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ON THE DEFINITENESS OF QUADRATIC FORMS WHICH OBEY CONDITIONS OF SYMMETRY

by HANS LEWY

According to classical Algebra a real quadratic form is positive definite whenever certain determinants of its coefficient matrix are positive. This test is in general non-linear in the coefficients and its application is made cumbersome by the difficulty of evaluating determinants. Any hope of simplification must depend therefore on special properties of the coefficient matrix. In the course of work on differential equations the A. was led to consider quadratic forms of the following type.

Denote by V_k or V_k^n the set of arbitrary combinations of k distinct elements of the set $\{1, 2, \dots, n\}$, counting as identical two combinations which differ only by the arrangement of their elements. Let $u(\omega)$, $\omega \in V_k$ be a real function of ω. We consider a real quadratic form $Q[u]$ subject to the only condition that it remain invariant when the numbers of $\{1, 2, \dots, n\}$ undergo an arbitrary permutation π, i. e. if $u(\omega)$ is replaced by $u(\pi\omega)$. Then the form $Q[u]$ is positive definite if and only if certain (explicit) inequalities hold which are linear in the coefficients; in fact it suffices to establish that $Q[u] > 0$ for certain finitely many choices of $u(\omega)$ which do not depend on the coefficients of Q.

NOTATION: In the sequel k, n are fixed positive integers and $n \geq k$; ω, ω_0 are elements of V_k^n; $\eta \supset \varepsilon$, $\eta \in V_r^n$, $\varepsilon \in V_s^n$ if the set η contains the set ε; $|\eta| = r$; $\eta \cap \varepsilon$ is that combination which consists of the elements of $\{1, \dots, n\}$ in common to η and ε.

In writing $\sum_{\eta \supset \varepsilon}$ or $\sum_{\eta \subset \varepsilon}$ we always understand summation over the first combination mentioned in the subscript, in this case η, the second, ε, being fixed, we also make this explicit by writing $\sum_{\eta, \eta \supset \varepsilon}$ etc... $\sum_{\omega}$ is an abbreviation

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for $\sum\limits_{\omega,\, \omega \in V_k}$; $\sum\limits_{\omega \supset \eta}$ an abbreviation for $\sum\limits_{\omega,\, \omega \in V_k,\, \omega \supset \eta}$ where η is fixed; $\sum\limits_{\omega \supset \eta \in V_s}$ means 0 if $s > k$, and means $\sum\limits_{\omega}$ if $s = 0$.

1. LEMMA 1.1. *Suppose* $1 < k,\, 2k - 1 \leq n$. *Given real* $v(\eta)$, $\eta \in V_{k-1}$, *the equations*

$$\sum_{\omega \supset \eta} u(\omega) = v(\eta), \quad \eta \in V_{k-1}$$

have a real solution $u(\omega)$, $\omega \in V_k$.

PROOF. If the Lemma were false there would exist real non-trivial $\lambda(\eta)$, $\eta \in V_{k-1}$ such that identically in $u(\omega)$

$$\sum_{\eta \in V_{k-1}} \lambda(\eta) \sum_{\omega \supset \eta} u(\omega) = 0.$$

Choosing an arbitrary $\omega_0 \in V_k$ put $u(\omega_0) = 1$, $u(\omega) = 0$ for $\omega \neq \omega_0$ and find

$$\sum_{\eta \in V_{k-1},\, \eta \subset \omega_0} \lambda(\eta) = 0, \quad \omega_0 \in V_k. \tag{1.1}$$

We shall prove that (1.1) implies that all $\lambda(\eta)$ vanish; whence the Lemma.

Take for ω_0 all elements of V_k contained in $\{1, 2, \ldots, k, k+1\}$ with $\omega_0 \supset \{1\}$, and add the equations (1.1) corresponding:

$$0 = \sum_{\omega_0,\, 1 \subset \omega_0 \subset \{1, \ldots, k+1\}} \; \sum_{\eta \subset \omega_0,\, \eta \in V_{k-1}} \lambda(\eta) = p \sum_{\eta \in V_{k-1},\, 1 \subset \eta \subset \{1, 2, \ldots, k+1\}} \lambda(\eta)$$

$$+ p' \sum_{\eta \in V_{k-1},\, 1 \not\subset \eta \subset \{1, 2, \ldots, k+1\}} \lambda(\eta)$$

with $p > 0$. The second sum on the right is

$$\sum_{\eta \in V_{k-1},\, \eta \subset \{2, \ldots, k+1\}} \lambda(\eta) = 0$$

by (1.1). Hence

$$\sum_{\eta \in V_{k-1},\, 1 \subset \eta \subset \{1, 2, \ldots, k+1\}} \lambda(\eta) = 0$$

or generally for $\{j_1, \ldots, j_{k+1}\} \in V_{k+1}$

$$\sum_{\eta \in V_{k-1},\, j_1 \subset \eta \subset \{j_1, \ldots, j_{k+1}\}} \lambda(\eta) = 0. \tag{1.2}$$

