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GOOD CHOICE SETS (*).

by J. C. E. DEKKER.

1. Introduction.

We are concerned with non-negative integers (*numbers*), collections of numbers (*sets*) and collections of sets (*classes*). The letters ε and o stand for the set of all numbers and the empty set of numbers respectively. We shall write (I_0, \dots, I_n) or $[I_0, \dots, I_n]$ for the collection consisting of the entities (i. e., numbers, sets, classes, or ordered pairs of numbers) I_0, \dots, I_n . Brackets will often be used instead of parentheses if this makes it easier to read a formula. Let $\{a_0, a_1, \dots\}$ be a sequence of numbers. Then we shall use « a_n » and « $a(n)$ » in the same sense. We write \subset for inclusion, proper or improper; proper inclusion is indicated by writing \subset_+ . A mapping from a subcollection of ε^n into ε is called a *function*; if f is a function, we denote its domain and its range by δf and ρf respectively. The sets α and β are *equivalent* [written: $\alpha \simeq \beta$], if there exists a one-to-one function f such that $\alpha \subset \delta f$ and $f(\alpha) = \beta$. Note that we may replace « $\alpha \subset \delta f$ » by « $\alpha = \delta f$ » without changing the meaning of « $\alpha \simeq \beta$ ». The sets α and β are *recursively equivalent* [written: $\alpha \simeq \beta$], if there exists a partial recursive one-to-one function p such that $\alpha \subset \delta p$ and $p(\alpha) = \beta$. Note that replacing « $\alpha \subset \delta p$ » by « $\alpha = \delta p$ » would change the meaning of « $\alpha \simeq \beta$ »; for $\sigma \simeq \sigma$ would become false for every set σ which is not r. e. (i. e., recursively enumerable), because δp is a r. e. set for every partial recursive function p of one variable. A possible definition of the cardinal number of a set α is: the class of all sets σ such that $\sigma \simeq \alpha$. Similarly we have defined the *RET* (i. e., *recursive equivalence type*) of a set α [written: $\text{Req}(\alpha)$] as the class of all sets σ such that $\sigma \simeq \alpha$. For a study of *RET*s the reader is referred to [2] and [5].

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Let a class of mutually disjoint, non-empty sets be called an *md-class*; such a class is therefore countable, i. e., finite or denumerable. We wish to show how the notion of the RET of a class of sets can be introduced for certain (though not all) *md*-classes. Throughout this paper S stands for an *md*-class and σ for the union of all sets in S . For every $x \in \sigma$ we denote the unique set α such that $x \in \alpha$ and $\alpha \in S$ by α_x .

DEFINITION. A set γ is a *choice set* of S , if

- (1) $\gamma \subset \sigma$,
- (2) γ contains exactly one element of each set in S .

A possible definition of the cardinal number of S is: the class of all sets σ such that $\sigma \supset \gamma$, for some choice set γ of S . Though any two choice sets of an *md*-class are equivalent, they need not be recursively equivalent. Let, for instance,

$$S = [(0, 1), (2, 3), (4, 5), \dots].$$

Then S has c choice sets (c denoting the cardinality of the continuum), while every non-zero RET contains exactly \aleph_0 sets; the c choice sets of S can therefore not all be recursively equivalent.

DEFINITION. A set γ is a *good choice set* (abbreviated: *gc-set*) of S , if it is a choice set for which there exists a partial recursive function $p(x)$ such that

- (3) $\sigma \subset \delta p$ and $(\forall x) [x \in \sigma \implies p(x) \in \gamma \cdot \alpha_x]$.

We shall prove in sections 3 and 5:

- (i) any two good choice sets of an *md*-class are recursively equivalent,
- (ii) among the c *md*-classes there are c which have a *gc-set* and c which have no *gc-set*.

NOTATION. $\zeta(S)$ is the class of all *gc*-sets of S .

DEFINITION. An *md*-class S is a *gc-class* if $\zeta(S)$ is non-empty. If S is a *gc-class*,

$$\text{RET}(S) = \text{Req}(\gamma), \quad \text{for any } \gamma \in \zeta(S).$$

In the trivial case that S is empty, $\zeta(S)$ contains exactly one set, namely \emptyset . The RET of the empty class is therefore 0. It is the purpose of this paper to prove a few propositions concerning *gc*-classes and their RETs.

While an *md*-class need not have an RET (since it need not be a *gc-class*), it is readily seen that every RET A is the RET of some *gc-class*. For let

$$\alpha \in A, \quad S = \{(x) \mid x \in \alpha\},$$

then S has exactly one choice set, namely α . Using the identity function we conclude that α is also a *gc*-set of S , hence $\text{RET}(S) = A$. In this case all sets in S have the same RET , namely 1. It is not hard to see that for any two non-zero RET s A and B there is a *gc*-class S such that

$$\text{RET}(S) = A, \quad (\forall \sigma)[\sigma \in S \implies \text{Req}(\sigma) = B].$$

For let $\alpha \in A, \beta \in B$. Suppose a_n is a one-to-one function ranging over α ; if α is a finite set of cardinality $k \geq 1$, we take $(0, \dots, k - 1)$ as the domain of a_n , but if α is infinite, we take ε as the domain of a_n . Put

$$j(x, y) = x + \frac{1}{2}(x + y)(x + y + 1), \quad j(p, \beta) = \{j(p, y) \mid y \in \beta\},$$

$$S = [j(a_0, \beta), j(a_1, \beta), \dots].$$

Obviously, every set in S is recursively equivalent to β , i. e., has $\text{RET } B$. Let $b \in \beta$ and let γ be the range of the function $j(a_n, b)$. Then $p(x) = j[k(x), b]$ is a recursive function such that

$$\sigma \subset \delta p \quad \text{and} \quad (\forall x)[x \in \sigma \implies p(x) \in \gamma \cdot \alpha_x].$$

This implies that γ is a *gc*-set of S and

$$\text{RET}(S) = \text{Req}(\gamma) = \text{Req}(\alpha) = A.$$

Note that $\text{Req}(\sigma) = A \cdot B$, because $\sigma = j(\alpha \times \beta)$. Hence

$$(4) \quad \text{Req}(\sigma) = \text{RET}(S) \cdot B.$$

The set σ is the union of all sets in the *gc*-class S , and all sets in S have the same RET , namely B . Relation (4) shows therefore that though our definition of $\text{RET}(S)$ may not be the only one possible, it is certainly natural, since in some sense

$$\underbrace{B + B + B + \dots}_{\ll A \text{ times} \gg} = A \cdot B.$$

2. Preliminaries.

The sets $\alpha_0, \dots, \alpha_n$ are *separable*, if there exist mutually disjoint r. e. sets β_0, \dots, β_n such that $\alpha_i \subset \beta_i$, for $0 \leq i \leq n$. We write $\alpha_0 \mid \alpha_1$ if α_0 and α_1 are separable. It is readily seen that $\alpha_0, \dots, \alpha_n$ are separable if and only

if there exists a partial recursive function $p(x)$ such that

$$(5) \quad \alpha_0 + \dots + \alpha_n \subset \delta p \quad \text{and} \quad \varrho p = (0, \dots, n),$$

$$(6) \quad \begin{cases} \text{for } x \in \alpha_0 + \dots + \alpha_n & \text{and} & 0 \leq i \leq n, \\ x \in \alpha_i \iff p(x) = i. \end{cases}$$

NOTATIONS.

$$\varrho_0 = 0,$$

$$\varrho_{x+1} = \begin{cases} [a(1), \dots, a(k)], \text{ where } a(1), \dots, a(k) \text{ are the} \\ \text{distinct numbers such that} \\ x + 1 = 2^{a(1)} + \dots + 2^{a(k)}, \end{cases}$$

$$r(x) = r_x = \text{card}(\varrho_x).$$

The class Q of all finite sets is enumerated without repetitions in the sequence $\varrho_0, \varrho_1, \dots$; the function r_x is clearly recursive.

A one-to-one function t_n from ε into ε is *regressive*, if there exists a partial recursive function $p(x)$ such that

$$(7) \quad \varrho t \subset \delta p,$$

$$(8) \quad p(t_0) = t_0 \quad \text{and} \quad (\forall n)[p(t_{n+1}) = t_n].$$

A function from ε into ε is *retraceable*, if it is strictly increasing and regressive. A set is *regressive* (or *retraceable*) if it is finite or the range of a regressive (respectively, retraceable) function.

For every regressive function t_n there also exists a partial recursive function which satisfies besides (7) and (8) the conditions

$$(9) \quad \varrho p \subset \delta p,$$

$$(10) \quad (\forall x)[x \in \delta p \implies (\exists y)[p^{y+1}(x) = p^y(x)]].$$

Every partial recursive function $p(x)$ related to the regressive function t_n by (7), (8), (9) and (10) is called a function which *regresses* t_n or a *regressing* function of t_n ; in the special case that t_n is strictly increasing, we call $p(x)$ a function which *retraces* t_n or a *retracing* function of t_n . If $p(x)$ is a regressing function of the regressive function t_n , then the function $p^*(x)$ de-

finned by

$$(11) \quad p^*(x) = (\mu y)[p^{y+1}(x) = p^y(x)], \quad \text{for } x \in \delta p,$$

is a partial recursive extension of t_x^{-1} .

