely assume that $A(x, t)$ and $B(x, t)$ are locally bounded from above, uniformly in t :

$$(2') \quad A(x, t) \leq \bar{A}(x); \quad B(x, t) \leq \bar{B}(x)$$

REMARK 2. For $\varrho > 0$ let $\Gamma_\varrho = \{x: |x| < \varrho\}$. The theorem may be strengthened slightly by requiring only that u be a positive solution of $Lu + cu - u_t \leq 0$ in $\Gamma_\varrho \times (0, \infty)$. Then the conclusion (3) holds for $x \in \Gamma_\varrho$ provided that ϱ is large enough (depending on L and c). In fact, it will be clear from the proof that ϱ may be chosen $\geq R$, where R is defined following (7) below.

COROLLARY. Let $c(x, t)$ and $u(x, t)$ be as in Theorem 1. If $\frac{B(x, t)}{A(x, t)} \leq 2 + \varepsilon(|x|)$ for large enough x and for all t , where ε is such that

$$\int_0^\infty \exp \left\{ - \int_s^r \varepsilon(s) ds/s \right\} dr/r = \infty,$$

then the conclusion (3) follows.

The corollary follows because under these conditions an antibarrier is constructed explicitly in [6].

PROOF OF THEOREM 1. It follows from Nirenberg's strong maximum principle that $u(x, t) > 0$ for $t > 0$. By shifting the origin of the t axis if necessary, we may assume with no loss of generality that $u > 0$ for $t = 0$ as well.

We say that a function $f(x, t)$ has *Property P* if it is continuous, has bounded piecewise continuous first derivatives, and is twice continuously differentiable in x except on a finite number of smooth surfaces. Furthermore, the directional derivative in the direction of traversal of such a surface suffers a *nonpositive* jump discontinuity. The following simple extension of the maximum principle holds, as noted for example by Il'in [4, 5].

Maximum principle: Let S be a bounded domain in $x - t$ space contained between the planes $t = 0$ and $t = t_1$. Let $u(x, t)$ be a function defined in S satisfying Property P and $Lu + cu - u_t \leq 0$ wherever u is regular. If $u \geq 0$ on the boundary of S exclusive of points with $t = t_1$, then $u \geq 0$ throughout S .

We shall need the following lemma, whose proof will be given in section 3.

LEMMA 1. *Under the hypotheses of Theorem 1, there is a function $V_1(x)$ defined for $|x| \geq \delta$ satisfying Property P and also:*

$$(4) \quad LV_1(x) < 0$$

for all t and all regular points of V_1 ;

$$(5) \quad V_1(x) = 0 \text{ for } |x| = \delta;$$

$$V_1(x) \rightarrow \infty \text{ as } |x| \rightarrow \infty.$$

$$\text{Set } w(x) = \begin{cases} \beta_1 - |x|^2 & \text{for } |x| \leq \delta, \\ \beta_1 - \delta^2 - \beta_2 V_1(x) & \text{for } |x| \geq \delta, \end{cases}$$

where β_1 will be given below, and β_2 is taken small enough so that $(-w)$ will satisfy Property P .

For $|x| < \delta$,

$$(6) \quad Lw + cw = -2B(x, t) + cw \geq -2B + c_1(\beta_1 - \delta^2).$$

Now choose β_1 so large that $\beta_1 - \delta^2 > 0$, and also

$$(7) \quad Lw + cw \geq 1$$

for $|x| < \delta$. Since $V_1 \rightarrow \infty$, there exists a number $R > 2\delta$ such that $w < 0$ for $|x| \geq R/2$.

Let Ω be the set of points in E^n for which $w(x) > 0$.

LEMMA 2. Let $m(t) = \inf_{\Omega} \frac{u(x, t)}{w(x)}$. There are constants $K > 0$, $\gamma > 0$ such that $m(t) \geq Ke^{\gamma t}$.