Suppose we have proved for $r = r_0 - 1$, $r_0 \leq k - 1$ that given $\varepsilon \in V_r$, $\delta \in V_{r+k}$, $\delta \supset \varepsilon$

$$\sum_{\eta \in V_{k-1},\, \varepsilon \subset \eta \subset \delta} \lambda(\eta) = 0, \tag{1.3}$$

which coincides with (1.2) for $r = 1$. We shall prove the same relation with r_0 instead of r. Take $\varepsilon = \{1, 2, \ldots, r\}$, $\delta' = \{1, 2, \ldots, r + k + 1\} \in V_{r+k}$ and obtain from the last relation

$$\sum_{\delta \in V_{r+k},\, \varepsilon \subset \delta \subset \delta'} \; \sum_{\eta \in V_{k-1},\, \varepsilon \subset \eta \subset \delta} \lambda(\eta) = 0$$

Here left hand can be split thus:

$$p \sum_{\{1, 2, \ldots, r+1\} \subset \eta \in V_{k-1},\, \eta \subset \delta'} \lambda(\eta) + p' \sum_{\{1, 2, \ldots r\} \subset \eta \in V_{k-1},\, \eta \subset \delta',\, \eta \not\ni r+1} \lambda(\eta) = 0$$

with p a positive integer, while the factor of p'

$$\sum_{\{1, \ldots, r\} \subset \eta \in V_{k-1},\, \eta \subset \{1, \ldots, r, r+2, \ldots, r+k+1\}} \lambda(\eta) = 0$$

by (1.3). Hence

$$\sum_{\{1, 2, \ldots, r+1\} \subset \eta \in V_{k-1},\, \eta \subset \{1, \ldots, r+1+k\}} \lambda(\eta) = 0,$$

or, more generally, relation (1.3) with r replaced by $r + 1 = r_0$. Note that we made use of the inequality $r_0 + k \leq n$ in granting the existence of $\delta' \in V_{r_0+k}$. Writing (1.3) for $r_0 = k - 1$ we obtain with $\varepsilon \in V_{k-1}$, $\delta \in V_{2k-1}$, $\delta \supset \varepsilon$

$$0 = \sum_{\eta \in V_{k-1},\, \varepsilon \subset \eta \subset \delta} \lambda(\eta) = \lambda(\varepsilon),\ \varepsilon \in V_{k-1}$$

which proves Lemma 1.1.

Note that there is an obvious interpretation of the statement of Lemma 1.1 which makes it correct even for $k = 1$

LEMMA 1.2. Suppose $1 \leq r < k$, $2k - 1 \leq n$. For any given real $v(\eta)$, $\eta \in V_r$ the equations

$$\sum_{\omega \supset \eta} u(\omega) = v(\eta),\ \eta \in V_r$$

have a real solution $u(\omega)$, $\omega \in V_k$.

PROOF. By Lemma 1.1 the assertion holds for $r = k - 1$. Suppose it proved for $k - 1 \geq r \geq r_0 > 1$ we prove it for $r = r_0 - 1$. The previous Lemma with k replaced by r_0 shows that for given real $v(\eta)$, $\eta \in V_{r_0-1}$ there are real $v'(\eta')$, $\eta' \in V_{r_0}$ such that

$$\sum_{\eta',\, V_{r_0} \ni \eta' \supset \eta} v'(\eta') = v(\eta), \;\; \eta \in V_{r_0-1}.$$

We now solve by induction

$$\sum_{\omega \supset \eta'} u(\omega) = v'(\eta'), \;\; \eta' \in V_{r_0}.$$

Then for these $u(\omega)$

$$\sum_{\eta',\, V_{r_0} \ni \eta' \supset \eta} \; \sum_{\omega \supset \eta'} u(\omega) = v(\eta), \;\; \eta \in V_{r_0-1}.$$

But left hand amounts to $p \sum_{\omega \supset \eta} u(\omega)$ where $p > 0$. The Lemma follows.

LEMMA 1.3. *Let $2k \leq n$. There are real $u(\omega)$, $\omega \in V_k$ such that*

$$\sum_{\omega \supset \eta} u(\omega) = 0, \;\; \eta \in V_r,$$

with fixed r in $0 \leq r < k$, but for at least one K in V_{r+1}

$$\sum_{\omega \supset K} u(\omega) \neq 0.$$

PROOF. Solve non-trivially $\sum_{K,\, V_{r+1} \ni K \supset \eta} v(K) = 0, \eta \in V_r$; and $\sum_{\omega \supset K} u(\omega) = v(K)$ according to Lemma 1.2.