Consider the following proposition. Let the sets α and β and the partial recursive functions $f(x)$ and $g(x)$ be related by the conditions

- (a) $\alpha \subset \delta f$ and $f(\alpha) = \beta$ and f is 1 — 1 on α ,
- (b) $\beta \subset \delta g$ and $g(\beta) = \alpha$ and g is 1 — 1 on β ,
- (c) $gf(x) = x, \quad \text{for } x \in \alpha.$

Then there exists a partial recursive one-to-one function $h(x)$ such that

- (d) $\alpha \subset \delta h \quad \text{and} \quad h(\alpha) = \beta,$
- (e) $h(x) = f(x), \quad \text{for } x \in \alpha.$

The proof is almost immediate. Under the hypothesis,

$$\sigma = \{x \in \delta f \mid gf(x) = x\}$$

is a r. e. set; hence, if $h(x)$ is the restriction of $f(x)$ to σ , then $h(x)$ satisfies the requirements. This proposition will be used in the following form:

$$(12) \quad \left\{ \begin{array}{l} \text{If } \alpha \text{ and } \beta \text{ are sets for which there exist partial} \\ \text{recursive functions } f(x) \text{ and } g(y) \text{ which satisfy} \\ \text{(a), (b), (c) above, then } \alpha \simeq \beta. \end{array} \right.$$

3. Elementary properties.

PROPOSITION P1. *Every two gc-sets of an md-class are recursively equivalent.*

PROOF. Let γ and δ be gc-sets of the md-class S . If the class S is finite, $\gamma \simeq \delta$ because γ and δ are finite sets of the same cardinality. Now assume that S, γ and δ are infinite. There exist partial recursive functions p and q such that

$$(13) \quad \sigma \subset \delta p \quad \text{and} \quad (\forall x)[x \in \sigma \implies p(x) \in \gamma \cdot \alpha_x],$$

$$(14) \quad \sigma \subset \delta q \quad \text{and} \quad (\forall x)[x \in \sigma \implies q(x) \in \delta \cdot \alpha_x].$$

It follows that

$$(15) \quad \delta \subset \delta p \text{ and } p(\delta) = \gamma \text{ and } p \text{ is } 1-1 \text{ on } \delta,$$

$$(16) \quad \gamma \subset \delta q \text{ and } q(\gamma) = \delta \text{ and } q \text{ is } 1-1 \text{ on } \gamma,$$

$$(17) \quad qp(x) = x, \quad \text{for } x \in \delta.$$

The last three relations imply $\gamma \simeq \delta$ by (12).

P1 guarantees that the notion $\text{RET}(S)$ is well-defined for any *gc*-class S . Let us consider the special case where S is a non-empty *md*-class which contains exactly k sets *all of which are finite*. It is readily seen that

(a) every choice set of S is a *gc*-set,

(b) S is a *gc*-class and $\text{RET}(S) = k$.

A finite *md*-class need not be a *gc*-class. For let $T = (\tau, \tau')$, where τ is any non-recursive set and τ' the complement of τ with respect to ε . If T had a *gc*-set, we would have $\tau \mid \tau'$, and τ would be recursive. Let us now take for τ an immune set with an immune complement. For every $k \geq 3$ we can decompose τ' into $k-1$ immune sets $\tau_1, \dots, \tau_{k-1}$. Then $B = (\tau, \tau_1, \dots, \tau_{k-1})$ is an *md*-class which contains exactly k sets, but B is not a *gc*-class. We conclude that for every $k \geq 2$, there exists an *md*-class of cardinality k which is not a *gc*-class.

PROPOSITION P2. *The non-empty finite md-class $S = (\alpha_0, \dots, \alpha_n)$ is a gc-class if and only if $\alpha_0, \dots, \alpha_n$ are separable; if S is a gc-class, each choice set of S is a gc-set and $\text{RET}(S)$ equals the cardinality of S .*

PROOF. Let $S = (\alpha_0, \dots, \alpha_n)$.

(a) Let $\alpha_0, \dots, \alpha_n$ be separable, say $\alpha_i \subset \beta_i$, $0 \leq i \leq n$, for mutually disjoint r. e. sets β_0, \dots, β_n . Put $\beta = \beta_0 + \dots + \beta_n$. Let $\gamma = (c_0, \dots, c_n)$ with $c_i \in \alpha_i$, $0 \leq i \leq n$, be any choice set of S . Then the function p defined by

$$\delta p = \beta, \quad (\forall x)(\forall i \leq n)[x \in \beta_i \implies p(x) = c_i],$$

is a partial recursive function which maps any element $x \in \sigma$ onto the number c_i such that $x \in \alpha_i$. Hence $\gamma \in \zeta(S)$ and S is a *gc*-class.

(b) Let S be a *gc*-class, $\delta = (d_0, \dots, d_n)$ with $d_i \in \alpha_i$, $0 \leq i \leq n$, a *gc*-set of S and q a partial recursive function such that

$$\sigma \subset \delta q, \quad (\forall x)(\forall i \leq n)[x \in \alpha_i \implies q(x) = d_i].$$

Then the function h defined by

$$\delta h = \delta, \quad (\forall i \leq n)[h(d_i) = i],$$

is partial recursive, hence so is the function hq . Moreover,

$$\sigma \subset \delta(hq), \quad \varrho(hq) = (0, \dots, n) \text{ and} \\ (\forall x) (\forall i \leq n) [x \in \alpha_i \iff hq(x) = i].$$

We conclude that $\alpha_0, \dots, \alpha_n$ are separable.

c) Let S be a gc -class. Then $\alpha_0, \dots, \alpha_n$ are separable by (b) and every choice set of S is a gc -set by our proof of (a).

Each choice set γ of S has cardinality $n + 1$, hence $\text{RET}(S) = \text{Req}(\gamma) = n + 1$.

REMARK. It is readily seen that every subclass of a gc -class is again a gc class. For let S be a gc -class with union σ and gc -set γ and let $p(x)$ be a partial recursive function such that

$$\sigma \subset \delta p \quad \text{and} \quad (\forall x) [x \in \sigma \implies p(x) \in \gamma \cdot \alpha_x].$$

Assume that $T \subset S$, where T has union τ . Then $\gamma \cdot \tau$ is a choice set of T , in fact a gc -set. For $\tau \subset \delta p$, since $\tau \subset \sigma \subset \delta p$; moreover, for $x \in \tau$, $p(x)$ is not only the unique element of $\gamma \cdot \alpha_x$, but also of $(\gamma \cdot \tau) \cdot \alpha_x$. Hence T is a gc -class.

Let S be a non-empty md -class. One of the basic propositions concerning such an md -class is: σ is finite if and only if S is a finite class of finite sets. This proposition will now be generalized.

DEFINITION. An md -class is *isolated* if it is a gc -class of which every (or, equivalently, at least one) gc -set is isolated. In other words: an md -class is *isolated* if it is a gc -class whose RET is an *isol*.

PROPOSITION P3. *Let S be a non-empty gc -class. Then σ is an isolated set if and only if S is an isolated class of isolated sets.*

PROOF. Let S be a non-empty gc -class.

(a) Assume that σ is isolated. Every set which belongs to S or $\zeta(S)$ is a subset of σ , hence again isolated. Thus S is an isolated class of isolated sets.

(b) Assume that S is an isolated class of isolated sets. Let $\gamma \in \zeta(S)$ and let p be a partial recursive function such that

$$\sigma \subset \delta p \quad \text{and} \quad (\forall x) [x \in \sigma \implies p(x) \in \gamma \cdot \alpha_x].$$

We wish to prove that σ has no infinite r. e. subset. Let β be any r. e. subset of σ . If β is empty we are through; if β is non-empty, so is $p(\beta)$. Note that $p(\beta)$ is a r. e. subset of $p(\sigma)$, i. e., of γ . Thus, γ being isolated, $p(\beta)$ must be finite. The set $p(\beta)$ consists of all elements in γ which represent sets in S with which β has a non-empty intersection. Let T consist of all $\alpha \in S$ for which $\alpha \cdot \beta \neq o$. Since the set $p(\beta)$ is finite, but non-empty, T is a non-empty, finite subclass of S . However, S is a *gc-class*, hence so is T . The sets in T are separable, because T is a finite *gc-class*. Let

$$T = (\delta_0, \dots, \delta_k), \quad \tau = \delta_0 + \dots + \delta_k.$$

The sets $\delta_0, \dots, \delta_k$ are isolated, since they belong to S ; thus τ is isolated, because it is the union of $k + 1$ separable, isolated sets. Recall that $\delta_0, \dots, \delta_k$ are the only sets in S with which β has a non-empty intersection. This implies $\beta \subset \tau$; hence β is finite, because β is r. e. and τ isolated.

DEFINITION. The classes S_1 and S_2 with unions σ_1 and σ_2 respectively are *separable* [written: $S_1 \mid S_2$], if $\sigma_1 \mid \sigma_2$.