Proof:

Let α be a real number such that $0 < \alpha \leq 1$ and α is small enough that

$$(8) \quad v + 2\alpha \leq 0 \text{ for } |x| \geq R;$$

$$(9) \quad Lw = -\beta_2 LV_1 \geq \alpha \text{ for } \delta < |x| < R.$$

Let $v(x, t) = w(x) + \alpha t$, and let $S = \{(x, t) : v(x, t) > 0; 0 < t \leq 2\}$. One sees that in S ,

$$(10) \quad Lv + cv - v_t \geq 0.$$

In fact, by (7) we know that for $|x| < \delta$, $Lv + cv - v_t = Lw + cw + \alpha t - \alpha \geq 1 - \alpha \geq 0$. And by (9), $Lv + cv - v_t \geq Lw - \alpha \geq \alpha - \alpha = 0$ in the remainder of S as well (note that (8) insures that S is contained in the cylinder $\{|x| < R\}$).

Let $\partial_1 S$ be the lateral boundary of S ; i. e., the part of the boundary for which $0 < t < 2$; and let $\partial_0 S$ be the base $\Omega \times \{t = 0\}$.

Let $\zeta(x, t) = \frac{u(x, t)}{m(0)} - v(x, t)$. By the definition of $m(0)$, $\zeta \geq 0$ on $\partial_0 S$. On $\partial_1 S$, $\zeta = u/m(0) \geq 0$ by assumption. Finally in S , we know from (10) that $L\zeta + c\zeta - \zeta_t \leq 0$ except at the irregular points of ζ . However $-v$, hence ζ , has Property P , so we conclude that $\zeta \geq 0$ in S ; i. e.

$$\frac{u}{m(0)} \geq w + \alpha t,$$

or

$$\frac{u(x, t)}{w(x)} \geq m(0) \left(1 + \frac{\alpha t}{w(x)}\right) \geq m(0) \left(1 + \frac{\alpha t}{\beta_1}\right). \text{ Setting } t \geq 1$$

and taking the infimum with respect to x in Ω , we obtain the conclusion

$$m(t) \geq m(0) \left(1 + \frac{\alpha}{\beta_1}\right) = km(0) \text{ for } 1 \leq t \leq 2.$$

Applying this result successively yields $m(t) \geq k^N m(0)$, for $N \leq t \leq N + 1$. The conclusion of the lemma follows immediately, for some K and γ which could be found in terms of k and $m(0)$. We now state a lemma which will be proved in section 3.

LEMMA 3. Let L satisfy (1) and (2'), and let $u(x, t) \geq 0$ be a solution of $Lu - u_t \leq 0$ in $E^n \times [0, 1]$. Let $\delta > 0$. Then there is a positive function $\chi(x)$ depending only on κ, A, B , and δ such that

$$u(x, 1) \geq u_0 \chi(x)$$

provided $u(x, 0) \geq u_0$ for $|x| \leq \delta$.

The proof of Theorem 1 is now completed by combining Lemmas 2 and 3 as follows. For fixed t set $u_0 = (\beta_1 - \delta^2) Ke^{rt}$. Thus for $|x| \leq \delta$, $u(x, t) \geq \geq Ke^{rt} w(x) \geq u_0$. Hence by Lemma 3, $u(x, t+1) \geq \psi(x) e^{r(t+1)}$, where we have set $\psi(x) = (\beta_1 - \delta^2) Ke^{-r} \chi(x)$. This finishes the proof.

THEOREM 2. Assume the hypotheses of Theorem 1 to hold, except that it is no longer required for an antibarrier to exist. There is a number C depending only on κ, M , and δ such that the conclusion (3) of Theorem 1 holds provided that $c_1 \geq C$.

REMARK: The remarks following Theorem 1 apply here as well. In this case ϱ may be chosen arbitrarily, except that $\varrho > \delta$; then C will depend on ϱ also. To see this one chooses β_1 below so that $0 < \beta_1 - \delta^2 < \beta_2 \varphi(\varrho)$.

PROOF of THEOREM 2. The proof utilizes the following lemma.

LEMMA 4. Assume hypotheses (1) and (2') of Theorem 1. Then for any $\delta > 0$ there exists a function $\varphi(r)$ defined for $r \geq \delta$, satisfying $\varphi(\delta) = 0$, $\varphi'(r) > 0$ for $r \geq \delta$, and $L\varphi(|x|) < 0$.

PROOF. Let $\tau(r) = \max_{\substack{|x|=r \\ \text{all } t}} \frac{B(x, t)}{A(x, t)}$, and set

$$\varphi'(r) = \exp \left[- \int_{\delta}^r (\tau(\varrho)/\varrho) d\varrho \right] > 0,$$

from which $\varphi(r)$ may be determined. We find that

$$L\varphi(|x|) = A(x, t) \left[\varphi'' + \left(\frac{B}{A} - 1 \right) \varphi' / |x| \right] < A(x, t) \left[\varphi'' + \frac{\tau(|x|)}{|x|} \right] \varphi' = 0,$$

which establishes Lemma 4.