The first is possible since $\binom{n}{r+1}$ is the number of unknowns and the number of equations is $\binom{n}{r}$ which, for $0 \leq r < k \leq \frac{n}{2}$ is $< \binom{n}{r+1}$. Then at least one $v(K) \neq 0$, but

$$p \sum_{\omega \supset \eta} u(\omega) \equiv \sum_{K \supset \eta} \sum_{\omega \supset K} u(\omega) = \sum_{K \supset \eta} v(K) = 0, \;\; \eta \in V_r$$

where p is a positive integer.

2. Let $Q[u]$ be a quadratic form in real $u(\omega)$, $\omega \in V_k$, such that Q remains invariant if the $u(\omega)$, $\omega \in V_k$ are replaced by $u(\pi\omega)$ where π is an arbitrary permutation of $(1, 2, \dots, n)$. Then we can write, with $|\omega_i \cap \omega_j| =$ = number of digits in common to ω_i and ω_j,

$$Q[u] = \Sigma\, c_{ij}\, u(\omega_i)\, u(\omega_j)$$

$$= \sum_{r=0}^{k} \sum_{|\omega_i \cap \omega_j| = r} c_{ij}\, u(\omega_i)\, u(\omega_j)$$

Applying to $Q[u]$ all permutations π we obtain

$$Q[u] = \sum_{r=0}^{k} c_r \sum_{|\omega_i \cap \omega_j| = r} u(\omega_i)\, u(\omega_j) \tag{2.1}$$

where we intend, interpreting the $\Sigma\, u(\omega_i)\, u(\omega_j)$ for $\omega_i \neq \omega_j$, that the sum should contain both terms $u(\omega_i)\, u(\omega_j)$ and $u(\omega_j)\, u(\omega_i)$.

It is possible to write, for $2k \leq n$,

$$Q[u] = \sum_{r=0}^{k} d_r \sum_{\eta \in V_r} \left(\sum_{\omega \supset \eta} u(\omega)\right)^2. \tag{2.2}$$

In fact we find that (2.2) is of form (2.1) with

$$\begin{aligned} c_k &= d_0 + \binom{k}{1} d_1 + \binom{k}{2} d_2 + \dots + d_k \\ c_{k-1} &= d_0 + \binom{k-1}{1} d_1 + \binom{k-1}{2} d_2 + \dots + d_{k-1} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ c_1 &= d_0 + d_1 \\ c_0 &= d_0 \end{aligned} \tag{2.3}$$

whose resolution with respect to d_j evidently exists; it is precisely

$$d_r = c_r - \binom{r}{1} c_{r-1} + \binom{r}{2} c_{r-2} - + \dots + (-)^r c_0\,, \quad 0 \leq r \leq k \tag{2.4}$$

It suffices to note that relations (2.3) yield if $d_\nu = \alpha^\nu$, the $\nu - th$ power of an indeterminate α, that $c_\nu = (1 + \alpha)^\nu = \beta^\nu$; in this case $\alpha^r = (\beta - 1)^r$ which then coincides with (2.4); but the powers of β are linearly independent, hence the solution in this special case must have the same coefficients as in the general case.

THEOREM 2.1. *If $n \geq 2k$, the quadratic form*

$$Q[u] = \sum_{r=0}^{k} d_r \sum_{\eta \in V_r} \left(\sum_{\omega \supset \eta} u(\omega) \right)^2,$$

in real $u(\omega)$, $\omega \in V_k$ is positive definite if and only if the linear inequalities hold

$$\sum_{r=\sigma}^{k} d_r \binom{k-\sigma}{r-\sigma} \binom{n-r-\sigma}{k-r} > 0, \quad \sigma = 0, \quad 1, \ldots, k. \tag{2.5}$$

PROOF. If all d_r are positive, the statement is trivial. We may therefore assume that for some r, $d_r < 0$. Note, however, that $d_k > 0$ is necessary for positive definiteness of Q. For (by Lemma 1.3) we can choose real $u(\omega)$, not all zero, such that

$$\sum_{\omega \supset \eta} u(\omega) = 0, \qquad \eta \in V_{k-1}.$$

Then $\sum_{\omega \supset \eta} u(\omega) = 0$, $\eta \in V_r$, $0 \leq r < k-1$ follows; $Q[u] = d_k \sum_{\omega} u^2(\omega) > 0$, whence $d_k > 0$, which is the inequality (2.5) for $\sigma = k$. Set

$$d_r^- = \begin{cases} -d_r, & \text{if } d_r < 0, \\ 0, & \text{if } d_r \geq 0, \end{cases}$$

$$d_r^+ = \begin{cases} d_r, & \text{if } d_r \geq 0, \\ 0, & \text{if } d_r < 0. \end{cases}$$

The proof of the Theorem involves maximizing the quotient

$$\lambda = \frac{\sum_{r=0}^{k-1} d_r^- \sum_{\eta \in V_r} \left(\sum_{\omega \supset \eta} u(\omega) \right)^2}{\sum_{r=0}^{k} d_r^+ \sum_{\eta \in V_r} \left(\sum_{\omega \supset \eta} u(\omega) \right)^2}. \tag{2.6}$$