PROPOSITION P4. Let S_1 and S_2 be separable *md-classes*. Then $S_1 + S_2$ is an *md-class* and

- (a) $S_1 + S_2$ is a *gc-class* if and only if both S_1 and S_2 are *gc-classes*,
- (b) if $S_1 + S_2$ is a *gc-class*,

$$\text{RET}(S_1 + S_2) = \text{RET}(S_1) + \text{RET}(S_2).$$

PROOF. Let S_1 and S_2 be separable *md-classes* with unions σ_1 and σ_2 respectively. Then $S_1 + S_2$ is an *md class* with union $\sigma_1 + \sigma_2$. Let $\sigma_1 \subset \tau_1$, $\sigma_2 \subset \tau_2$, where τ_1 and τ_2 are disjoint r. e. sets.

(1) Assume that S_1 and S_2 are *gc-classes*. Let $\gamma_1 \in \zeta(S_1)$, $\gamma_2 \in \zeta(S_2)$, then $\gamma_1 + \gamma_2$ is obviously a choice set of $S_1 + S_2$. For $x \in \sigma_1$ we denote the unique set α such that $x \in \alpha$ and $\alpha \in S_1$ by α_x ; for $x \in \sigma_2$ we denote the unique set β such that $x \in \beta$ and $\beta \in S_2$ by β_x . Suppose p_1 and p_2 are partial recursive functions such that

$$\sigma_1 \subset \delta p_1 \quad \text{and} \quad (\forall x) [x \in \sigma_1 \implies p_1(x) \in \gamma_1 \cdot \alpha_x],$$

$$\sigma_2 \subset \delta p_2 \quad \text{and} \quad (\forall x) [x \in \sigma_2 \implies p_2(x) \in \gamma_2 \cdot \beta_x].$$

Let the function p_3 be defined by

$$\delta p_3 = \tau_1 \cdot \delta p_1 + \tau_2 \cdot \delta p_2,$$

$$\text{for } x \in \delta p_3, \quad p_3(x) = \begin{cases} p_1(x), & \text{if } x \in \tau_1 \cdot \delta p_1, \\ p_2(x), & \text{if } x \in \tau_2 \cdot \delta p_2. \end{cases}$$

The sets τ_1 and τ_2 are r. e. and disjoint, while the sets δp_1 and δp_2 are r. e. It follows that $\tau_1 \cdot \delta p_1$ and $\tau_2 \cdot \delta p_2$ are disjoint and r. e.: thus δp_3 is a r. e. set and $p_3(x)$ a partial recursive function. Clearly, $\sigma_1 + \sigma_2 \subset \delta p_3$ and for $x \in \sigma_1 + \sigma_2$,

$$x \in \sigma_1 \implies p_3(x) = p_1(x) \in \gamma_1 \cdot \alpha_x,$$

$$x \in \sigma_2 \implies p_3(x) = p_2(x) \in \gamma_2 \cdot \beta_x,$$

$$\gamma_1 \cdot \alpha_x \subset (\gamma_1 + \gamma_2) \cdot \alpha_x; \quad \gamma_2 \cdot \beta_x \subset (\gamma_1 + \gamma_2) \cdot \beta_x.$$

We conclude that $\gamma_1 + \gamma_2$ is a *gc*-set of $S_1 + S_2$.

(2) Assume that $S_1 + S_2$ is a *gc*-class. Then S_1 and S_2 are *gc*-classes, since they are subclasses of $S_1 + S_2$. Let $\gamma \in \zeta(S_1 + S_2)$ and let p be a partial recursive function such that

$$\sigma_1 + \sigma_2 \subset \delta p \quad \text{and} \quad (\forall x)[x \in \sigma_1 + \sigma_2 \implies p(x) \in \gamma \cdot (\sigma_1 + \sigma_2)].$$

Putting $\gamma_1 = \gamma \cdot \sigma_1$ and $\gamma_2 = \gamma \cdot \sigma_2$ we see that γ_1 and γ_2 are *gc*-sets of S_1 and S_2 respectively; moreover, p is a partial recursive function related to γ_1 , S_1 and γ_2 , S_2 in the desired manner.

(3) Let $S_1 + S_2$ be a *gc*-class. Then S_1 and S_2 are *gc*-classes by (a). Also, in view of our proof of (a),

$$\gamma_1 \in \zeta(S_1) \ \& \ \gamma_2 \in \zeta(S_2) \implies \gamma_1 + \gamma_2 \in \zeta(S_1 + S_2).$$

The relations $\gamma_1 \subset \sigma_1$, $\gamma_2 \subset \sigma_2$, $\sigma_1 \mid \sigma_2$ imply $\gamma_1 \mid \gamma_2$. Hence

$$\begin{aligned} \text{RET}(S_1 + S_2) &= \text{Req}(\gamma_1 + \gamma_2) = \text{Req}(\gamma_1) + \text{Req}(\gamma_2) \\ &= \text{RET}(S_1) + \text{RET}(S_2). \end{aligned}$$

NOTATION. For any two classes A and B ,

$$A \times B = \{j(\alpha \times \beta) \mid \alpha \in A \ \& \ \beta \in B\},$$

where $j(\alpha \times \beta) = \{j(x, y) \mid x \in \alpha \ \& \ y \in \beta\}$.

REMARK. Let A and B have unions σ_A and σ_B respectively and let $\sigma_{A \times B}$ be the union of $A \times B$. Then it is readily seen that

$$\sigma_{A \times B} = j(\sigma_A \times \sigma_B).$$

We also note that for arbitrary sets $\sigma_1, \sigma_2, \tau_1, \tau_2$,

$$j(\sigma_1 \times \tau_1) \cdot j(\sigma_2 \times \tau_2) = j(\sigma_1 \cdot \sigma_2 \times \tau_1 \cdot \tau_2).$$

We finally observe that for any two non-empty countable classes A and B of non-empty sets, $A \times B$ is an *md*-class, if both A and B are *md*-classes.

PROPOSITION P5. *Let S_1 and S_2 be two non empty md-classes. Then $S_1 \times S_2$ is a non-empty md-class and*

- (a) $S_1 \times S_2$ is a *gc*-class if and only if both S_1 and S_2 are *gc*-classes,
- (b) if $S_1 \times S_2$ is a *gc*-class,

$$\text{RET}(S_1 \times S_2) = \text{RET}(S_1) \cdot \text{RET}(S_2).$$

PROOF. Let S_1 and S_2 be two non-empty *md*-classes with unions σ_1 and σ_2 respectively. We already know that $S_1 \times S_2$ is a non-empty *md*-class. Let for $x \in \sigma_1, y \in \sigma_2$,

$\alpha_x =$ the set α such that $x \in \alpha$ and $\alpha \in S_1$,

$\beta_y =$ the set β such that $y \in \beta$ and $\beta \in S_2$.

Note that the union of $S_1 \times S_2$ is $j(\sigma_1 \times \sigma_2)$, while the relation $j(x, y) \in j(\sigma_1 \times \sigma_2)$ implies

$j(\alpha_x \times \beta_y) =$ the set δ such that $j(x, y) \in \delta$ and $\delta \in S_1 \times S_2$.

(1) Assume that S_1 and S_2 are *gc*-classes with *gc*-sets γ_1 and γ_2 respectively. Let p_1 and p_2 be partial recursive functions such that

$$\sigma_1 \subset \delta p_1 \quad \text{and} \quad (\forall x) [x \in \sigma_1 \implies p_1(x) \in \gamma_1 \cdot \alpha_x],$$

$$\sigma_2 \subset \delta p_2 \quad \text{and} \quad (\forall y) [y \in \sigma_2 \implies p_2(y) \in \gamma_2 \cdot \beta_y].$$

Then the mapping p_3 defined by

$$p_3(z) = j[p_1 k(z), p_2 l(z)], \quad \text{for } z \in j(\delta p_1 \times \delta p_2),$$

is a partial recursive function such that $j(\sigma_1 \times \sigma_2) \subset \delta p_3$. Also,

for $z = j(x, y) \in j(\sigma_1 \times \sigma_2)$,

$$p_3(z) = j[p_1(x), p_2(y)] \in j(\alpha_x \times \beta_y).$$

Since $p_1(x) \in \gamma_1 \cdot \alpha_x$ and $p_2(y) \in \gamma_2 \cdot \beta_y$, we have

$$\begin{aligned} p_3(z) &\in j(\gamma_1 \cdot \alpha_x \times \gamma_2 \cdot \beta_y), \\ p_3(z) &\in j(\gamma_1 \times \gamma_2) \cdot j(\alpha_x \times \beta_y). \end{aligned}$$

Hence $j(\gamma_1 \times \gamma_2)$ is a *gc*-set of $S_1 \times S_2$.

(2) Assume that $S_1 \times S_2$ is a *gc*-class. Let for $z \in j(\sigma_1 \times \sigma_2)$ the set η such that $z \in \eta$ and $\eta \in S_1 \times S_2$ be denoted by η_z .

Suppose that δ is a *gc*-set of $S_1 \times S_2$ and that p is a partial recursive function such that

$$\begin{aligned} j(\sigma_1 \times \sigma_2) &\subset \delta p, \\ j(x, y) \in j(\sigma_1 \times \sigma_2) &\implies pj(x, y) \in \delta \cdot \eta_{j(x, y)}. \end{aligned}$$

Let $\beta \in S_2$ and $b \in \beta$. Then $S_1 \times [\beta]$ is a *gc*-class, since it is a subclass of the *gc*-class $S_1 \times S_2$. Put

$$\begin{aligned} \delta_\beta &= \{pj(x, b) \mid x \in \sigma_1\}, \text{ i. e.,} \\ \delta_\beta &= \{pj(x, y) \mid x \in \sigma_1 \ \& \ y \in \beta\}. \end{aligned}$$

Then δ_β is a *gc*-set of $S_1 \times [\beta]$ and $k(\delta_\beta)$ a choice set of S_1 .