Let

$$w(x) = \begin{cases} \beta_1 - |x|^2 & \text{for } |x| \leq \delta, \\ \beta_1 - \delta^2 - \beta_2 \varphi(|x|) & \text{for } |x| \geq \delta, \end{cases}$$

where β_2 is chosen small enough so that $(-w)$ satisfies Property P , and β_1 is chosen so that $0 < \beta_1 - \delta^2 < \beta_2 \lim_{r \rightarrow \infty} \varphi(r)$. Now observe that (6) holds. It remains only to choose c_1 large enough so that (7) holds for $|x| < \delta$. By virtue of the choice of β_1 , there still exists a finite R such that $w < 0$ for $|x| \geq R/2$. The remainder of the proof of Theorem 1 is repeated verbatim.

THEOREM 3. *Let $u(x, t)$ be a regular solution of*

$$(11) \quad L_1 u \equiv Lu + cu - u_t \leq 0$$

in $S_0 = E^n \times (0, \infty)$. We assume that

(i) *there exist $A', B', C' > 0$, $0 \leq \alpha \leq 2$ such that*

$$|a_{ij}| \leq A'(|x|^{2-\alpha} + 1), |b_i| \leq B'(|x| + 1), |c| \leq C'(|x|^\alpha + 1),$$

$$(ii) \quad a_{ij} \xi_i \xi_j \geq \kappa |\xi|^2, \quad \kappa = \text{const} > 0,$$

(iii) *there exist constants β, γ ($\gamma > 0$, $\beta > \beta_0$, β_0 being a positive number which depends among others on A', B', C' and κ) and a point $\bar{x} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_n)$ such that for large t 's, say $t \geq t_0 \geq 0$, the following inequality is satisfied:*

$$\beta^2 \sum a_{ij} (x_i - \bar{x}_i)(x_j - \bar{x}_j) - \beta [\sum a_{ii} + \sum b_i (x_i - \bar{x}_i)] + c - \gamma \geq 0,$$

$$(iv) \quad u(x, t) \geq -M \exp(K|x|^\alpha), \text{ if } 0 < \alpha \leq 2,$$

and

$$u(x, t) \geq -M_0(|x|^{K_0} + 1), \text{ if } \alpha = 0,$$

for some

$$M, K, M_0, K_0 > 0,$$

$$(v) \quad u(x, 0) \geq 0,$$

$$(vi) \quad u(x, 0) \not\equiv 0.$$

Then $\lim_{t \rightarrow \infty} u(x, t) = +\infty$, the convergence being of exponential order and uniform on every compact x -set. More precisely, there exists $\lambda > 0$ such that

$$(12) \quad u(x, t) \geq \lambda \exp \left[-\frac{\beta}{2} |x - \bar{x}|^2 + \gamma(t - t_0) \right] \text{ for } t \geq t_0.$$

REMARK: The following example shows the necessity of assumption (iii): the equation

$$u_{xx} + (2 - 4x^2)u - u_t = 0$$

has the bounded solution $u = e^{-x^2}$. In this example (iii) is not fulfilled.

PROOF. We shall make use of the following Theorem *T* (see [1] Theorem 1): If $u(x, t)$ is a regular solution of $L_1 u \leq 0$ in S_0 satisfying (iv) and (v) and if the coefficients satisfy (i) and $a_{ij} \xi_i \xi_j \geq 0$, then $u(x, t) \geq 0$ in S . (Note that the both-sided growth condition for u in the first part of Theorem I in [1] and the stronger growth condition in case of $\alpha = 0$ are not essential. Furthermore, Theorem *T* is valid for any domain D contained in S_0 ; this again follows from Bodanko's paper).

By this theorem ($u(x, t) \geq 0$ in S_0). Now by (vi) and by Nirenberg's strong maximum principle, $u(x, t) > 0$ in S_0 .