The existence of a maximum is certain in view of $d_k > 0$; for it suffices to restrict the competition to the compact set of those $u(\omega)$ which make the denominator equal to 1. Now if $\lambda = \lambda_{\max} \geq 1$ then $Q[u] = (1 - \lambda_{\max}) \sum_{r=0}^{k} d_r^{+} \sum_{\eta \in V_r} \left(\sum_{\omega \supset \eta} u(\omega) \right)^2 \leq 0$ and $Q[u]$ is not positive definite. On the other hand if $Q[u]$ can assume non-positive values, the quotient λ can become 1 or ≥ 1, whence certainly $\lambda_{\max} \geq 1$. Hence $\lambda_{\max} < 1$ implies, and is implied by, the positive definiteness of $Q[u]$ (notation $Q >> 0$).

We shall now indicate $k+1$ ways of choosing non-trivial $u_i(\omega)$, $i = 0, 1, \dots, k$ such that the corresponding inequalities $Q[u_i(\omega)] > 0$ are our necessary conditions for $Q >> 0$. These choices will afterwards be shown to certainly contain one for which $\lambda = \lambda_{\max}$ and for which then $\lambda_{\max} < 1$.

First choice is all $u(\omega) = u_0(\omega) = 1$. Then $\sum_{\omega \supset \eta \in V_r} u_0(\omega) = \binom{n-r}{k-r}$ and

$$0 < Q[u_0] = \sum_{r=0}^{k} d_r \binom{n}{r} \binom{n-r}{k-r}^2 = \binom{n}{k} \sum_{r=0}^{k} d_r \binom{k}{n} \binom{n-r}{k-r}$$

or

$$0 < \sum_{r=0}^{k} d_r \binom{k}{r} \binom{n-r}{k-r}$$

and

$$\lambda_0 = \sum_r d_r^{-} \binom{k}{r} \binom{n-r}{k-r} \Big/ \sum_r d_r^{+} \binom{k}{r} \binom{n-r}{k-r} < 1$$

The next choice, $u_1(\omega)$, is made as follows: we single out the numbers 1 and 2 and set

$$u_1(\omega) = 0, \quad \text{if} \quad \omega \supset (1 \cup 2) \quad \text{or if} \quad \omega \cap (1 \cup 2) = \varnothing.$$

We put

$$u_1(\omega) = \begin{cases} 1, & \text{if} \quad \omega \supset (2), \quad \omega \not\supset (1 \cup 2) \\ -1, & \text{if} \quad \omega \supset (1), \quad \omega \not\supset (1 \cup 2). \end{cases}$$

Then we easily verify

$$\sum_{\omega \supset \eta \in V_r} u_1(\omega) = \begin{cases} 0, & \text{if} \quad \eta \supset (1 \cup 2) \quad \text{or} \quad \eta \cap (1 \cup 2) = \varnothing \\ \binom{n-1-r}{k-r}, & \text{if} \quad \eta \supset (2), \quad \text{but} \quad \eta \not\supset (1 \cup 2) \\ -\binom{n-1-r}{k-r}, & \text{if} \quad \eta \supset (1), \quad \text{but} \quad \eta \not\supset (1 \cup 2). \end{cases}$$

Accordingly

$$0 < Q[u_1] = 2 \sum_{r=1}^{k} d_r \binom{n-2}{r-1} \binom{n-1-r}{k-r}^2 =$$

$$= 2 \binom{n-2}{k-1} \sum_{1}^{k} d_r \binom{k-1}{r-1} \binom{n-1-r}{k-r}$$

or

$$0 < \sum_{r=1}^{k} d_r \binom{k-1}{r-1} \binom{n-1-r}{k-r}$$

and

$$\lambda_1 = \sum_{r=1}^{k} d_r^- \binom{k-1}{r-1} \binom{n-1-r}{k-r} \Big/ \sum_{r=1}^{k} d^+ \binom{k-1}{r-1} \binom{n-1-r}{k-1} < 1.$$

We choose $u_i(\omega)$ for $1 \leq i \leq \frac{n}{2}$ as follows. We separate the numbers $1, \dots, 2i$ and define

$$u_i(\omega) = 0 \quad \text{if} \quad |\,\omega \cap \{1, 2, \dots, 2i\}\,| > i \quad \text{or} \quad |\,\omega \cap \{1, 2, \dots, 2i\}\,| < i.$$