Now assume $x \in \sigma_1$. Since $b \in \beta \subset \sigma_2$ we have $j(x, b) \in j(\sigma_1 \times \sigma_2) \subset \delta p$, hence

$$x \in \sigma_1 \implies pj(x, b) \in \delta \cdot \eta_{j(x, b)}.$$

The last relation implies

$$\begin{aligned} x \in \sigma_1 &\implies pj(x, b) \in \delta_\beta \cdot j(\alpha_x \times \beta_b) \\ &\implies kpj(x, b) \in k(\delta_\beta) \ \& \ kpj(x, b) \in \alpha_x \\ &\implies kpj(x, b) \in k(\delta_\beta) \cdot \alpha_x. \end{aligned}$$

The set σ_1 is included in the domain of the partial recursive function $kpj(x, b)$ of x . Hence $k(\delta_\beta)$ is a *gc*-set of S_1 . Similarly one can prove that S_2 has a *gc*-set. Hence S_1 and S_2 are *gc*-classes.

(3) Let $S_1 \times S_2$ be a *gc*-class. Then S_1 and S_2 are *gc*-classes by (a). Let γ_1 and γ_2 be *gc*-sets of S_1 and S_2 respectively. Then $j(\gamma_1 \times \gamma_2)$ is a *gc*-set of $S_1 \times S_2$ in view of our proof of (a). Thus

$$\begin{aligned} \text{RET}(S_1 \times S_2) &= \text{Req } j(\gamma_1 \times \gamma_2) \\ &= \text{Req } (\gamma_1) \cdot \text{Req } (\gamma_2) = \text{RET}(S_1) \cdot \text{RET}(S_2). \end{aligned}$$

4. The class $\text{Bin}(\alpha)$.

NOTATIONS. For any set α and any number k ,

$$\gamma(\alpha, k) = \{n \mid \varrho_n \subset \alpha \ \& \ r_n = k\},$$

$$\text{Bin}(\alpha) = \{\gamma(\alpha, k) \mid k \geq 1\}.$$

The class $\text{Bin}(\alpha)$ is an *md*-class for any set α . If α is a finite set of cardinality $m \geq 1$, then $\text{Bin}(\alpha)$ consists of m finite sets; in this case $\text{Bin}(\alpha)$ is a *gc*-class which has the number m as its cardinality and its RET . This is still true in case $m = 0$, for then $\text{Bin}(\alpha)$ is empty. For any infinite set α , $\text{Bin}(\alpha)$ is a denumerable *md*-class of infinite sets. The next proposition tells us when $\text{Bin}(\alpha)$ is a *gc*-class. We write R for $\text{Req}(\varepsilon)$, i. e., for the RET which consists of all infinite r. e. sets.

PROPOSITION P6. *Let $A = \text{Req}(\alpha)$. Then*

- (a) *if α has an infinite r. e. subset, $\text{Bin}(\alpha)$ is a *gc*-class of RET R ,*
- (b) *if α is regressive, $\text{Bin}(\alpha)$ is a *gc*-class of RET A ,*
- (c) *if α is immune, but not regressive, $\text{Bin}(\alpha)$ is not a *gc*-class.*

PROOF. Let α be any set. For any number x such that $\varrho_x \neq \emptyset$ and $\varrho_x \subset \alpha$, we write γ_x for the unique set in $\text{Bin}(\alpha)$ which contains x . Hence $\gamma_x = \gamma(\alpha, r_x)$, since r_x denotes the cardinality of ϱ_x .

(a) Let α have an infinite r. e. subset, say β .

Suppose b_n is a one-to-one recursive function ranging over β , and $c_{n+1} = b_n$ for every number n . Then $\beta = (c_1, c_2, \dots)$ and there exists a recursive function d such that $d(0) = 0$ and $\varrho_{d(n+1)} = (c_1, \dots, c_{n+1})$. Let δ consist of the numbers $d(1), d(2), \dots$. Since $rd(n) = n$ we see that δ is a choice set of $\text{Bin}(\alpha)$. Denoting the union of $\text{Bin}(\alpha)$ by σ we have for $x \in \sigma$,

$$d(r_x) \in \delta \cdot \gamma(\alpha, r_x), \quad \text{i. e.,} \quad d(r_x) \in \delta \cdot \gamma_x.$$

Thus δ is a *gc*-set of $\text{Bin}(\alpha)$ and the RET of $\text{Bin}(\alpha)$ is $\text{Req}(\delta) = R$.

(b) Let the set α be regressive. If α is finite we are through. Now assume that α is infinite. Let a_n be a regressive function ranging over α and p a regressing function of a_n . Then there exists a partial recursive function q such that

$$\delta q = \delta p \quad \text{and} \quad (\forall x)(\forall n)[x = a_{n+1} \implies \varrho_{q(x)} = (a_0, \dots, a_n)].$$

Let δ consist of $q(a_1), q(a_2), \dots$, then δ is a choice set of $\text{Bin}(\alpha)$, because $q(a_n) \in \gamma(\alpha, n)$, for $n \geq 1$. To show that δ is a *gc*-set of $\text{Bin}(\alpha)$ we define a function f as follows :

first of all, $\delta f = \sigma$; secondly, let an element $x \in \sigma$ be given. Then the numbers $r(x), i_0, \dots, i_{r(x)-1}$ such that

$$\begin{aligned} \varrho_x &= [a(i_0), \dots, a(i_{r(x)-1})], \\ i_0 &< i_1 < \dots < i_{r(x)-1}, \end{aligned}$$

can be computed. Clearly, $i_{r(x)-1} \geq r(x) - 1$. By regressing the function a_n from $a(i_{r(x)-1})$ to $a_{r(x)-1}$ we can therefore compute the unique number y such that

$$\varrho_y = (a_0, \dots, a_{r(x)-1}).$$

This number y we call $f(x)$. For $x \in \sigma$,

$$f(x) = q(a_{r(x)}) \in \delta \cdot \alpha_x.$$

It is readily verified that f has a partial recursive extension. Thus δ is a *gc*-set of $\text{Bin}(\alpha)$. It remains to be shown that $\delta \simeq \alpha$. For every $y \in \alpha$ there exists a unique number n such that $y = a_n$; let us call this number n the *a-rank* of y ; it can be effectively computed from y , since it equals $p^*(y)$. Let for $x \in \delta$,

$$g(x) = \text{the element of highest } a\text{-rank in } \varrho_x,$$

then we have for $n \geq 1, x \geq 0$,

$$x = q(a_n) \implies \varrho_x = (a_0, \dots, a_{n-1}) \implies g(x) = a_{n-1}.$$

The function g therefore maps δ one-to-one onto α . It is readily proved that both g and g^{-1} have partial recursive extensions. Thus $\delta \simeq \alpha$ by (12).

c) Throughout this part of the proof α denotes an infinite set. We call a set *recursively infinite* (abbreviated: r. i.), if it has an infinite r. e. subset. Thus, if α is not r. i., α is immune. Consider the two statements :

(I) if $\text{Bin}(\alpha)$ is a *gc*-class, there is a regressive *gc*-set δ of $\text{Bin}(\alpha)$ such that the function d_n defined by « $d_n \in \delta \cdot \gamma(\alpha, n)$, for $n \geq 1$ » has the property : $\varrho_{a(1)} \mathbf{C}_+ \varrho_{a(2)} \mathbf{C}_+ \dots$,

(II) if $\text{Bin}(\alpha)$ has a *gc*-set δ with the properties listed under (I), then either α is r. i. or $\alpha \simeq \delta$.

These two statements imply

$$\text{Bin}(\alpha) \text{ a } gc\text{-class} \implies \alpha \text{ r. i. or } \alpha \text{ regressive,}$$

i. e., the contrapositive of (c). It therefore suffices to establish (I) and (II).

Re (I). Let γ be a *gc*-set of $\text{Bin}(\alpha)$ and let for $n \geq 1$, the unique element of $\gamma \cdot \gamma(\alpha, n)$ be denoted by $c(n)$. Since $\varrho_{c(n)}$ has cardinality n , there exist numbers $c_{11}, c_{21}, c_{22}, c_{31}, c_{32}, c_{33}, \dots$ such that

$$\varrho_{c(1)} = (c_{11}),$$

$$\varrho_{c(2)} = (c_{21}, c_{22}), \text{ where } c_{21} < c_{22},$$

$$\varrho_{c(3)} = (c_{31}, c_{32}, c_{33}), \text{ where } c_{31} < c_{32} < c_{33},$$

$$\vdots$$

Put $e_1 =$ the first number occurring in $c_{11}, c_{21}, c_{22}, c_{31}, \dots$,
 $e_{n+1} =$ the first number occurring in $c_{11}, c_{21}, c_{22}, c_{31}, \dots$,
 which does not belong to (e_1, \dots, e_n) .