At first we shall show that there are $\lambda, \beta_0 > 0$ such that

$$(13) \quad u(x, t_0) \geq \lambda \exp\left(-\frac{\beta_0}{2} |x - \bar{x}|^2\right) \text{ for } x \in E^n.$$

For this purpose consider the function

$$v(x, t) = \exp\left(-\frac{\mu |x - \bar{x}|^2 - \nu}{t - t_0 + \eta}\right), \quad 0 < \eta < t_0.$$

The positive constants μ, ν can be chosen so that $L_1 v \geq 0$ in $E^n \times [t_0 - \eta, t_0]$. Indeed, we have

$$\begin{aligned} L_1 v = & \frac{v}{(t - t_0 + \eta)^2} [4\mu^2 \sum a_{ij} (x_i - \bar{x}_i)(x_j - \bar{x}_j) - 2\mu(t - t_0 + \eta) \sum a_{ii} \\ & - 2\mu(t - t_0 + \eta) \sum b_i (x_i - \bar{x}_i) + c(t - t_0 + \eta)^2 - \mu |x - \bar{x}|^2 + \nu]. \end{aligned}$$

Now we use the inequalities

$$t - t_0 + \eta \leq \eta, \quad |x_i| \leq |x|, \quad |\bar{x}| \cdot |x - \bar{x}| \leq \frac{1}{2} (|\bar{x}|^2 + |x - \bar{x}|^2),$$

$$|x|^2 \leq 2|x - \bar{x}|^2 + 2|\bar{x}|^2$$

to derive

$$\begin{aligned} L_1 v \geq & \frac{v}{(t - t_0 + \eta)^2} \left\{ [4\mu^2 - (8\eta nA' + 4\eta nB' + 1)\mu - 4\eta^2 C'] |x - \bar{x}|^2 - \right. \\ & \left. - (8\eta n\mu A' + \eta n \mu B' + 4\eta^2 C') |\bar{x}|^2 - 4\eta \mu A' - \eta n \mu B' - 2\eta^2 C' + \nu \right\}. \end{aligned}$$

Let $\mu_0(\eta)$ be the largest root of the equation

$$4\pi\mu^2 - (8\eta nA' + 4\eta nB' + 1)\mu - 4\eta^2 C' = 0.$$

Putting

$$\nu_0 = (8\eta n\mu_0 A' + \eta n\mu_0 B' + 4\eta^2 C') |\bar{x}|^2 + 4\eta n\mu_0 A' + \eta n\mu_0 B' + 2\pi^2 C',$$

we see that the function

$$\nu_0(x, t) = \exp\left(-\frac{\mu_0 |x - \bar{x}|^2 - \nu_0}{t - t_0 + \eta}\right)$$

satisfies $L_1 \nu_0 \geq 0$ in $E^n \times [t_0 - \eta, t_0]$. Let $R > \sqrt{\frac{\mu_0}{\nu_0}}$ be fixed. We define

$$\lambda = \min_{\substack{|x - \bar{x}| \leq R \\ t \in [t_0 - \eta, t_0]}} u(x, t).$$

Since $u(x, t) > 0$ in S , λ is a positive number.

The function

$$w(x, t) = u(x, t) - \lambda \nu_0(x, t)$$

satisfies the inequality $L_1 w \leq 0$ in $E^n \times [t_0 - \eta, t_0]$. Furthermore $w(x, t) \geq 0$ for $|x - \bar{x}| = R$, $t \in [t_0 - \eta, t_0]$ and for $|x - \bar{x}| \geq R$, $t = t_0 - \eta$. Now we are able to use Theorem *T* again to conclude that $w(x, t) \geq 0$ in the region $|x - \bar{x}| \geq R$, $t_0 - \eta \leq t \leq t_0$. Since

$$\nu_0(x, t) \geq \exp\left(\frac{\mu_0 |x - \bar{x}|^2}{t - t_0 + \eta}\right),$$

we have

$$(14) \quad u(x, t) \geq \lambda \exp\left(-\frac{\mu_0 |x - \bar{x}|^2}{t - t_0 - \eta}\right) \text{ for } |x - \bar{x}| \geq R, t_0 - \eta \leq t \leq t_0.$$

From the definition of λ it follows that inequality (14) remains true in the whole strip $E^n \times [t_0 - \eta, t_0]$. Substituting, in particular, $t = t_0$ and putting

$$\frac{\mu_0}{\eta} = \frac{\beta_0}{2} \text{ we get (13).}$$

Now the function

$$z(x, t) = u(x, t) - \lambda \exp\left[-\frac{\beta}{2} |x - \bar{x}|^2 + \gamma(t - t_0)\right], \beta \geq \beta_0, t \geq t_0,$$

satisfies, by assumption (iii), $L_1 z \leq 0$ for $t \geq t_0$. Furthermore, from (13), it follows that $z(x, t_0) \geq 0$. Consequently, by Theorem T, $z(x, t) \geq 0$ for $t \geq t_0$, $x \in E^n$, which was to be proved.