Furthermore, $u_i(\omega) = 0$ even though $|\,\omega \cap \{1, 2, \dots, 2i\}\,| = i$ unless ω contains exactly one element, say a_1, of the pair $\{1, 2\}$, one, say a_2, of the pair $\{3, 4\}, \dots$, and one, say a_i, of the pair $\{2i-1, 2i\}$. In this case we put

$$u_1(\omega) = (-1)^{a_1 + a_2 + \dots a_i}.$$

Now let $\eta \in V_r$, $|\,\eta \cap \{1, 2, \dots, 2i\}\,| \lesseqgtr i$. Then $\sum_{\omega \supset \eta} u_i(\omega) = 0$, either trivially because η contains more than one element of any pair above, whence $u(\omega) = 0$ for $\omega \supset \eta$; or if η does not contain any element of one such pair $\{a, a'\}$ then each ω with $u_i(\omega) \neq 0$ must contain either a or a' and then the pairing of an $u_i(\omega)$ containing a with one containing a' will give zero sum. There remains the possibility that $r \geq i$ and η contains of each pair exactly one element, say $\eta \supset \{a_1, a_2, \dots, a_i\}$. Then

$$\sum_{\omega \supset \eta \in V_r} u_i(\omega) = (-1)^{a_1 + a_2 + \dots + a_i} \binom{n-i-r}{k-r}$$

and

$$Q[u_i] = \sum d_r \sum_{\eta \in V_r} \left(\sum_{\omega \supset \eta} u_i(\omega) \right)^2$$

$$= 2^i \sum_{r=i}^{k} d_r \binom{n-2i}{r-i} \binom{n-i-r}{k-r}^2$$

$$= 2^i \binom{n-2i}{k-i} \sum_{r=i}^{k} d_r \binom{k-i}{r-i} \binom{n-i-r}{k-r} > 0,$$

$$\lambda_i = \sum_{r \geq i}^{k} d_r^- \binom{k-i}{r-i} \binom{n-i-r}{k-r} \Big/ \sum_{r \geq i}^{k} d_r^+ \binom{k-i}{r-i} \binom{n-i-r}{k-r} < 1 \tag{2.7}$$

with $i = 0, 1, \ldots, k-1$. These inequalities, together with $d_k > 0$, are therefore consequences of $Q >> 0$.

We now turn the proof of sufficiency. For it we need the derivative of λ in (2.6) with respect to $u(\omega_0)$ where ω_0 is a fixed element of V_k. We have

$$\frac{\partial}{\partial u(\omega_0)} \sum_{\eta} \left(\sum_{\omega \supset \eta \in V_r} u(\omega) \right)^2 = 2 \sum_{\eta \subset \omega_0} \sum_{\omega \supset \eta \in V_r} u(\omega);$$

hence λ is stationary if and only if

$$\lambda \sum d_r^+ \sum_{\eta \subset \omega_0} \sum_{\omega \supset \eta \in V_r} u(\omega) = \sum d_r^- \sum_{\eta \subset \omega_0} \sum_{\omega \supset \eta \in V_r} u(\omega), \quad \omega_0 \in V_k. \tag{2.8}$$

Summing over all $\omega_0 \in V_k$ we obtain

$$\lambda \sum_r d_r^+ \sum_{\omega_0 \in V_k} \sum_{\eta \subset \omega_0} \sum_{\omega \supset \eta \in V_r} u(\omega) = \sum_r d_r^- \sum_{\omega_0 \in V_k} \sum_{\eta \subset \omega_0} \sum_{\omega \supset \eta \in V_r} u(\omega).$$

The factor of $\lambda\, d_r^+$ is the special case $\sigma = 0$ of

$$\sum_{\omega_0 \supset \delta_\sigma} \sum_{\eta \subset \omega_0} \sum_{\omega \supset \eta \in V_r} u(\omega) \equiv \sum f_r(\omega)\, u(\omega)$$

where σ is an integer $0 \leq \sigma \leq k$, ω_0, ω are in V_k, and δ_σ is an element of V_σ.

Let $|\omega \cap \delta_\sigma| = t$ so that $0 \leq t \leq \sigma$. In ω are contained $\binom{t}{\tau}\binom{k-t}{r-\tau}$ distinct $\eta \in V_r$ with $|\eta \cap \delta_\sigma| = \tau$ for $0 \leq \tau \leq t$. For each such η, the number of ω_0 containing η and also containing δ_σ is

$$\binom{n-r-\sigma+\tau}{k-r-\sigma+\tau}.$$

Hence

$$f_r(\omega) = \sum_{\tau \geq 0}^{t} \binom{t}{\tau} \binom{k-t}{r-\tau} \binom{n-r-\sigma+\tau}{k-r-\sigma+\tau}. \tag{2.9}$$

We now formulate the following hypothesis H_σ for $\sigma \geq 1$:

$$\sum_{\omega \supset \eta} u(\omega) = 0, \qquad \eta \in V_r, \qquad 0 \leq r \leq \sigma - 1, \tag{2.10}$$

but there is a $\delta \in V_\sigma$, called δ_σ, such that

$$\sum_{\omega \supset \delta_\sigma} u(\omega) \neq 0.$$

Note that (2.10) implies by summation

$$\Sigma f_s(\omega)\, u(\omega) = \sum_{\omega_0 \supset \delta_\sigma} \sum_{\eta \subset \omega_0} \sum_{\omega \supset \eta \in V_s} u(\omega) = 0, \quad \text{for} \quad 0 \leq s \leq \sigma - 1.$$