There clearly exists a one-to-one function d such that

$$\varrho_{d(0)} = o, \quad \varrho_{d(n+1)} = (e_1, \dots, e_{n+1}).$$

We recall that $\varrho_i \subset_+ \varrho_k$ implies $i < k$. Thus the function d is strictly increasing. Let δ consist of the numbers $d(1), d(2), \dots$, then $d(n)$ is the unique element of $\delta \cdot \gamma(\alpha, n)$, for $n \geq 1$. The set δ is therefore a choice set of $\text{Bin}(\alpha)$. We now prove that γ is a regressive set. Let p be a partial recursive function such that $\sigma \subset \delta p$ and for $x \in \sigma$, $p(x) \in \gamma \cdot \gamma_x$, i. e., $p(x) \in \gamma \cdot \gamma(\alpha, r_x)$. Let any element of γ be given, say $c(n+1)$. Then we can compute the numbers $c_{n+1,1}, \dots, c_{n+1,n+1}$ such that

$$\varrho_{c(n+1)} = (c_{n+1,1}, \dots, c_{n+1,n+1}),$$

hence also the number i such that

$$\varrho_i = (c_{n+1,1}, \dots, c_{n+1,n}).$$

Then $p(i) = c(n)$, since $i \in \gamma(\alpha, n)$. The number $c(n)$ can therefore be effectively computed from the number $c(n+1)$. Hence γ is the range of some regressive function, e. g., of the function \bar{c} defined by $\bar{c}(n) = c(n+1)$, for $n \in \varepsilon$. We conclude that γ is a regressive set. We proceed to show that δ is a *gc*-set of $\text{Bin}(\alpha)$. Given any number $x \in \sigma$ we can compute the numbers $n = r(x)$ and $c(n) = p(x)$, hence also the numbers $c(1), \dots, c(n)$ and the finite sequence

$$(\Sigma) \quad c_{11}, c_{21}, c_{22}, c_{31}, c_{32}, c_{33}, \dots, c_{n1}, \dots, c_{nn}.$$

The last n elements of (Σ) are distinct, hence from (Σ) we can compute the numbers e_1, \dots, e_n and the number $d(n)$ such that $\varrho_{d(n)} = (e_1, \dots, e_n)$. However, for $n \geq 1$ we have $d(n) \in \delta \cdot \gamma_x$, i. e., $d(n) \in \delta \cdot \gamma(\alpha, n)$. Thus δ is a *gc*-set of $\text{Bin}(\alpha)$. It follows that $\gamma \simeq \delta$ by PI and that δ is regressive, because γ is regressive. This completes the proof of (I). We observe in passing that δ is a retraceable set. For first of all, $d(1) < d(2) < \dots$. Secondly, given $d(n+1)$ we know an $(n+1)$ -element subset of α , hence also an n -element subset of α , i. e., a number in $\gamma(\alpha, n)$, say t ; from t we can compute the unique element of $\delta \cdot \gamma(\alpha, n)$, i. e., $d(n)$.

Re (II). Let δ be a *gc*-set of $\text{Bin}(\alpha)$ with the properties mentioned in (I). Define for $n \geq 1$,

$$d(n) = \text{unique element of } \delta \cdot \gamma(\alpha, n),$$

$$e_1 = \text{unique element of } \varrho_{d(1)},$$

$$e_{n+1} = \text{unique element of } \varrho_{d(n+1)} - \varrho_{d(n)}.$$

Thus $\varrho_{d(n)} = (e_1, \dots, e_n)$, for $n \geq 1$. Let the set η consist of e_1, e_2, \dots , then $\eta \subset \alpha$, since

$$\eta = \sum_{n=1}^{\infty} \varrho_{d(n)} \quad \text{and} \quad (\forall n) [n \geq 1 \implies \varrho_{d(n)} \subset \alpha].$$

We distinguish two cases:

$$(IIa) \quad \eta \subset_+ \alpha, \qquad (IIb) \quad \eta = \alpha.$$

Re (IIa). Let $t \in \alpha - \eta$ and let q be a partial recursive function such that $\sigma \subset \delta q$ and $q(x) \in \delta \cdot \gamma(\alpha, r_x)$, for $x \in \sigma$. We may assume the number $d(1)$, hence also the number e_1 , as known. We have $\varrho_{d(1)} = (e_1)$; here $e_1 \neq t$, since $t \in \alpha - \eta$. We now know a two-element subset of α , namely (e_1, t) and can compute its canonical index, say a and also the number $q(a)$, i. e., the canonical index of (e_1, e_2) . Since $t \notin (e_1, e_2)$ we can compute the canonical index of a three-element subset of α , namely (e_1, e_2, t) ; let b be this canonical index. Then $q(b)$ is the canonical index of (e_1, e_2, e_3) etc. This effective procedure does not terminate, since $t \in \alpha - \eta$. Hence η is an infinite r. e. subset of α and α is r. i.

Re (IIb). Assume $\eta = \alpha$. We wish to prove $\alpha \simeq \delta$, i. e., $\eta \simeq \delta$. Let h be the mapping from δ onto η such that $h(d_n) = e_n$, for $n \geq 1$. It follows from the definition of the function e in terms of the function d that h has a partial recursive extension. Let any number of η be given, say e_n ; then $d_n = h^{-1}(e_n)$ can be computed in the following manner. From the one-

element subset (e_n) of α we can compute d_1 and e_1 . If upon comparing e_1 and e_n we find out that $e_n = e_1$, we know that $n = 1$ and we have found d_n , since in this case $d_n = d_1$. If, on the other hand, $e_n \neq e_1$, we know that $n \neq 1$ and we have a two-element subset of α , namely (e_1, e_n) ; this enables us to compute the numbers d_2 and e_2 . We continue this procedure until it terminates, i. e., until we have found d_n and (e_1, \dots, e_n) ; this must happen after a finite number of steps. It is readily proved that the function h^{-1} has a partial recursive extension. We conclude by (12) that $\delta \simeq \eta$, i. e., $\delta \simeq \alpha$. Since δ is regressive, so is α .

COROLLARY. *There exist exactly c md -classes; among these c are gc -classes and c are not.*

PROOF. There are at most c md -classes, since every md -class is countable. Let A be called a *Bin-class*, if $A = \text{Bin}(\alpha)$, for some α . There are exactly c immune sets; among these c are regressive and c are not. Thus there are exactly c *Bin-classes* of immune sets; among these c are gc -classes and c are not. It readily follows that there exist c md -classes of immune sets; among these c are gc -classes and c are not. This is slightly stronger than the corollary.

An isol is called *regressive*, if it consists entirely of regressive sets, (or equivalently, if it contains at least one regressive set). Let A_R denote the collection of all regressive isols. It is proved in [3] that A_R is neither closed under addition nor under multiplication, but that the $\min(x, y)$ function from ε^2 into ε can be extended in a natural manner to a $\min(X, Y)$ function from A_R^2 into A_R . However, it is not true that $\min(X, Y) = X$ or $\min(X, Y) = Y$, for any two regressive isols X and Y .

PROPOSITION P7. *Let α, β be two non-empty, isolated sets, $A = \text{Req}(\alpha)$, $B = \text{Req}(\beta)$ and*

$$S = \{j(\xi \times \eta) \mid (\exists n)[n \geq 1 \ \& \ \xi = \gamma(\alpha, n) \ \& \ \eta = \gamma(\beta, n)]\}.$$

If α and β are regressive, i. e., if $A, B \in A_R$, then S is a gc -class and $\text{RET}(S) = \min(A, B)$.

PROOF. Assume the hypothesis. If A or B is finite, so is $\min(A, B)$ and the desired conclusion holds. From now on we assume that α and β are infinite regressive sets. Suppose γ and δ are gc -sets of $\text{Bin}(\alpha)$ and $\text{Bin}(\beta)$ respectively and that for $n \geq 1$, $c_n \in \gamma \cdot \gamma(\alpha, n)$ and $d_n \in \delta \cdot \gamma(\beta, n)$. Let the unions of the classes $\text{Bin}(\alpha)$, $\text{Bin}(\beta)$, S be denoted by σ_1 , σ_2 , σ_3 respecti-

vely. Consider partial recursive functions p and q and the set μ such that

$$\sigma_1 \subset \delta p \quad \text{and} \quad (\forall x) [x \in \sigma_1 \implies p(x) = c_{r(x)}],$$

$$\sigma_2 \subset \delta q \quad \text{and} \quad (\forall y) [y \in \sigma_2 \implies q(y) = d_{r(y)}],$$

$$\mu = [j(c_1, d_1), j(c_2, d_2), \dots].$$

The set μ is a choice set of S , because

$$S = [j[\gamma(\alpha, 1) \times \gamma(\beta, 1)], j[\gamma(\alpha, 2) \times \gamma(\beta, 2)], \dots].$$