2. A Countertheorem.

THEOREM 4. *Let the operator L satisfy (1), (2') of Theorem 1. Also assume it has a barrier near infinity; i. e. a positive function $W(x)$ defined for $|x| \geq X$ such that $LW < 0$, and $W(x) \rightarrow 0$ as $|x| \rightarrow \infty$. Finally assume the coefficients a_{ij} and b_i are Hölder continuous functions of all their arguments. Then there exists a smooth function $c(x) \geq 0$ with $c(x) \not\equiv 0$, and a positive but bounded solution $u(x, t)$ of $Lu + cu - u_t = 0$.*

REMARK. This theorem shows a result by Szybiak [7] to be incorrect.

COROLLARY. *Let L satisfy (1), (2') and have Hölder continuous coefficients as in Theorem 4. If $\frac{B(x, t)}{A(x, t)} \geq 2 + \varepsilon(|x|)$ for large enough x and for*

all t , where $\int \exp \left\{ - \int_s^r \varepsilon(s) ds/s \right\} dr/r < \infty$, then the conclusion of Theorem 4 holds.

The corollary follows because under these conditions a barrier is constructed explicitly in [6].

The proof of Theorem 4 will employ the following lemma.

LEMMA 5. *Assume hypotheses (1) and (2'). Then for any $\delta > 0$ there exists a function $\varphi_1(r)$ defined for $r \geq \delta$, satisfying $\varphi_1(\delta) = 0$, $\varphi_1'(r) > 0$ for $r \geq \delta$, and $L\varphi_1(|x|) \geq 0$.*

PROOF: The proof is the same as that of Lemma 4, except that now we set

$$(15) \quad \tau(r) = \min_{\substack{|x|=r \\ \text{all } t}} \left(\frac{B(x, t)}{A(x, t)} - 1 \right).$$

PROOF OF THEOREM 4. Let $m = \min_{|x|=X} W(x)$ and $R_0 > X$ a number such that $W(x) \leq m/2$ for $|x| = R_0$. Let $\delta < X$ and σ be positive numbers so small that $L(|x|^2) = 2(a_{ii} + b_i x_i) \geq \sigma > 0$ for $|x| < \delta$.

Let

$$W_1(x) = \begin{cases} \alpha - \beta |x|^2, & |x| \leq \delta, \\ \min \{ \alpha - \beta \delta^2 - \varepsilon \varphi_1(|x|), W(x) \}, & |x| \geq \delta, \end{cases}$$

where φ_1 is the function obtained in Lemma 5, and $\alpha, \beta, \varepsilon$ are positive numbers chosen as follows. First ε is chosen so that $\varepsilon \varphi_1(R_0) = m/2$; then β is chosen so small that W_1 has Property P near $|x| = \delta$; finally α is chosen so that $\alpha - \beta\delta^2 = m$. Thus by construction $W_1(x)$ has Property P for all x , and satisfies

$$LW_1 \leq -\beta\sigma < 0, \quad |x| < \delta;$$

$$LW_1 \leq 0, \quad |x| \geq \delta.$$

Let $c(x) \geq 0$ be a function such that $c \equiv 0$ for $|x| \geq \delta$. Clearly

$$(16) \quad LW_1 + cW_1 - (W_1)_t \begin{cases} \leq -\beta + c\alpha & \text{for } |x| < \delta \\ \leq 0 & \text{for } |x| \geq \delta. \end{cases}$$

We require that $c \not\equiv 0$, but that c be small enough so that $LW_1 + cW_1 - (W_1)_t \leq 0$ for all x where defined.