We have the

PROPOSITION: Under H_σ

$$\Sigma f_r(\omega)\, u(\omega) = \binom{k-\sigma}{r-\sigma} \binom{n-r-\sigma}{k-r} \sum_{\omega \supset \delta_\sigma} u(\omega).$$

LEMMA 2.1. *As $f_r(\omega)$ depends by* (2.9) *on* $|\omega \cap \delta_\sigma| = t$ *rather than on* ω, *we put* $f_r(\omega) = A_t$ *and have*

$$A_\sigma - \binom{\sigma}{1} A_{\sigma-1} + \binom{\sigma}{2} A_{\sigma-2} - + \dots + (-)^\sigma A_0 = \binom{k-\sigma}{r-\sigma} \binom{n-r-\sigma}{k-r}.$$

PROOF. Consider left hand, expressed with the aid of (2.9); for fixed τ, the coefficient of $\binom{n-r-\sigma+\tau}{k-r-\sigma+\tau}$ is

$$\binom{\sigma}{\tau}\binom{k-\sigma}{r-\tau} - \binom{\sigma}{1}\binom{\sigma-1}{\tau}\binom{k-\sigma+1}{r-\tau} + \binom{\sigma}{2}\binom{\sigma-2}{\tau}\binom{k-\sigma+2}{r-\tau} - + \dots$$

$$= \binom{\sigma}{\tau}\left[\binom{k-\sigma}{r-\tau} - \binom{\sigma-\tau}{1}\binom{k-\sigma+1}{r-\tau} + \binom{\sigma-\tau}{2}\binom{k-\sigma+2}{r-\tau} - + \dots\right] =$$

$$= (-)^{\sigma-\tau} \binom{\sigma}{\tau}\binom{k-\sigma}{r-\sigma},$$

in view of the binomial relation

$$\binom{a}{b} - \binom{c}{1}\binom{a+1}{b} + \binom{c}{2}\binom{a+2}{b} - + \dots = (-)^c \binom{a}{b-c}. \tag{2.11}$$

Consequently

$$A_\sigma - \binom{\sigma}{1} A_{\sigma-1} + \binom{\sigma}{2} A_{\sigma-2} - + \dots = \binom{k-\sigma}{r-\sigma} \sum_{\tau=0}^{\sigma} (-)^{\sigma-\tau} \binom{\sigma}{\tau} \binom{n-r-\sigma+\tau}{k-r-\sigma+\tau}.$$

Now we have the binomial relation

$$(2.12)\qquad \binom{a}{b}-\binom{c}{1}\binom{a+1}{b+1}+\binom{c}{2}\binom{a+2}{b+2}\ldots=(-)^c\binom{a}{b+c}.$$

Thus

$$A_\sigma-\binom{\sigma}{1}A_{\sigma-1}+\binom{\sigma}{2}A_{\sigma-2}+\ldots=\binom{k-\sigma}{r-\sigma}\binom{n-r-\sigma}{k-r}\text{ q.e.d.}$$

LEMMA 2.2. Under H_σ,

$$(2.13)\qquad \sum_\omega t^\varrho u(\omega)=0,\ \varrho=0,1,,\ldots,\sigma-1,\ t=|\omega\cap\delta_\sigma|,\quad\text{for}\quad \sigma\leq k\leq\frac{n}{2}.$$

PROOF. By (2.9), $f_s(\omega)$ is a polynomial in t of degree s; the highest coefficient, (i.e. that of t^s), is

$$\sum_{\tau\geq 0}^{s}(-)^{\tau+s}\frac{1}{\tau!\,(s-\tau)!}\binom{n-\sigma+\tau-s}{k-\sigma+\tau-s}=$$

$$=\frac{(-1)^s}{s!}\sum_{\tau\geq 0}^{s}\binom{s}{\tau}\binom{n-\sigma+\tau-s}{k-\sigma+\tau-s}(-)^\tau=\frac{1}{s!}\binom{n-\sigma-s}{k-\sigma}$$

in wiew of (2.12).

For $s<\sigma\leq k\leq\frac{n}{2}$, this coefficient is $\neq 0$. Moreover, (2.13) holds for $\varrho=0$, since $\sigma>0$. Suppose it to hold for $0,1,\ldots,\varrho_0<\sigma-1$, then it holds for ϱ_0+1. In fact from H_σ we know $\Sigma f_{\varrho_0+1}(\omega)\,u(\omega)=0$ and $f_{\varrho_0+1}(\omega)$ is a polynomial in t of degree ϱ_0+1 whose highest coefficient does not vanish. Thus subtracting a suitable linear combination of $\Sigma f_j(\omega)\,u(\omega)=0$, $j=$ $=0,1,\ldots,\varrho_0$, we obtain (2.13) for $\varrho=\varrho_0+1$, q.e.d.