Assume $j(x, y) \in \sigma_3$. Then

$$j[p(x), q(y)] = j[c_{r(x)}, d_{r(y)}] \in j[\gamma(\alpha, r_x) \times \gamma(\beta, r_y)],$$

where the third set mentioned in the last formula is the unique set in S which contains $j(x, y)$. Put

$$h(z) = j[pk(z), ql(z)], \quad \text{for } z \in j(\delta p \times \delta q),$$

then h is a partial recursive function related to σ_3 and μ in the usual manner. Thus μ is a gc -set of S and

$$\text{RET}(S) = \text{Req}(\mu) = \min(A, B).$$

REMARK. Under the hypothesis of P7, $\text{RET}(S)$ is a regressive isol, since A_R^2 is closed under the minimum function. Note that $\text{Bin}(\alpha)$, $\text{Bin}(\beta)$ and S are all gc -classes. The isolated sets α and β can, however, be chosen in such a manner that S is a gc -class, while only one of the two classes $\text{Bin}(\alpha)$ and $\text{Bin}(\beta)$ is a gc -class. This can be shown by the following example due to J. Barback. Let τ_1 be an immune, regressive set and τ_2 an immune, indecomposable set. Put

$$\alpha = \{2x \mid x \in \tau_1\}, \quad \eta = \{2x + 1 \mid x \in \tau_2\}, \quad \beta = \alpha + \eta.$$

Then $\alpha \simeq \tau_1$, $\eta \simeq \tau_2$, hence α is also immune and regressive, while η is also immune and indecomposable. The set β is immune, because it is the sum of two separable, immune sets. Note that $\eta \subset \beta$ and $\eta \mid \beta - \eta$. Thus, if β were regressive, η would be regressive by [3, P5]; however, η is indecomposable, while every infinite regressive set is decomposable. Thus β is immune, but not regressive. While $\text{Bin}(\alpha)$ is a gc -class, $\text{Bin}(\beta)$ is not a gc -class. Let γ be a gc -set of $\text{Bin}(\alpha)$ and let p be a partial recursive func-

tion related to γ and σ_1 in the usual manner. Let $c_n \in \gamma \cdot \gamma(\alpha, n)$, for $n \geq 1$. Put

$$\theta = [j(c_1, c_1), j(c_2, c_2), \dots],$$

$$h(z) = j[pk(z), pk(z)], \quad \text{for } z \in j(\delta p \times \varepsilon).$$

Then θ is a choice set of S and for $n \geq 1$,

$$j(x, y) \in j[\gamma(\alpha, n) \times \gamma(\beta, n)] \implies hj(x, y) = j(c_n, c_n) \in j[\gamma(\alpha, n) \times \gamma(\beta, n)].$$

Moreover, $\sigma_3 \subset \delta h$, hence h is a partial recursive function related to θ and σ_3 in the usual manner. Thus θ is a *gc*-set of S . We conclude that of the three classes $\text{Bin}(\alpha)$, $\text{Bin}(\beta)$ and S , exactly two are *gc*-classes, namely $\text{Bin}(\alpha)$ and S . We observe in passing that

$$\text{RET}(S) = \text{Req}(\theta) = \text{Req}(\gamma) = \text{Req}(\alpha) = A.$$

5. Characterization of *gc*-classes.

DEFINITIONS. Let $p(x)$ be a partial recursive function and S a *gc*-class. Then $p(x)$ is a *gc*-function of S , if

$$(18) \quad \sigma \subset \delta p \quad \text{and} \quad p(\sigma) \in \zeta(S),$$

$$(19) \quad (\forall x)[x \in \sigma \implies p(x) \in p(\sigma) \cdot \alpha_x],$$

$$(20) \quad \varrho p \subset \delta p \quad \text{and} \quad (\forall x)[x \in \delta p \implies p^2(x) = p(x)].$$

A *gc*-function is a partial recursive function which is a *gc*-function of at least one *gc*-class.

Every *gc*-class has at least one *gc*-function. For let S be a *gc*-class. Then every partial recursive function $p(x)$ related to S by (18) and (19) has a partial recursive restriction $p_1(x)$ such that

$$\varrho p_1 \subset \delta p_1 \quad \text{and} \quad (\forall x)[x \in \delta p_1 \implies p_1^2(x) = p_1(x)].$$

NOTATION. For any partial recursive function $p(x)$,

$$\text{Gen}(p) = \{p^{-1}(y) \mid y \in \varrho p\}.$$

For every partial recursive function $p(x)$, $\text{Gen}(p)$ is an *md*-class; it is empty if and only if $p(x)$ is nowhere defined; moreover, it is a r. e. class of r. e. sets; in fact, it is r. e. without repetitions.

PROPOSITION P8. *A partial recursive function $p(x)$ is a gc function if and only if it satisfies (20). Moreover, if $p(x)$ satisfies (20), it is a gc-function of the class $S = \text{Gen}(p)$ with $\sigma = \delta p$ and $p(\sigma) = \varrho p \in \zeta(S)$.*

PROOF. One direction of the biconditional is trivial. Let $p(x)$ be a partial recursive function which satisfies (20). Observe that (20) is equivalent to

$$(21) \quad \varrho p \subset \delta p \quad \text{and} \quad (\forall y)[y \in \varrho p \implies p(y) = y].$$

For let $p(x)$ satisfy (20). Assume $y_1 \in \varrho p$, say $y_1 = p(x_1)$. Then $p(y_1) = p^2(x_1) = p(x_1) = y_1$. Conversely, assume that $p(x)$ satisfies (21). Let $x_1 \in \delta p$ and put $y_1 = p(x_1)$. Then $y_1 \in \varrho p$ and $p^2(x_1) = p(y_1) = y_1 = p(x_1)$. We may therefore assume that $p(x)$ satisfies both (20) and (21). Let $S = \text{Gen}(p)$. Then $\sigma = p^{-1}(\varrho p) = \delta p$, and $p(\sigma) = p(\delta p) = \varrho p$. We claim

- (i) $\varrho p \subset \sigma$,
- (ii) $\alpha \in S \implies \varrho p \cdot \alpha$ contains exactly one element,
- (iii) $(\forall x)[x \in \sigma \implies p(x) \in \varrho p \cdot \alpha_x]$.

Note that (i) and (ii) imply that ϱp is a choice set of S , while (i), (ii), (iii) and $\sigma = \delta p$ imply that ϱp is a gc-set of S .

Re (i). $\varrho p \subset \delta p$ and $\delta p = \sigma$; thus $\varrho p \subset \sigma$.

Re (ii). Let $\alpha \in S$, say $\alpha = p^{-1}(y_1)$, for some $y_1 \in \varrho p$. Then $y_1 \in \varrho p$ implies $p(y_1) = y_1$, hence $y_1 \in p^{-1}(y_1) = \alpha$; thus $y_1 \in \varrho p \cdot \alpha$. Moreover, y_1 is the only element of $\varrho p \cdot \alpha$. For assume $y_2 \in \varrho p \cdot \alpha$. Then $p(y_2) = y_2$ because $y_2 \in \varrho p$, and $p(y_2) = y_1$ because $y_2 \in \alpha$; thus $y_1 = y_2$.

Re (iii). Let $s \in \sigma$. Then $s \in \delta p$; put $y_1 = p(s)$. Hence $s \in p^{-1}(y_1)$ and $\alpha_s = p^{-1}(y_1)$. We now have $s \in \alpha_s$, $\alpha_s \in S$ and $\alpha_s = p^{-1}(y_1)$. According to (ii), y_1 is the only element in $\varrho p \cdot \alpha_s$. However, $y_1 = p(s)$, hence $p(s) \in \varrho p \cdot \alpha_s$.

PROPOSITION P9. *Let $p(x)$ be a gc-function of the gc-class S . Then $\sigma = \delta p$ if and only if $S = \text{Gen}(p)$.*

PROOF. Let $p(x)$ be a gc-function of the gc-class S . The «if part» is immediate, for $S = \text{Gen}(p)$ implies

$$\sigma = \Sigma \{p^{-1}(y) \mid y \in \varrho p\} = \delta p.$$

Now assume $\sigma = \delta p$. Let $T = \text{Gen}(p)$; denote the union of all sets in T by τ . We know by P8 that $p(x)$ is a gc-function of T with $\tau = \delta p$. Thus

$\sigma = \tau$, since both σ and τ are equal to δp . It clearly suffices to prove $S = T$, i. e., (i) $S \subset T$, and (ii) $T \subset S$.

Re (i). Let $\alpha \in S$. Let x_0 be any element of α ; put $y_0 = p(x_0)$. Then

$$x \in \alpha \implies p(x) = p(x_0) \implies p(x) = y_0,$$

i. e., $\alpha \subset p^{-1}(y_0)$. Denoting $p^{-1}(y_0)$ by β we see that $\alpha \subset \beta$ and $\beta \in T$. The inclusion $\alpha \subset \beta$ must be improper. For suppose $b \in \beta - \alpha$. Then $b \in \sigma$, since $b \in \beta$ and $\beta \in T$, while $\sigma = \tau$. We claim

$$(22) \quad [\bar{\alpha} \in S \ \& \ \bar{\alpha} \neq \alpha] \implies b \notin \bar{\alpha}.$$

For assuming the hypothesis of (22),

$$[x \in \bar{\alpha} \ \& \ \bar{\alpha} \neq \alpha] \implies p(x) \neq y_0,$$

$$b \in \beta \implies b \in p^{-1}(y_0) \implies p(b) = y_0,$$

so that $x \neq b$. Combining (22) with the hypothesis $b \notin \bar{\alpha}$ we obtain

$$b \notin \Sigma\{\alpha \mid \alpha \in S\}, \text{ i. e., } b \notin \sigma.$$

The assumption $\alpha \subset_+ \beta$ leads therefore to the contradiction: $b \in \sigma$ and $b \notin \sigma$. Hence $\alpha = \beta$, and $\alpha \in T$ because $\beta \in T$.