Let $\varphi_0(x)$ be a smooth function satisfying $0 < \varphi_0(x) \leq W_1(x)$. We shall construct a solution $u(x, t)$ satisfying $0 < u(x, t) \leq W_1(x)$, $u(x, 0) = \varphi_0(x)$. For any $R > 0$, let Γ_R be the ball $\{|x| < R\}$, and let $u_R(x, t)$ be the solution in $\Gamma_R \times (0, \infty)$ of $Lu_R + cu_R - (u_R)_t = 0$; $u_R(x, 0) = \varphi_0(x)$; $u_R(x, t) = 0$ for $|x| = R$. Since $(W_1 - u_R)$ satisfies Property P and is nonnegative on the boundary, we have that $u_R(x, t) \leq W_1(x)$ for all R , and for $(x, t) \in \Gamma_R \times (0, \infty)$. Let $R \rightarrow \infty$; then the u_R form an increasing bounded sequence approaching some limit $u(x, t) \leq W_1(x)$. By the Schauder estimates [2], for each bounded set the derivatives $(u_R)_{ij}$ and $(u_R)_t$ are equicontinuous. A subsequence of the u_R is therefore termwise differentiable to these orders of differentiation. It follows that $Lu + cu - u_t = 0$, $u(x, 0) = \varphi_0(x)$, and $0 < u < W_1$. This completes the proof.

3. Proofs of the Lemmas.

PROOF OF LEMMA 1. Suppose the antibarrier $V(x)$ is defined for $|x| \geq X > \delta$. Let $\varphi(|x|)$ be the function constructed in Lemma 4, and let $m_1 = \varphi(X)$, $m_2 = \varphi(X+1) > m_1$. Let $V_2(x) = a_1 V(x) + a_2$, where $a_1 > 0$ and a_2 are constants chosen so that

$$(17) \quad m_1 < V_2(x) < m_2$$

for $X \leq |x| \leq X+1$. $V_2(x)$ is likewise an antibarrier. Let

$$V_3(x) = \begin{cases} \varphi(|x|), & \delta \leq |x| \leq X, \\ \text{Min}[\varphi(|x|), V_2(x)], & X \leq |x| \leq X+1, \\ V_2(x), & |x| \geq X+1. \end{cases}$$

Condition (17) assures that $V_3(x)$ will be continuous, and the fact that it is the minimum of two regular functions guarantees the proper jump relation for $V_3(x)$ to satisfy Property P . Finally let $V_1(x) = V_3(x) + \varphi(|x|)$; (4) follows since $L\varphi < 0$.

PROOF OF LEMMA 3. It follows from the strong maximum principle that there exists such a function χ depending on L ; our task will be to find one depending only on $\kappa, \bar{A}, \bar{B}$, and δ . For this we use two auxiliary lemmas.

LEMMA 6. *Let δ and u be as in Lemma 3. There is a number $\sigma > 0$ depending only on $\kappa, \bar{A}, \bar{B}$, and δ such that*

$$(18) \quad u(x, t) \geq \sigma u_0$$

for $|x| \leq \sigma\delta$, $0 \leq t \leq 1$.

PROOF: We shall construct a function $v(x, t) = h(|x|)e^{-\lambda t}$ satisfying $Lv \geq 0$, $v = 0$ for $|x| = \delta$, and $v > 0$ for $|x| < \delta$. Setting $r = |x|$, we calculate

$$Lv - v_* = e^{-\lambda t} \{A(x, t)(h'' - h'/r) + (h'/r)B(x, t) + \lambda h\},$$

Hence

$$Lv - v_* \geq e^{-\lambda t} \{\kappa(x)h'' + (\bar{B}(x) - \kappa(x))h'/r + \lambda h\}$$

wherever $h'' \geq 0$, $h' \leq 0$; and

$$Lv - v_* \geq e^{-\lambda t} \{\bar{A}(x)h'' + (\bar{B}(x) - \kappa(x))h'/r + \lambda h\}$$

wherever $h'' \leq 0$, $h' \leq 0$. We set $N_1 = \sup_{|x| \leq \delta} (\bar{B} - \kappa)$, $N_2 = \sup_{|x| \leq \delta} \bar{A}(x)$,

$$N_3 = \inf_{|x| \leq \delta} \kappa(x),$$

and define

$$h(r) = \begin{cases} \alpha - \beta r^2, & 0 \leq r \leq r_0 \\ (\delta - r)^2, & r_0 \leq r \leq \delta, \\ 0 & r \geq \delta, \end{cases}$$

where $r_0 = \text{Max} \left[\frac{\delta}{2}, \delta - \frac{N_3}{N_1} \right]$, and α and β are chosen so that h and h' are continuous at $r = r_0$. Then for $r_0 \leq r \leq \delta$,

$$Lv - v_* \geq e^{-\lambda t} [2N_3 - 2N_1(\delta - r_0) + \lambda h] \geq 0,$$

and for $0 \leq r \leq r_0$,

$$Lv - v_* \geq e^{-\lambda t} [-2\beta N_2 - 2\beta N_1 + \lambda(\alpha - \beta r_0^2)].$$