LEMMA. 2.3. Under H_σ,

$$\sum_\omega t^\sigma u(\omega)=\sigma!\sum_{\omega\supset\delta_\sigma}u(\omega)\quad\text{if}\quad\sigma\leq k\leq\frac{n}{2}.$$

PROOF. From the previous Lemma we conclude

$$\frac{1}{\sigma!}\sum_\omega t^\sigma u(\omega)=\sum_\omega\binom{t}{\sigma}u(\omega),\quad t=|\omega\cap\delta_\sigma|.$$

But $\binom{t}{\sigma}=0$ for $t=0,1,\ldots,\sigma-1$

whence

$$\Sigma\, t^{\sigma}\, u(\omega) = \sigma\,! \sum_{|\,\omega \cap \delta_\sigma| = \sigma} u(\omega) = \sigma!\, \sum_{\omega \supset \delta_\sigma} u(\omega)\,.$$

PROOF OF PROPOSITION: Let $A(t)$ be a polynomial in t of degree σ, such that $A(\nu) = f_\nu(\omega)$ for $|\,\omega \cap \delta_\sigma| = \nu$, $\nu = 0, 1, \ldots, \sigma$.

Put $A(t) = \tilde{A}\binom{t}{\sigma} + p(t)$ where the degree of the polynomial $p(t)$ is $\leq \sigma - 1$. It follows from Lemma 2.3. that under H_σ

$$\sum_{\omega} u(\omega) f_r(\omega) = \tilde{A} \sum_{\omega \supset \delta_\sigma} u(\omega)\,.$$

But the σ-th difference

$$\mathrm{A}(\sigma) - \binom{\sigma}{1} A(\sigma - 1) + \binom{\sigma}{2} A(\sigma - 2) + \ldots = \tilde{A}$$

since this difference of a polynomial of degree $\leq \sigma - 1$ vanishes. Hence the Proposition follows from Lemma 2.1. Returning to the proof of sufficiency in Theorem 2.1, let us consider $\lambda = \lambda_{\max}$ and (2.8) with maximizing $u(\omega)$. Either $\sum\limits_{\omega} u(\omega) \neq 0$ or there is a $\sigma \leq k$ for with hypothesis H_σ applies; for certainly not all $u(\omega) = 0$, $\omega \in V_k$. In case $\Sigma\, u(\omega) \neq 0$ we sum (2.8) over all $\omega_0 \in V_k$ and obtain

$$\Sigma\, f_r(\omega)\, u(\omega) = \binom{k}{r}\binom{n-r}{k-r} \Sigma\, u(\omega)\,,$$

$$\left[\lambda_{\max}\, \Sigma\, d_r^{+} \binom{k}{r}\binom{n-r}{k-r} - \Sigma\, d_r^{-} \binom{k}{r}\binom{n-r}{k-r}\right] \Sigma\, u(\omega) = 0$$

and $\lambda_{\max} = \lambda_0 < 1$ by hypothesis (2.5) for $\sigma = 0$. But if $\Sigma\, u(\omega) = 0$, H_σ applies for some $\sigma > 0$. Summing (2.8) over all ω_0 containing δ_σ we obtain by the Proposition

$$\left(\lambda_{\max}\, \Sigma\, d_r^{+} \binom{k-\sigma}{r-\sigma}\binom{n-r-\sigma}{k-r} - \Sigma\, d_r^{-} \binom{k-\sigma}{r-\sigma}\binom{n-r-\sigma}{k-r}\right) \sum_{\omega \supset \delta_\sigma} u(\omega) = 0$$

and $\sum\limits_{\omega \supset \delta_\sigma} u(\omega) \neq 0$ yields that $\lambda_{\max} = \lambda_\sigma < 1$ by (2.5). Hence $Q >> 0$. The theorem is proved.

3. If the inequality $n \geq 2k$ of Theorem 2.1 does not apply we can nevertheless obtain a corresponding theorem by the following simple device.