Re (ii). Let $\beta \in T$, say $\beta = p^{-1}(y_1)$, where $y_1 \in \rho p$. Then $\beta \subset \sigma$, since $\beta \subset \tau$ and $\tau = \sigma$. Note that $\beta = p^{-1}(y_1)$ implies $p(\beta) = (y_1)$; combining this with $\beta \subset \sigma$, we see that β must be included in some set of S , say α . The set β is non-empty, for it belongs to the md -class T ; let $b \in \beta$. We claim that $a \in \alpha$ implies $a \in \beta$. For assume $a \in \alpha$. Then $a, b \in \alpha$, since $b \in \beta$, $\beta \subset \alpha$; this implies $p(a) = p(b)$. On the other hand, $\beta = p^{-1}(y_1)$, hence $b \in \beta$ implies $p(b) = y_1$. Thus $p(a) = y_1$ and $a \in p^{-1}(y_1)$, i. e., $a \in \beta$. We have therefore proved that $\alpha \subset \beta$. Since we also have $\beta \subset \alpha$, we conclude that $\alpha = \beta$. Hence $\beta \in S$, since $\alpha \in S$. We have proved that $T \subset S$.

DEFINITION I. A class S is *primitive*, if it satisfies one of the following three conditions:

(i) S is empty, (ii) S is a non-empty, finite md -class of r. e. sets, (iii) S is a denumerable md -class of r. e. sets and there exists a recursive function $a(n, x)$ such that if $\alpha_n = \rho a(n, x)$, for $n \in \varepsilon$, then $\alpha_0, \alpha_1, \dots$ are distinct and $S = (\alpha_0, \alpha_1, \dots)$.

DEFINITION II. A class S is *primitive*, if it is a gc -class with a gc -function $p(x)$ such that $S = \text{Gen}(p)$.

DEFINITION III. A class S is *primitive*, if $S = \text{Gen}(p)$, for some partial recursive function $p(x)$.

PROPOSITION P10. *The three definitions of a primitive class are equivalent.*

PROOF. Let S be an *md*-class. We shall establish the three conditionals

- (a) S I-primitive $\implies S$ II-primitive,
- (b) S II-primitive $\implies S$ III-primitive,
- (c) S III-primitive $\implies S$ I-primitive.

Since (b) is trivial we shall restrict our attention to (a) and (c).

Re (a). Let S be I-primitive. We distinguish three cases.

Case 1. S is empty. Let $p(x)$ be the partial recursive function which is nowhere defined. Then $S = \text{Gen}(p)$ and $p(x)$ is a *gc*-function of S .

Case 2. S is a non-empty, finite *md*-class of r. e. sets, say $S = (\alpha_0, \dots, \alpha_n)$. Note that $\alpha_0, \dots, \alpha_n$ are non empty and mutually disjoint. Let $a_i \in \alpha_i$, for $0 \leq i \leq n$. Define a function $p(x)$ by

$$\delta p = \alpha_0 + \dots + \alpha_n, \quad (\forall x)(\forall i \leq n)[x \in \alpha_i \implies p(x) = a_i].$$

Then $p(x)$ is partial recursive and $\varrho p = (a_0, \dots, a_n)$. Hence

$$\text{Gen}(p) = \{p^{-1}(y) \mid y \in \varrho p\} = [p^{-1}(a_0), \dots, p^{-1}(a_n)] = (\alpha_0, \dots, \alpha_n) = S.$$

It also follows from the definition of $p(x)$ that

$$\begin{aligned} p(\sigma) &= p(\alpha_0 + \dots + \alpha_n) = (a_0, \dots, a_n) \in \zeta(S), \\ (\forall x)[x \in \sigma &\implies p(x) \in p(\sigma) \cdot \alpha_x], \\ \varrho p \subset \sigma, &\quad \text{hence} \quad \varrho p \subset \delta p, \\ (\forall x)[x \in \sigma &\implies p^2(x) = p(x)]. \end{aligned}$$

Hence $p(x)$ is a *gc*-function of S .

Case 3. S is a denumerable *md*-class of r. e. sets and there exists a recursive function $a(n, x)$ such that the sets $\alpha_0 = \varrho a(0, x)$, $\alpha_1 = \varrho a(1, x)$, ... are distinct and $S = (\alpha_0, \alpha_1, \dots)$. Define a function $p(x)$ by

$$\delta p = \alpha_0 + \alpha_1 + \dots, \quad \text{and} \quad (\forall x)(\forall i)[x \in \alpha_i \implies p(x) = a(i, 0)].$$

The set δp is therefore r. e., and given any $x \in \delta p$ we can compute the unique number i such that $x \in \alpha_i$. Thus $p(x)$ is a partial recursive function with $[a(0, 0), a(1, 0), \dots]$ as its range. Also,

$$\text{Gen}(p) = \{p^{-1}(y) \mid y \in \varrho p\} = [p^{-1}a(0, 0), p^{-1}a(1, 0), \dots] = (\alpha_0, \alpha_1, \dots) = S.$$

We can verify as we did in Case 2 that $p(x)$ is a *gc*-function of S . In each of the three cases, $S = \text{Gen}(p)$, where $p(x)$ is a *gc*-function of S , i. e., S is II-primitive.

Re (c). Let S be III-primitive, say $S = \text{Gen}(p)$ for some partial recursive function p . We distinguish three cases.

Case 1. $p(x)$ is nowhere defined. Then S is empty.

Case 2. ϱp is non empty, but finite, say $\varrho p = (c_0, \dots, c_k)$, where c_0, \dots, c_k are distinct. Then

$$S = \text{Gen}(p) = [p^{-1}(c_0), \dots, p^{-1}(c_k)],$$

where $p^{-1}(c_0), \dots, p^{-1}(c_k)$ are mutually disjoint, because c_0, \dots, c_k are distinct. For $0 \leq i \leq k$, $p^{-1}(c_i)$ is a r. e. set, since $p(x)$ is a partial recursive function. Hence S is a finite class which consists of $k + 1$ mutually disjoint r. e. sets.

Case 3. ϱp is infinite. Since ϱp is also r. e., there exists a one-to-one recursive function which ranges over ϱp , say e_n . Thus

$$S = \text{Gen}(p) = [p^{-1}(c_0), p^{-1}(c_1), \dots].$$

Given any number n we can effectively find a (definition of a) recursive function $a_n(x)$ of x which ranges over $p^{-1}(c_n)$. The sets $p^{-1}(c_0), p^{-1}(c_1), \dots$ are mutually disjoint, since c_0, c_1, \dots are distinct. Put $a(n, x) = a_n(x)$, for $n \in \varepsilon$, then $a(n, x)$ is a recursive function; also, the sets $\varrho a(0, x), \varrho a(1, x), \dots$ are distinct and S consists of $\varrho a(0, x), \varrho a(1, x), \dots$. In each of the three cases S is I-primitive.

COROLLARY. *A class S is primitive if and only if it is a *gc*-class with a *gc*-function $p(x)$ such that $\delta p = \sigma$.*

PROOF. By P9 and P10.

DEFINITION. An *md*-class T is a *restriction* of a *gc*-class S , if

- (a) for every $\beta \in T$, there is an α_β such that $\beta \subset \alpha_\beta$ and $\alpha_\beta \in S$,
- (b) there is a $\gamma \in \zeta(S)$ such that $\beta \in T$ implies $\gamma \cdot \alpha_\beta \subset \beta$.

REMARK. Let the *md*-class T be a restriction of the *gc*-class S . Then every set $\beta \in T$ uniquely determines the set α_β such that $\beta \subset \alpha_\beta$ and $\alpha_\beta \in S$, since β is non-empty and the sets in S are mutually disjoint.

It is clear that every subclass of a *gc*-class S is a restriction of S . We observed in section 3 that every subclass of a *gc* class is again a *gc*-class. This last statement will now be generalized.

PROPOSITION P11. *Every restriction of a gc-class is again a gc-class.*

PROOF. Let $\gamma \in \zeta(S)$ and let the *md*-class T be related to S and γ by (a) and (b). Suppose $p(x)$ is a partial recursive function related to σ and γ in the usual manner. Let S_0 be the class of all sets α_β , for $\beta \in T$. Denote the unions of S_0 and T by σ_0 and τ respectively, and let $\gamma_0 = \gamma \cdot \sigma_0$. The relation $S_0 \subset S$ implies first of all that $\gamma_0 \in \zeta(S_0)$ and secondly that $p(x)$ is related to σ_0 and γ_0 in the usual manner. The class T can now be obtained from S_0 by replacing every $\alpha \in S_0$ by a set β such that $\gamma_0 \cdot \alpha \subset \beta \subset \alpha$. Thus

$$\begin{aligned} (\forall x)[x \in \alpha \ \& \ \alpha \in S_0 \implies p(x) \in \gamma_0 \cdot \alpha] \quad \text{implies} \\ (\forall x)[x \in \beta \ \& \ \beta \in T \implies p(x) \in \gamma_0 \cdot \beta]. \end{aligned}$$

Hence $\gamma_0 \in \zeta(T)$ and $p(x)$ is related to τ and γ_0 in the usual manner. Note that $\gamma \cdot \sigma_0 = \gamma \cdot \tau$; we could therefore also have defined γ_0 as $\gamma \cdot \tau$.