Since the coefficient of λ is positive, we may choose λ large enough so that $Lv \geq 0$ throughout.

Now let $V(x, t) = \frac{u_0}{\alpha} v(x, t)$; clearly $V \leq u$ for $t = 0$ and also for $r = \delta$.

Hence $u(x, t) \geq V(x, t)$ for $r \leq \delta, t \geq 0$; in particular $u(x, t) \geq \frac{u_0}{\alpha} e^{-\lambda} h(r) \geq \sigma u_0$ for $r \leq \sigma\delta, 0 \leq t \leq 1$, and an appropriately chosen σ .

LEMMA 7. *Given any numbers $R > 0$ and $p > 0$, there is a number p_1 depending on $\kappa, \bar{A}, \bar{B}, R$, and p , such that if $u(x, t) > u_0 e^{-p|t|}$, for $|x| = R, 0 \leq t \leq 1$, then $u(x, t) \geq u_0 e^{-p_1|t|}$ for $R \leq r \leq R + 1, 0 \leq t \leq 1$.*

PROOF: We define $v(x, t) = u_0 f(t(R + 2 - r))$, where $f(s) = e^{-p_1/s}$, and p_1 will be determined later. Then

$$Lv - v_* = A(x, t)(t^2 f'' + t f' / r) - \left[\frac{B(x, t)t}{r} + (R + 2 - r) \right] f'.$$

We assume $p_1 \geq 4$; then $f'(s) \geq 0$ and $f''(s) \geq 0$ for $0 \leq s \leq 2$, and we have, setting $s = t(R + 2 - r)$,

$$\begin{aligned} Lv - v_* &\geq \kappa t^2 f'' + \left[(\kappa - \bar{B}) \frac{t}{r} - 2 \right] f' \\ &= \frac{\kappa}{(R + 2 - r)^2} \left[s^2 f'' - \frac{2 + (\bar{B} - \kappa)t/r}{\kappa} (R + 2 - r)^2 f' \right] \\ &\geq \frac{\kappa}{(R + 2 - r)^2} [s^2 f'' - K f'] \end{aligned}$$

where

$$K = 2 \text{ Max}_{R \leq r \leq R+2} \frac{2 + (\bar{B}(x) - \kappa(x))/r}{\kappa(x)}.$$

But $s^2 f'' - Kf' = (p_1/s)(-2 + (p_1 - K)/s)f \geq p_1(-4 + p_1 - K)f/2s$ for $0 \leq s \leq 2$, $p_1 \geq K$. Hence $Lv - v_* \geq 0$ for $p_1 \geq K + 4$. We now set

$$p_1 = \text{Max} [2p, K + 4]$$

so that $Lv - v_* \geq 0$ for $R \leq r \leq R + 2$, $0 \leq t \leq 1$; $v(x, t) = u_0 e^{-p_1/2t} \leq u(x, t)$ for $r = R$; and $v(x, t) = 0 \leq u(x, t)$ for $t = 0$, and for $r = R + 2$. Hence $u(x, t) \geq v(x, t)$ in the annular cylinder $R \leq r \leq R + 2$, $0 \leq t \leq 1$. In particular for $r \leq R + 1$,

$$u(x, t) \geq u_0 e^{-p_1/t}.$$

This completes the proofs of Lemmas 6 and 7.

Lemma 3 is proved now as follows. By assumption $u(x, 0) > u_0 > 0$ in some interval; suppose it is the interval $r \leq \delta$. Then Lemma 6 provides a lower bound for u in the region $r \leq \sigma\delta$, $t \leq 1$. Now apply Lemma 7 successively with $R_\nu = \sigma\delta + \nu$, $\nu = 0, 1, 2, \dots$ to obtain the conclusion.

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