Let $n \leq 2k \leq 2n$. Denote by ω' the complement of ω in $\{1, 2, \ldots, n\}$ so that $\omega' \in V_{n-k}$. If we put

$$u(\omega) \equiv u'(\omega') \tag{3.1}$$

then a quadratic form in $u(\omega)$ becomes a quadratic form in $u'(\omega')$; invariance under permutation of $\{1, 2, \ldots, n\}$ of one form implies that of the other. If we put $k' = n - k$, we have $2k' \leq n$ so that we can apply to $Q'[u']$, defined by $Q'[u'] \equiv Q[u]$, the theorem 2.1. This requires, of course, evaluation of the coefficients d'_r, $0 \leq r \leq k'$, of Q'. We can also obtain a more direct version of Theorem if we recall the definition of $u_i(\omega)$. Observe that the definition of $u_i(\omega)$ is such that

$$u_i(\omega) = u'_i(\omega')(-1)^i$$

We have therefore the

THEOREM 3.1. A quadratic form $Q[u]$ of real $u(\omega)$, $\omega \in V_k$, $1 \leq k \leq n$ is positive definite if and only if

$$Q[u_i(\omega)] > 0 \tag{3.2}$$

for $i = 0, 1, \ldots, k$ when $2k \leq n$; and $Q >> 0$ if and only if (3.2) holds for $i = 0, 1, \ldots, n-k$ when $2k \geq n$.

The conditions of positive definiteness of $Q[u]$, if $2k \geq n$, may be made explicit in an analogous manner as for $2k \leq n$. With

$$Q[u] = \sum_{r=0}^{k} c_r \sum_{|\omega_j \cap \omega_j| = r} u(\omega_i)\, u(\omega_j), \quad 2k \geq n \tag{3.3}$$

we have necessarily $c_r = 0$ for $r < 2k - n$. With the aid of (3.1) we find

$$Q[u] = Q'[u'] = \sum_{r'=0}^{n-k} c'_{r'} \sum_{|\omega'_i \cap \omega'_j| = r'} u'(\omega'_i)\, u'(\omega'_j).$$

Between c'_r and c_r we have the relation

$$c'_{r'} = c_r \quad \text{for} \quad r - r' = 2k - n.$$

For this relation holds certainly if $r = k$, $r' = n - k$; and a reduction of r by one unit entails a like reduction of r'. By (2.4) we can construct the quantities $d'_{r'}$ of $Q'[u']$, replacing k by $k' = n - k$; i.e. $r = r' + 2k - n$ and

$$d'_{r'} = c_r - \binom{r'}{1} c_{r-1} + \binom{r'}{2} c_{r-2} - + \dots + (-)^{r'} c_{r-r'}, \quad 0 \leq r' \leq n - k \tag{3.4}$$

THEOREM 3.2. For $2k \geq n$, $Q[u]$ of (3.3) is positive definite if and only if

$$\sum_{r'=\sigma}^{k'} d'_{r'} \binom{k'-\sigma}{r'-\sigma} \binom{n-r'-\sigma}{k'-r'} > 0, \quad \sigma = 0, 1, \dots, k' = n - k$$

where $d'_{r'}$ and c_r are related by (3.4).

4. If n is allowed to be infinite, i.e. if $\omega \in V_k^\infty$ is a k-combination of arbitrary k distinct positive integers, the quadratic form $Q[u]$ of (2.1) will certainly converge if $\{u(\omega)\}$ is an element of the Banach space of real $u(\omega)$ with norm

$$\| u \| = \sum_\omega | u(\omega) | . \tag{4.1}$$

For $0 \leq r \leq k$ we have

$$\binom{k}{r} \| u \| = \sum_{\eta \in V_r} \sum_{\omega \supset \eta} | u(\omega) | \tag{4.2}$$

whence

$$\sum_{\eta \in V_r} \sum_{\omega \supset \eta} u^2(\omega) \leq \sum_{\eta \in V_r} \Big(\sum_{\omega \supset \eta} | u(\omega) | \Big)^2 < \infty$$

the last inequality being a consequence of (4.2). It is thus possible to reduce $Q[u]$ to the form (2.2) with d_k and c_k related through (2.3). Clearly the conditions (2.5) are necessary conditions of $Q >> 0$ for every $n \geq 2k$ since $V_n^k \subset V_k^\infty$ and an obvious specialization of $u(\omega)$ reduces to the finite case. Now the σ-th inequality in (2.5) is a polynomial in n of degree $k - \sigma$ and highest coefficient d_σ. Dividing by $n^{k-\sigma}$ and letting $n \to \infty$ we obtain from (2.5)

$$d_\sigma \geq 0, \quad \sigma = 0, 1, \dots, k$$

as necessary conditions for the positive definiteness of $Q[u]$.

Taking n finite and $\geq 2k$ we find as before that $d_k > 0$ is necessary for $Q >> 0$.

THEOREM 4.1. The quadratic form $Q[u]$ of (2.1) on the Banach space of norm (4.1) is positive definite if and only if

$$d_k > 0, \ d_\sigma \geq 0, \ \sigma = 0, 1, \dots, k-1 \tag{4.3}$$

with d_r as in (2.4).

PROOF. Necessity having been proved, we must only prove sufficiency. By (2.2), (4.3), $Q[u] \geq d_k \sum_\omega u^2(\omega)$; if $\|u\| = \Sigma |u(\omega)| > 0$, then $\Sigma u^2(\omega) > 0$, $Q[u] > 0$, q.e.d.

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