PROPOSITION P12. *Let T be a gc-class. For every gc-function $p(x)$ of T , T is a restriction of the primitive gc-class $\text{Gen}(p)$.*

PROOF. Let T be a *gc*-class and let $p(x)$ be one of its *gc*-functions. Put $S = \text{Gen}(p)$. In view of P8 the class S is a primitive class with $p(x)$ as a *gc*-function; also $\sigma = \delta p$ and $p(\sigma) = \varrho(p) \in \zeta(S)$. We wish to prove that T is a restriction of S , i. e.,

- (a) for every $\beta \in T$, there is an α_β such that $\beta \subset \alpha_\beta$ and $\alpha_\beta \in S$,
- (b) there is a $\gamma \in \zeta(S)$ such that $\beta \in T \implies \gamma \cdot \alpha_\beta \subset \beta$.

Re (a). Let $\beta \in T$. Then $\beta \neq \sigma$; let $b \in \beta$, $c = p(b)$ and $\alpha_\beta = p^{-1}(c)$. Also, $c \in \varrho p$, hence

$$\alpha_\beta = p^{-1}(c) \in \text{Gen}(p) = S.$$

Since the element b of β is mapped by p onto c , p maps every element of β onto c . Hence $\beta \subset p^{-1}(c)$, i. e., $\beta \subset \alpha_\beta$.

Re (b). Put $\gamma = p(\sigma)$, then $\gamma \in \zeta(S)$. Let $\beta \in T$. Define b, c, α_β as in the proof of (a). Then $c \in \varrho p = p(\sigma)$, hence $c \in \gamma$. Moreover, since $p(x)$ is a *gc*-function, we have

$$c \in \varrho p \implies p(c) = c \implies c \in p^{-1}(c) \implies c \in \alpha_\beta.$$

We conclude that $c \in \gamma \cdot \alpha_\beta$. Since $\gamma \in \zeta(S)$ and $\alpha_\beta \in S$, the set $\gamma \cdot \alpha_\beta$ contains only one element, hence $\gamma \cdot \alpha_\beta = (c)$. Finally, $b \in \beta$ and $\beta \in T$ imply $p(b) \in \beta$, hence $c \in \beta$. Thus $\gamma \cdot \alpha_\beta \subset \beta$.

It follows from P11 and P12 that: *an md-class is a gc-class if and only if it is a restriction of some primitive gc-class.* Let us therefore compare

gc-classes in general with primitive *gc*-classes. We have seen in section 4 that there are exactly c *gc*-classes and in section 1 that for every RET A , there is a *gc*-class with A as its RET. On the other hand, we immediately see from the definition of a primitive class that there are exactly \aleph_0 primitive classes and that a primitive class can only have one of $0, 1, \dots$ or R as its RET.

The restrictions of a primitive class can be simply described. For let S be a primitive class. Then every restriction T of S can be obtained as follows: choose a $\gamma \in \zeta(S)$ and form a subclass T of S by treating each $\alpha \in S$ in the following manner: either delete α altogether or replace α by any set β such that $\gamma \cdot \alpha \subset \beta \subset \alpha$.

It remains to characterize the *gc*-sets of any primitive class P . If P is finite, the *gc*-sets of P are simply the choice sets of P . Now assume P is infinite; let $a(n, x)$ be a recursive function such that

- (i) $n \neq m \implies \rho a(n, x)$ disjoint from $\rho a(m, x)$,
- (ii) S consists of $\rho a(0, x), \rho a(1, x), \dots$.

Then γ is a *gc*-set of P if and only if γ equals $\rho a(f_n, u_n)$, for some recursive permutation f_n and some recursive function u_n .

6. Miscellaneous remarks.

(A). We have not yet introduced a relation of recursive equivalence between *md*-classes. This can, however, be done in a natural manner.

NOTATION. For every *md*-class S ,

$$R(S) = \{(x, y) \in \sigma^2 \mid \alpha_x = \alpha_y\}.$$

DEFINITION. Let S_1 and S_2 be *md*-classes with unions σ_1 and σ_2 respectively. Let $R_1 = R(S_1)$ and $R_2 = R(S_2)$. Then S_1 is *recursively equivalent* to S_2 [written: $S_1 \simeq S_2$], if there exists a partial recursive one-to-one function $p(x)$ such that

- (a) $\sigma_1 \subset \delta p$ and $p(\sigma_1) = \sigma_2$,
- (b) $x R_1 y \iff p(x) R_2 p(y)$, for $x, y \in \sigma_1$.

This \simeq relation between *md*-classes is clearly reflexive, symmetric and transitive. We also have for *md*-classes S_1 and S_2 : $S_1 \simeq S_2$ implies $\sigma_1 \simeq \sigma_2$. The following five properties of the \simeq relation between *md*-classes are readily verified.

- (i) Let $S_1 \simeq S_2$. Then S_1 is a *gc*-class if and only if S_2 is a *gc*-class.
- (ii) Let $S_1 \simeq S_2$, where S_1 and S_2 are *gc*-classes. Then

$$\gamma_1 \in \zeta(S_1) \ \& \ \gamma_2 \in \zeta(S_2) \implies \gamma_1 \simeq \gamma_2.$$

- (iii) Let $S_1 \simeq S_1^*$, $S_2 \simeq S_2^*$, $S_1 | S_2$ and $S_1^* | S_2^*$. Then $S_1 + S_2 \simeq S_1^* + S_2^*$.
- (iv) $S_1 \simeq S_1^*$ & $S_2 \simeq S_2^* \implies S_1 \times S_2 \simeq S_1^* \times S_2^*$.
- (v) $\alpha \simeq \beta \implies \text{Bin}(\alpha) \simeq \text{Bin}(\beta)$.

(B) Let us say that an *md*-class has *property* π , if there exists a partial recursive function $q(x, y)$ such that

$$(23) \quad \sigma^2 \subset \delta q \quad \text{and} \quad \varrho q \subset (0, 1),$$

$$(24) \quad \begin{cases} \alpha_x = \alpha_y \implies q(x, y) = 1, & \text{for } x, y \in \sigma, \\ \alpha_x \neq \alpha_y \implies q(x, y) = 0, & \text{for } x, y \in \sigma. \end{cases}$$

Intuitively speaking S has *property* π , if there is an effective procedure which enables us to decide for any two numbers in σ whether or not they belong to the same set in S . It is readily seen that

$$(25) \quad S \text{ a gc-class} \implies S \text{ has property } \pi.$$

For assume the hypothesis. Let $\gamma \in \zeta(S)$ and let $p(x)$ be a partial recursive function related to γ and σ in the usual manner. Then the partial recursive function $q(x, y)$ defined by

$$q(x, y) = \overline{sg} | p(x) - p(y) |, \quad \text{for } x, y \in \delta p,$$

satisfies (23) and (24). We claim that the converse of (25) is false. For let $S = \text{Bin}(\alpha)$ for a set α which is immune, but not regressive. Then S is not a *gc*-class by P6. On the other hand, S has *property* π , since the recursive function

$$q(x, y) = \overline{sg} | r_x - r_y |, \quad \text{for } x, y \in \varepsilon,$$

satisfies (23) and (24).

(C) We recall the definition of $\Phi_f(T)$.

NOTATION. Let $f(x)$ be any one-to-one function from ε into ε and let $T \in \mathcal{A}_R - \varepsilon$. Then

$$\Phi_f(T) = \text{Req } \varrho t_{f(n)},$$

where t_n is any regressive function ranging over any set in T .

It is readily seen that if $f(x)$ is a strictly increasing, recursive function, Φ_f maps $\mathcal{A}_R - \varepsilon$ into itself. Several other properties of the mapping Φ_f are discussed in [6] and [7]. Let us assume that $f(x)$ is a strictly increasing recursive function such that $f(0) = 0$. We wish to show how one can associate with every $T \in \mathcal{A}_R - \varepsilon$ a simple *gc*-class of finite sets which has

$\Phi_f(T)$ as its RET. Let $\tau \in T$ and let t_n be a regressive function ranging over τ . Put

$$\tau_n = \{t_x \mid f(n) \leq x < f(n+1)\}, \quad \text{for } n \in \varepsilon,$$

$$S = (\tau_0, \tau_1, \dots), \quad \gamma = (t_{f(0)}, t_{f(1)}, \dots).$$

Then S is an m -class of finite sets with τ as its union and γ as a choice set. Let

$$g(x) = f(\mu n) [f(n) \leq x < f(n+1)], \quad \text{for } x \in \varepsilon,$$

$$q(x) = t_{g(n)}(x), \quad \text{for } x \in \tau.$$

The function $g(x)$ is recursive and

$$(\forall x) (\forall n) [x \in \tau_n \implies q(x) \in \gamma \cdot \tau_n].$$

Given any $x \in \tau$ we can compute the unique number n such that $x = t_n$, i. e., the number $n = t^{-1}(x)$, hence also the number $g(n)$. However, $g(n)$ is less than or equal to n ; this enables us to compute $q(x) = t_{g(n)}$ from t_n . It is readily proved that $q(x)$ has a partial recursive extension. Thus $\gamma \in \zeta(S)$ and

$$\text{RET}(S) = \text{Req}(\gamma) = \text{Req } \rho t_{f(n)} = \Phi_f(T).$$

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